ECONOMIC GROWTH AND SOCIAL CAPITAL

By

SUSUMU NISHIBE *and* YUKIHIKO KIYOKAWA*

Introduction

Recently the importance of social capital has been recognized more and more in its various economic aspects. We shall analyse the role of social capital good in economic growth from the viewpoint of heterogeneity of goods.

In our economy there exist two capital goods, being heterogeneous with each other. Their functions in a production process and markets are crucially different from each other. The one may be called the private capital good, and the other the social capital good. The latter has the following characteristics, as seen in Section II; It is made through the transformation function from the output produced by the production function. It has to be financed by the taxation of government, since it has no market for selling and buying itself.

However, we must begin with clarifying the concept of heterogeneity. In relation to the so-called "Hahn problem" [2], the question of heterogeneous capital goods has been tackled by some economists in the context of economic growth theory [3] [6] [7]. And the various definitions on heterogeneity may be found in them. In Shell-Stiglitz's paper [7], for example, two capital goods are perfectly substitutable in the flow dimension, but not in the stock dimension. Their concept of heterogeneity is closely related to the non-shiftability of capital goods.

This paper proposes its own definition on heterogeneity. That is, heterogeneity is measured by the malleability between goods. If two goods are heterogeneous, i.e. different in shape, quality, function and so on, we must pay some cost to transform from one to another. It implies that perfect malleability indicates homogeneity and perfect substitutability. It is assumed for simplicity that the cost for transformation between heterogeneous goods will evaporate from the good in a transformation process.

Section I confirms characteristics and assumptions of the model. We analyse in Section II the influence of existence of the social capital good on the optimal tax rates as the quasi-golden rule under the specification of Cobb-Douglas production function. It shall be shown that the optimal tax rates and the equilibrium capital-labor ratios are functions of the degree of heterogeneity. The possibility of existence and stability of the long-run steady growth path shall be investigated by the optimal growth model.

* The authors are Research Fellow of the Japan Economic Research Center and Assistant (*Joshu*) at the Hitotsubashi University.

I. *The One-Sector Two-Capital Model*

In the analysis that follows this paper shall work with a one-sector economy having two kinds of capital, private and social capital goods. Private capital formation comes from the private savings in some specified forms of saving functions, through private decisions, i.e. profit-maximizing behaviors, while the size of social capital invested is determined by the governmental decision through taxation, which also influences private decisions.[1]

Let K_1 be the private capital good, K_2 be the social capital good and L be the labor existing in an economy. Then there exists a well-behaved, smooth production function,

$$Y=F(K_1, K_2, L), \tag{1}$$

where F is strictly concave, twice differentiable, and it has everywhere positive first-derivatives. It is assumed that F is homogeneous of degree one, hence,

$$\frac{Y}{L}=F\left(\frac{K_1}{L}, \frac{K_2}{L}, 1\right)\equiv f(k_1, k_2), \tag{2}$$

where $k_1=\frac{K_1}{L}$, $k_2=\frac{K_2}{L}$.

Furthermore, it is assumed for simplicity that both capital goods do not depreciate, and that labor force grows at a constant rate n,

$$\dot{L}=nL, \tag{3}$$

where $\dot{L}$ means dL/dt. This notation applies correspondingly to such cases hereafter.

By the assumption of well-behavedness, the social capital K_2 is indispensable for this economy. Since the size of social capital is determined by the government, entrepreneurs try to maximize their profit under given K_2. They can, therefore, determine sizes of their firms, because F is decreasing returns to scale with respect to K_1 and L.

We shall suppose that there is no market for selling and renting a social capital good, and that entrepreneurs may use it without cost in production, although they have to pay taxes to the government.

Then it may be imagined that the social capital good could be regarded as the public goods, like roads, bridges, weather-forecasting system, ports, etc., or that they could be expenses to remove disgoods resulted from production activities or to maintain a certain quality level of production factors.

Having merely one production function in an economy, transformation must occur from the output Y to other heterogeneous goods indispensable for an economy through a transformation function. The transformation function is expressed in general as follows:

$$H(Y;\ C,\ \dot{K}_1,\ \dot{K}_2)=0, \tag{4}$$

or in a explicit function form,

$$\dot{K}_2=G(Y;\ C,\ \dot{K}_1). \tag{4'}$$

We may assume without loss of generality the output Y is homogeneous to the consumption

[1] Similar models are analysed in Arrow and Kurz [1] from the viewpoint of "controllability".

good C.

Heterogeneity between goods is defined as the malleability with cost. The degree of heterogeneity is measured by the cost for transformation. A part of this transformation cost could be regarded as the organizational cost in production. The cost being zero, both goods are homogeneous and perfectly substitutable.[2] The dotted-line pyramid in Fig. 1 shows the case where all goods are homogeneous with each other. When all of them are heterogeneous with each other, the transformation possibility frontier lies, as indicated in real lines, strictly inside the dotted-line pyramid, due to the cost for transformation.

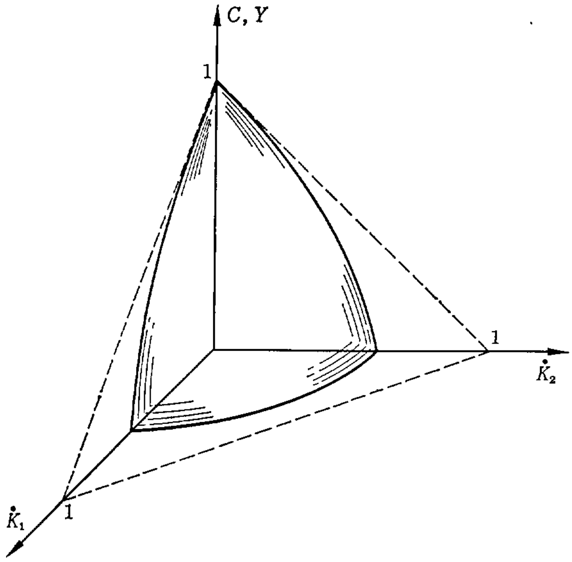

FIG. 1

It is assumed that the transformation function G is homogeneous of degree one, and the marginal rate of transformation is diminishing, in other words the transformation cost is increasing.

In Section II, the private capital good K_1 is considered to be homogeneous to C (see Fig. 2) in order to concentrate attention on the effects of optimal tax rates through the heterogeneity of K_2, although the assumption is fairly strong. It will be discussed more generally in Section III under an optimal growth model.

[2] "Homogeneity", "perfect substitutability", "perfect malleability" and "perfect shiftability" are closely connected with each other, and sometimes mixed up.

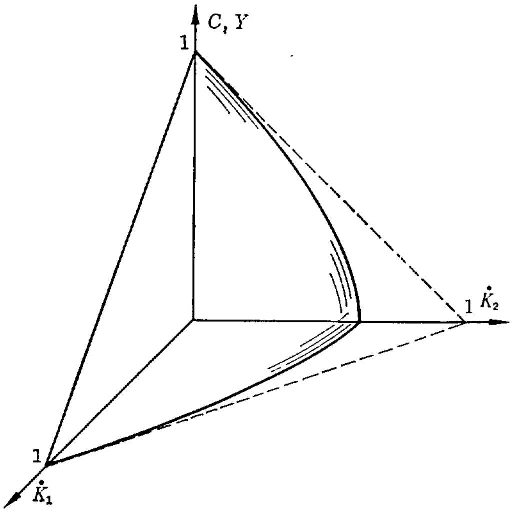

FIG. 2

II. *Steady Growth and Optimal Tax Rates*

1. Existence of a Unique-stable Steady Growth Path

As shown above, a social capital good K_2 is indispensable for production. Entrepreneurs may use it without paying its cost. However, there does not exist a market for it by its proper characteristics. The social capital must be financed through taxation by the government. This section analyses the existence and stability of a steady growth path and the relationship between consumption-maximizing tax rates and the degree of heterogeneity in K_2.

To make this analysis as unambiguous as possible, the Cobb-Douglas production function shall be used throughout this section.
Thus the production function is,

$$Y=F(K_1,\ K_2,\ L)=K_1^{\alpha}K_2^{\beta}L^{\gamma}, \tag{1$'$}$$

where $\alpha>0$, $\beta>0$, $\gamma=1-\alpha-\beta>0$.
By the linear-homogeneity,

$$\frac{Y}{L}=f(k_1,\ k_2)=k_1^{\alpha}k_2^{\beta}. \tag{2$'$}$$

Since the consumption good C and the private capital good K_1 are assumed to be homogeneous to the output Y,

$$Y=C+\dot{K}_1+Z, \tag{5}$$

where Z is the amount of $\dot{K}_2$ in terms of Y, namely, the amount before transformation. $\dot{K}_2$

cannot be directly added up in (5) because of heterogeneity.

The transformation function is (see Fig. 3),

$$\dot{K}_2=G(Z,\ Y),$$

where the role of Y in G may be interpreted as a kind of external effect. As G is homogeneous of degree one,

$$\frac{\dot{K}_2}{Y}=G\left(\frac{Z}{Y},\ 1\right)\equiv\varphi(x), \tag{6}$$

where $x=\dfrac{Z}{Y}$, $1>x\geqq 0$.

The diminishing marginal rate of transformation implies that

$$\varphi'(x)>0 \text{ and } \varphi''(x)<0. \tag{7}$$

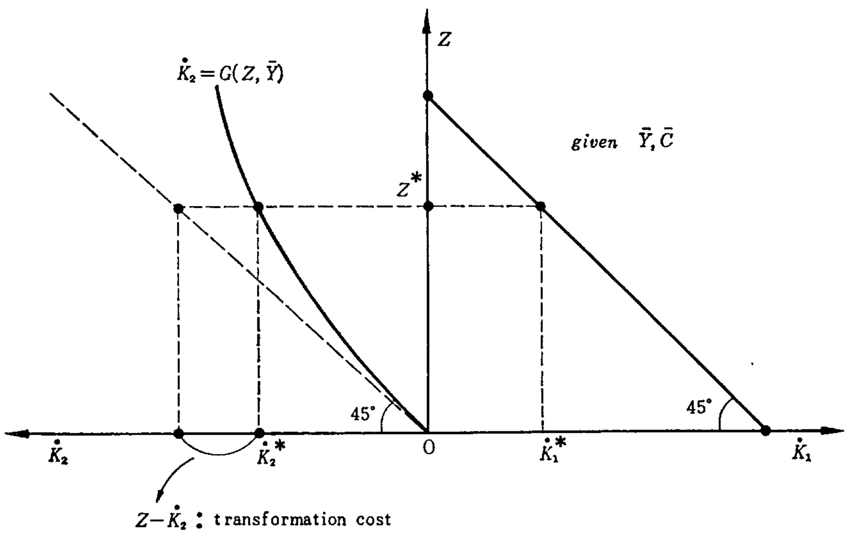

FIG. 3

Since entrepreneurs can use the social capital freely, profits are,

$$\Pi=Y-rK_1-wL. \tag{8}$$

where r is the real rental-rate with respect to K_1 and w is the real wage-rate. Entrepreneurs try to maximize profits, its maximizing conditions $\left(\dfrac{\partial\Pi}{\partial K_1}=0 \text{ and } \dfrac{\partial\Pi}{\partial L}=0\right)$ require,

$$r=f_1=\alpha k_1^{\alpha-1}k_2^{\beta}, \tag{9}$$

and

$$w=f-k_1f_1-k_2f_2=(1-\alpha-\beta)k_1^{\alpha}k_2^{\beta}, \tag{10}$$

where $f_1=\frac{\partial f}{\partial k_1}$ and $f_2=\frac{\partial f}{\partial k_2}$.[3]

The government collects taxes (T) from profits (Π) and factor incomes of capitalists and wage-workers (rK_1, wL),

$$T=\tau_c\Pi+\tau_s rK_1+\tau_w wL, \tag{11}$$

where τ_c, τ_s, τ_w are tax rates for profits, quasi-rentals and wage-bills respectively. It should be noted that the marginal conditions (9) (10) just derived are not modified at all after taxation upon profits.

It is assumed for simplicity that capitalists and entrepreneurs do not consume, and wage-workers do not save. Thus

$$S=(1-\tau_s)rK_1+(1-\tau_c)\Pi, \tag{12}$$

and

$$C=(1-\tau_w)wL. \tag{13}$$

In equilibrium, we must have

$$Z=T, \tag{14}$$

and

$$\dot{K}_1=S. \tag{15}$$

By the aid of (9) and (10),

$$\dot{K}_1=S=L\{f(k_1, k_2)-w-\tau_s k_1 f_1-\tau_c f_2 k_2\}, \tag{16}$$

so we obtain,

$$\begin{aligned}\frac{\dot{k}_1}{k_1}&=\frac{\dot{K}_1}{K_1}-\frac{\dot{L}}{L}=\frac{1}{k_1}\{f(k_1, k_2)-w-\tau_s k_1 f_1-\tau_c f_2 k_2\}-n,\\ &=k_1^{\alpha-1}k_2^{\beta}\{\alpha+\beta-\alpha\tau_s-\beta\tau_c\}-n.\end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned}\dot{K}_2&=Y\varphi\left(\frac{Z}{Y}\right)=Y\varphi\left(\frac{T}{Y}\right),\\ &=Lf(k_1, k_2)\varphi\left(\frac{\{\tau_w w+\tau_s f_1 k_1+\tau_c f_2 k_2\}}{f(k_1, k_2)}\right);\end{aligned}$$

hence,

$$\begin{aligned}\frac{\dot{k}_2}{k_2}&=\frac{f(k_1, k_2)}{k_2}\varphi\left(\frac{\{\tau_w w+\tau_s f_1 k_1+\tau_c f_2 k_2\}}{f(k_1, k_2)}\right)-n,\\ &=k_1^{\alpha}k_2^{\beta-1}\varphi(\alpha\tau_s+\beta\tau_c+(1-\alpha-\beta)\tau_w)-n.\end{aligned} \tag{18}$$

The long-run equilibrium is attained when $\dot{k}_1=\dot{k}_2=0$. Then, we obtain, from (17) and (18) respectively,

$$k_2=\left[\frac{n}{\alpha+\beta-\alpha\tau_s-\beta\tau_c}\right]^{\frac{1}{\beta}}\cdot k_1^{\frac{1-\alpha}{\beta}}, \tag{19}$$

and

$$k_2=\left[\frac{n}{\varphi(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)}\right]^{-\frac{1}{1-\beta}}\cdot k_1^{\frac{\alpha}{1-\beta}}. \tag{20}$$

Shapes of these curves guarantee the unique-stable steady growth path,

[3] Sizes of firms can be determined by the decreasing returns to scale. Their production functions are, however, linearly homogeneous like the transformation functions. Thus the consistent aggregation to the macro function is possible.

$$\left.\frac{dk_2}{dk_1}\right|_{\dot{k}_1=0}=\frac{1-\alpha}{\beta}\frac{k_2}{k_1}; \quad \frac{1-\alpha}{\beta}>1, \tag{21}$$

and

$$\left.\frac{dk_2}{dk_1}\right|_{\dot{k}_2=0}=\frac{\alpha}{1-\beta}\frac{k_2}{k_1}; \quad 1>\frac{\alpha}{1-\beta}>0. \tag{22}$$

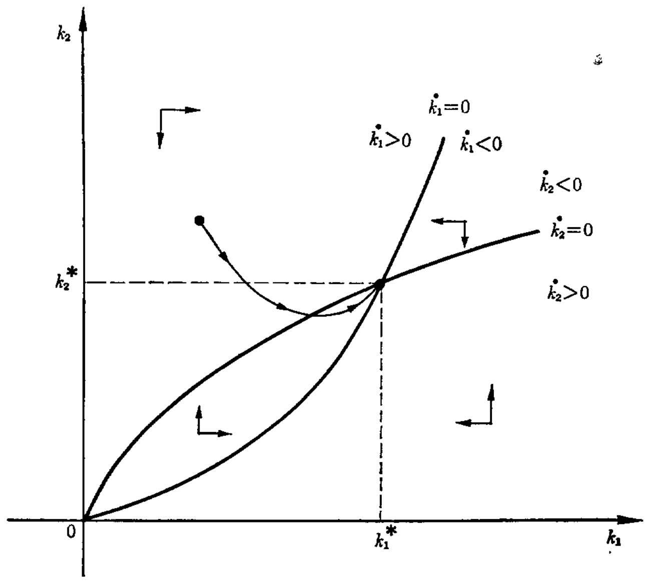

FIG. 4

Arrows in Fig. 4 show the steady growth path to be stable. Thus we may solve (19) and (20) with respect to k_1 and k_2,

$$k_1{}^*=n^{-\frac{1}{\gamma}}\varphi(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)^{\frac{\beta}{\gamma}}\cdot[\alpha+\beta-\alpha\tau_s-\beta\tau_c]^{\frac{1-\beta}{\gamma}}, \tag{23}$$

and

$$k_2{}^*=n^{-\frac{1}{\gamma}}\varphi(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)^{\frac{1-\alpha}{\gamma}}\cdot[\alpha+\beta-\alpha\tau_s-\beta\tau_c]^{\frac{\alpha}{\gamma}}. \tag{24}$$

2. Determination of Optimal Tax Rates

So far the existence of unique-stable steady growth path for given tax rates (τ_c, τ_s, τ_w) has been proved. There exists a corresponding steady growth path to each set of tax rates (τ_c, τ_s, τ_w). Hence, the tax rates which maximize per capita consumption on the steady growth paths may be chosen and determined.

Per capita consumption on a steady growth path is,

$$\begin{aligned} c^* &= (1-\tau_w)w^*, \\ &= (1-\tau_w)(1-\alpha-\beta)k^*{}_1{}^{\alpha}k_2{}^{*\beta}, \end{aligned}$$

where an asterisk (*) means equilibrium values. Substitute (23) and (24) into it, we derive,

$$c^*=\gamma n^{-\frac{\alpha+\beta}{\gamma}}(1-\tau_w)\varphi(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)^{\frac{\beta}{\gamma}}[\alpha+\beta-(\alpha\tau_s+\beta\tau_c)]^{\frac{\alpha}{\gamma}}, \tag{25}$$

which is a function of τ_w, τ_c and τ_s.
Maximize c^* with respect to τ_w, τ_c and τ_s,
$\frac{\partial c^*}{\partial \tau_w}=0$ gives

$$\frac{\varphi'(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)}{\varphi(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)}=\frac{1}{\beta(1-\tau_w)}. \tag{26}$$

And both $\frac{\partial c^*}{\partial \tau_c}=0$ and $\frac{\partial c^*}{\partial \tau_s}=0$ provide the same condition:

$$\frac{\varphi'(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)}{\varphi(\alpha\tau_s+\beta\tau_c+\gamma\tau_w)}=\frac{\alpha}{\beta}\cdot\frac{1}{[\alpha+\beta-(\alpha\tau_s+\beta\tau_c)]}. \tag{27}$$

Define $\tau_p=\alpha\tau_s+\beta\tau_c$, then (26) and (27) may be solved with respect to τ_w and τ_p.

Next, the concept of elasticity is introduced to make our analysis more clear-cut. The elasticity of transformation (σ) is defined as follows,

$$\sigma=\frac{x}{\varphi(x)}\cdot\frac{d\varphi(x)}{dx}=[\tau_p+\gamma\tau_w]\frac{\varphi'(\tau_p+\gamma\tau_w)}{\varphi(\tau_p+\gamma\tau_w)}, \tag{28}$$

where $1\geqq\sigma>0$, as indicated in Fig. 3.
If $\dot{K}_2$ is homogeneous to $\dot{K}_1$, then no transformation cost is needed: $Z=\dot{K}_2$, so $\sigma=1$. Using the concept of elasticity, solve (26) and (27) with respect to τ_w and $\tau_p=\alpha\tau_s+\beta\tau_c$.

We can derive the following relations,

$$\tau_w^*=\frac{-\beta(1-\sigma)}{1-\beta(1-\sigma)} \tag{29}$$

$$\tau_p^*=\alpha\tau_w^*+\beta\tau_c^*=\frac{\sigma\beta(\alpha+\beta)+\beta\gamma}{1-\beta(1-\sigma)}, \tag{30}$$

and

$$\tau_p^*=\alpha\tau_w^*+\beta. \tag{31}$$

These relationships show that the optimal tax rates τ_w^*, τ_p^* are a function of the elasticity of transformation. Thus some interesting properties may be deduced. First, the optimal tax rates of capitalists and entrepreneurs can not be determined, but the linear combination τ_p^* of those alone can be determined. Its components τ_c^*, τ_s^* are left to the discretion of government. Secondly, the optimal tax rate of wage-workers is always non-positive. That is to say, they may receive subsidies, as long as K_2 is not a homogeneous good to K_1. Note that except for the homogeneous case, the elasticity of transformation (σ) is in the interval (0,1) by the diminishing marginal rate of transformation (7).

$$\tau_w^*=-\frac{\beta(1-\sigma)}{1-\beta(1-\sigma)}\leqq 0 \qquad \text{for } \forall\sigma, \tag{32}$$

since $0<\sigma\leqq 1$. Thirdly, as an extreme case, when and only when K_2 is homogenous to K_1, the optimal tax rate of wage-workers is zero ($\tau_w^*=0$), and that of non-wage-workers is equal to the elasticity of output with respect to K_2 ($\tau_p^*=\beta$). Assume furthermore all savings by capitalists go to private capital formation ($\tau_s^*=0$), then all profits should be collected as profit tax ($\tau_c^*=1$). This is the standard golden rule, namely,

$$\frac{\dot{K}_1}{Y}=\alpha \quad \text{and} \quad \frac{\dot{K}_2}{Y}=\beta.$$

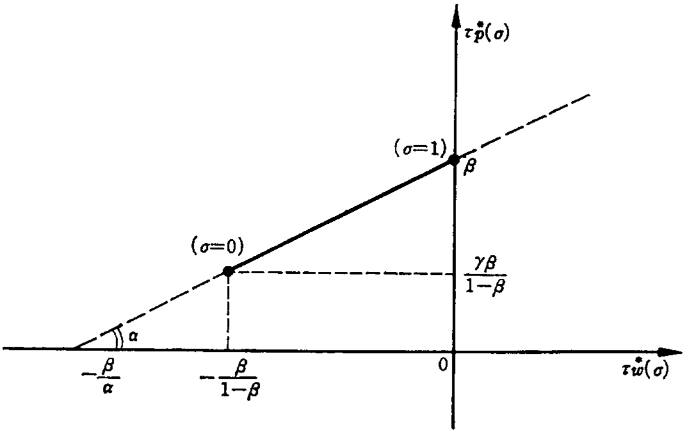

FIG. 5

Fourthly, the stronger the degree of heterogeneity is, the lower the optimal tax rates, i.e.,

$$\frac{d\tau_w^*}{d\sigma}>0, \quad \frac{d\tau_p^*}{d\sigma}>0. \tag{33}$$

Finally, we shall investigate the effects of heterogeneity on equilibrium capital-labor ratios through the optimal tax rates.

$$k_1^*=n^{-\frac{1}{\gamma}}\varphi(\tau_p^*+\gamma\tau_w^*)^{\frac{\beta}{\gamma}}[\alpha+\beta-\tau_p^*]^{\frac{1-\beta}{\gamma}},$$

$$=n^{-\frac{1}{\gamma}}\varphi\left(\frac{\beta\sigma}{1-\beta(1-\sigma)}\right)^{\frac{\beta}{\gamma}}\left[\frac{\alpha}{1-\beta(1-\sigma)}\right]^{\frac{1-\beta}{\gamma}},$$

similarly,

$$k_2^*\doteq n^{-\frac{1}{\gamma}}\varphi\left(\frac{\beta\sigma}{1-\beta(1-\sigma)}\right)^{\frac{1-\alpha}{\gamma}}\left[\frac{\alpha}{1-\beta(1-\sigma)}\right]^{\frac{\alpha}{\gamma}}.$$

Thus the equilibrium capital-labor ratios are a function of the elasticity of transformation. Hence the effects of the elasticity of transformation on the equilibrium capital-labor ratios are,

$$\frac{dk_1^*}{d\sigma}=0, \quad \frac{dk_2^*}{d\sigma}>0. \tag{34}$$

The equilibrium capital-labor ratio of K_1 is independent of heterogeneity, and that of K_2 decreases as the degree of heterogeneity increases. These results may easily be conjectured by decomposing into two components, together with (33).

Differentiate (23) and (24) with respect to τ_p^* and τ_w^*. Then,

$$\frac{\partial k_1{}^*}{\partial \tau_p{}^*}<0^{4} \quad \text{and} \quad \frac{\partial k_2{}^*}{\partial \tau_p{}^*}>0. \tag{35}$$

Similarly,

$$\frac{\partial k_1{}^*}{\partial \tau_w{}^*}>0 \quad \text{and} \quad \frac{\partial k_2{}^*}{\partial \tau_w{}^*}>0. \tag{36}$$

(see Figs. 6 and 7)

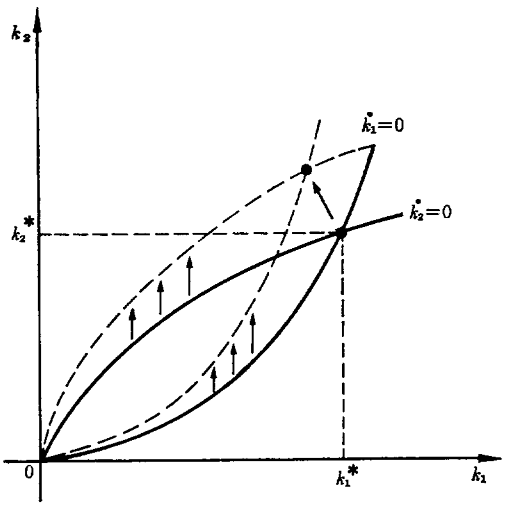

FIG. 6. EFFECTS OF CHANGES IN $\tau_p{}^*$

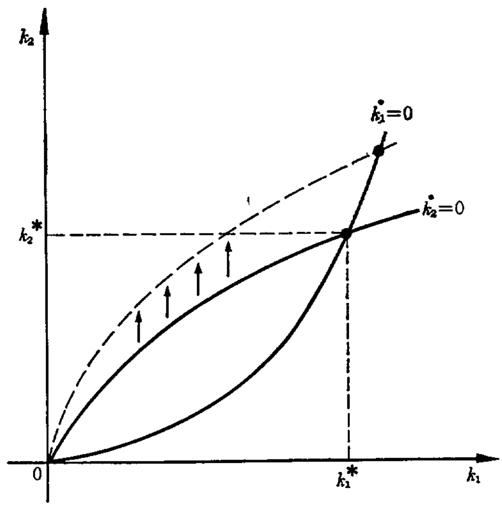

FIG. 7. EFFECTS OF CHANGES IN $\tau_w{}^*$

Combining (35) and (36) with (33), e.g. $\dfrac{dk_1{}^*}{d\sigma}=\dfrac{\partial k_1{}^*}{\partial \tau_p{}^*}\dfrac{d\tau_p{}^*}{d\sigma}$, we may eventually reach the result (34).

3. Digression: Proportional Saving Function

So far we have assumed the extreme case of classical saving function, namely wage-workers do not save except for their tax payments, and non-wage-workers do not consume at all. To analyse to what extent our conclusions just obtained depend upon the above specifications, the proportional saving function shall be introduced.

Let

$$Y_D=(1-\tau_f)\cdot(wL+rK_1), \tag{37}$$

[4] More exactly, $\dfrac{\partial k_1{}^*}{\partial \tau_p{}^*}$ means $\left.\dfrac{\partial k_1{}^*}{\partial \tau_p}\right|_{\substack{\tau_w=\tau_w^* \\ \tau_p=\tau_p^*}}$.

$\dfrac{\partial k^*}{\partial \tau_p}\gtreqless 0$ according as $\sigma \lesseqgtr \dfrac{1-\beta}{\beta}\cdot\dfrac{\tau_p+\gamma\tau_w}{\alpha+\beta-\tau_p}$.

Thus

$$\frac{1-\beta}{\beta}\cdot\frac{\tau_p{}^*+\gamma\tau_w{}^*}{\alpha+\beta-\tau_p{}^*}=\frac{1-\beta}{\beta}\cdot\frac{\beta}{\alpha}\sigma>\sigma, \qquad \therefore\ \frac{\partial k_1{}^*}{\partial \tau_p{}^*}<0.$$

$$S=sY_D+(1-\tau_c)\Pi, \tag{38}$$
$$C=(1-s)Y_D, \tag{39}$$

and

$$T=\tau_f(wL+rK_1)+\tau_c\Pi,^5 \tag{40}$$

where Y is the disposable factor income, it implies that the tax rates for wage-workers and for capitalists are common (τ_f). The saving ratio is s.

Equilibrium requires,

$$\dot{K}_1=S \quad \text{and} \quad Z=T.$$

Thus we gain, as in the case of classical saving function,

$$\frac{\dot{k}_1}{k_1}=\{s(1-\tau_f)(1-\beta)+\beta(1-\tau_c)\}k_1^{\alpha-1}k_2^{\beta}-n \tag{41}$$

and

$$\frac{\dot{k}_2}{k_2}=\varphi(\tau_c\beta+(1-\beta)\tau_f)k_1^{\alpha}k_2^{\beta-1}-n. \tag{42}$$

The long-run equilibrium is realized when $\dot{k}_1=\dot{k}_2=0$. Then (41) and (42) provide, respectively,

$$k_2=\left[\frac{n}{s(1-\tau_f)(1-\beta)+\beta(1-\tau_c)}\right]^{\frac{1}{\beta}}k_1^{\frac{1-\alpha}{\beta}}, \tag{43}$$

and

$$k_2=\left[\frac{n}{\varphi(\tau_c\beta+(1-\beta)\tau_f}\right]^{-\frac{1}{1-\beta}}k_1^{\frac{\alpha}{1-\beta}}. \tag{44}$$

They ensure the unique-stable long-run equilibrium, in which the capital-labor ratios are,

$$k_1{}^*=n^{-\frac{1}{\gamma}}\varphi(\tau_c\beta+(1-\beta)\tau_f)^{\frac{\beta}{\gamma}}[s(1-\tau_f)(1-\beta)+\beta(1-\tau_c)]^{\frac{1-\beta}{\gamma}}, \tag{45}$$

and

$$k_2{}^*=n^{-\frac{1}{\gamma}}\varphi(\tau_c\beta+(1-\beta)\tau_f)^{\frac{1-\alpha}{\gamma}}[s(1-\tau_f)(1-\beta)+\beta(1-\tau_c)]^{\frac{\alpha}{\gamma}}. \tag{46}$$

Hence, per capita consumption on a steady growth path is

$$\begin{aligned} c^* &= (1-s)(1-\tau_f)(1-\beta)k_1{}^{*\alpha}k_2{}^{*\beta} \\ &= \gamma n^{-\frac{\alpha+\beta}{\gamma}}(1-\beta)(1-s)(1-\tau_f)\varphi(\tau_c\beta+(1-\beta)\tau_f)^{\frac{\beta}{\gamma}}\cdot[s(1-\tau_f)(1-\beta)+\beta(1-\tau_c)]^{\frac{\alpha}{\gamma}}. \end{aligned} \tag{47}$$

Differentiating (47) with respect to τ_f and τ_c, the optimal tax rates are obtained,

$$\tau_f{}^*=1-\frac{1-\alpha-\beta}{(1-s)(1-\beta)[1-\beta(1-\sigma)]}, \tag{48}$$

and

$$\tau_c{}^*=1-\frac{\alpha-s(1-\beta)}{\beta(1-s)[1-\beta(1-\sigma)]}. \tag{49}$$

These results reveal the fact that some conclusions obtained in II-1 and II-2 depend partly upon the assumption of saving functions.

---

[5] We may consider also the saving tax, then the taxation policy is not controllable. The sales tax, instead of the profit tax, provides almost the same results.

III. *An Optimal Growth Model*

In Section II we discussed a normative property of the quasi-golden rule under a positive model, where, therefore, only steady growth paths were compared. We shall investigate now the existence of a long-run equilibrium and the conditions satisfied to reach it by a controllable optimal growth model. Notations are common to those in the previous sections, unless otherwise notated.

The production function,

$$Y=F(K_1, K_2, L), \tag{1}$$

satisfies the same properties as those mentioned in Section I. It is assumed the proportional saving function,

$$S=sF(K_1, K_2, L), \tag{50}$$

and

$$Y=C+S, \tag{51}$$

where S is considered as the homogeneous good to the output Y and the consumption good C.

The transformation function is more general than used in Section II;

$$\dot{K}_2=G(S, \dot{K}_1), \tag{52}$$

namely $\dot{K}_1$ and $\dot{K}_2$ are simultaneously transformed from S, all of which are heterogeneous with each other. The function G is assumed to be linear-homogeneous, so

$$\frac{\dot{K}_2}{S}=G\left(1, \frac{\dot{K}_1}{S}\right)\equiv g(x), \tag{53}$$

where $x=\dfrac{\dot{K}_1}{S}$, $0\leqq x<1$.

Note that $\dfrac{\dot{K}_1}{S}$, $\dfrac{\dot{K}_2}{S}$ and $\dfrac{\dot{K}_1}{S}+\dfrac{\dot{K}_2}{S}$ are strictly smaller than 1, because of the transformation cost.

Then our problem is:

$$\text{to maximize } \int_0^\infty u\left(\frac{C}{L}\right)e^{-\delta t}dt,$$

$$\text{subject to } \quad Y=F(K_1, K_2, L),$$

$$\dot{K}_2=G(S, \dot{K}_1),$$

$$\text{and} \qquad S=sF(K_1, K_2, L),$$

where $u(\cdot)$ is a utility function and δ is a discount rate. Hence investments are expressed as follows,

$$\dot{K}_1=xS=xsLf(k_1, k_2),$$

$$\dot{K}_2=g(x)S=sLf(k_1, k_2)g(x).$$

Thus the fundamental system of two differential equations is obtained,

$$\dot{k}_1=xsf(k_1, k_2)-nk_1, \tag{54}$$

$$\dot{k}_2=sg(x)f(k_1, k_2)-nk_2. \tag{55}$$

They show the state variables (k_1, k_2) and the control variables (x, s).

According to the Pontryagin's maximum principle, the Hamiltonian function is defined as,

$$\begin{aligned} H &= u[(1-s)f(k_1, k_2)] + \varphi_1 \dot{k}_1 + \varphi_2 \dot{k}_2 \\ &= u[(1-s)f(k_1, k_2)] + \varphi_1[xsf(k_1, k_2) - nk_1] + \varphi_2[sg(x)f(k_1, k_2) - nk_2], \end{aligned} \tag{56}$$

where φ_1 and φ_2 are the shadow prices.

And the price equations become,

$$\begin{aligned} \dot{\varphi}_1 &= \delta\varphi_1 - \frac{\partial H}{\partial k_1}, \\ &= (\delta+n)\varphi_1 - \{u'(1-s) + s[\varphi_1 x + \varphi_2 g(x)]\} f_1, \end{aligned} \tag{57}$$

and

$$\begin{aligned} \dot{\varphi}_2 &= \delta\varphi_2 - \frac{\partial H}{\partial k_2}, \\ &= (\delta+n)\varphi_2 - \{u'(1-s) + s[\varphi_1 x + \varphi_2 g(x)]\} f_2. \end{aligned} \tag{58}$$

We need the conditions that the control variables x, s maximize H, which are,

$$\begin{aligned} \frac{\partial H}{\partial x} &= sf(k_1, k_2)[\varphi_1 + \varphi_2 g'(x)] = 0, \\ &\therefore\ \varphi_1 + \varphi_2 g'(x) = 0 \end{aligned} \tag{59}$$

and

$$\begin{aligned} \frac{\partial H}{\partial s} &= f(k_1, k_2)[\varphi_1 x + \varphi_2 g(x) - u'] = 0, \\ &\therefore\ \varphi_1 x + \varphi_2 g(x) = u'. \end{aligned} \tag{60}$$

Defining the ratios m, ϕ,

$$m = \frac{k_1}{k_2} \quad \text{and} \quad \phi = \frac{\varphi_1}{\varphi_2},$$

then

$$\frac{\dot{m}}{m} = \frac{\dot{k}_1}{k_1} - \frac{\dot{k}_2}{k_2} = sf(k_1, k_2)\frac{x}{k_2}\left[\frac{1}{m} - \frac{g(x)}{x}\right], \tag{61}$$

and

$$\frac{\dot{\phi}}{\phi} = \frac{\dot{\varphi}_1}{\varphi_1} - \frac{\dot{\varphi}_2}{\varphi_2} = \{u' \cdot (1-s) + s[\varphi_1 x + \varphi_2 g(x)]\} \frac{f_1}{\varphi_1}\left[\phi - \frac{f_1}{f_2}\right]. \tag{62}$$

The necessary conditions[6] for the long-run steady growth path are,

---

[6] The necessary and sufficient conditions are as follows;

First, $\dot{k}_1 = \dot{k}_2 = 0$ give, respectively,

(1) $\dfrac{\dot{K}_1}{K_1} = n$

and

(2) $\dfrac{\dot{K}_2}{K_2} = n.$

Secondly, $\dot{\varphi}_1 = \dot{\varphi}_2 = 0$ give, respectively,

(3) $\varphi_1 = \dfrac{u' \cdot (1-s) \cdot f_1}{(\delta+n) - s[f_1 x + f_2 g(x)]},$

and

(4) $\varphi_2 = \dfrac{u' \cdot (1-s) \cdot f_2}{(\delta+n) - s[f_1 x + f_2 g(x)]}.$

$$\dot{m}=\dot{\phi}=0, \tag{63}$$

therefore, we get,

$$x=mg(x), \tag{64}$$

and

$$\phi=\frac{f_1}{f_2}. \tag{65}$$

These conditions shall now be investigated more exactly to see their characteristics. From (59) we obtain,

$$g'(x)=-\frac{\varphi_1}{\varphi_2}=-\phi, \tag{66}$$

which implies that x is a function of ϕ, i.e. $x=x(\phi)$. Differentiate (66) with respect to ϕ,

$$g''(x)\frac{dx}{d\phi}=-1.$$

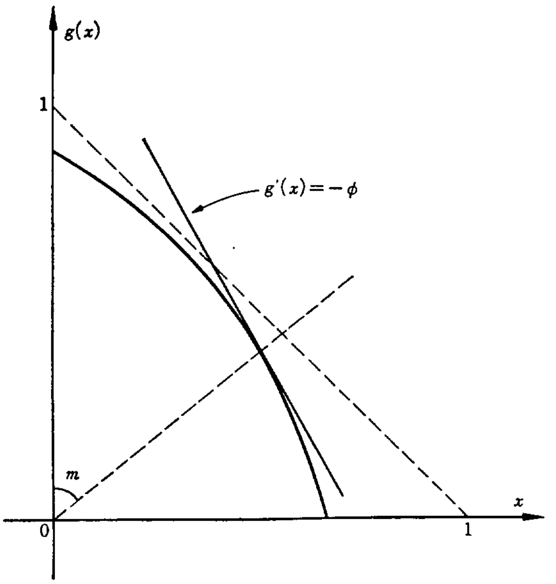

FIG. 8

Hence

$$x'(\phi)=-\frac{1}{g''(x)}>0, \tag{67}$$

since $g'(x)<0$ and $g''(x)<0$, as shown in Fig. 8.

Thus the slope of (64) is, by differentiating it with respect to ϕ,

$$\frac{d\phi}{dm}=\frac{g[x(\phi)]}{x'(\phi)\{1-mg'[x(\phi)]\}}>0, \tag{68}$$

by $x'(\phi)>0$ and $g'(x)<0$.

On the other hand, to derive more definite results, specify the production function f as the CES function. Then the ratio of marginal productivities is a decreasing function of the ratio of capital-labor ratios,

$$\frac{f_1}{f_2}=\omega\left(\frac{k_1}{k_2}\right)=\omega(m). \tag{69}$$

By (65), $\phi=\omega(m)$, which has the property

$$\frac{d\phi}{dm}=\omega'(m)<0. \tag{70}$$

It can be typically illustrated in the following example. If we adopt the Cobb-Douglas function $f=k_1{}^{\alpha}k_2{}^{\beta}$ as a special case of the CES function, then,

$$\frac{f_1}{f_2}=\frac{\alpha}{\beta}\frac{1}{m}=\phi,$$

and

$$\frac{d\phi}{dm}=-\frac{\alpha}{\beta}\frac{1}{m^2}<0.$$

The possibility of existence for a long-run equilibrium depends on whether the necessary conditions (64) (65) are satisfied. The condition (64) means simply,

$$\frac{\dot{K}_1}{K_1}=\frac{\dot{K}_2}{K_2} \tag{71}$$

Another condition requires,

$$-\frac{d\dot{K}_2}{d\dot{K}_1}\bigg|_{\dot{s}=0}=\frac{f_1}{f_2},^{7} \tag{72}$$

since $\dfrac{f_1}{f_2}=\phi=-g'(x)$.

In other words, the marginal rate of substitution on the transformation function must be equal to the ratio of marginal productivities. Whether (72) is satisfied depends on shapes of the production function f and of the transformation function g. This is a rather strong condition for existence, compared with the usual balanced growth conditions, like (71). If K_1 and K_2 are homogeneous, then $g'(x)\equiv -1$, and (65) becomes

$$f_1=f_2,$$

which might be looser than (72).

Suppose that the necessary conditions (64), (65) are satisfied and the interior solution is ensured. The saddle point (m^*, ϕ^*) solution by properties of (68) and (70) can then be obtained. The stable branches of this saddle point depict the optimal growth path. In infinite time-horizon planning, the convergence of shadow-prices is not a necessary condition, but a sufficient condition.

[7] This is independent of the ratio of shadow prices. In other words, the shadow prices are controllable so long as this condition in real terms is satisfied.

However, we have the almost definite conjecture, from properties and results in finite time-horizon planning, that the stable optimal growth path is unique. By controlling shadow prices φ_1, φ_2, we may asympotically approach the unique long-run equilibrium along the stable branches.

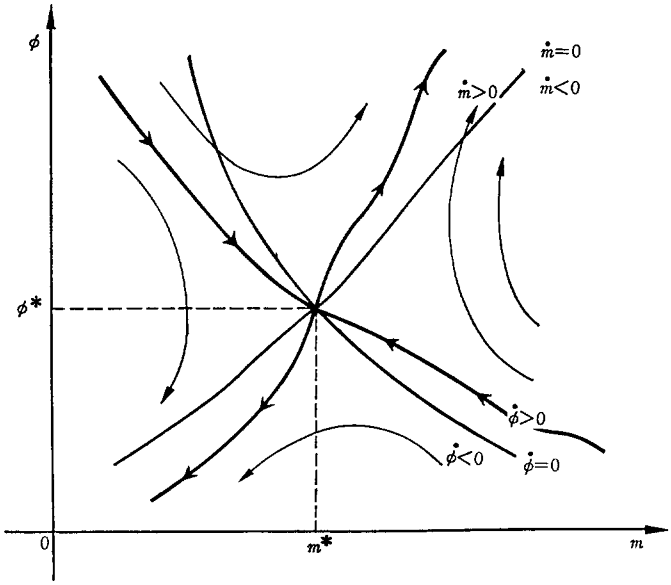

FIG. 9

Concluding Remarks

The models just discussed focuses on the role of heterogeneous capital good as the social capital in economic growth. As no attempt will be made to summarize this essay, the question of transformation function should be mentioned.

In order to simplify the analysis, some assumptions were imposed. Significances and limitations of some of them were pointed out in the context or foot-notes. The assumption about the transformation cost, however, has not been touched upon as yet. It should be noted that in our analysis the concept of transformation of the micro-economic theory is borrowed and applied to the macro-model without any modification.

As a necessary result of it, we must assume the transformation cost to be radioactives, although this radioactive law could be partly justified by interpreting it as the organizational cost in production. We should, however, have grasped at the macro level that inputs of primary factors of production, e.g. labor, are indispensable to the process of transformation, and may receive their rewards for it. This suggests to us to introduce another sector engaged in transforming activities. Then our paradoxical result will be solved; the paradoxical result is that the more heterogeneous goods are, the lower are optimal tax-rates and the equilibrium capital-labor ratio. Refer to (33) and (34). This kind of extension is the next step to our analysis, until then, our approach might be regarded as the first approximation.

REFERENCES

[1] Arrow, K. J. and M. Kurz: "Optimal Public Investment Policy and Controllability with Fixed Private Savings Ratio," *Journal of Economic Theory*, Aug. 1969.

[2] Hahn, F. H.: "Equilibrium Dynamics with Heterogeneous Capital Goods," *Quarterly Journal of Economics*, Nov. 1966.

[3] Kurz, M.: "The General Instability of a Class of Competitive Growth Processes," Technical Report No. 147, Institute for Mathematical Studies in the Social Sciences, Stanford, Dec. 1966.

[4] Ohta, M.: "Optimal Policy of Social Capital Accumulation," *Economic Studies Quarterly*, Aug. 1969.

[5] Pontryagin, L. S. and others: *The Mathematical Theory of Optimal Processes*, New York, 1962.

[6] Samuelson, P. A.: "Indeterminacy of Development in a Heterogeneous-Capital Model with Constant Saving Propensity," in K. Shell (ed.), *Essays on the Theory of Optimal Economic Growth*, Mass., 1967.

[7] Shell, K. and J. E. Stiglitz: "The Allocation of Investment in a Dynamic Economy," *Quarterly Journal of Economics*, Nov. 1967.