A NOTE ON TECHNICAL PROGRESS*

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One of the production functions which are in "disembodied" technical progress is shown by

(1)......... $Y=F(A(t)K, B(t)L)$,

where Y=output, K=the existing stock of capital, L=the number of labour employed, t=time, $A(t)$=capital-augmenter and $B(t)$=labour-augmenter. $A(t)$ and $B(t)$ are respectively some non-decreasing function of t. Choosing suitable units, we may put $A(0)=B(0)=1$ without loss of generality. The only condition which we impose on the production function (1) is that the first derivatives of (1) are all positive, namely

(2)......... $\dfrac{\partial Y}{\partial K}>0$ and $\dfrac{\partial Y}{\partial L}>0$.

Then let us call that a technical progress is "purely capital-augmenting" if $B(t)$ is independent of t, namely

(3)......... $Y=F(A(t)K, L)$

and "purely labour-augmenting" if $A(t)$ is independent of t, namely

(4)......... $Y=F(K, B(t)L)$.

Now we want to prove the following

Theorem 1: In order for a technical progress which is purely capital-augmenting to be also purely labour-augmenting, it is necessary and sufficient that the production function is derscribed by

$$Y=\Psi(C(t)K^{\alpha}L^{\beta}),$$

where Ψ *=any differentiable function,* $C(t)$ *=an increasing function of* t*, and* α *and* β *=some constants.*

Sufficiency is self-evident. To prove the necessity of the theorem, it would be useful to define $A(t)K\equiv X_1$ and $B(t)L\equiv X_2$. Thus

(5)......... $Y=F(X_1, X_2)$.

Let us further put $\log Y=y$, $\log X_1=x_1$ and $\log X_2=x_2$. Thus it follows

(6)......... $y=f(x_1, x_2)$.

The first derivatives of (6) with respect to x_1 and x_2 are denoted by f_1 and f_2 respectively. Once again we put $\log A(t)=\phi_1(t)$, $\log B(t)=\phi_2(t)$, $\log K=k$ and $\log L=l$.

Proof of Necessity: Using the above notations, we have

(7)......... $f(x_1, l)=f(k, x_2)$.

The first differentiation of this equation with respect to t gives us

(8)......... $\phi'_1(t)\cdot f_1(x_1, l)=\phi'_2(t)\cdot f_2(k, x_2)$,

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where $\phi_1'(t)$ is the first derivative of $\phi_1(t)$ with respect to t. Let us also differentiate (7) with respect to x_2. Then it follows

(9)........$f_2(x_1, l)=f_2(k, x_2)$,

because $l=x_2-\phi_2(t)$. Inserting (9) into (8), we get

(10)........$\dfrac{\phi_1'(t)}{\phi_2'(t)}=\dfrac{f_1(x_1, l)}{f_2(x_1, l)}$.

Thus it must follow

(11)........$\dfrac{f_1(x_1, l)}{f_2(x_1, l)}=\text{constant}$

or

(12)........$\dfrac{1}{\alpha}\dfrac{\partial f}{\partial x_1}=\dfrac{1}{\beta}\dfrac{\partial f}{\partial l}$,

where α and β=some constants and $f=f(x_1, l)$. The solution of (12) is given by

(13)........$f(x_1, l)=f(\alpha x_1+\beta l)$.

Because of $\alpha x_1+\beta l=\alpha\phi_1(t)+\alpha k+\beta l$, we get

(14)........$y=f(\alpha\phi_1(t)+\alpha k+\beta l)$,

or, taking anti-logarithm,

(15)........$Y=\Psi(C(t)K^{\alpha}L^{\beta})$,

where Ψ=any differentiable function and $C(t)$=anti-log $\alpha\phi_1(t)$. (Q.E.D.)

Theorem 2: If the production function in Theorem 1 is homogeneous of m-th degree, it must be

$$Y=D(t)K^{\alpha'}L^{\beta'},$$

where $\alpha'+\beta'=m$ and $D(t)$ is an increasing function of t.

Proof: Being (15) homogeneous of m-th degree, we get

(16)........$\lambda^m Y=\Psi(C(t)(\lambda K)^{\alpha}(\lambda L)^{\beta})=\Psi(C(t)K^{\alpha}L^{\beta}\lambda^{\alpha+\beta})$,

where λ is any real number. Let us put

(17)........$\lambda^{\alpha+\beta}=(C(t)K^{\alpha}L^{\beta})^{-1}$.

Then we have

(18)........$\lambda^m=(C(t)K^{\alpha}L^{\beta})^{-\frac{m}{\alpha+\beta}}$.

Putting (17) and (18) into (16), it must follow

(19)........$(C(t)K^{\alpha}L^{\beta})^{-\frac{m}{\alpha+\beta}}\cdot Y=\Psi(1)$

or

(20)........$Y=\Psi(1)\cdot C(t)^{\frac{m}{\alpha+\beta}}\cdot K^{\frac{\alpha}{\alpha+\beta}m}L^{\cdot\frac{\beta}{\alpha+\beta}m}$

or

(21)........$Y=D(t)K^{\alpha'}L^{\beta'}$

where $D(t)=\Psi(1)C(t)^{\frac{m}{\alpha+\beta}}$, $\alpha'=\dfrac{\alpha}{\alpha+\beta}m$ and $\beta'=\dfrac{\beta}{\alpha+\beta}m$. It should be apparent that $\alpha'+\beta'=m$.

(Q.E.D.)