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A NOTE ON TECHNICAL PROGRESS*

By Kenjiro Ara**

One of the production functions which are in "disembodied" technical progress is shown by

\[ Y = F(A(t)K, B(t)L), \]

where \( Y \) = output, \( K \) = the existing stock of capital, \( L \) = the number of labour employed, \( t \) = time, \( A(t) \) = capital-augmenter and \( B(t) \) = labour-augmenter. \( A(t) \) and \( B(t) \) are respectively some non-decreasing function of \( t \). Choosing suitable units, we may put \( A(0) = B(0) = 1 \) without loss of generality. The only condition which we impose on the production function (1) is that the first derivatives of (1) are all positive, namely

\[ \frac{\partial Y}{\partial K} > 0 \quad \text{and} \quad \frac{\partial Y}{\partial L} > 0. \]

Then let us call that a technical progress is "purely capital-augmenting" if \( B(t) \) is independent of \( t \), namely

\[ (3) \quad Y = F(A(t)K, L) \]

and "purely labour-augmenting" if \( A(t) \) is independent of \( t \), namely

\[ (4) \quad Y = F(K, B(t) L). \]

Now we want to prove the following

Theorem 1: In order for a technical progress which is purely capital-augmenting to be also purely labour-augmenting, it is necessary and sufficient that the production function is described by

\[ Y = \Psi(C(t)K^\alpha L^\beta), \]

where \( \Psi \) = any differentiable function, \( C(t) \) = an increasing function of \( t \), and \( \alpha \) and \( \beta \) = some constants.

Sufficiency is self-evident. To prove the necessity of the theorem, it would be useful to define \( A(t)K = X_1 \) and \( B(t)L = X_2 \). Thus

\[ (5) \quad Y = F(X_1, X_2). \]

Let us further put \( \log Y = y, \log X_1 = x_1 \) and \( \log X_2 = x_2 \). Thus it follows

\[ (6) \quad y = f(x_1, x_2). \]

The first derivatives of (6) with respect to \( x_1 \) and \( x_2 \) are denoted by \( f_1 \) and \( f_2 \) respectively. Once again we put \( \log A(t) = \phi_1(t), \log B(t) = \phi_2(t), \log K = k \) and \( \log L = l \).

Proof of Necessity: Using the above notations, we have

\[ (7) \quad f(x_1, x_2) = f(k, l). \]

The first differentiation of this equation with respect to \( t \) gives us

\[ (8) \quad \phi_1(t) f_1(x_1, t) + \phi_2(t) f_2(x_2, t), \]

* In writing this note, I owe very much to Professor S. Nabeya, especially as to the way of proof of Theorem 1. The remaining errors which may exist, however, must be attributed solely to the author.

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where \( \phi_i(t) \) is the first derivative of \( \phi_i(t) \) with respect to \( t \). Let us also differentiate (7) with respect to \( x_2 \). Then it follows
\[
(9) \quad f_2(x_1, t) = f_2(k, x_2),
\]
because \( t = x_2 - \phi_d(t) \). Inserting (9) into (8), we get
\[
(10) \quad \frac{\phi_i'(t)}{\phi_i(t)} = \frac{f_1(x_1, t)}{f_2(x_1, t)}.
\]
Thus it must follow
\[
(11) \quad \frac{f_1(x_1, t)}{f_2(x_1, t)} = \text{constant}
\]
or
\[
(12) \quad \frac{1}{\alpha} \frac{\partial f}{\partial x_1} + \frac{1}{\beta} \frac{\partial f}{\partial t} = 0,
\]
where \( \alpha \) and \( \beta \) = some constants and \( f = f(x_1, t) \). The solution of (12) is given by
\[
(13) \quad f(x_1, t) = f(\alpha x_1 + \beta t).
\]
Because of \( \alpha x_1 + \beta t = \alpha \phi_i(t) + \alpha k + \beta t \), we get
\[
(14) \quad y = f(\alpha \phi_i(t) + \alpha k + \beta t),
\]
or, taking anti-logarithm,
\[
(15) \quad y = \Psi(C(t)K^{n+\alpha}),
\]
where \( \Psi \) = any differentiable function and \( C(t) = \text{anti-log} \alpha \phi_i(t) \). (Q.E.D.)

**Theorem 2:** If the production function in Theorem 1 is homogeneous of \( m \)-th degree, it must be
\[
y = D(t)K^{n+\alpha},
\]
where \( \alpha' + \beta' = m \) and \( D(t) \) is an increasing function of \( t \).

**Proof:** Being (15) homogeneous of \( m \)-th degree, we get
\[
(16) \quad \lambda^m y = \Psi(C(t)(\lambda K)^n(\lambda L)^\alpha) = \Psi(C(t)K^{n+\alpha}),
\]
where \( \lambda \) is any real number. Let us put
\[
(17) \quad \lambda^{n+\alpha} = (C(t)K^{n+\alpha})^{-1}.
\]
Then we have
\[
(18) \quad \lambda^m = (C(t)K^{n+\alpha})^{-m}. \quad \text{Putting (17) and (18) into (16), it must follow}
\]
\[
(19) \quad (C(t)K^{n+\alpha})^{-m} \cdot y = \Psi(1)
\]
or
\[
(20) \quad y = \Psi(1) \cdot C(t)^{n+\alpha} \cdot K^{n+\alpha} \cdot L^{m} \cdot \frac{m}{n+\alpha}
\]
or
\[
(21) \quad y = D(t)K^{n+\alpha}
\]
where \( D(t) = \Psi(1)C(t)^{n+\alpha} \), \( \alpha' = \frac{\alpha}{\alpha + \beta} m \) and \( \beta' = \frac{\beta}{\alpha + \beta} m \). It should be apparent that \( \alpha' + \beta' = m \). (Q.E.D.)