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<th>On CES Production Function</th>
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Let the production function be denoted by

\[ Y = F(X_1, X_2, \ldots, X_s), \]

where \( Y \) is output and \( X_i \) is the \( i \)-th input. In what follows we assume that

1. \( \frac{\partial Y}{\partial X_i} > 0 \) (\( i = 1, 2, \ldots, s \)),
2. \( \alpha_i \left( \frac{\partial Y}{\partial X_i} \right)^{\sigma} X_i = \alpha_j \left( \frac{\partial Y}{\partial X_j} \right)^{\sigma} X_j \) \( (i, j = 1, 2, \ldots, s) \),

where \( \alpha_i > 0 \) and \( \sigma \geq 0 \) are respectively constants. As is easily seen, assumption (ii) means that we are dealing with a CES (Constant Elasticity of Substitution) production function because of

\[ \sigma = \frac{g \left( \frac{X_j}{X_i} \right)}{g \left( \frac{\partial Y}{\partial X_i} \cdot \frac{\partial Y}{\partial X_j} \right)}, \]

where \( g(x) = \frac{dx}{x} \). Notice that the production function (1) is not necessary to be an homogeneous function, still less linear homogeneous. For the sake of convenience, we use the notation

\[ \sigma = \frac{1}{1+\rho}. \]

Now we can prove the following

**Theorem 1:** In case of \( \rho \neq 0 \) (\( \therefore \sigma \neq 1 \)) the production function can be always expressed by the form

\[ Y = \psi(B(\beta_1 X_1^\sigma + \beta_2 X_2^\sigma + \cdots + \beta_s X_s^\sigma)), \]

where \( \psi \) is any differentiable function and \( B \) and \( \beta \) (\( \beta_i > 0 \) and \( \sum \beta_i = 1 \)) are respectively constants.

**Proof:** Assuming that \( \rho \neq 0 \), let \( X_i \) be transformed into \( Z_i \) in such a way that

\[ Z_i = X_i^{\frac{1}{\rho}} \quad (i = 1, 2, \ldots, s), \]

where \( \gamma_i = \alpha_i^{\frac{1}{\rho}}. \) From this, we have

\[ \frac{dZ_i}{dX_i} = \frac{1}{\gamma_i} X_i^{(1+\rho)} \quad (i = 1, 2, \ldots, s). \]

Because of (2), the production function can be rewritten by the equation

\[ Y = f(Z_1, Z_2, \ldots, Z_s), \]

where \( f \) is any differentiable function. Totally differentiating (4), we have

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(5) \[ dY = \frac{\partial Y}{\partial Z_1} dZ_1 + \frac{\partial Y}{\partial Z_2} dZ_2 + \ldots + \frac{\partial Y}{\partial Z_s} dZ_s. \]

Now let us notice that

(6) \[ \frac{\partial Y}{\partial Z_i} = \frac{\partial Y}{\partial Z_i} - \frac{\partial Y}{\partial X_i} \frac{1}{\gamma_t} X_i^{(1+s)} \quad (i=1, 2, \ldots, s), \]
then we have

(7) \[ \frac{\partial Y}{\partial Z_1} = \gamma_1 \frac{\partial Y}{\partial X_1} X_1^{1+s} = \gamma_1 \frac{\partial Y}{\partial X_1} X_1^{1+s} = \frac{\partial Y}{\partial Z_1}, \]

or

(8) \[ \frac{\partial Y}{\partial Z_1} = \frac{\partial Y}{\partial Z_2} = \ldots = \frac{\partial Y}{\partial Z_s} = \frac{\partial Y}{\partial Z}. \]

Putting (8) into (5), we get

(9) \[ dY = \frac{\partial Y}{\partial Z} (dZ_1 + dZ_2 + \ldots + dZ_s). \]

For the moment let \( Y \) be given at \( Y = Y_0 \). Then \( dY_0 = 0 \). By assumption, \( \frac{\partial Y}{\partial Z} > 0 \) for any level of \( Y \). Therefore it follows

(10) \[ dZ_1 + dZ_2 + \ldots + dZ_s = 0. \]

Integrating (10), we have

\[ Z_1 + Z_2 + \ldots + Z_s = c_0, \]

where \( c_0 \) is a constant of integration at \( Y = Y_0 \). In the same way we have

\[ Z_1 + Z_2 + \ldots + Z_s = c_i \]

at \( Y = Y_i \), where \( Y_i \) is any given level of \( Y \) and \( c_i \) is a constant associated with \( Y = Y_i \). Thus we have generally

\[ Z_1 + Z_2 + \ldots + Z_s = \phi(Y) \]

or

\[ Y = \Psi (Z_1 + Z_2 + \ldots + Z_s) \]

or

(11) \[ Y = \Psi (B \{ \beta_1 X_1^{r} + \beta_2 X_2^{r} + \ldots + \beta_s X_s^{r} \}), \]

where

\[ \beta_i = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_s} > 0 \]

\[ B = -\frac{1}{\rho} \left\{ \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_s} \right\}. \]

Thus we could prove Theorem I. The case of \( \rho = 1 \) will be discussed in the connection with Theorem III.

Theorem II: In case of \( \rho \neq 0 \) (\( \therefore \rho = 1 \)), the CES production function can be always expressed by the form

\[ Y = A \cdot \left[ \beta_1 X_1^{r} + \beta_2 X_2^{r} + \ldots + \beta_s X_s^{r} \right]^{\frac{m}{m-r}} \]

if it is a homogeneous function of \( m \)-th degree, where \( A \) is a constant.

Proof: For the sake of simplicity, let us put

(12) \[ Y = B \{ \beta_1 X_1^{r} + \beta_2 X_2^{r} + \ldots + \beta_s X_s^{r} \}, \]
then we have \( Y = \Psi (X) \). Because of homogeneity of \( m \)-th degree, we have

(13) \[ Y^m = \Psi (p^r \cdot X), \]
where $p$ is any real number. Let $p^{-x} = X^{-1}$ (i.e. $p^m = X^m$), then it follows
\[(14) \ldots \ldots X^m \cdot Y = \Psi(1)\]
or
\[(15) \ldots \ldots Y = \Psi(1) X^{-m},\]
where $\Psi(1)$ is a constant. Thus we have finally
\[(16) \ldots \ldots Y = A \cdot [\beta_1 X_1^r + \beta_2 X_2^r + \ldots + \beta_s X_s^r]^{-m},\]
where
\[A = \Psi(1) \cdot B^{m/r} = \text{constant.}\]

**Theorem III**: The production function (16) can be expressed by the following equations
\[
(i) \quad \lim_{r \to 0} Y = A^* \cdot X_1^{m/1} \cdot X_2^{m/2} \ldots \cdot X_s^{m/s},
\]
\[
(ii) \quad \lim_{r \to \infty} Y = A^{**} \min [X_1^m, X_2^m, \ldots, X_s^m],
\]
where $A^*$ and $A^{**}$ are respectively constants.

**Proof**: From (16) it follows
\[(17) \ldots \ldots \log Y = \log A - m \frac{\log \left[ \beta_1 X_1^r + \ldots + \beta_s X_s^r \right]}{\rho},\]
Now let us first consider the case of $\rho = 0$ (i.e. $\sigma = 1$), namely
\[(18) \ldots \ldots \lim_{r \to 0} \log Y = \log A^* = \lim_{r \to 0} m \frac{\log \left[ \beta_1 X_1^r + \ldots + \beta_s X_s^r \right]}{\rho},\]
where $\log A^* = \log A$. As is easily seen, the second term of the right side of (18) is reduced to 0 (remember that $\log \Sigma \beta = \log 1 = 0$), then we can apply the l'Hospital's rule
\[(19) \ldots \ldots \lim_{x \to a} \frac{h(x)}{f(x)} = \lim_{x \to a} \frac{h'(x)}{f'(x)} \quad \text{for} \quad f'(x) \neq 0\]
to this case. Remembering that
\[\frac{d}{d\rho} m \frac{\log \left[ \beta_1 X_1^r + \ldots + \beta_s X_s^r \right]}{\beta_1 X_1^r + \ldots + \beta_s X_s^r} = m \frac{\beta_1 X_1^r \log X_1 + \ldots + \beta_s X_s^r \log X_s}{\beta_1 X_1^r + \ldots + \beta_s X_s^r},\]
we have finally
\[(20) \ldots \ldots \lim_{r \to 0} \log Y = \log A^* + m \left[ \beta_1 \log X_1 + \ldots + \beta_s \log X_s \right]\]
or, taking antilogarithm,
\[(21) \ldots \ldots \lim_{r \to 0} Y = A^* \cdot X_1^{m/1} \cdot X_2^{m/2} \ldots \cdot X_s^{m/s}.\]
This is obviously a Cobb-Douglasian production function with homogeneity of $m$-th degree.

Next we consider the case of $\rho \to \infty$ (i.e. $\sigma = 0$), namely
\[(22) \ldots \ldots \lim_{r \to \infty} \log Y = \log A^{**} = \lim_{r \to \infty} m \frac{\log \left[ \beta_1 X_1^r + \ldots + \beta_s X_s^r \right]}{\rho},\]
where $\log A^{**} = \log A$. Now that the second term of the right side of (22) is reduced to $-\infty$, we can again apply the l'Hospital's rule (19) to this case. Thus we have
\[(23) \ldots \ldots \lim_{r \to \infty} \log Y = \log A^{**} + \lim_{r \to \infty} m \frac{\beta_1 X_1^r \log X_1 + \ldots + \beta_s X_s^r \log X_s}{\beta_1 X_1^r + \ldots + \beta_s X_s^r}.\]
Without loss of generality let us assume that
\[X_1 < X_j \quad (j = 2, 3, \ldots, s),\]
then we have
\begin{equation*}
\lim_{\rho \to -\infty} \frac{\beta_1 X_1^\rho \log X_1 + \cdots + \beta_s X_s^\rho \log X_s}{\beta_1 X_1^\rho + \cdots + \beta_s X_s^\rho} = \frac{\beta_1 \log X_1 + \beta_2 \left( \frac{X_2}{X_1} \right)^\rho \log X_2 + \cdots}{\beta_1 + \beta_2 \left( \frac{X_2}{X_1} \right)^\rho + \cdots} = m \log X_1,
\end{equation*}

namely
\[ \lim_{\rho \to -\infty} Y = A^{**} X_1^m. \]

Next let us assume that 
\[ X_1 = X_2 < X_j \quad (j = 3, 4, \ldots, s), \]
then we have
\begin{equation*}
\lim_{\rho \to -\infty} \frac{\beta_1 X_1^\rho \log X_1 + \cdots + \beta_s X_s^\rho \log X_s}{\beta_1 X_1^\rho + \cdots + \beta_s X_s^\rho} = \frac{\beta_1 \log X_1 + \beta_2 \log X_2 + \beta_3 \left( \frac{X_3}{X_1} \right)^\rho \log X_3 + \cdots}{\beta_1 + \beta_2 + \beta_3 \left( \frac{X_3}{X_1} \right)^\rho + \cdots} = m \frac{\beta_1}{\beta_1 + \beta_2} \log X_1 + m \frac{\beta_3}{\beta_1 + \beta_2} \log X_2 = m \log X_1 = m \log X_2,
\end{equation*}

namely
\[ \lim_{\rho \to -\infty} Y = A^{**} X_1^m = A^{**} X_2^m. \]

The procedure is the same in other cases. Thus we could prove that
\[ (24) \ldots \ldots \lim_{\rho \to -\infty} Y = A^{**} \min \{X_1^m, X_2^m, \ldots, X_s^m\}. \]

This is obviously a limitational production function with homogeneity of \(m\)-th degree.

(1965, Oct.)