<table>
<thead>
<tr>
<th>Title</th>
<th>Internal and External Matrix Multipliers in the Input-Output Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Miyazawa, Kenichi</td>
</tr>
<tr>
<td>Citation</td>
<td>Hitotsubashi Journal of Economics, 7(1): 38-55</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1966-06</td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://doi.org/10.15057/8073">http://doi.org/10.15057/8073</a></td>
</tr>
</tbody>
</table>
INTERNAL AND EXTERNAL MATRIX MULTIPLIERS
IN THE INPUT-OUTPUT MODEL*

By Kenichi Miyazawa**

Contents

I. Introduction
II. Partitioned Matrix Multipliers
III. Interdependent Model of Goods-Producing Sectors and Service Sectors
IV. Interregional Repercussion Model
V. Some Extensions of the Model
  1) The extension to the number of partitioned groups
  2) The inclusion of the income formation process

I. Introduction

There are many useful applications if we divide $n$ industries in the input-output tables into two or more strategic industry groups and trace back to the interaction between these groups. To mention a few such examples, we have the interactions between the goods-producing sectors and service sectors, between the primary growth sectors and the supplementary or derived growth sectors, and between two regions which have structural different characters. Another example is found in the necessity of distinguishing industries subject to capacity limitations from those which have plenty of capacity. Examples of this sort might be given by the hundreds.

In the usual input-output analysis, the $n \times n$ inverse matrix shows the ultimate total effects of interindustrial propagation, but it cannot tell us the disjoined effects separating into these two distinguished activities. This paper is an attempt to clarify the problems of this type by means of the formulation of partitioned matrix multipliers and their relationships, and to apply our formula to the input-output data in two cases: 1) the interdependent model of goods-producing sectors and service sectors, 2) interregional repercussion model of the Japanese economy.


** Professor (Kyōju) of Economics.
II. Partitioned Matrix Multipliers

We divide $n$ industries in the usual input-output table into two subgroups designated P sector which consists of $l$ industries and S sector which consists of $m$ industries. Then, the $n \times n$ matrix of input coefficients is

$$A* = \begin{bmatrix}
A & A_1 \\
S_1 & S
\end{bmatrix} = \begin{bmatrix}
l & m \\
l & m
\end{bmatrix}$$

where $A, A_1$ are submatrices of coefficients showing the input of P sector’s products in the P and S sectors respectively, and $S_1, S$ are submatrices of coefficients showing the input of S sector’s products in the P and S sectors respectively. Among these submatrices, $A$ and $S$ are square having the order $l \times l$ and $m \times m$, $A_1$ and $S_1$ are rectangular having the order $l \times m$ and $m \times l$.

Since the $n \times n$ Leontief inverse $B^* = (I - A*)^{-1}$ tells us only the total ultimate effects but not the disjoined interdependence of the above two activities, we must introduce some device consisting of partitioned matrix multipliers. In order to solve this problem, we decompose the elements of the Leontief inverse into three sides of propagation aspects, i.e.,

(i) internal propagation activities inside P sector’s industries,
(ii) internal propagation activities inside S sector’s industries,
(iii) intersectoral propagation activities between P and S sectors’ industries.

For aspects (i) and (ii), we may term the $l \times l$ inverse

$$B = (I - A)^{-1}$$

the internal matrix multiplier of the P sector and the $m \times m$ inverse

$$T = (I - S)^{-1}$$

the internal matrix multiplier of the S sector, then these two “internal matrix multipliers” show the interindustrial propagation effects in the inside of each sector. Of course, each internal matrix multiplier does not operate independently under its own power, but is able to operate with the other sector’s industrial activity.

So that, according to the economic causal process, the intersectoral propagations accompanied by the operation of internal multipliers $B$ and $T$ can be written as the form of four rectangular sub-matrix-multipliers, which express the aspect (iii), i.e.,

$B_1 = S_1 B$ — S-goods-input in P sector induced by internal propagation in P sector’s industries ($m \times l$).

$B_2 = B A_1$ — internal propagation in P sector’s industries induced by P-goods-input in S sector ($l \times m$).

$T_1 = A_1 T$ — P-goods-input in S sector induced by internal propagation in S sector’s industries ($l \times m$).

$T_2 = T S_1$ — internal propagation in S sector’s industries induced by S-goods-input in P sector ($m \times l$).

These sub-multipliers $B_1, B_2, T_1$ and $T_2$ show the coefficients of induced effects on output or input activities between two sectors and call themselves the production-generating process in succession.
Such a repercussion process due to these induced effects naturally leads to the intersectoral multiplier between the P and S sectors. If we select the coefficients of the induced effect on production (i.e., \( B_2 \) and \( T_2 \)) as the base of this intersectoral multiplier, then it will take the form:

\[
K = (I - T_2 B_2)^{-1}
\]

or alternatively

\[
L = (I - B_2 T_2)^{-1}.
\]

We could define the matrix \( K \) as the external matrix multiplier of the S sector, and the matrix \( L \) as the external matrix multiplier of the P sector according to their economic meanings. Of course \( K \) has the order \( m \times m \), and \( L \) has \( l \times l \), because the multiplications of rectangular matrices \( T_2 B_2 \) or \( B_2 T_2 \) make the new square matrices having the order \( m \times m \) or \( l \times l \) respectively.\(^1\)

Then, we have now arrived at the fact that the total of the propagation effects in P and S sectors’ industries, each generated by its own sector’s activities, are expected to take the values \( LB \) and \( KT \) respectively, i.e., “the internal matrix multiplier” premultiplied by “the external matrix multiplier”. So, if we put

\[
KT = M
\]

\[
LB = N
\]

then we can prove the following formula:

\[
B^* = (I - A^*)^{-1} = \begin{bmatrix}
B + B_2 MB_1 & B_2 M \\
MB_1 & M
\end{bmatrix}
\]

or

\[
= \begin{bmatrix}
N & NT_1 \\
T_2 N & T_2 T_2 NT_1
\end{bmatrix}
\]

In the other words, we can partition off the original Leontief inverse \( B^* = (I - A^*)^{-1} \) in terms of the combined effects of internal and external matrix multipliers and their induced sub-matrix-multipliers.\(^2\)

The proof of the formula (6) is as follows:

\[
\begin{bmatrix}
B + B_2 MB_1 & B_2 M \\
MB_1 & M
\end{bmatrix}
\begin{bmatrix}
I - A & -A_1 \\
-S_1 & I - S
\end{bmatrix}
= \begin{bmatrix}
I & O \\
O & I
\end{bmatrix}
\]

\[
\therefore \quad B(I - A) + B_2 MB_1(I - A) - B_2 MS_1
\]

\[
= I + B_2 MS_1(B(I - A) - B_2 MS_1)
\]

\[
= I + B_2 MS_1 - B_2 MS_1 = I
\]

\[
MB_1(I - A) - MS_1
\]

\[
= MS_1 B(I - A) - MS_1
\]

\[
= MS_1 - MS_1 = O
\]

\[
-BA_1 - B_2 MB_1 A_1 + B_2 M(I - S)
\]

\[
= -B_2 - B_2 KT_1 B_2 + B_2 KT(I - S)
\]

\[
= -B_2 - B_2 KT_1 B_2 + B_2 K = -B_2 (I + KT_1 B_2 - K)
\]

\[
= -B_2 (I - K(I - T_1 B_2)) = O
\]

\[\text{Another formulation of the external matrix multipliers based on the coefficients of induced effect on intersectoral input activities (i.e., } T_1 \text{ and } B_1 \text{) are:} \]

\[
\bar{K} = (I - B_1 T_1)^{-1} \quad \text{(4')} \]

and

\[
\bar{L} = (I - T_1 B_1)^{-1} \quad \text{(5')} \]

where \( \bar{K} \) has the order \( m \times m \), and \( \bar{L} \) has the order \( l \times l \).

The existence of these inverses (external multipliers \( K, \bar{K}, L, \bar{L} \) and \( \bar{L} \)) as well as the existence of internal multipliers \( (B \) and \( T) \) is warranted by the existence of the original Leontief inverse matrix.
\[ -MB_1A_1 + M(I - S) \]
\[ = -KTB_1A_1 + KT(I - S) \]
\[ = -KT_2B_2 + K \]
\[ = K(I - T_1B_2) = I. \]

In exactly the same manner, we have
\[
\begin{bmatrix}
\frac{N}{T_2N} & \frac{NT_1}{T_2NT_1} \\
I - A & -A_1 \\
-S_1 & I - S
\end{bmatrix} = \begin{bmatrix} I & O \end{bmatrix}.
\]

The same result can be obtained by solving the following system according to economic causal reasoning.
\[
\begin{align*}
X_p &= AX_p + A_1X_s + Y_p \\
X_s &= SXX_p + SX_s + Y_s
\end{align*}
\]
where \(X_p\) is an output vector of \(P\) sector's industries, \(X_s\) is an output vector of \(S\) sector's industries, and \(Y_p, Y_s\) are the final demand vectors of the \(P\) and \(S\) sectors respectively. Thus the solution of this system is stated as
\[
\begin{bmatrix} X_p \\ X_s \end{bmatrix} = \begin{bmatrix} B + B_2MB_1 & B_2M \\ MB_1 & M \end{bmatrix} \begin{bmatrix} Y_p \\ Y_s \end{bmatrix}
\]
or
\[
\begin{bmatrix} X_p \\ X_s \end{bmatrix} = \begin{bmatrix} N \\ T_2N \end{bmatrix} \begin{bmatrix} NT_1 \\ T_2NT_1 \end{bmatrix} \begin{bmatrix} Y_p \\ Y_s \end{bmatrix}.
\]

From which, it is easily seen that the total effects in the \(P\) and \(S\) sectors originated in its own sector's activities and can be written in the additive form \(B + B_2MB_1\) or \(T + T_2NT_1\) as well as the multiplied form \(LB\) or \(KT\).\(^2\) Thus, the partitioned intersectoral activities may be viewed in two ways: (a) the first expression of the formula (6) shows it from the viewpoint of \(P\) sector side and (b) the second expression shows the same fact from the viewpoint of \(S\) sector side. These expressions go hand in hand to make the general formulation applicable to the various problems.

\(^2\) Using the notation in note 1, we can prove the following identities:
\[
KT = TK = M
\]
\[
LB = BL = N
\]
that is, the expression that the internal multiplier \textit{postmultiplied} by the external multiplier is also possible as well as the \textit{premultiplied} expression.

The proof of the latter identity is that:
\[
LB = (I - B_2T_2)^{-1}B = [I + \sum_{m=1}^{\infty} (B_2T_2)^m]B = (I + B_2MS_1)B
\]
\[
BL = B(I - T_1B_1)^{-1} = B[I + \sum_{m=1}^{\infty} (T_1B_1)^m] = B(I + A_1MB_1)
\]
because
\[
\sum_{m=1}^{\infty} (B_2T_2)^m = \sum_{m=1}^{\infty} B_2(T_2B_2)^{m-1}T_2 = B_2[\sum_{m=0}^{\infty} (T_2B_2)^m]T_2
\]
\[
= B_2(I - T_2B_2)^{-1}T_2 = B_2KTS_1 = B_2MS_1
\]
and
\[
\sum_{m=1}^{\infty} (T_1B_1)^m = \sum_{m=1}^{\infty} (A_1TS_1B_1)^m = \sum_{m=1}^{\infty} A_1(TS_1B_1)^{m-1}TS_1B = A_1[I + \sum_{m=0}^{\infty} (T_2B_2)^m]TB_1
\]
\[
= A_1(I - T_2B_2)^{-1}TB_1 = A_1KTB_1 = A_1MB_1.
\]
So, we obtain
\[
LB = (I + B_2MS_1)B = B + B_2MB_1
\]
\[
= B[I + A_1MB_1] = BL\quad\text{(\textasteriskcentered)}
\]

In exactly the same manner, we get
\[
KT = TK = M.
\]
\(^3\) For this equality between the multiplied and additive forms, see the equation (\textasteriskcentered) in the above note 2).
One more alternative expression of the Leontief inverse by the partitioned matrix multipliers is

\[ B^* = (I - A^*)^{-1} = \begin{bmatrix} LB & LB_T \\ KTzB & KT \end{bmatrix} \approx \begin{bmatrix} LB & LBTz \\ KT & KT_b \end{bmatrix} \]

(6a)

We may easily prove the identity between this expression and the equation (6).4

Mathematically, our formula also gives us a method of the reduction of the order in the matrix-calculations when the inversion of matrices of high order is not suitable for available computational equipment.

III. Interdependent Model of Goods-Producing Sectors and Service Sectors

An empirical application of our model is made for the interindustry data of the Japanese economy (54 sectors) published by Japanese Government under the co-operation of Economic Planning Agency and five other Ministries (the Ministry of Agriculture and Forestry, the Ministry of International Trade and Industry, the Ministry of Construction, the Statistics Bureau of the Prime Minister's Office, and the Administrative Management Agency), which consists of 50 goods-producing sectors and 4 service sectors. In formula (6), we put the P sector for the goods-producing sectors and S sector for the service sectors, i.e., \( l = 50 \) and \( m = 4 \).

Now let us divide the elements of the internal matrix multiplier of the goods-producing sectors \( B (50 \times 50) \) calculated from the above equation by the elements of \( 50 \times 50 \) part in the published Leontief inverse \( B^* = (I - A^*)^{-1} \). We then obtain the values which may be called “the inside propagation ratio of goods-producing sectors”. By the numerical test on the row elements of these \( 50 \times 50 \) ratios, we have arrived at the conclusion that the industries having many higher values of those ratios are the less service-dependent sectors, and vice versa.

The Table 1 is a summarized version of this test, and shows industry-categories of goods-producing sectors by type of the degree of dependence on service activity. Those in category A have characteristics relatively independent of service activity, and those in category D are at the other extreme. Roughly speaking, categories from A-1 to D-2 may be thought of as successive stages of dependency on service sectors.

In Group A, the “inside propagation ratios” of each industry take the value predominantly more than 0.9 (in A-1 group), or more than 0.8-0.9 (in A-2, 3 groups). Those in Group B are in the range of 0.7-0.9. In Group C, those ratios are spread far and wide in the range of about 0.5-0.9, and among this category the ratios in C-2 group concentrate in values 0.7-0.8. In Group D, the inside propagation ratios take a lower value ranging about from 0.4-0.5 to 0.7, and the industries in this group are the most service-dependent sectors.

The reason why the above industrial differential-pattern occurs may be traced by the discerning the difference between the values of the elements in the \( 50 \times 50 \) part of Leontief inverse \( B^* \) and the values of the elements in the internal matrix multiplier \( B \), which equals to

\[ \text{For example, the identity } KT_zB = T_zN \text{ is shown as follows:} \]

\[ T_zN = T_z(B + B_2MB) = TS_b + T_zB_2KTB_1 = (1 + T_zB_2K)TB_1 \]

in which \( I + T_zB_2K = K \) because \( (I - T_zB_2K)K = I \), so we obtain

\[ T_zN = KTB_1 = KTS_bB = KT_zB. \]
TABLE 1. Industry Groups by Type of the Degree of Dependence on the Service Activity

<table>
<thead>
<tr>
<th>Groups</th>
<th>Names of Goods-producing Industries*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group A</td>
<td>1 Basic chemicals, Non-metallic minerals.</td>
</tr>
<tr>
<td></td>
<td>2 Electricity, Intermediate chemicals, Pig iron, ferro-alloys and crude steel, Metallic ores, Non-metallic mineral products.</td>
</tr>
<tr>
<td></td>
<td>3 Rolled steel, Natural fibre yarns, Coal and lignite, Non-ferrous metal ingots, Chemical fibre yarns, Metal products, Forestry, Coal products.</td>
</tr>
<tr>
<td>Group B</td>
<td>1 Machinery and instruments (except electric), Steel casting and forging, Miscellaneous crops.</td>
</tr>
<tr>
<td></td>
<td>2 Primary non-ferrous metal products, Saw-mills and plywood, Chemical fertilizers, Fabrics, Rubber products, Pulp.</td>
</tr>
<tr>
<td>Group C</td>
<td>1 Leather and leather products, Livestock, Furniture and wood products, Rice, wheat and barley, Electric machinery and equipment.</td>
</tr>
<tr>
<td></td>
<td>2 Starch, sugar, seasonings, etc., Miscellaneous textile products, Crude petroleum and natural gas, Paper and paper products, Miscellaneous processed foods, City gas and water services, Repair and maintenance of machines, buildings and structures, Petroleum products.</td>
</tr>
<tr>
<td>Group D</td>
<td>1 Rice and barley polishing and grain-flour mills, Miscellaneous manufactures, Fisheries, Printing and publishing.</td>
</tr>
<tr>
<td></td>
<td>2 Drugs, soap and cosmetics, Transport equipment, Manufactured tobacco and beverages.</td>
</tr>
</tbody>
</table>

* Excluding the dummy industries such as Business consumption, Office supplies, Scraps, and Undistributed.
** Service sectors other than the above goods-producing sectors are Wholesale and retail trade, Transportation and communication, Real estate and ownership of dwellings, and Banking, insurance and services.
*** The order of listing is that the industries in Group A are the most service-independent sectors, and those in Group D are the most service-dependent sectors.

$B_2MB_1$ as shown the formula. So, we must discuss the relative weight between $B_1$, $B_2$, and $M$ in their propagation process.

By inquiring into Table 2 and 3, we can see what sort of goods-producing industries has more inducible power for service activity (see Table of values of $B_1$), or what sort of service sectors has more inducible power for goods-producing activity (see Table of values of $B_2$). A general feature of these figures is of particular interest; the comparison of values of these two intersectoral sub-multipliers may suggest that the weight of $B_2$ in propagation activity is smaller than that of $B_1$ on the whole. The number of values having more than 3% in the Table of $B_2$ are less than could be counted on the fingers of both hands (exclude the Undistributed sector), while the Table of $B_1$ has many number of values more than 3%. In other words, the inducible power of one sector to another is more powerful in the case of goods-producing sector than the case of service sector. It does not need to be said that there are different effects from industry to industry as tolding the Tables.

One comment is needed because of the weakness in the data of the service sector which leads to the estimation errors in the original Leontief inverse matrix. If this data weakness is not negligible, our method explained the above must be reread in such a way that the proportion of errors in the elements of the Leontief inverse is actually due to a shortcoming of the service sector's data. For example, the reliability of the inverse-elements may be judged by means of Table 1 such that those in Group A-1 are the most reliable and those in Group D-2 are the most unreliable.
### Table 2. Coefficients of Service-Input Induced by Internal Propagation in Goods-Producing Sector

\[ B'_1 = (S, B') \]  
(单位: \(10^{-6}\))

<table>
<thead>
<tr>
<th>Sector</th>
<th>Trade</th>
<th>Transportation and Communication</th>
<th>Real estate</th>
<th>Banking, Insurance and Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rice, wheat and barley</td>
<td>28382</td>
<td>10876</td>
<td>347</td>
<td>12243</td>
</tr>
<tr>
<td>Miscellaneous crops</td>
<td>50790</td>
<td>18595</td>
<td>614</td>
<td>15910</td>
</tr>
<tr>
<td>Livestock</td>
<td>61239</td>
<td>27121</td>
<td>897</td>
<td>24374</td>
</tr>
<tr>
<td>Forestry</td>
<td>13009</td>
<td>10888</td>
<td>1830</td>
<td>23862</td>
</tr>
<tr>
<td>Fisheries</td>
<td>54357</td>
<td>21589</td>
<td>6341</td>
<td>40338</td>
</tr>
<tr>
<td>Coal and lignite</td>
<td>22829</td>
<td>32782</td>
<td>4228</td>
<td>31772</td>
</tr>
<tr>
<td>Crude petroleum and natural gas</td>
<td>25045</td>
<td>30833</td>
<td>27628</td>
<td>39486</td>
</tr>
<tr>
<td>Metallic ores</td>
<td>28316</td>
<td>26165</td>
<td>8939</td>
<td>20216</td>
</tr>
<tr>
<td>Non-metallic minerals</td>
<td>20540</td>
<td>21651</td>
<td>4514</td>
<td>15567</td>
</tr>
<tr>
<td>Rice and barley polishing and grain-flour mills</td>
<td>55069</td>
<td>16636</td>
<td>765</td>
<td>14102</td>
</tr>
<tr>
<td>Starch, sugar, seasonings, etc.</td>
<td>136074</td>
<td>26157</td>
<td>2085</td>
<td>32963</td>
</tr>
<tr>
<td>Manufactured tobacco and beverages</td>
<td>33969</td>
<td>15115</td>
<td>767</td>
<td>19457</td>
</tr>
<tr>
<td>Miscellaneous processed foods</td>
<td>95402</td>
<td>38199</td>
<td>2984</td>
<td>40303</td>
</tr>
<tr>
<td>Natural fibre yarns</td>
<td>28558</td>
<td>16786</td>
<td>1944</td>
<td>26100</td>
</tr>
<tr>
<td>Chemical fibre yarns</td>
<td>52525</td>
<td>51397</td>
<td>5743</td>
<td>50883</td>
</tr>
<tr>
<td>Fabrics</td>
<td>55940</td>
<td>34169</td>
<td>5023</td>
<td>43814</td>
</tr>
<tr>
<td>Miscellaneous textile products</td>
<td>71816</td>
<td>23592</td>
<td>4105</td>
<td>46784</td>
</tr>
<tr>
<td>Saw-mills and plywood</td>
<td>18568</td>
<td>15909</td>
<td>2841</td>
<td>34709</td>
</tr>
<tr>
<td>Furniture and wood products</td>
<td>37096</td>
<td>39130</td>
<td>3036</td>
<td>49755</td>
</tr>
<tr>
<td>Pulp</td>
<td>32338</td>
<td>38650</td>
<td>3631</td>
<td>27412</td>
</tr>
<tr>
<td>Paper and paper products</td>
<td>46449</td>
<td>45912</td>
<td>2881</td>
<td>36863</td>
</tr>
<tr>
<td>Printing and publishing</td>
<td>52511</td>
<td>63314</td>
<td>2326</td>
<td>87273</td>
</tr>
<tr>
<td>Coal products</td>
<td>27008</td>
<td>147083</td>
<td>3056</td>
<td>27483</td>
</tr>
<tr>
<td>Petroleum products</td>
<td>11974</td>
<td>10508</td>
<td>7081</td>
<td>17553</td>
</tr>
<tr>
<td>Basic chemicals</td>
<td>40572</td>
<td>96450</td>
<td>4175</td>
<td>35066</td>
</tr>
<tr>
<td>Chemical fertilizers</td>
<td>45497</td>
<td>67430</td>
<td>2948</td>
<td>39844</td>
</tr>
<tr>
<td>Intermediate chemicals</td>
<td>83199</td>
<td>56255</td>
<td>5679</td>
<td>48159</td>
</tr>
<tr>
<td>Drugs, soap and cosmetics</td>
<td>82332</td>
<td>44149</td>
<td>5863</td>
<td>128004</td>
</tr>
<tr>
<td>Rubber products</td>
<td>39029</td>
<td>27059</td>
<td>3449</td>
<td>37856</td>
</tr>
<tr>
<td>Leather and leather products</td>
<td>39011</td>
<td>17434</td>
<td>2685</td>
<td>26308</td>
</tr>
<tr>
<td>Non-metallic mineral products</td>
<td>66153</td>
<td>66697</td>
<td>4087</td>
<td>36925</td>
</tr>
<tr>
<td>Pig iron, ferro-alloys, crude steel</td>
<td>24552</td>
<td>57757</td>
<td>2506</td>
<td>16538</td>
</tr>
<tr>
<td>Steel casting and forging</td>
<td>41669</td>
<td>46669</td>
<td>3209</td>
<td>26740</td>
</tr>
<tr>
<td>Rolled steel</td>
<td>29437</td>
<td>36458</td>
<td>3383</td>
<td>25969</td>
</tr>
<tr>
<td>Non-ferrous metal ingots</td>
<td>26665</td>
<td>38975</td>
<td>5811</td>
<td>15111</td>
</tr>
<tr>
<td>Primary non-ferrous metal products</td>
<td>30039</td>
<td>37476</td>
<td>4037</td>
<td>21291</td>
</tr>
<tr>
<td>Metal products</td>
<td>44866</td>
<td>42068</td>
<td>2468</td>
<td>26247</td>
</tr>
<tr>
<td>Machinery and instruments</td>
<td>55789</td>
<td>37272</td>
<td>6127</td>
<td>34995</td>
</tr>
<tr>
<td>Electric machinery and equipment</td>
<td>74690</td>
<td>38973</td>
<td>2560</td>
<td>34970</td>
</tr>
<tr>
<td>Transport equipment</td>
<td>76278</td>
<td>41917</td>
<td>2868</td>
<td>29493</td>
</tr>
<tr>
<td>Repair and maintenance of machinery, etc.</td>
<td>70881</td>
<td>68647</td>
<td>3204</td>
<td>25575</td>
</tr>
<tr>
<td>Miscellaneous manufactures</td>
<td>73958</td>
<td>37389</td>
<td>3834</td>
<td>47743</td>
</tr>
<tr>
<td>Electricity</td>
<td>23340</td>
<td>51460</td>
<td>6996</td>
<td>29435</td>
</tr>
<tr>
<td>City gas and water services</td>
<td>35028</td>
<td>54657</td>
<td>2783</td>
<td>40696</td>
</tr>
<tr>
<td>Business consumption expenditure</td>
<td>65213</td>
<td>236774</td>
<td>816</td>
<td>453851</td>
</tr>
<tr>
<td>Building construction</td>
<td>69495</td>
<td>59666</td>
<td>3624</td>
<td>33835</td>
</tr>
<tr>
<td>Miscellaneous construction</td>
<td>53675</td>
<td>68526</td>
<td>3734</td>
<td>37094</td>
</tr>
<tr>
<td>Office supplies</td>
<td>145474</td>
<td>56522</td>
<td>3146</td>
<td>57429</td>
</tr>
<tr>
<td>Scraps</td>
<td>8902</td>
<td>12466</td>
<td>612</td>
<td>4975</td>
</tr>
<tr>
<td>Undistributed</td>
<td>60336</td>
<td>36822</td>
<td>15166</td>
<td>68993</td>
</tr>
</tbody>
</table>

* This Table is shown in transposed form, interchanging rows and columns of matrix \(B_1\) for convenience.
### Table 3. Coefficients of Internal Propagation in Goods-Producing Sector induced by Input in Service Sector

<table>
<thead>
<tr>
<th>Sector</th>
<th>Trade</th>
<th>Transportation and Communication</th>
<th>Real estate</th>
<th>Banking, Insurance and Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rice, wheat and barley</td>
<td>3089</td>
<td>2242</td>
<td>783</td>
<td>29239</td>
</tr>
<tr>
<td>Miscellaneous crops</td>
<td>3990</td>
<td>4058</td>
<td>4637</td>
<td>19022</td>
</tr>
<tr>
<td>Livestock</td>
<td>1169</td>
<td>1079</td>
<td>571</td>
<td>14622</td>
</tr>
<tr>
<td>Forestry</td>
<td>11461</td>
<td>15325</td>
<td>12491</td>
<td>10104</td>
</tr>
<tr>
<td>Fisheries</td>
<td>2193</td>
<td>1156</td>
<td>417</td>
<td>18837</td>
</tr>
<tr>
<td>Coal and lignite</td>
<td>3005</td>
<td>29780</td>
<td>3805</td>
<td>8463</td>
</tr>
<tr>
<td>Crude petroleum and natural gas</td>
<td>73</td>
<td>1616</td>
<td>60</td>
<td>152</td>
</tr>
<tr>
<td>Metallic ores</td>
<td>430</td>
<td>1642</td>
<td>2057</td>
<td>938</td>
</tr>
<tr>
<td>Non-metallic minerals</td>
<td>692</td>
<td>2136</td>
<td>2169</td>
<td>1191</td>
</tr>
<tr>
<td>Rice and barley polishing and grain-flour mills</td>
<td>2725</td>
<td>1559</td>
<td>582</td>
<td>22837</td>
</tr>
<tr>
<td>Starch, sugar, seasonings, etc.</td>
<td>3396</td>
<td>3126</td>
<td>754</td>
<td>13335</td>
</tr>
<tr>
<td>Manufactured tobacco and beverages</td>
<td>7770</td>
<td>1782</td>
<td>708</td>
<td>39452</td>
</tr>
<tr>
<td>Miscellaneous processed foods</td>
<td>7226</td>
<td>8824</td>
<td>2185</td>
<td>51582</td>
</tr>
<tr>
<td>Natural fibre yarns</td>
<td>3240</td>
<td>4244</td>
<td>1263</td>
<td>4274</td>
</tr>
<tr>
<td>Chemical fibre yarns</td>
<td>1420</td>
<td>1770</td>
<td>374</td>
<td>1610</td>
</tr>
<tr>
<td>Fabrics</td>
<td>4038</td>
<td>4231</td>
<td>912</td>
<td>4454</td>
</tr>
<tr>
<td>Miscellaneous textile products</td>
<td>3241</td>
<td>4515</td>
<td>465</td>
<td>4332</td>
</tr>
<tr>
<td>Saw-mills and plywood</td>
<td>11377</td>
<td>17033</td>
<td>15608</td>
<td>5156</td>
</tr>
<tr>
<td>Furniture and wood products</td>
<td>7463</td>
<td>3619</td>
<td>1387</td>
<td>2959</td>
</tr>
<tr>
<td>Pulp</td>
<td>7519</td>
<td>3273</td>
<td>749</td>
<td>4573</td>
</tr>
<tr>
<td>Paper and paper products</td>
<td>27839</td>
<td>11564</td>
<td>2640</td>
<td>16491</td>
</tr>
<tr>
<td>Printing and publishing</td>
<td>26221</td>
<td>10531</td>
<td>2566</td>
<td>28052</td>
</tr>
<tr>
<td>Coal products</td>
<td>1142</td>
<td>13318</td>
<td>2587</td>
<td>4038</td>
</tr>
<tr>
<td>Petroleum products</td>
<td>1739</td>
<td>46832</td>
<td>1743</td>
<td>3608</td>
</tr>
<tr>
<td>Basic chemicals</td>
<td>1608</td>
<td>2455</td>
<td>1079</td>
<td>4008</td>
</tr>
<tr>
<td>Chemical fertilizers</td>
<td>579</td>
<td>578</td>
<td>465</td>
<td>3375</td>
</tr>
<tr>
<td>Intermediate chemicals</td>
<td>5821</td>
<td>7316</td>
<td>4482</td>
<td>18662</td>
</tr>
<tr>
<td>Drugs, soap and cosmetics</td>
<td>4300</td>
<td>1521</td>
<td>462</td>
<td>21077</td>
</tr>
<tr>
<td>Rubber products</td>
<td>1497</td>
<td>8185</td>
<td>2142</td>
<td>4592</td>
</tr>
<tr>
<td>Leather and leather products</td>
<td>492</td>
<td>445</td>
<td>215</td>
<td>1294</td>
</tr>
<tr>
<td>Non-metallic mineral products</td>
<td>2238</td>
<td>7239</td>
<td>16036</td>
<td>5135</td>
</tr>
<tr>
<td>Pig iron, ferro-alloys, crude steel</td>
<td>3897</td>
<td>15750</td>
<td>19479</td>
<td>7558</td>
</tr>
<tr>
<td>Steel casting and forging</td>
<td>1117</td>
<td>4956</td>
<td>12340</td>
<td>2441</td>
</tr>
<tr>
<td>Rolled steel</td>
<td>4982</td>
<td>20187</td>
<td>22222</td>
<td>9382</td>
</tr>
<tr>
<td>Non-ferrous metal ingots</td>
<td>1038</td>
<td>3919</td>
<td>4924</td>
<td>2325</td>
</tr>
<tr>
<td>Primary non-ferrous metal products</td>
<td>1621</td>
<td>7347</td>
<td>8507</td>
<td>2412</td>
</tr>
<tr>
<td>Metal products</td>
<td>3532</td>
<td>7256</td>
<td>17153</td>
<td>6619</td>
</tr>
<tr>
<td>Machinery and instruments</td>
<td>1924</td>
<td>7657</td>
<td>15969</td>
<td>5234</td>
</tr>
<tr>
<td>Electric machinery and equipment</td>
<td>1800</td>
<td>4128</td>
<td>11355</td>
<td>1690</td>
</tr>
<tr>
<td>Transport equipment</td>
<td>539</td>
<td>15919</td>
<td>4408</td>
<td>9116</td>
</tr>
<tr>
<td>Repair and maintenance of machinery, etc.</td>
<td>12516</td>
<td>51220</td>
<td>168849</td>
<td>13629</td>
</tr>
<tr>
<td>Miscellaneous manufactures</td>
<td>5627</td>
<td>1773</td>
<td>1358</td>
<td>5097</td>
</tr>
<tr>
<td>Electricity</td>
<td>6751</td>
<td>15826</td>
<td>3510</td>
<td>14921</td>
</tr>
<tr>
<td>Gas and water services</td>
<td>1427</td>
<td>2142</td>
<td>544</td>
<td>6594</td>
</tr>
<tr>
<td>Business consumption expenditure</td>
<td>81569</td>
<td>18212</td>
<td>7124</td>
<td>41333</td>
</tr>
<tr>
<td>Building construction</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Miscellaneous construction</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Office supplies</td>
<td>37665</td>
<td>4861</td>
<td>1496</td>
<td>8926</td>
</tr>
<tr>
<td>Scraps</td>
<td>3396</td>
<td>7789</td>
<td>9057</td>
<td>6812</td>
</tr>
<tr>
<td>Undistributed</td>
<td>45826</td>
<td>82680</td>
<td>16110</td>
<td>42035</td>
</tr>
</tbody>
</table>
Of course, from the viewpoint of the goods-producing sector, the sub-multiplier \( B_i \) operates on that sector only in an indirect manner in the sense that it needs a medium operator expressed by \( M = KT \) as shown by the equation (6). The values of elements of \( K \) and \( T \) are summarized in the Table 4 which shows the powers of dispersion of service sectors internally and externally. On the whole, many values of the elements in the internal multiplier \( T \) are somewhat higher than those in the external multiplier \( K \) (except Real estate's column), but the difference between the values of these two multipliers is not so large. This fact means again that the weight of dependence of the service sector on the goods-producing sector is considerably large in its character.

**Table 4. Internal and External Multipliers in Service Sector**

(1) Internal Multiplier of Service Sector: \( T \)

<table>
<thead>
<tr>
<th></th>
<th>Trade</th>
<th>Transportation and Communication</th>
<th>Real estate</th>
<th>Banking, Insurance and Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trade (wholesale and retail)</td>
<td>1.006382</td>
<td>14208</td>
<td>1618</td>
<td>40780</td>
</tr>
<tr>
<td>Transportation and communication</td>
<td>49969</td>
<td>1.020766</td>
<td>1205</td>
<td>30004</td>
</tr>
<tr>
<td>Real estate</td>
<td>20839</td>
<td>4942</td>
<td>1.000402</td>
<td>12839</td>
</tr>
<tr>
<td>Banking, insurance and services</td>
<td>61886</td>
<td>54474</td>
<td>31979</td>
<td>1.042842</td>
</tr>
</tbody>
</table>

(2) External Multiplier of Service Sector: \( K \)

<table>
<thead>
<tr>
<th></th>
<th>Trade</th>
<th>Transportation and Communication</th>
<th>Real estate</th>
<th>Banking, Insurance and Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trade (wholesale and retail)</td>
<td>1.018500</td>
<td>14691</td>
<td>13090</td>
<td>21801</td>
</tr>
<tr>
<td>Transportation and communication</td>
<td>28365</td>
<td>1.015694</td>
<td>10241</td>
<td>21579</td>
</tr>
<tr>
<td>Real estate</td>
<td>2004</td>
<td>2499</td>
<td>1.000991</td>
<td>2099</td>
</tr>
<tr>
<td>Banking, insurance and services</td>
<td>49081</td>
<td>20060</td>
<td>8373</td>
<td>1.034038</td>
</tr>
</tbody>
</table>

Here, we switch our topics from the quantity-determination model in the input-output system to the price-determination model in that system and turn to a study of the cost-push effects of service-prices on the prices of P sector's products.

Obviously, the prices of P sector's products are given by the equation:

\[
P_p = A'P_p + S'_p + v_p
\]  

(8)

where \( P_p, P_t \) are vectors of prices of P sector's products and S sector's service-outputs respectively, \( v_p \) is the vector of value-added per unit of output in P sector's industries, and the coefficient matrices \( A' \) and \( S'_t \) are the transpose of the matrices \( A \) and \( S_t \) in the quantity model.

This price formation equation (8) is a part of the following larger model:

\[
\begin{align*}
P_p &= A'P_p + S'_p + v_p \\
P_t &= A_t'P_p + S'_t + v_t
\end{align*}
\]  

(9)

In this system, we take \( P_t \) and \( v_p \) as data, and \( P_p \) and \( v_t \) as variables. The variations of \( P_p \) is due to cost-push effects, and, if we wish, the variations of \( v_t \) could be viewed as the resultant change in wages or profits in the S sector due to the variation in prices of P sector's
products, but here we omit this latter relation. Of course, the selection between data and variables is dependant upon the setting of the problems.

Then, price-determination in the goods-producing sectors is given by the equation:

\[ P_p = (I - A)^{-1} \{ S^t P + v_p \} = B^t \{ S^t P + v_p \} \]

where \( B^t \) is the transpose of the internal matrix multiplier of the \( P \) sector in the quantity model.

If service-prices rise from \( P^* \) to \( P^* + \Delta P^* \), the resultant price-rise in \( P \) sector's products will be

\[ \Delta P_p = B^t S^t P^* (\Delta P^*) = B^t S^t P^* \]

Thus in order to know the cost-push effect of a rise in service prices on the prices of \( P \) sector's products, all that is needed is the transposition of the sub-matrix-multiplier \( B^t \).

Going back to Table 2 and rereading it from the viewpoint of cost-push effects, we may discover some new facts. Especially we see: (a) relatively more stimulated effects are brought by the rise of the service-price in the Trade sector and the Transportation and communication sector than by the price rise in the other service sectors and (b) the resultant higher price rise is concentrated into some particular commodities such as Starch, sugar, seasonings and Miscellaneous processed foods in the case of Trade service-cost, and Coal products and Basic chemicals in the case of Transportation and communication cost.

A comment is needed to evaluate the figures in Table 2 viewed as the cost-push effects. There is somewhat of a tendency to overvalue those figures more than the theoretically expected ones, because most of the arguments for constant input coefficients depend on the absence of variation in relative price, and changes in the relative prices bring the substitution effects between inputs and set limits on price rises of the cost-push type. However, on the contrary, the rising trend in the service-input coefficients in recent Japanese industries leads to an undervaluation of the actual values of \( B^t \), because the Table's figures are based on somewhat old data.

IV. Interregional Repercussion Model

The main purpose of the interregional input-output model developed by the works of Isard, Leontief, Moses, Chenery and others, is to analyze the interrelations among trade and production in two or more regions. Our internal-and-external matrix multiplier model may be well applied to this purpose in a somewhat extended form, because the usual inverse of interregional input-output model tells us only the ultimate total effects but not disjoined effects separating into interdependence between regional internal and external multipliers.

One example of an interregional input-output table in Japan is the data published by the Hokkaido Development Bureau which divides the Japanese economy into two regions having

---

same number of industries (30 sectors and 2 regions). In our formula (6), we use the $P$ sector as Hokkaido and the $S$ sector as the Rest of Japan. For convenience we call the latter region Honshū, the main island of Japan. Of course, here $l=m=30$.

Table 5 is concerned with the internal and external matrix multipliers in each region, but only cites the column sum or row sum of the elements of the matrices because of limited space. The economic meaning of the column sum is to summarize the pattern of "the power of dispersion" of industries in each region, and the meaning of the row sum is to express

<table>
<thead>
<tr>
<th>TABLE 5. SUMMARY TABLE OF INTERNAL AND EXTERNAL MULTIPLIERS OF AN INTERREGIONAL MODEL OF THE JAPANESE ECONOMY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Internal multiplier of Hokkaido, $B$</strong></td>
</tr>
<tr>
<td>-----------------------------------------</td>
</tr>
<tr>
<td>row sum</td>
</tr>
<tr>
<td>Public utilities</td>
</tr>
<tr>
<td>Metal mining</td>
</tr>
<tr>
<td>Non-metal mining</td>
</tr>
<tr>
<td>Petroleum and natural gas</td>
</tr>
<tr>
<td>Coal mining</td>
</tr>
<tr>
<td>Processed foods</td>
</tr>
<tr>
<td>Textiles</td>
</tr>
<tr>
<td>Saw-mill and plywood</td>
</tr>
<tr>
<td>Pulp, paper and products</td>
</tr>
<tr>
<td>Chemicals</td>
</tr>
<tr>
<td>Coal products</td>
</tr>
<tr>
<td>Rubber products</td>
</tr>
<tr>
<td>Leather and products</td>
</tr>
<tr>
<td>Nonmetallic mineral products</td>
</tr>
<tr>
<td>Iron and steel</td>
</tr>
<tr>
<td>Nonferrous metal products</td>
</tr>
<tr>
<td>Steel products</td>
</tr>
<tr>
<td>Machinery</td>
</tr>
<tr>
<td>Lumber and products</td>
</tr>
<tr>
<td>Printing and publishing</td>
</tr>
<tr>
<td>Miscellaneous manufactures</td>
</tr>
<tr>
<td>Forestry</td>
</tr>
<tr>
<td>Fishing</td>
</tr>
<tr>
<td>Agriculture</td>
</tr>
<tr>
<td>Dummy sector</td>
</tr>
<tr>
<td>Service</td>
</tr>
<tr>
<td>Business consumption</td>
</tr>
<tr>
<td>Trade</td>
</tr>
<tr>
<td>Transportation</td>
</tr>
<tr>
<td>Undistributed</td>
</tr>
<tr>
<td><strong>(Average)</strong></td>
</tr>
</tbody>
</table>
"the sensitivity of dispersion" for industries in each region.

As shown by the figures in the Table, the internal propagation in Hokkaido (B) has a multiplier effect of 1.77 on the average, and it calls in turn the round about external repercussion through Honshū's industrial activity (L) to the amount of about a 0.7%-up effect on the average, so that the total effect equals to $1.77 \times 1.007 = 1.782$ on the average. On the contrary, the internal multiplier effect in Honshū (T) has a considerably higher value of 2.53, but the round about external multiplier (K) shows only a 0.3%-up effect on the average. This industrial differential that is shown by the values of the external multipliers L and K suggests the characteristics of each region's industrial activity according to its role in the national economy.

**Table 6. Some Coefficients of Inducement to Production per Unit of Input in the Other Region**

<table>
<thead>
<tr>
<th>(a) Row sum of elements of $T_2$</th>
<th>(c) Row sum of elements of $B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Honshū's industry</strong></td>
<td><strong>Hokkaido's industry</strong></td>
</tr>
<tr>
<td>Iron and steel</td>
<td>Coal mining</td>
</tr>
<tr>
<td>Textiles</td>
<td>.0812</td>
</tr>
<tr>
<td>Chemicals</td>
<td>Pulp, paper and products</td>
</tr>
<tr>
<td>Agriculture</td>
<td>.0614</td>
</tr>
<tr>
<td>Machinery</td>
<td>Iron and steel</td>
</tr>
<tr>
<td>Metal mining</td>
<td>.0448</td>
</tr>
<tr>
<td>Leather and products</td>
<td>Forestry</td>
</tr>
<tr>
<td>Coal products</td>
<td>.0284</td>
</tr>
<tr>
<td>Pulp, paper and products</td>
<td>Transportation</td>
</tr>
<tr>
<td>(Average)</td>
<td>.0277</td>
</tr>
<tr>
<td>(b) Column sum of elements of $T_2$</td>
<td>(d) Column sum of elements of $B_2$</td>
</tr>
<tr>
<td><strong>Hokkaido's industry</strong></td>
<td><strong>Honshū's industry</strong></td>
</tr>
<tr>
<td>Steel products</td>
<td>Printing and publishing</td>
</tr>
<tr>
<td>Leather and products</td>
<td>.0577</td>
</tr>
<tr>
<td>Machinery</td>
<td>Coal products</td>
</tr>
<tr>
<td>Nonmetallic mineral products</td>
<td>.0520</td>
</tr>
<tr>
<td>Textiles</td>
<td>Fishing</td>
</tr>
<tr>
<td>Miscellaneous manufactures</td>
<td>.0419</td>
</tr>
<tr>
<td>Rubber products</td>
<td>Public utilities</td>
</tr>
<tr>
<td></td>
<td>.0374</td>
</tr>
<tr>
<td>(Average)</td>
<td>Lumber products</td>
</tr>
<tr>
<td></td>
<td>.0357</td>
</tr>
<tr>
<td></td>
<td>Pulp, paper and products</td>
</tr>
<tr>
<td></td>
<td>.0319</td>
</tr>
<tr>
<td></td>
<td>Nonmetallic mineral products</td>
</tr>
<tr>
<td></td>
<td>.0255</td>
</tr>
<tr>
<td></td>
<td>Steel products</td>
</tr>
<tr>
<td></td>
<td>.0211</td>
</tr>
<tr>
<td></td>
<td>Processed foods</td>
</tr>
<tr>
<td></td>
<td>.0168</td>
</tr>
<tr>
<td>(Average)</td>
<td>(Average) .0159</td>
</tr>
</tbody>
</table>

* Sectors listed here are the industries having row sum or column sum values higher than the average.

** Table (a) or (c) lists the names of the industry receiving the induced effects, and (b) or (d) lists the names of the industry giving the induced effects.
To see this point more clearly, the "inside propagation ratios" (its definition is the same as before) of Hokkaido's industries are calculated in the $30 \times 30$ matrix base. Although the table of the calculated figures is omitted, from it we find that the most self-sufficing industries in Hokkaido are those in the light industry group such as textiles, rubber products, leather and leather products, printing and publishing, miscellaneous manufactures, and those in the non-manufacture group such as services, trade, public utilities.

The industries in this category are relatively independent of the Honshū's industrial activity, and their "inside propagation ratios" all take values more than 0.9. At the other extreme,

**Table 7. Some Coefficients of Inducement to Input by Internal Propagation in the Other Region**

<table>
<thead>
<tr>
<th>(a) Row sum of elements of $B_1$</th>
<th>(c) Row sum of elements of $T_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Honshū's industry</strong></td>
<td><strong>Hokkaido's industry</strong></td>
</tr>
<tr>
<td>Iron and steel</td>
<td>Coal mining</td>
</tr>
<tr>
<td>Machineries</td>
<td>Pulp paper and products</td>
</tr>
<tr>
<td>Chemicals</td>
<td>Iron and steel</td>
</tr>
<tr>
<td>Textiles</td>
<td>Processed foods</td>
</tr>
<tr>
<td>Leather and products</td>
<td>Transportation</td>
</tr>
<tr>
<td>Processed foods</td>
<td>Fishing</td>
</tr>
<tr>
<td>Metal mining</td>
<td>Agriculture</td>
</tr>
<tr>
<td>Agriculture</td>
<td>Nonmetallic mineral products</td>
</tr>
<tr>
<td>Coal products</td>
<td></td>
</tr>
<tr>
<td>Pulp, paper and products</td>
<td></td>
</tr>
<tr>
<td>Steel products</td>
<td></td>
</tr>
<tr>
<td><strong>(Average)</strong></td>
<td><strong>(Average)</strong></td>
</tr>
<tr>
<td></td>
<td>.1847</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b) Column sum of elements of $B_1$</th>
<th>(d) Column sum of elements of $T_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Honshū's industry</strong></td>
<td><strong>Hokkaido's industry</strong></td>
</tr>
<tr>
<td>Steel products</td>
<td>Iron and steel</td>
</tr>
<tr>
<td>Nonmetallic mineral products</td>
<td>Coal products</td>
</tr>
<tr>
<td>Leather and products</td>
<td>Printing and publishing</td>
</tr>
<tr>
<td>Machineries</td>
<td>Fishing</td>
</tr>
<tr>
<td>Textiles</td>
<td>Pulp, paper and products</td>
</tr>
<tr>
<td>Miscellaneous manufactures</td>
<td>Public utilities</td>
</tr>
<tr>
<td>Fishing</td>
<td>Nonmetallic mineral products</td>
</tr>
<tr>
<td>Nonferrous metal products</td>
<td>Steel products</td>
</tr>
<tr>
<td>Chemicals</td>
<td>Lumber products</td>
</tr>
<tr>
<td><strong>(Average)</strong></td>
<td><strong>(Average)</strong></td>
</tr>
<tr>
<td></td>
<td>.1847</td>
</tr>
</tbody>
</table>

* Sectors listed here are the industries having row sum or column sum values higher than the average.
** Table (a) or (c) lists the names of the industry receiving the induced effects, and (b) or (d) lists the names of the industry giving the induced effects.
there is a group highly dependent on Honshū’s industrial activity which includes such heavy industries as iron and steel, non-ferrous metal products, and the resource industries such as metal mining, non-metal mining, pulp, paper and paper products, fishing, etc.

Such internal propagation patterns together with the external input patterns of interindustrial activity in each region depict the characteristics of interregional repercussions whose estimated results are summarized in Tables 6 and 7. In these tables the coefficients of interregional inducement relations are shown in a summary form designating the column sum and row sum of the elements of four sub-multiplier matrices $B_2$, $T_2$, $B_1$ and $T_1$. The names of industries listed here are only those having higher values than the average.

The sub-multipliers $B_2 = BA_1$ and $T_2 = TS_1$ are concerned with the propagation of production activities in each region induced by the input activity in the other region. Reflecting the high dependence of Hokkaido’s activities on Honshū’s industries, the elements of the multiplier $T_2$ have higher values than those of the multiplier $B_2$, and their average values take 0.3806 versus 0.0159. A similar situation is found in the comparison between $B_1 = S_1B$ and $T_1 = A_1T$ which shows the input inducement effects of one region on the other, and their values take 0.1847 versus 0.0207 on the average. These results suggest that a development program in Hokkaido gives rise to many leakages in the interregional production process and generates much benefit to Honshū’s industries.

We cite here only one point for example: the production of Honshū’s iron and steel industry induced by Hokkaido’s input is the extremely high row sum value of 2.303 as shown the Table 6-(a). This high value has its origin in Hokkaido’s industries such as steel products and machinery as shown the column sum figures in the Table 6-(b). Of course, to see the details of cross-effects of this sort, it is necessary to trace back the test of the figures of elements in these matrices themselves instead of the test of column sum or row sum values.

In any case, such analyses play a role which elucidates the inherent properties of the inter-regional industrial relationships, and we may be expected a fruitful application of this method (combining the extended model in the next section) to the forthcoming comprehensive data of the Japanese interregional input-output table compiled by the Ministry of International Trade and Industry. This will enable us to make the various combination of nine regions of the Japanese economy.7

V. Some Extensions of the Model

The method which was suggested in this paper express one direction of the extension of the input-output model that may be called the “intensive type”. The another direction of the extension may be the “extensive type” which combines the input-output model in other various models such as the macro-econometric model, the linear programming model, and so forth. But, in this section, we are again concerned with the intensive type, and limit our interest to the extension of the above internal-and-external matrix multiplier model on the following two points.

7 The report on this MITI 9 blocks-interregional input-output table that took three prepared years will be published in the autumn of this year. The nine regions are Hokkaido, Tōhoku, Hokuriku, Kantō, Tōkai, Kinki, Chūgoku, Shikoku and Kyūshū.
1) The Extension to the Number of Partitioned Groups

The line of reasoning which was employed in section 11 could be extended to increase the number of partitioned industry groups (or regions), that is, the two-partitioned model must be enlarged to three or more partitioned models. However, the straightforward procedure of dividing the model into a large number of industry-groups (or regions) degenerates into formalism, or apt to be too complex. We must not adhere too much to form at the expense of the empirical spirit. So, we prefer to employ “a method of localizable partition” in which the redivision of groups limited to some strategic particular part of industries (or regions).

The procedure of the extension employed here consists of two steps.

Step 1. The first step is the re-partition of some “internal matrix multiplier”, say the internal matrix multiplier of S sector, \(T\). If we redivide the S sector into two sub-sections designated section S-1 and section S-2, it will lead to the repartitioned matrix of input coefficients as follows:

\[
A^{*} = \begin{bmatrix}
S_{1} & S_{2} \\
S_{1} & S_{2}
\end{bmatrix}
\begin{bmatrix}
P \\
S_{1} \\
S_{2}
\end{bmatrix}
\]

From the above coefficient matrix, we get the following relationships for S sector’s industrial activities:

i) Internal multipliers of section S-1 and S-2 in the S sector;

\[
Q = (I - u)^{-1} j \times j \quad (13)
\]

\[
R = (I - v)^{-1} k \times k \quad (14)
\]

ii) Intersectional (not intersectoral) sub-multipliers showing the inducement to production between sections S-1 and S-2 inside the S sector;

\[
Q_{u_{i}} = a j \times k \quad (15)
\]

\[
R_{v_{i}} = P k \times j \quad (16)
\]

iii) Localized external multipliers of section S-1 and S-2 inside the S sector (i.e., the meaning of “external” in this case is limited to the inside of S sector and not extends beyond it);

\[
U = (I - \alpha \beta)^{-1} j \times j \quad (17)
\]

\[
V = (I - \beta \alpha)^{-1} k \times k \quad (18)
\]

Then, according to the formula (6a), we have the internal multiplier of the whole S sector in the redivided form, i.e.,

\[
T = (I - S)^{-1} = \begin{bmatrix}
I - u & -u_{i} \\
-v_{i} & I - v
\end{bmatrix}^{-1} = \begin{bmatrix}
UQ & UaR \\
V\beta Q & VR
\end{bmatrix} \quad (19)
\]

Step 2. The second procedure is the generalization of the above localized multipliers into the relationships including the interaction between P and S sectors. The pattern of this inter-and-intra sectoral relations is given by the following Chart.

In this Chart, \(B, Q, R\) inside the three circles denote the internal multipliers of the P sector, S-1 section, S-2 section respectively. \(L, U, V\) outside the circles denote the external multipliers of P sector, S-1 section, and S-2 section respectively.
With the aid of this Chart, we get six routes of inter-and-intra sectoral inducement relationships shown as follows:

<table>
<thead>
<tr>
<th>&lt;inducement routes&gt;</th>
<th>&lt;sub-multipliers&gt;</th>
<th>&lt;order of matrices&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) S-2 → S-1</td>
<td>α = Qu₁</td>
<td>jxk</td>
</tr>
<tr>
<td>S-1 → S-2</td>
<td>β = Rv₁</td>
<td>kxj</td>
</tr>
<tr>
<td>(b) P → S-1</td>
<td>σ = Q(u₀ + u₁Rv₀)</td>
<td>jxl</td>
</tr>
<tr>
<td>P → S-2</td>
<td>τ = R(v₀ + v₁Qu₀)</td>
<td>kxl</td>
</tr>
<tr>
<td>(c) S-1 → P</td>
<td>λ = B(a₁U + a₂VRv₁)</td>
<td>lxj</td>
</tr>
<tr>
<td>S-2 → P</td>
<td>μ = B(a₁V + a₂UQu₁)</td>
<td>lxk</td>
</tr>
</tbody>
</table>

The coming into existence of these six sub-multipliers showing induced effects on production activity may be easily verified by tracing the repercussion routes between the above sectors or sections in the Chart.

We have now arrived at a formula of the partitioned matrix multiplier in this case which can be stated as follows:

\[
(I - A^*)^{-1} = \begin{bmatrix}
    LB & L(\lambda Q, \mu R) \\
    \left( \begin{array}{c}
        Uα \\
        Vτ 
    \end{array} \right) & \begin{bmatrix}
        UQ & UαR \\
        VβQ & VR + \left( \begin{array}{c}
            Uα \\
            Vτ 
        \end{array} \right)L(\lambda Q, \mu R) 
    \end{bmatrix}
\end{bmatrix}
\]

or

\[
(I - A^*)^{-1} = \begin{bmatrix}
    LB & L(\lambda Q, \mu R) \\
    \left( \begin{array}{c}
        UαLB \\
        VτLB 
        \end{array} \right) & \begin{bmatrix}
        UQ & UαR(L(\lambda Q, \mu R)) \\
        VβQ & VR + UαLQ + U(α + αLμ)R 
    \end{bmatrix}
\end{bmatrix}
\]

The main course of the derivation of the formula (20) is that the system

\[
\begin{align*}
    X_p &= AX_p + a_0X_u + a_1X_v + Y_p \\
    X_u &= a_0X_p + uX_u + u_1X_v + Y_u \\
    X_v &= v_0X_p + v_1X_u + vX_v + Y_v
\end{align*}
\]

can be solved in a partiality form for the production level of S sector (regarded as equations (2) and (3)) by considering the economic causal succession on routes of the induced effects. The result is

\[
\begin{bmatrix}
    X_u \\
    X_v \\
    Y_u \\
    Y_v
\end{bmatrix} = \begin{bmatrix}
    Uα & UQ & UαR \\
    Vτ & VβQ & VR
\end{bmatrix} \begin{bmatrix}
    X_p \\
    Y_p
\end{bmatrix}.
\]
Substituting this equation into (1) and collecting terms gives the formula (20).

From this formula, we may see that the external (not localized) matrix multipliers of sections \( S-1 \) and \( S-2 \) are equal to \( U(I+\alpha L\alpha) \) and \( V(I+\tau L\mu) \) respectively.

In application of the formula for the practical problems, we may have two advantages:

1. the numbers of industries in each partitioned sector (or section) is not necessary the same (i.e., \( l \neq j \neq k \)), and
2. the above treatment can be adapted to the further subdivision of the particular part of strategic sectors or sections in succession, so we get a method of studying the various characters of industry groups (or regions) according to their differing roles in the national economy.

2) The Inclusion of the Income Formation Process

The next extension of our model is the inclusion of the income generation process which is omitted in the usual input-output model. This omission is justified only if the level of income and its use do not depend on the composition of production, because in this case a disaggregation of income generated by sector will add nothing to an analysis of the aggregated Keynesian type. But, under less rigid assumptions this procedure is no longer valid, especially in the interregional model. The location of production depends on the location of consumption, and the latter cannot be determined separately from the calculation of the income generated in each region.

In the usual extension of this, the household sector is transferred to the processing sectors from exogenous sectors and is regarded as an industry whose output is labor and whose inputs are consumption goods as shown in an actual example of Chenery's Italian regional model. But a more correct procedure in dealing with consumption is not to regard it as a fictitious production activity, but to introduce the consumption function of a Keynesian type in a disaggregated form.

As shown in another of the author's papers, this latter procedure means, by implication, combining the Leontief propagation process and the Keynesian propagation process in a disaggregated form. In its formulated multiplier equation it takes the shape of an "extended matrix multiplier" using the Leontief inverse multiplied by the subjoined inverse matrix showing the effects of endogenous changes in consumption demand. This proposal distinguishing the inverse reflecting production activities from the inverse reflecting consumption activities, may be well combined with our internal-and-external matrix multiplier model.

A summarized version of this combination is that: if we term \( E \) the "enlarged matrix multiplier" including the income formation process and take a case as example in which the economy consists of three partitioned industry groups (or regions) having the number of industries \( l, j, \) and \( k \) respectively, then we have the extended multiplier equation in this case as follows:

---

\[ E = B^* (I - C H B^*)^{-1} \]
\[ = B^* (I + C Z H B^*) \]  \hspace{1cm} (21)

where \( B^* = (I - A^*)^{-1} \) is the Leontief inverse having the order \((l+j+k) \times (l+j+k)\), \( C \) is the matrix of consumption coefficients having \(l+j+k\) rows and three columns, and \( H \) is the matrix of the value-added ratios having three rows and \(l+j+k\) columns. The subjoined inverse matrix \((I - C H B^*)^{-1}\) in the first expression of the equation (21) shows the effects of endogenous changes in each group’s (or region’s) consumption expenditure, and it can be converted into the form which is shown as the second expression of the equation (21).

In the latter form, the matrix \( Z \) is to be written the expression that
\[ Z = (I - H B^* C)^{-1} \]  \hspace{1cm} (22)

which can be called the “multi-sector income multiplier” having three rows and three columns. The economic meaning of this matrix \( Z \) is that its elements show how much income in one group (or region) is generated by the expenditure from 1 unit of additional income in the other group (or region), but the details on the above formulations and the characters of the matrix \( Z \) are omitted in this paper.\(^{11}\)

\(^{11}\) See, K. Miyazawa and S. Masegi. op. cit., especially pp. 91-97.