INTERNAL AND EXTERNAL MATRIX MULTIPLIERS IN THE INPUT-OUTPUT MODEL*

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I. *Introduction*

There are many useful applications if we divide n industries in the input-output tables into two or more strategic industry groups and trace back to the interaction between these groups. To mention a few such examples, we have the interactions between the goods-producing sectors and service sectors, between the primary growth sectors and the supplementary or derived growth sectors, and between two regions which have structural different characters. Another example is found in the necessity of distinguishing industries subject to capacity limitations from those which have plenty of capacity. Examples of this sort might be given by the hundreds.

In the usual input-output analysis, the $n \times n$ inverse matrix shows the ultimate total effects of interindustrial propagation, but it cannot tell us the disjoined effects separating into these two distinguished activities. This paper is an attempt to clarify the problems of this type by means of the formulation of partitioned matrix multipliers and their relationships, and to apply our formula to the input-output data in two cases: 1) the interdependent model of goods-producing sectors and service sectors, 2) interregional repercussion model of the Japanese economy.

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* The mathematical part of this paper is a summarized version of Chap. 4 and Appendix C in the writer's *The Structural Interdependence Analysis of Economy* (in Japanese), 1963, and the empirical applications are based on the author's articles (in Japanese): "Interdependent Structure of Goods-Producing Sectors and Service Sectors", *Economic Review (Keizai Kenkyu)*, Vol. 14, No. 3, July 1963, and "An Interregional Input-Output Model and its Application", *Monthly Survey of Japan Industrial Structure Institute*, No. 52, Jan. 1965. I have introduced some improvements in this rewriting.

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II. *Partitioned Matrix Multipliers*

We divide n industries in the usual input-output table into two subgroups designated P sector which consists of l industries and S sector which consists of m industries. Then, the $n\times n$ matrix of input coefficients is

$$A^*=\left[\begin{array}{c|c} \overbrace{A}^{l} & \overbrace{A_1}^{m} \\ \hline S_1 & S \end{array}\right]\begin{array}{l} \}l \\ \}m \end{array} \qquad l+m=n \tag{1}$$

where A, A_1 are submatrices of coefficients showing the input of P sector's products in the P and S sectors respectively, and S_1, S are submatrices of coefficients showing the input of S sector's products in the P and S sectors respectively. Among these submatrices, A and S are square having the order $l\times l$ and $m\times m$, A_1 and S_1 are rectangular having the order $l\times m$ and $m\times l$.

Since the $n\times n$ Leontief inverse

$$B^*=(I-A^*)^{-1}$$

tells us only the total ultimate effects but not the disjoined interdependence of the above two activities, we must introduce some device consisting of partitioned matrix multipliers. In order to solve this problem, we decompose the elements of the Leontief inverse into three sides of propagation aspects, i. e.,

(i) internal propagation activities inside P sector's industries,

(ii) internal propagation activities inside S sector's industries,

(iii) intersectoral propagation activities between P and S sectors' industries.

For aspects (i) and (ii), we may term the $l\times l$ inverse

$$B=(I-A)^{-1} \tag{2}$$

the *internal matrix multiplier of the* P *sector* and the $m\times m$ inverse

$$T=(I-S)^{-1} \tag{3}$$

the *internal matrix multiplier of the* S *sector*, then these two "internal matrix multipliers" show the interindustrial propagation effects in the inside of each sector. Of course, each internal matrix multiplier does not operate independently under its own power, but is able to operate with the other sector's industrial activity.

So that, according to the economic causal process, the intersectoral propagations accompanied by the operation of internal multipliers B and T can be written as the form of four rectangular *sub-matrix-multipliers*, which express the aspect (iii), i. e.,

$B_1=S_1B$ —S-goods-input in P sector induced by internal propagation in P sector's industries $(m\times l)$.

$B_2=BA_1$—internal propagation in P sector's industries induced by P-goods-input in S sector $(l\times m)$.

$T_1=A_1T$—P-goods-input in S sector induced by internal propagation in S sector's industries $(l\times m)$.

$T_2=TS_1$ —internal propagation in S sector's industries induced by S-goods-input in P sector $(m\times l)$.

These sub-multipliers B_1, B_2, T_1 and T_2 show the coefficients of induced effects on output or input activities between two sectors and call themselves the production-generating process in succession.

Such a repercussion process due to these induced effects naturally leads to the intersectoral multiplier between the P and S sectors. If we select the coefficients of the induced effect on production (i. e., B_2 and T_2) as the base of this intersectoral multiplier, then it will take the form:

$$K=(I-T_2B_2)^{-1} \tag{4}$$

or alternatively

$$L=(I-B_2T_2)^{-1}. \tag{5}$$

We could define the matrix K as the *external matrix multiplier of the* S *sector*, and the matrix L as the *external matrix multiplier of the* P *sector* according to their economic meanings. Of course K has the order $m \times m$, and L has $l \times l$, because the multiplications of rectangular matrices T_2B_2 or B_2T_2 make the new square matrices having the order $m \times m$ or $l \times l$ respectively.[1]

Then, we have now arrived at the fact that the total of the propagation effects in P and S sectors' industries, each generated by its own sector's activities, are expected to take the values LB and KT respectively, i. e., "the internal matrix multiplier" premultipied by "the external matrix multiplier". So, if we put

$$KT=M$$
$$LB=N$$

then we can prove the following formula:

$$B^*=(I-A^*)^{-1}=\left[\begin{array}{c|c} B+B_2MB_1 & B_2M \\ \hline MB_1 & M \end{array}\right]$$

or

$$=\left[\begin{array}{c|c} N & NT_1 \\ \hline T_2N & T+T_2NT_1 \end{array}\right]. \tag{6}$$

In the other words, we can partition off the original Leontief inverse $B^*=(I-A^*)^{-1}$ in terms of the combined effects of internal and external matrix multipliers and their induced sub-matrix-multipliers.[2]

The proof of the formula (6) is as follows:

$$\left[\begin{array}{c|c} B+B_2MB_1 & B_2M \\ \hline MB_1 & M \end{array}\right]\left[\begin{array}{c|c} I-A & -A_1 \\ \hline -S_1 & I-S \end{array}\right]=\left[\begin{array}{c|c} I & O \\ \hline O & I \end{array}\right]$$

$$\begin{aligned}
\because\ & B(I-A)+B_2MB_1(I-A)-B_2MS_1 \\
&\quad =I+B_2MS_1B(I-A)-B_2MS_1 \\
&\quad =I+B_2MS_1-B_2MS_1=I \\
& MB_1(I-A)-MS_1 \\
&\quad =MS_1B(I-A)-MS_1 \\
&\quad =MS_1-MS_1=O \\
& -BA_1-B_2MB_1A_1+B_2M(I-S) \\
&\quad =-B_2-B_2KT_2B_2+B_2KT(I-S) \\
&\quad =-B_2-B_2KT_2B_2+B_2K=-B_2(I+KT_2B_2-K) \\
&\quad =-B_2[I-K(I-T_2B_2)]=O
\end{aligned}$$

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1 Another formulation of the external matrix multipliers based on the coefficients of induced effect on intersectoral input activities (i. e., T_1 and B_1) are

$$\bar{K}=(I-B_1T_1)^{-1} \tag{4'}$$

and

$$\bar{L}=(I-T_1B_1)^{-1} \tag{5'}$$

where $\bar{K}$ has the order $m \times m$, and $\bar{L}$ has the order $l \times l$.

The existence of these inverses (external multipliers K, L, $\bar{K}$ and $\bar{L}$) as well as the existence of internal multipliers (B and T) is warranted by the existence of the original Leontief inverse matrix.

$$
\begin{aligned}
-MB_1A_1+M(I-S)& \\
&=-KTB_1A_1+KT(I-S)\\
&=-KT_2B_2+K\\
&=K(I-T_2B_2)=I.
\end{aligned}
$$

In exactly the same manner, we have

$$\left[\begin{array}{c|c} N & NT_1 \\ \hline T_2N & T+T_2NT_1 \end{array}\right]\left[\begin{array}{c|c} I-A & -A_1 \\ \hline -S_1 & I-S \end{array}\right]=\left[\begin{array}{c|c} I & O \\ \hline O & I \end{array}\right].$$

The same result can be obtaind by solving the following system according to economic causal reasoning.

$$\begin{cases} X_p=AX_p+A_1X_s+Y_p \\ X_s=S_1X_p+SX_s+Y_s \end{cases} \tag{7}$$

where X_p is an output vector of P sector's industries, X_s is an output vector of S sector's industries, and Y_p, Y_s are the final demand vectors of the P and S sectors respectively. Thus the solution of this system is stated as

$$\left[\begin{array}{c} X_p \\ \hline X_s \end{array}\right]=\left[\begin{array}{c|c} B+B_2MB_1 & B_2M \\ \hline MB_1 & M \end{array}\right]\left[\begin{array}{c} Y_p \\ \hline Y_s \end{array}\right]$$

or

$$=\left[\begin{array}{c|c} N & NT_1 \\ \hline T_2N & T+T_2NT_1 \end{array}\right]\left[\begin{array}{c} Y_p \\ \hline Y_s \end{array}\right].$$

From which, it is easily seen that the total effects in the P and S sectors originated in its own sector's activities and can be written in *the additive form* $B+B_2MB_1$ or $T+T_2NT_1$ as well as the *multiplied form* LB or KT.[3] Thus, the partitioned intersectoral activities may be viewed in two ways: (a) the first expression of the formula (6) shows it from the viewpoint of P sector side and (b) the second expression shows the same fact from the viewpoint of S sector side. These expressions go hand in hand to make the general formulation applicable to the various problems.

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[2] Using the notation in note 1), we can prove the following identities:

$$KT=T\bar{K}=M$$
$$LB=B\bar{L}=N$$

that is, the expression that the internal multiplier *postmultiplied* by the external multiplier is also possible as well as the *premultiplied* expression.

The proof of the latter identity is that:

$$LB=(I-B_2T_2)^{-1}B=[I+\sum_{m=1}^{\infty}(B_2T_2)^m]B=(I+B_2MS_1)B$$

$$B\bar{L}=B(I-T_1B_1)^{-1}=B[I+\sum_{m=1}^{\infty}(T_1B_1)^m]=B(I+A_1MB_1)$$

because

$$
\begin{aligned}
\sum_{m=1}^{\infty}(B_2T_2)^m&=\sum_{m=1}^{\infty}B_2(T_2B_2)^{m-1}T_2=B_2[\sum_{m=0}^{\infty}(T_2B_2)^m]T_2\\
&=B_2(I-T_2B_2)^{-1}T_2=B_2KT_2=B_2KTS_1=B_2MS_1
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{m=1}^{\infty}(T_1B_1)^m&=\sum_{m=1}^{\infty}(A_1TS_1B)^m=\sum_{m=1}^{\infty}A_1(TS_1BA_1)^{m-1}TS_1B=A_1[\sum_{m=0}^{\infty}(T_2B_2)^m]TB_1\\
&=A_1(I-T_2B_2)^{-1}TB_1=A_1KTB_1=A_1MB_1.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
LB&=(I+B_2MS_1)B=B+B_2MB_1\\
&=B(I+A_1MB_1)=B\bar{L}.
\end{aligned} \tag{*}
$$

In exactly the same manner, we get

$$KT=T\bar{K}=M.$$

[3] For this equality between the multiplied and additive forms, see the equation (*) in the above note 2).

One more alternative expression of the Leontief inverse by the partitioned matrix multipliers is

$$B^*=(I-A^*)^{-1}=\left[\begin{array}{c|c} LB & LB_2T \\ \hline KT_2B & KT \end{array}\right]=\left[\begin{array}{c|c} LB & LBT_1 \\ \hline KTB_1 & KT \end{array}\right]. \tag{6a}$$

We may easily prove the identity between this expression and the equation (6).[4]

Mathematically, our formula also gives us a method of the reduction of the order in the matrix-calculations when the inversion of matrices of high order is not suitable for available computational equipment.

III. *Interdependent Model of Goods-Producing Sectors and Service Sectors*

An empirical application of our model is made for the interindustry data of the Japanese economy (54 sectors) published by Japanese Government under the co-operation of Economic Planning Agency and five other Ministries (the Ministry of Agriculture and Forestry, the Ministry of International Trade and Industry, the Ministry of Construction, the Statistics Bureau of the Prime Minister's Office, and the Administrative Management Agency), which consists of 50 goods-producing sectors and 4 service sectors. In formula (6), we put the P sector for the goods-producing sectors and S sector for the service sectors, i.e., $l=50$ and $m=4$.

Now let us divide the elements of the internal matrix multiplier of the goods-producing sectors B (50×50) calculated from the above equation by the elements of 50×50 part in the published Leontief inverse $B^*=(I-A^*)^{-1}$. We then obtain the values which may be called "the inside propagation ratio of goods-producing sectors". By the numerical test on the row elements of these 50×50 ratios, we have arrived at the conclusion that the industries having many higher values of those ratios are the less service-dependent sectors, and *vice versa*.

The Table 1 is a summarized version of this test, and shows industry-categories of goods-producing sectors by type of the degree of dependence on service activity. Those in category A have characteristics relatively independent of service activity, and those in category D are at the other extreme. Roughly speaking, categories from A-1 to D-2 may be thought of as successive stages of dependency on service sectors.

In Group A, the "inside propagation ratios" of each industry take the value predominantly more than 0.9 (in A-1 group), or more than 0.8~0.9 (in A-2, 3 groups). Those in Group B are in the range of 0.7~0.9. In Group C, those ratios are spread far and wide in the range of about 0.5~0.9, and among this category the ratios in C-2 group concentrate in values 0.7~0.8. In Group D, the inside propagation ratios take a lower value ranging about from 0.4~0.5 to 0.7, and the industries in this group are the most service-dependent sectors.

The reason why the above industrial differential-pattern occurs may be traced by the discerning the difference between the values of the elements in the 50×50 part of Leontief inverse B^* and the values of the elements in the internal matrix multiplier B, which equals to

[4] For example, the identity $KT_2B=T_2N$ is shown as follows:

$$T_2N=T_2(B+B_2MB_1)=TS_1B+T_2B_2KTB_1=(I+T_2B_2K)TB_1$$

in which $I+T_2B_2K=K$ because $(I-T_2B_2)K=I$, so we obtain

$$T_2N=KTB_1=KTS_1B=KT_2B.$$

TABLE 1. INDUSTRY GROUPS BY TYPE OF THE DEGREE OF DEPENDENCE ON THE SERVICE ACTIVITY

Groups		Names of Goods-producing Industries*
Group A	1	Basic chemicals, Non-metallic minerals.
	2	Electricity, Intermediate chemicals, Pig iron, ferro-alloys and crude steel, Metallic ores, Non-metallic mineral products.
	3	Rolled steel, Natural fibre yarns, Coal and lignite, Non-ferrous metal ingots, Chemical fibre yarns, Metal products, Forestry, Coal products.
Group B	1	Machinery and instruments (except electric), Steel casting and forging, Miscellaneous crops.
	2	Primary non-ferrous metal products, Saw-mills and plywood, Chemical fertilizers, Fabrics, Rubber products, Pulp.
Group C	1	Leather and leather products, Livestock, Furniture and wood products, Rice, wheat and barley, Electric machinery and equipment.
	2	Starch, sugar, seasonings, etc., Miscellaneous textile products, Crude petroleum and natural gas, Paper and paper products, Miscellaneous processed foods, City gas and water services, Repair and maintenance of machines, buildings and structures, Petroleum products.
Group D	1	Rice and barley polishing and grain-flour mills, Miscellaneous manufactures, Fisheries, Printing and publishing.
	2	Drugs, soap and cosmetics, Transport equipment, Manufactured tobacco and beverages.

* Excluding the dummy industries such as Business consumption, Office supplies, Scraps, and Undistributed.

** Service sectors other than the above goods-producing sectors are Wholesale and retail trade, Transportation and communication, Real estate and ownership of dwellings, and Banking, insurance and services.

*** The order of listing is that the industries in Group A are the most service-independent sectors, and those in Group D are the most service-dependent sectors.

B_2MB_1 as shown the formula. So, we must discuss the relative weight between B_1, B_2 and M in their propagation process.

By inquiring into Table 2 and 3, we can see what sort of goods-producing industries has more inducible power for service activity (see Table of values of B_1), or what sort of service sectors has more inducible power for goods-producing activity (see Table of values of B_2). A general feature of these figures is of particuler interest; the comparision of values of these two intersectoral sub-multipliers may suggest that the weight of B_2 in propagation activity is smaller than that of B_1 on the whole. The number of values having more than 3% in the Table of B_2 are less than could be counted on the fingers of both hands (exclude the Undistributed sector), while the Table of B_1 has many number of values more than 3%. In other words, the inducible power of one sector to another is more powerful in the case of goods-producing sector than the case of service sector. It does not need to be said that there are different effects from industry to industry as tolding the Tables.[5]

[5] One comment is needed because of the weakness in the data of the service sector which leads to the estimation errors in the original Leontief inverse matrix. If this data weekness is not negligible, our method explained the above must be reread in such a way that the proportion of errors in the elements of the Leontief inverse is actually due to a shortcoming of the service sector's data. For example, the reliability of the inverse-elements may be judged by means of Table 1 such that those in Group A–1 are the most reliable and those in Group D–2 are the most unreliable.

TABLE 2. COEFFICIENTS OF SERVICE-INPUT INDUCED BY INTERNAL PROPAGATION IN GOODS-PRODUCING SECTOR*

$B_1' = (S_1 B)'$ (unit: 10^{-6})

Sector	Trade	Tarnsportation and Communication	Real estate	Banking, Insurance and Services
Rice, wheat and barley	28382	10876	347	12243
Miscellaneous crops	50790	18595	614	15910
Livestock	61239	27121	897	24374
Forestry	13509	10888	1830	23862
Fisheries	54357	21559	6341	40338
Coal and lignite	22829	32782	4228	31772
Crude petroleum and natural gas	25045	30837	27628	39486
Metallic ores	28316	26165	8939	20216
Non-metallic minerals	20540	21651	4514	15567
Rice and barley polishing and grain-flour mills	55809	16636	755	14102
Starch, sugar, seasonings, etc.	116074	26157	2085	32663
Manufactured tobacco and beverages	33969	15115	767	19457
Miscellaneous processed foods	95402	38199	2984	40303
Natural fibre yarns	28258	16786	1944	26100
Chemical fibre yarns	52525	51397	5743	50083
Fabrics	55940	34169	5023	40814
Miscellaneous textile products	71816	32592	4105	46784
Saw-mills and plywood	18568	15909	2841	34709
Furniture and wood products	37096	39130	3036	49755
Pulp	32338	35650	3631	27412
Paper and paper products	46449	45912	2881	36683
Printing and publishing	52511	63314	2326	87273
Coal products	27008	147083	3056	27438
Petroleum products	11974	10508	7081	17553
Basic chemicals	40572	96450	4175	35066
Chemical fertilizers	45497	67430	2948	39844
Intermediate chemicals	83199	56255	5579	48159
Drugs, soap and cosmetics	82932	44149	5963	128804
Rubber products	39029	27059	3449	37856
Leather and leather products	39011	17434	2685	26308
Non-metallic mineral products	66153	66697	4087	36925
Pig iron, ferro-alloys, crude steel	24252	57757	2506	16368
Steel casting and forging	41869	46669	3209	26740
Rolled steel	29437	56458	3383	23905
Non-ferrous metal ingots	26865	38975	5811	15111
Primary non-ferrous metal products	30039	37476	4037	21291
Metal products	44686	42068	2468	26247
Machinery and instruments	56789	37272	6127	34695
Electric machinery and equipment	74690	38973	2560	34970
Transport equipment	76278	41917	2686	29493
Repair and maintenance of machinery, etc.	70881	46847	3204	25575
Miscellaneous manufactures	73958	37369	3834	47743
Electricity	23340	51460	6996	29435
City gas and water services	38028	54657	2783	40696
Business consumption expenditure	65213	236774	816	453851
Building construction	69495	59666	3624	33835
Miscellaneous construction	53675	68526	3734	37094
Office supplies	145474	56522	3146	57429
Scraps	8902	12466	612	4975
Undistributed	60336	36822	15166	68993

* This Table is shown in transposed form, interchanging rows and columns of matrix B_1 for convenience.

TABLE 3. COEFFICIENTS OF INTERNAL PROPAGATION IN GOODS-PRODUCING SECTOR INDUCED BY INPUT IN SERVICE SECTOR

$B_2=BA_1$ (unit: 10^{-6})

Sector	Trade	Transportation and Communication	Real estate	Banking, Insurance and Services
Rice, wheat and barley	3089	2242	783	29239
Miscellaneous crops	3990	4058	4637	19022
Livestock	1169	1079	571	14622
Forestry	11461	15325	12491	10104
Fisheries	2103	1156	417	18837
Coal and lignite	3005	29780	3805	8463
Crude petroleum and natural gas	73	1616	60	152
Metallic ores	430	1642	2057	938
Non-metallic minerals	692	2136	2169	1191
Rice and barley polishing and grain-flour mills	2725	1959	582	28287
Starch, sugar, seasonings, etc.	3396	3126	754	13335
Manufactured tobacco and beverages	7770	1782	708	39452
Miscellaneous processed foods	7226	8824	2185	51582
Natural fibre yarns	3240	4244	1263	4274
Chemical fibre yarns	1420	1770	374	1610
Fabrics	4038	4231	912	4434
Miscellaneous textile products	3241	4515	465	4332
Saw-mills and plywood	11377	17033	15608	5156
Furniture and wood products	7463	3619	1387	2959
Pulp	7519	3273	749	4573
Paper and paper products	27839	11564	2640	16491
Printing and publishing	29721	10536	2566	28052
Coal products	1142	13318	2857	4028
Petroleum products	1739	46832	1743	3608
Basic chemicals	1608	2455	1079	4008
Chemical fertilizers	579	578	465	3375
Intermediate chemicals	5821	7316	4482	18662
Drugs, soap and cosmetics	4200	1521	462	21077
Rubber products	1497	8185	2142	4592
Leather and leather products	492	445	215	1204
Non-metallic mineral products	2238	7239	16036	5135
Pig iron, ferro-alloys, crude steel	3897	15750	19479	7558
Steel casting and forging	1117	4956	12340	2441
Rolled steel	4982	20187	22222	9382
Non-ferrous metal ingots	1038	3919	4924	2325
Primary non-ferrous metal products	1621	7347	8507	2412
Metal products	3532	7256	17153	6619
Machinery and instruments	1924	7657	15969	5234
Electric machinery and equipment	1800	4128	11355	1660
Transport equipment	539	15919	4408	9116
Repair and maintenance of machinery, etc.	12516	51220	168849	13629
Miscellaneous manufactures	5627	1773	1358	5097
Electricity	6751	15826	3510	14921
City gas and water services	1427	2142	544	6594
Business consumption expenditure	81569	18212	7124	41333
Building construction	0	0	0	0
Miscellaneous construction	0	0	0	0
Office supplies	37665	4861	1496	8926
Scraps	3396	7789	9057	6812
Undistributed	45826	82680	16110	42035

Of course, from the viewpoint of the goods-producing sector, the sub-multiplier B_1 operates on that sector only in an indirect manner in the sense that it needs a medium operator expressed by $M=KT$ as shown by the equation (6). The values of elements of K and T are summarized in the Table 4 which shows the powers of dispersion of service sectors internally and externally. On the whole, many values of the elements in the internal multiplier T are somewhat higher than those in the external multiplier K (except Real estate's column), but the difference between the values of these two multipliers is not so large. This fact means again that the weight of dependence of the service sector on the goods-producing sector is considerably large in its character.

TABLE 4. INTERNAL AND EXTERNAL MULTIPLIERS IN SERVICE SECTOR

(1) Internal Multiplier of Service Sector: T

	Trade	Transportation and Communication	Real estate	Banking, Insurance and Services
Trade (wholesale and retail)	1.006382	14208	1618	40780
Transportation and communication	49969	1.020766	1205	30004
Real estate	20839	4942	1.000402	12839
Banking, insurance and services	61886	54474	31979	1.042842

(2) External Multiplier of Service Sector: K

	Trade	Transportation and Communication	Real estate	Banking, Insurance and Services
Trade (wholesale and retail)	1.018500	14691	13090	21801
Transportation and communication	28305	1.015694	10241	21579
Real estate	2004	2499	1.000991	2099
Banking, insurance and services	49081	20060	8373	1.034038

Here, we switch our topics from the quantity-determination model in the input-output system to the price-determination model in that system and turn to a study of the cost-push effects of service-prices on the prices of P sector's products.

Obviously, the prices of P sector's products are given by the equation:

$$P_p=A'P_p+S_1'P_s+v_p \tag{8}$$

where P_p, P_s are vectors of prices of P sector's products and S sector's service-outputs respectively, v_p is the vector of value-added per unit of output in P sector's industries, and the coefficient matrices A' and S_1' are the transpose of the matrices A and S_1 in the quantity model.

This price formation equation (8) is a part of the following larger model:

$$\begin{cases} P_p=A'P_p+S_1'P_s+v_p \\ P_s=A_1'P_p+S'P_s+v_s. \end{cases} \tag{9}$$

In this system, we take P_s and v_p as data, and P_p and v_s as variables. The variations of P_p is due to cost-push effects, and, if we wish, the variations of v_s could be viewed as the resultant change in wages or profits in the S sector due to the variation in prices of P sector's products, but here we omit this latter relation. Of course, the selection between data and variables is dependant upon the setting of the problems.

Then, price-determination in the goods-producing sectors is given by the equation:

$$P_p=(I-A')^{-1}\{S_1'P_s+v_p\}$$
$$=B'\{S_1'P_s+v_p\} \qquad (10)$$

where B' is the transpose of the internal matrix multiplier of the P sector in the quantity model.

If service-prices rise from P_s to $P_s+\varDelta P_s$, the resultant price-rise in P sector's products will be

$$\varDelta P_p=B'S_1'\varDelta P_s=(S_1B)'\varDelta P_s=B_1'\varDelta P_s. \qquad (11)$$

Thus in order to know the cost-push effect of a rise in service prices on the prices of P sector's products, all that is needed is the transposition of the sub-matrix-multiplier B_1.

Going back to Table 2 and rereading it from the viewpint of cost-push effects, we may discover some new facts. Especially we see: (a) relatively more stimulated effects are brought by the rise of the service-price in the Trade sector and the Transportation and communication sector than by the price rise in the other service sectors and (b) the resultant higher price rise is concentrated into some particular commodities such as Starch, sugar, seasonings and Miscellaneous processed foods in the case of Trade service-cost, and Coal products and Basic chemicals in the case of Transportation and communication cost.

A comment is needed to evaluate the figures in Table 2 viewed as the cost-push effects. There is somewhat of a tendency to overvalue those figures more than the theoretically expected ones, because most of the arguments for constant input coefficients depend on the absence of variation in relative price, and changes in the relative prices bring the substitution effects between inputs and set limits on price rises of the cost-push type. However, on the contrary, the rising trend in the service-input coefficients in recent Japanese industries leads to an undervaluation of the actual values of B_1', because the Table's figures are based on somewhat old data.

IV. *Interregional Repercussion Model*

The main purpose of the interregional input-output model developed by the works of Isard, Leontief, Moses, Chenery and others,[6] is to analyze the interrelations among trade and production in two or more regions. Our internal-and-external matrix multiplier model may be well applied to this purpose in a somewhat extended form, because the usual inverse of interregional input-output model tells us only the ultimate total effects but not disjoined effects separating into interdependence between regional internal and external multipliers.

One example of an interregional input-output table in Japan is the data published by the Hokkaido Development Bureau which divides the Japanese economy into two regions having

[6] W. Isard, "Interregional and Regional Input-Output Analysis", *Review of Economics and Statistics*, Nov. 1951; W. W. Leontief, "Interregional Theory", in *Studies in the Structure of the American Economy* by W. W. Leontief and others, 1953; H. B. Chenery, "Regional Analysis", in *The Structure and Growth of the Italian Economy*, by H. B. Chenery, P. G. Clark and V. Cao-Pinna, 1953; L. N. Moses, "The Stability of Interregional Trading Patterns and Input-Output Analysis", *American Economic Review*, Dec. 1955; H. B. Chenery "Interregional and International Input-Output Analysis", in *The Structural Interdependence of the Economy*, ed. by T. Barna, 1956.

same number of industries (30 sectors and 2 regions). In our formula (6), we use the P sector as Hokkaido and the S sector as the Rest of Japan. For convenience we call the latter region Honshū, the main island of Japan. Of course, here $l=m=30$.

Table 5 is concerned with the internal and external matrix multipliers in each region, but only cites the column sum or row sum of the elements of the matrices because of limited space. The economic meaning of the column sum is to summarize the pattern of "the power of dispersion" of industries in each region, and the meaning of the row sum is to express

TABLE 5. SUMMARY TABLE OF INTERNAL AND EXTERNAL MULTIPLIERS OF AN INTERREGIONAL MODEL OF THE JAPANESE ECONOMY

	Internal multiplier of Hokkaido, B		External multiplier of Hokkaido, L		Internal multiplier of Honshū, T		External multiplier of Honshū, K	
	row sum	column sum	row sum	column sum	row sum	column sum	row sum	column sum
Public utilities	2.0757	1.5281	1.0053	1.0014	2.1371	2.0574	1.0016	1.0073
Metal mining	1.5973	1.5558	1.0103	1.0033	1.4232	1.8264	1.0041	1.0015
Non-metal mining	1.1008	1.5049	1.0010	1.0051	1.2693	1.4445	1.0005	1.0006
Petroleum and natural gas	1.3339	1.9988	1.0013	1.0077	1.7521	2.2020	1.0017	1.0001
Coal mining	2.5418	1.4247	1.0269	1.0040	1.7799	1.8384	1.0013	1.0007
Processed foods	1.7938	2.1285	1.0074	1.0017	2.5493	2.4767	1.0036	1.0025
Textiles	1.2630	1.5018	1.0012	1.0058	3.9697	2.9991	1.0069	1.0014
Saw-mill and plywood	1.4638	2.0173	1.0021	1.0001	1.5687	2.7364	1.0006	1.0009
Pulp, paper and products	2.1617	2.2074	1.0170	1.0037	3.1503	2.9812	1.0050	1.0056
Chemicals	1.8464	1.9450	1.0058	1.0048	4.2311	2.9287	1.0069	1.0026
Coal products	1.5500	2.2040	1.0073	1.0011	2.3138	2.4280	1.0052	1.0103
Rubber products	1.0272	1.5886	1.0000	1.0054	1.2967	2.0640	1.0014	1.0007
Leather and products	1.0242	1.2081	1.0000	1.0126	1.1281	2.9118	1.0001	1.0011
Nonmetallic mineral products	1.1340	1.8448	1.0007	1.0050	1.4833	2.4099	1.0015	1.0030
Iron and steel	2.8235	2.7949	1.0626	1.0028	5.3194	3.8007	1.0163	1.0066
Nonferrous metal products	1.0109	1.7924	1.0031	1.0093	2.0035	2.9756	1.0028	1.0108
Steel products	1.0922	1.3679	1.0005	1.0637	1.4305	3.0454	1.0027	1.0032
Machinery	1.0824	1.8586	1.0015	1.0267	1.8852	2.9986	1.0069	1.0010
Lumber and products	1.1370	1.9050	1.0008	1.0044	1.1408	2.5479	1.0002	1.0044
Printing and publishing	1.1219	2.0063	1.0003	1.0061	2.2198	2.7236	1.0016	1.0100
Miscellaneous manufactures	1.0409	1.7745	1.0000	1.0045	1.2726	2.9778	1.0015	1.0009
Forestry	2.5630	1.2480	1.0057	1.0008	3.4962	2.0360	1.0016	1.0001
Fishing	1.5319	1.9547	1.0026	1.0061	1.4503	2.2795	1.0002	1.0162
Agriculture	2.5916	1.4334	1.0097	1.0011	3.3701	1.7911	1.0047	1.0009
Dummy sector	2.1319	1.6158	1.0063	1.0113	4.8711	2.8317	1.0061	1.0007
Service	3.0585	1.4978	1.0046	1.0020	4.3531	1.8817	1.0030	1.0014
Business consumption	2.2541	2.4622	1.0047	1.0029	2.4325	3.1556	1.0016	1.0012
Trade	2.2341	1.4880	1.0069	1.0008	3.6191	1.9028	1.0031	1.0000
Transportation	2.6468	1.5602	1.0095	1.0046	2.1017	2.1303	1.0024	1.0017
Undistributed	2.8650	1.6818	1.0056	1.0018	4.7226	3.3584	1.0045	1.0018
(Average)	1.7699	1.7699	1.0070	1.0070	2.5247	2.5247	1.0033	1.0032

"the sensitivity of dispersion" for industries in each region.

As shown by the figures in the Table, the internal propagation in Hokkaido (B) has a multiplier effect of 1.77 on the average, and it calls in turn the round about external repercussion through Honshū's industrial activity (L) to the amount of about a 0.7%-up effect on the average, so that the total effect equals to $1.77 \times 1.007 = 1.782$ on the average. On the contrary, the internal multiplier effect in Honshū (T) has a considerably higher value of 2.53, but the round about external multiplier (K) shows only a 0.3%-up effect on the average. This industrial differential that is shown by the values of the external multipliers L and K suggests the characteristics of each region's industrial activity according to its role in the national economy.

TABLE 6. SOME COEFFICIENTS OF INDUCEMENT TO PRODUCTION PER UNIT OF INPUT IN THE OTHER REGION*

(a) Row sum of elements of T_2

Honshū's industry	
Iron and steel	2.3033
Textiles	.8631
Chemicals	.8006
Agriculture	.5751
Machinery	.4907
Metal mining	.4317
Leather and products	.4097
Coal products	.4016
Pulp, paper and products	.4016
(Average)	.3806

(c) Row sum of elements of B_2

Hokkaido's industry	
Coal mining	.0812
Pulp, paper and products	.0614
Iron and steel	.0448
Forestry	.0284
Transportation	.0277
Fishings	.0267
Agriculture	.0252
Processed foods	.0208
Trade	.0165
(Average)	.0159

(b) Column sum of elements of T_2

Hokkaido's industry	
Steel products	2.2120
Leather and products	1.2278
Machinery	1.0868
Nonmetallic mineral products	.8064
Textiles	.6211
Miscellaneous manufactures	.4637
Rubber products	.3936
(Average)	.3806

(d) Column sum of elements of B_2

Honshū's industry	
Printing and publishing	.0577
Coal products	.0520
Fishing	.0502
Iron and steel	.0419
Public utilities	.0374
Lumber products	.0357
Pulp, paper and products	.0319
Nonmetalic mineral products	.0255
Steel products	.0211
Processed foods	.0168
(Average)	.0159

* Sectors listed here are the industries having row sum or column sum values higher than the average.

** Table (a) or (c) lists the names of the industry receiving the induced effects, and (b) or (d) lists the names of the industry giving the induced effects.

To see this point more clearly, the "inside propagation ratios" (its definition is the same as before) of Hokkaido's industries are calculated in the 30×30 matrix base. Although the table of the calculated figures is omitted, from it we find that the most self-sufficing industries in Hokkaido are those in the light industry group such as textiles, rubber products, leather and leather products, printing and publishing, miscellaneous manufactures, and those in the non-manufacture group such as services, trade, public utilities.

The industries in this category are relatively independent of the Honshū's industrial activity, and their "inside propagation ratios" all take values more than 0.9. At the other extreme,

TABLE 7. SOME COEFFICIENTS OF INDUCEMENT TO INPUT BY INTERNAL PROPAGATION IN THE OTHER REGION*

(a) Row sum of elements of B_1

Honshū's industry	
Iron and steel	1.0728
Machinery	.5409
Chemicals	.4335
Textiles	.4243
Leather and products	.3962
Processed foods	.3675
Metal mining	.3232
Agriculture	.2978
Coal products	.2906
Pulp, paper and products	.2396
Steel products	.2349
(Average)	.1847

(c) Row sum of elements of T_1

Hokkaido's industry	
Coal mining	.1521
Pulp paper and products	.1034
Iron and steel	.0865
Processed foods	.0462
Transportation	.0339
Fishing	.0336
Agriculture	.0241
Nonmetallic mineral products	.0214
(Average)	.0207

(b) Column sum of elements of B_1

Hokkaido's industry	
Steel products	.6243
Nonmetallic mineral products	.4534
Leather and products	.4517
Mashinery	.3721
Textiles	.2629
Miscellaneous manufactures	.2387
Fishing	.2354
Nonferrous metal products	.1960
Chemicals	.1951
(Average)	.1847

(d) Column sum of elements of T_1

Honshū's industry	
Iron and steel	.0474
Coal products	.0451
Printing and publishing	.0427
Fishing	.0410
Pulp, paper and products	.0368
Public utilities	.0353
Nonmetallic mineral products	.0320
Steel products	.0303
Lumber products	.0275
Nonferrous metal products	.0212
(Average)	.0207

* Sectors listed here are the industries having row sum or column sum values higher than the average.

** Table (a) or (c) lists the names of the industry receiving the induced effects, and (b) or (d) lists the names of the industry giving the induced effects.

there is a group highly dependent on Honshū's industrial activity which includes such heavy industries as iron and steel, non-ferrous metal products, and the resourse industries such as metal mining, non-metal mining, pulp, paper and paper products, fishing, etc.

Such internal propagation patterns together with the external input patterns of interindustrial activity in each region depict the characteristics of interregional repercussions whose estimated results are summarized in Tables 6 and 7. In these tables the coefficients of interregional inducement relations are shown in a summary form designating the column sum and row sum of the elements of four sub-multiplier matrices B_2, T_2, B_1 and T_1. The names of industries listed here are only those having higher values than the average.

The sub-multipliers $B_2=BA_1$ and $T_2=TS_1$ are concerned with the propagation of production activities in each region induced by the input activity in the other region. Reflecting the high dependence of Hokkaido's activities on Honshū's industries, the elements of the multiplier T_2 have higher values than those of the multiplier B_2, and their average values take 0.3806 versus 0.0159. A similar situation is found in the comparison between $B_1=S_1B$ and $T_1=A_1T$ which shows the input inducement effects of one region on the other, and their values take 0.1847 versus 0.0207 on the average. These results suggest that a development program in Hokkaido gives rise to many leakages in the interregional production process and generates much benefit to Honshū's industries.

We cite here only one point for example: the production of Honshū's iron and steel industry induced by Hokkaido's input is the extremely high row sum value of 2.303 as shown the Table 6-(a). This high value has its origin in Hokkaido's industries such as steel products and machinery as shown the column sum figures in the Table 6-(b). Of course, to see the details of cross-effects of this sort, it is necessary to trace back the test of the figures of elements in these matrices themselves instead of the test of column sum or row sum values.

In any case, such analyses play a role which elucidates the inherent properties of the inter-and-intra regional industrial relationships, and we may be expected a fruitful application of this method (combining the extended model in the next section) to the forthcoming comprehensive data of the Japanese interregional input-output table compiled by the Ministry of International Trade and Industry. This will enable us to make the various combination of nine regions of the Japanese economy.[7]

V. *Some Extensions of the Model*

The method which was suggested in this paper express one direction of the extension of the input-output model that may be called the "intensive type". The another direction of the extension may be the "extensive type" which combines the input-output model in other various models such as the macro-econometric model, the linear programming model, and so forth. But, in this section, we are again concerned with the intensive type, and limit our interest to the extension of the above internal-and-external matrix multiplier model on the following two points.

[7] The report on this MITI 9 blocks-interregional input-output table that took three prepared years will be published in the autumn of this year. The nine regions are Hokkaido, Tōhoku, Hokuriku, Kantō. Tōkai, Kinki, Chūgoku, Shikoku and Kyūshū.

1) The Extension to the Number of Partitioned Groups

The line of reasoning which was employed in section II could be extended to increase the number of partitioned industry groups (or regions), that is, the two-partitioned model must be enlarged to three or more partitioned models. However, the straightforward procedure of dividing the model into a large number of industry-groups (or regions) degenerates into formalism, or apt to be too complex. We must not adhere too much to form at the expense of the empirical spirit. So, we prefer to employ "a method of localizable partition" in which the redivision of groups limited to some strategic particular part of industries (or regions).

The procedure of the extension employed here consists of two steps.

Step 1. The first step is the re-partition of some "internal matrix multiplier", say the internal matrix multiplier of S sector, T. If we redivide the S sector into two sub-sections designated section S-1 and section S-2, it will lead to the repartitioned matrix of input coefficients as follows:

$$A^*=\left[\begin{array}{c|c} A & A_1 \\ \hline S_1 & S \end{array}\right]=\begin{array}{c} \begin{array}{c|cc} \text{P sector} & \text{S-1} & \text{S-2} \end{array} \\ \left[\begin{array}{c|c:c} A & a_0 & a_1 \\ \hline u_0 & u & u_1 \\ \hdashline v_0 & v_1 & v \end{array}\right] \begin{array}{l} \}\,l \\ \}\,j \\ \}\,k \end{array} \\ \begin{array}{ccc} \underbrace{}_{l} & \underbrace{}_{j} & \underbrace{}_{k} \end{array} \end{array} \qquad \begin{array}{l} j+k=m \\ l+m=n \end{array} \tag{12}$$

From the above coefficient matrix, we get the following relationships for S sector's industrial activities:

i) Internal multipliers of section S-1 and S-2 in the S sector;

$$Q=(I-u)^{-1} \qquad j\times j \tag{13}$$

$$R=(I-v)^{-1} \qquad k\times k \tag{14}$$

ii) Intersectional (not intersectoral) sub-multipliers showing the inducement to production between sections S-1 and S-2 inside the S sector;

$$Qu_1=\alpha \qquad j\times k \tag{15}$$

$$Rv_1=\beta \qquad k\times j \tag{16}$$

iii) Localized external multipliers of section S-1 and S-2 inside the S sector (i.e., the meaning of "external" in this case is limited to the inside of S sector and not extends beyond it);

$$U=(I-\alpha\beta)^{-1} \qquad j\times j \tag{17}$$

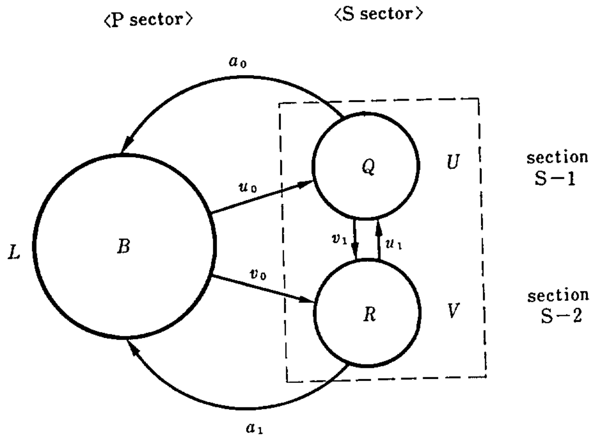
$$V=(I-\beta\alpha)^{-1} \qquad k\times k \tag{18}$$

Then, according to the formula (6a), we have the internal multiplier of the whole S sector in the redivided form, i. e.;

$$T=(I-S)^{-1}=\left[\begin{array}{c|c} I-u & -u_1 \\ \hline -v_1 & I-v \end{array}\right]^{-1}=\left[\begin{array}{c|c} UQ & U\alpha R \\ \hline V\beta Q & VR \end{array}\right]. \tag{19}$$

Step 2. The second procedure is the generalization of the above localized multipliers into the relationships including the interaction between P and S sectors. The pattern of this inter-and-intra sectoral relations is given by the following Chart.

In this Chart, B, Q, R inside the three circles denote the internal multipliers of the P sector, S-1 section, S-2 section respectively. L, U, V outside the circles denote the external multipliers of P sector, S-1 section, and S-2 section respectively.



With the aid of this Chart, we get six routes of inter-and-intra sectoral inducement relationships shown as follows:

<inducement routes>	<sub-multipliers>	<order of matrices>
(a) S-2 → S-1	$\alpha=Qu_1$	$j\times k$
S-1 → S-2	$\beta=Rv_1$	$k\times j$
(b) P → S-1	$\sigma=Q(u_0+u_1Rv_0)$	$j\times l$
P → S-2	$\tau=R(v_0+v_1Qu_0)$	$k\times l$
(c) S-1 → P	$\lambda=B(a_0U+a_1VRv_1)$	$l\times j$
S-2 → P	$\mu=B(a_1V+a_0UQu_1)$	$l\times k$

The coming into existence of these six sub-multipliers showing induced effects on production activity may be easily verified by tracing the repercussion routes between the above sectors or sections in the Chart.

We have now arrived at a formula of the partitioned matrix multiplier in this case which can be stated as follows:

$$(I-A^*)^{-1}=\left[\begin{array}{c|c} LB & L(\lambda Q,\ \mu R) \\ \hline \begin{pmatrix} U\sigma \\ V\tau \end{pmatrix}LB & \begin{bmatrix} UQ & U\alpha R \\ V\beta Q & VR \end{bmatrix}+\begin{pmatrix} U\sigma \\ V\tau \end{pmatrix}L(\lambda Q,\ \mu R) \end{array}\right]$$

or

$$=\begin{bmatrix} LB & L\lambda Q & L\mu R \\ U\sigma LB & U(I+\sigma L\lambda)Q & U(\alpha+\sigma L\mu)R \\ V\tau LB & V(\beta+\tau L\lambda)Q & V(I+\tau L\mu)R \end{bmatrix}. \qquad (20)$$

The main course of the derivation of the formula (20) is that the system

$$\begin{cases} X_p=AX_p+a_0X_u+a_1X_v+Y_p & ① \\ X_u=u_0X_p+uX_u+u_1X_v+Y_u & ② \\ X_v=v_0X_p+v_1X_u+vX_v+Y_v & ③ \end{cases}$$

can be solved in a partiality form for the production level of S sector (regarded as equations ② and ③) by considering the economic causal succession on routes of the induced effects. The result is

$$\begin{bmatrix} X_u \\ X_v \end{bmatrix}=\begin{bmatrix} U\sigma & UQ & U\alpha R \\ V\tau & V\beta Q & VR \end{bmatrix}\begin{bmatrix} X_p \\ Y_u \\ Y_v \end{bmatrix}.$$

Substituting this equation into ① and collecting terms gives the formula (20).

From this formula, we may see that the external (not localized) matrix multipliers of sections S-1 and S-2 are equal to $U(I+\sigma L\lambda)$ and $V(I+\tau L\mu)$ respectively.

In application of the formula for the practical problems, we may have two advantages: (1) the numbers of industries in each partitioned sector (or section) is not necessary the same (i.e., $l \neq j \neq k$), and (2) the above treatment can be adapted to the further subdivision of the particular part of strategic sectors or sections in succession, so we get a method of studying the various characters of industry groups (or regions) according to their differing roles in the national economy.

2) *The Inclusion of the Income Formation Process*

The next extension of our model is the inclusion of the income generation process which is omitted in the usual input-output model. This omission is justified only if the level of income and its use do not depend on the composition of production, because in this case a disaggregation of income generated by sector will add nothing to an analysis of the aggregated Keynesian type. But, under less rigid assumptions this procedure is no longer valid, especially in the interregional model. The location of production depends on the location of consumption, and the latter cannot be determined separately from the calculation of the income generated in each region.

In the usual extension of this, the household sector is transferred to the processing sectors from exogenous sectors and is regarded as an industry whose output is labor and whose inputs are consumption goods as shown in an actual example of Chenery's Italian regional model.[8] But a more correct procedure in dealing with consumption is not to regard it as a fictitious production activity, but to introduce the consumption function of a Keynesian type in a disaggregated form.

As shown in another of the author's papers,[9] this latter procedure means, by implication, combining the Leontief propagation process and the Keynesian propagation process in a disaggregated form. In its formulated multiplier equation it takes the shape of an "extended matrix multiplier" using the Leontief inverse multiplied by the *subjoined inverse matrix* showing the effects of endogenous changes in consumption demand.[10] This proposal distinguishing the inverse reflecting production activities from the inverse reflecting consumption activities, may be well combined with our internal-and-external matrix multiplier model.

A summarized version of this combination is that: if we term E the "enlarged matrix multipier" including the income formation process and take a case as example in which the economy consists of three partitioned industry groups (or regions) having the number of industries l, j, and k respectively, then we have the extended multiplier equation in this case as follows:

[8] See H.B. Chenery, "Regional Analysis", in *op. cit.*, 1953, or H.B. Chenery and P.G. Clark, *Interindustry Economics*, 1959, Chap. 8.

[9] K. Miyazawa. "Foreign Trade Multiplier, Input-Ontput Analysis and the Consumption Function", *Quarterly Journal of Economics*, Vol. Lxxiv, No.1, Feb. 1960. Some comments on this article are found in R. Artle, "On Some Methods and Problems in the Study of Metropolitan Economies", *Papers and Proceedings of the Regional Science Association*, Vol. vii, 1961.

[10] For an extended model including the effects of income-distribution on the production and the income formation, see K. Miyazawa and S. Masegi, "Interindustry Analysis and the Structure of Income-Distribution", *Metroeconomica*, Vol. xv, Fas. 2, 3, Agosto-Dicembre, 1963.

$$E=B^*(I-CHB^*)^{-1}$$
$$=B^*(I+CZHB^*) \quad (21)$$

where $B^*=(I-A^*)^{-1}$ is the Leontief inverse having the order $(l+j+k)\times(l+j+k)$, C is the matrix of consumption coefficients having $l+j+k$ rows and three columns, and H is the matrix of the value-added ratios having three rows and $l+j+k$ columns. The subjoined inverse matrix $(I-CHB^*)^{-1}$ in the first expression of the equation (21) shows the effects of endogenous changes in each group's (or region's) consumption expenditure, and it can be coverted into the form which is shown as the second expression of the equation (21).

In the latter form, the matrix Z is to be written the expression that

$$Z=(I-HB^*C)^{-1} \quad (22)$$

which can be called the "multi-sector income multiplier" having three rows and three columns. The economic meaning of this matrix Z is that its elements show how much income in one group (or region) is generated by the expenditure from 1 unit of additional income in the other group (or region), but the details on the above formulations and the characters of the matrix Z are omitted in this paper.[11]

[11] See, K. Miyazawa and S. Masegi. *op. cit.*, especially pp. 91-97.