REGIONAL INPUT-OUTPUT ANALYSIS

——An Interpretation of Chenery's Model——

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1. *Introduction*

So far we have seen many important works on regional input-output analysis, although this analysis seems to belong to a rather recent study in input-output analysis.[1] In my last book[2] I set up one complete model putting its theoretical basis on the Chenery's idea.[3] In this paper I try to find out a relation between such a regional input-output model and the usual and original Leontief model which treats a national economy as a whole. I also intend to interpret the regional model in terms of linear programming technique, and reach a conclusion that it is identical with a model which minimize the total of labor inputs required to fulfil the given final demand.

2. *Formulation of Chenery's Idea*

As written before, a formulation of Chenery's regional analysis was tried in my recent book. The essential part of such a model is stated again in what follows.

A national economy is divided into two economies of Region A and Region B, and also into three economies of Sectors I, II and III. Table 1 shows regional input coefficients. The letter a represents an input coefficient of Region A, and the letter b an input coefficient of Region B. Table 2 shows regional supply coefficients. The letter p is related to the demand of Region A, in which p_a represents a supply coefficient from Region A and p_b a

Table 1

	Region A			Region B		
	I	II	III	I	II	III
I	a_{11}	a_{12}	a_{13}	b_{11}	b_{12}	b_{13}
II	a_{21}	a_{22}	a_{23}	b_{21}	b_{22}	b_{23}
III	a_{31}	a_{32}	a_{33}	b_{31}	b_{32}	b_{33}

Table 2

	Demand of Reg. A		Demand of Reg. B	
	Reg. A	Reg. B	Reg. A	Reg. B
I	p_{a1}	p_{b1}	q_{a1}	q_{b1}
II	p_{a2}	p_{b2}	q_{a2}	q_{b2}
III	p_{a3}	p_{b3}	q_{a3}	q_{b3}

1 Walter Isard, International and Interregional Input-Output Analysis; A Model of a Space-Economy, *Review of Economics and Statistics*, Vol. XXXIII, No. 4, Nov. 1951, pp. 318—328.
Wassily W. Leontief, Interregional Theory, *Studies in the Structure of the American Economy*, ed. by W. Leontief and Others, 1953, pp. 93—115.
Vera Cao-Pinna, Problems of Establishing and Using Regional Input-Output Accounting, *Regional Economic Planning*, ed. by Walter Isard and John H. Cumberland, 1961, pp. 305—338.

2 Isamu Yamada, *Theory and Application of Interindustry Analysis*, 1961.

3 Hollis B. Chenery, Regional Analysis, *The Structure and Growth of the Italian Economy*, ed. by Mutual Security Agency, 1953, pp. 97—116.

supply coefficient from Region B. In the same way, the letter q show the demand of Region B, in which q_a represents a supply coefficient from Region A and q_b a supply coefficient from Region B.

First of all, the following relations are held between regional supply coefficients.

$$\begin{aligned} &p_{a1}+p_{b1}=1, \qquad && q_{a1}+q_{b1}=1, \\ &p_{a2}+p_{b2}=1, && q_{a2}+q_{b2}=1, \\ &p_{a3}+p_{b3}=1, && q_{a3}+q_{b3}=1. \end{aligned} \tag{1}$$

Then the final demand and final supply are given in Table 3. In this table, the letter Y means the final demand and the letter S the final supply. A suffix a represents the final demand or the final supply of Region A, and a suffix b the final demand or the final supply of Region B. Likewise, suffixes 1, 2 and 3 represent the final demand or the final supply of Sectors I, II and III respectively. Also Y^o or S^o represents the demand or the supply which is final or the direct effect of the initial injection. In the same way, Y' or S' indicates the demand or the supply of the first indirect effect, and Y'' or S'' of the second indirect effect, and so forth.

Table 3

	Final Demand		Final Supply	
	Reg. A	Reg. B	Reg. A	Reg. B
I	$Y_{a1}{}^o$	$Y_{b1}{}^o$	$S_{a1}{}^o$	$S_{b1}{}^o$
II	$Y_{a2}{}^o$	$Y_{b2}{}^o$	$S_{a2}{}^o$	$S_{b2}{}^o$
III	$Y_{a3}{}^o$	$Y_{b3}{}^o$	$S_{a3}{}^o$	$S_{b3}{}^o$

It is clear that we can get the following relations:

$$\begin{aligned} &p_a Y_a{}^o + q_a Y_b{}^o = S_a{}^o, \\ &p_b Y_a{}^o + q_b Y_b{}^o = S_b{}^o, \end{aligned} \tag{2}$$

where

$$p_a=\begin{bmatrix} p_{a1} & 0 & 0 \\ 0 & p_{a2} & 0 \\ 0 & 0 & p_{a3} \end{bmatrix}, \quad p_b=\begin{bmatrix} p_{b1} & 0 & 0 \\ 0 & p_{b2} & 0 \\ 0 & 0 & p_{b3} \end{bmatrix},$$

$$q_a=\begin{bmatrix} q_{a1} & 0 & 0 \\ 0 & q_{a2} & 0 \\ 0 & 0 & q_{a3} \end{bmatrix}, \quad q_b=\begin{bmatrix} q_{b1} & 0 & 0 \\ 0 & q_{b2} & 0 \\ 0 & 0 & q_{b3} \end{bmatrix}.$$

and

$$Y_a{}^o=\begin{bmatrix} Y_{a1}{}^o \\ Y_{a2}{}^o \\ Y_{a3}{}^o \end{bmatrix}, \quad Y_b{}^o=\begin{bmatrix} Y_{b1}{}^o \\ Y_{b2}{}^o \\ Y_{b3}{}^o \end{bmatrix}, \quad S_a{}^o=\begin{bmatrix} S_{a1}{}^o \\ S_{a2}{}^o \\ S_{a3}{}^o \end{bmatrix}, \quad S_b{}^o=\begin{bmatrix} S_{b1}{}^o \\ S_{b2}{}^o \\ S_{b3}{}^o \end{bmatrix}.$$

Further, we define a and b as follows:

$$a=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad b=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

both of which are regional input coefficient matrices.

The first indirect effects of Region A and B are,

$$Y'_a=[ap_a \ aq_a]\begin{bmatrix} Y_a{}^o \\ Y_b{}^o \end{bmatrix},$$
$$Y'_b=[bp_b \ bq_b]\begin{bmatrix} Y_a{}^o \\ Y_b{}^o \end{bmatrix}. \tag{3}$$

Similarly, the second, the third,......indirect effects can be formulated in the same way as (3). Finally we can get the following results as the total of direct and indirect effects:

$$Y_a=Y_a{}^o+[ap_a \ aq_a][I-Q]^{-1}\begin{bmatrix} Y_a{}^o \\ Y_b{}^o \end{bmatrix},$$
$$Y_b=Y_b{}^o+[bp_b \ bq_b][I-Q]^{-1}\begin{bmatrix} Y_a{}^o \\ Y_b{}^o \end{bmatrix}, \tag{4}$$

where Y_a *and* Y_b show the total of the direct and indirect effects respectively, and[4]

$$Q=\begin{bmatrix} ap_a & aq_a \\ bp_b & bq_b \end{bmatrix}.$$

The formula (4) we finally obtained here can be extended to the case of more than two regions and more than three sectors.

3. *An interpretation of Regional Model in terms of Linear Programming Technique*

Here two region and two sector case will be considered. The result can, however, be extended to the general n region and m sector case.

As usually assumed in a static model, demand is not allowed to exceed its corresponding supply. Therefore, the following relations are easily obtained, taking (1) into consideration.

$$p_{a1}+p_{b1}\geqq 1, \quad q_{a1}+q_{b1}\geqq 1,$$
$$p_{a2}+p_{b2}\geqq 1, \quad q_{a2}+q_{b2}\geqq 1. \tag{5}$$

Next, the identities (2) can also be utilized in this case. Namely,

$$p_{a1}Y_{a1}{}^o+q_{a1}Y_{b1}{}^o=S_{a1}{}^o,$$
$$p_{a2}Y_{a2}{}^o+q_{a2}Y_{b2}{}^o=S_{a2}{}^o,$$
$$p_{b1}Y_{a1}{}^o+q_{b1}Y_{b1}{}^o=S_{b1}{}^o,$$
$$p_{b2}Y_{a2}{}^o+q_{b2}Y_{b2}{}^o=S_{b2}{}^o. \tag{6}$$

Making use of (5),

$$S_{a1}+S_{b1}=(p_{a1}+p_{b1})Y_{a1}+(q_{a1}+q_{b1})Y_{b1}$$
$$\geqq Y_{a1}+Y_{b1}, \tag{7}$$

where Y_{a1} and Y_{b1} are used instead of $Y_{a1}{}^o$ and $Y_{b1}{}^o$.

In addition, from (5),

$$(p_{a1}+p_{b1})+(q_{a1}+q_{b1})\geqq 2. \tag{8}$$

An objective function is set up as follows:

$$l_{a0}S_{a1}=l_{a0}(p_{a1}Y_{a1}+q_{a1}Y_{b1}),$$
$$l_{b0}S_{b1}=l_{b0}(p_{b1}Y_{a1}+q_{b1}Y_{b1}), \tag{9}$$

the sum of which is to be made minimum. In (9) l_{a0} and l_{b0} are the volumes of labor required to get a unit of S_{a1} and S_{b1}. The sum of the labor required in (9) is,

$$l_{a0}Y_{a1}(p_{a1}+p_{b1})+l_{b0}Y_{b1}(q_{a1}+q_{b1})$$
$$-(p_{b1}Y_{a1}-q_{a1}Y_{b1})(l_{a0}-l_{b0}).$$

Here we assume that $(p_{b1}Y_{a1}-q_{a1}Y_{b1})(l_{a0}-l_{b0})$ is negligibly small, then the objective function

[4] The necessary and sufficient condition of the existence of $[I-Q]^{-1}$ is, of course,
$|I-Q| \neq 0$.
Practically, we usually meet such a case in familiar input-output tables, except an extremely special one.

becomes,

$$l_{a0}Y_{a1}(p_{a1}+p_{b1})+l_{b0}Y_{b1}(q_{a1}+q_{b1})=\min. \tag{10}$$

A linear programming model of the regional analysis stated above is such that Y_{a1} and Y_{b1} given, the values of $p_{a1}+p_{b1}$ and $q_{a1}+q_{b1}$ are to be sought in order to make the objective function (10) minimum, under the constraints of (7) and (8). If $Y_{a1}>Y_{b1}$, so far as

$$\frac{Y_{b1}}{Y_{a1}}<\frac{l_{a0}}{l_{b0}}<1, \tag{11}$$

the following relations are derived from (7) and (8) both of which take the equality signs only,

$$p_{a1}+p_{b1}=1, \quad q_{a1}+q_{b1}=1. \tag{12}$$

Similarly,

$$p_{a2}+p_{b2}=1, \quad q_{a2}+q_{b2}=1. \tag{13}$$

Therefore, we are able to conclude that, even if we assume (5), only equal relations in (5) can be held, so far as the total of labor input is to be minimized.

4. *Relationship between Regional Model and General Input-Output Model*

Now we consider the general case, i.e., the case of m regions; A, B,..., M and n sectors; I, II,..., N, after expanding the two region and three sector case formulated in Section 2. Here we newly define,

$$Y=\begin{bmatrix} Y_a \\ Y_b \\ \vdots \\ Y_m \end{bmatrix}, \quad Y^o=\begin{bmatrix} Y_a{}^o \\ Y_b{}^o \\ \vdots \\ Y_m{}^o \end{bmatrix},$$

where Y_a, Y_b,..., Y_m and $Y_a{}^o$, $Y_b{}^o$,..., $Y_m{}^o$ have the same meanings as before. Then, the following simple equation can be derived, by adding respectively each side of $Y_a=Y_a{}^o+[ap_a \; aq_a \ldots ar_a][I-Q]^{-1}\begin{bmatrix} Y_a{}^o \\ \vdots \\ Y_m{}^o \end{bmatrix}; \cdots ; \; Y_m=Y_m{}^o+[mp_m \; mq_m \ldots mr_m][I-Q]^{-1}\begin{bmatrix} Y_a{}^o \\ \vdots \\ Y_m{}^o \end{bmatrix}$, where $r_a,\ldots, r_m$ are supply coefficients of the demand of Region M from Region A,..., M,

$$Y=[I-Q]^{-1}Y^o \tag{14}$$

Or,

$$[I-Q]Y=Y^o \tag{15}$$

As it is clear from the above equation (14), it is identical with the ordinary input-output model of one national economy, if Y^o is replaced by the final demand and Q by the usual input coefficient matrix and further Y by its corresponding output.

The next step is to examine supply coefficients, in order to identify Q with the ordinary input coefficient matrix. Here we assume that each region has the same input coefficient structure and the demand of each region is wholly met by the supply of each region, i.e., self-supplied. Then,

$$a=b=\cdots=m,$$

$$p_a=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \; p_b=0, \; \ldots, \; p_m=0,$$

$$q_a=0, \; q_b=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \; \ldots, \; q_m=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_a=0,\ r_b=0,\ \ldots,\ r_m=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}.$$

Therefore,

$$Q=\begin{bmatrix}a & 0\ldots0\\ 0 & b\ldots0\\ \multicolumn{2}{c}{\cdots\cdots\cdots}\\ 0 & 0\ldots m\end{bmatrix}=a\begin{bmatrix}I_n & & 0\\ & \ddots & \\ 0 & & I_n\end{bmatrix},$$

$$I_{mn}-Q=I_{mn}-a\begin{bmatrix}I_n & & 0\\ & \ddots & \\ 0 & & I_n\end{bmatrix},$$

where I_{mn} is a unit matrix of mn dimensions and I_n of n dimensions Multiplying the both sides of (15) by an addition row vector from the left,

$$[I_n-a][Y_a+Y_b+\ldots+Y_m]=[Y_a^o+Y_b^o+\ldots+Y_m^o]. \tag{16}$$

If we put

$$Y_a+Y_b+\ldots+Y_m=Y^*,$$
$$Y_a^o+Y_b^o+\ldots+Y_m^o=Y^{o*},$$

the equation (16) leads to

$$[I_n-a]Y^*=Y^{o*}. \tag{17}$$

The last equation is nothing but the familiar input-output model of one national economy.