A NOTE ON INPUT-OUTPUT MATRICES

By KENJIRO ARA

Assistant Professor of Economics

I

Let A be an indecomposable and non-negative matrix in the usual type of Input-Output Analysis. The object of this paper is to prove the following properties:

(i) The matrix A can be transformed into a matrix B by a non-singular transformation

$$PAP^{-1}=B>0,$$

where P is a non-negative diagonal matrix and B a matrix the sum-totals of each column vector of which are all the same.

(ii) Let us call the normalized P (in the sense that the diagonal elements are normalized) simply the Shadow-Price Matrix, then the diagonal elements of the Shadow-Price Matrix are proved to be composed of the elements of the normalized eigen vector which associates to the maximal positive root of A.

(iii) The sum-totals of each column vectors of B are all equal to the maximal positive root of A.

(iv) The Shadow-Price Matrix is always unique in the sense that there can't be any ways to obtain Shadow-Price Matrix other than the method as indicated in (ii).

II

By assumption, A' (the transposed matrix of A) is an indecomposable and non-negative matrix. Thus it is clear from the Frobeneous theorem [1, p. 598] that A' has a positive root λ_1 the absolute value of which is largest and to which a positive eigen-vector p_1 is associated, i.e.,

(1) .... $A'p_1=\lambda_1 p_1>0 \quad (p_1>0)$.

Without losing generality, let us assume that p_1 is already normalized and write by P the diagonal matrix the diagonal elements of which are the elements of the normalized p_1. Hence we may obtain

(2) .... $A'Pe=\lambda_1 Pe$.

Multiplying both sides by P^{-1}, it follows

(3) .... $[P^{-1}A'P-\lambda_1 E]e=0$,

where E and e denote the unit-matrix and-vector respectively. Let us put

(4) .... $P^{-1}A'P=B'$,

then the total-sums of each row-vectors of B' (and therefore those of each column-vectors of B) are all equal λ_1. Thus we could prove the validity of Theorem (i), (ii), and (iii).

For Theorem (iv) to be verified, it would be enough to prove that eigen-vectors of any roots of A other than λ_1 always contain at least one negative element.

Let λ_i be any root of A except λ_1. Thus we have

(5) .... $Aq_i=\lambda_i q_i$,

where q_i is the eigen-vector associated to λ_i. Multiplying by p_1, we have again

(6) .... $p'_1Aq_i=\lambda_i p'_1q_i=q'_iA'p_1=\lambda_1 q'_ip_1$.

Thus it follows

(7) .... $(\lambda_1-\lambda_i)p'_1q_i=0$.

By assumption, $\lambda_1\neq\lambda_i$, so that

(8) .... $p_1'q_i=0$.

However, $p_1>0$, so that at least one element of q_i must be negative. Thus the proof of Theorem (iv) is complete.

III

Let us suppose that the matrix A is open with respect to the profit-earning sector. Thus i-th column vector of A comes to denote the items of cost per unit of output in the i-th industry ($i=1,2,\dots,n$). Let

(9) .... $p'=(p_1,p_2,\dots,p_n)$

be a price-vector given at the market. With this price-vector given, the average rate of profit earned per unit of output in the i-th industry is denoted by

(10) .... $p_i-(p_1a_{1i}+p_2a_{2i}+\dots\dots+p_na_{ni})$,

where a_{ji} is the j-th element of the i-th column vector. This last equation can be again rewritten as

(11) .... $Pe-A'Pe=[E-A']Pe$.

As far as the competition is free, the average rate of profit per unit of output must be equal in every industries. Let r be such equilibrium rate, then it may be obtained

(12) .... $p'=p'A+rp'$

or alternatively

(13) .... $[E(1-r)-A']Pe=0$

under the general equilibrium condition of the economy.

By its economic implication, the price-vector p must be a positive vector, so that the following equation must be established:

$$(14) \quad \ldots\ldots\ldots\ldots \quad |(1-r)E-A'|=0.$$

Let us assume that A is indecomposable. As denoted above, only the maximal positive root of A can have the positive eigen-vector. If the maximal positive root of A is larger than unit, the equilibrium rate of profit must be negative. In this case entrepreneours will stop the production. If it is unit, the equilibrium rate of profit must be zero. This is really the neo-classical case of perfect competition. But if the maximal positive root of A is smaller than unit, it is always possible for the equilibrium rate of profit to prevail at some positive level. Whether an equilibrium rate of profit is possible at some positive level or not depends definitely upon the properties of the matrix A itself, not other way round.

It would be not difficult to see that the eigen-vector associated to the maximal positive root of A is nothing but the equilibrium price-vector which makes the average rate of profit equal all over the industries. Being P non-singular, it follows from (14)

$$(15) \quad \ldots\ldots\ldots\ldots |P^{-1}||E(1-r)-A'||P|=|E(1-r)-B'|=0,$$

where $P^{-1}A'P\equiv B'$. As indicated above, the total-sums of each column-vectors of B are all equal to $(1-r)$, namely equal to the average cost per unit of outputs, so that the maximal root of A is nothing but the average cost per unit of output under the condition of general equilibrium.[1] The uniqueness of equilibrium prices would be also easily confirmed by normalizing the equilibrium price-vector p.

REFERENCE

Debreu, G. and I. N. Herstein, "Nonnegative Square Matrices", *Econometrica*, 1953, Oct., pp. 597—607.

[1] Let us suppose that the n-th sector is wage-earners' sector. The return over the cost in this sector may be deemed either as saving by wage-earners or as the payment to profit-earners or both.