A SIMPLE MODEL OF MACRO-ECONOMIC DYNAMICS

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I

It is well known today that the modern macro-economic dynamics can be divided into two main approaches—Hicksian equilibrium approach and Harrodian disequilibrium approach.

Let I and S be respectively Investment (demand for new capital) and Saving (supply of new capital). The equilibrium approach regards the economic reality as the "moving equilibrium process" of $I=S$ and the equilibrium level of national income, which can be calculated by solving $I=S$, is supposed to represent the actual level of national income. The equilibrium approach is further characterised by its little or no concern with the process through which the national income or products are produced. Therefore, the effective demand is the only factor which determines the actual process of national economy. At any rate, if the equilibrium approach should be justified, this theory has to prove the stability of the moving equilibrium process.

On the other hand, the disequilibrium approach admits the possibility of $I=S$. But, once the equation $I=S$ fails to hold, this theory denies the existence of any mechanism which counterbalances the disequilibrium process instantly. On the contrary, according to this theory, such a counterbalancing effort has the effect of enlarging the disequilibrium position, and this disequilibrium process is itself supposed to be the phenomena of business fluctuations, i.e. if $I>S$, it means the prosperity, and if $S>I$, the depression.

In this connection, we remember that the disequilibrium theory is closely related with the theory of Wicksellian cumulative process resulting from the discrepancy between the natural and monetary interest rates. Of course, it is to be remarked that in Wicksell's theory the main variable is assumed to be the general price level of commodities. But, the identification of the lack of equilibrium as represented in $I=S$ with the business fluctuations of the national economy has certainly originated from Wicksell.

Then, why is the national economy not provided with any mechanism which could instantly counterbalance the lack of equilibrium? As will be shown in the following, this is explained by the consideration of the correspondence between the effective demand and effective supply restricted by capital. We find many difficulties involved in the equilibrium approach, because it does not take into consideration the supply side of the national income. Therefore, we are forced to give more weight to the disequilibrium theory, which takes up as subject of analysis the interrelation between the effective demand and supply. In what follows, we shall be concerned with a systematic presentation of some results obtained about macro dynamics from the standpoint of the disequilibrium theory.

II

In order to simplify the following observations, the influences of the change of commodity prices will not be taken into consideration. The variables to be used are therefore all real quantities deflated by some suitable commodity price indices.

In the first place, let $K(t)$ be the national capital at the beginning of the *t-th* period. Taking into consideration suitable idle capacity, we suppose $Y(t)$ will be obtained from $K(t)$, where $Y(t)$ denotes the national income during the *t-th* period. If there exists between $K(t)$ and $Y(t)$ such a linear and homogeneous relation that in case $K(t)$ is doubled, $Y(t)$ is doubled too, the relation in question is expressed as

$$Y(t)=\sigma K(t), \quad \cdots\cdots\cdots\cdots \quad (1)$$

where σ is the production coefficient of capital. Of course, this coefficient depends upon the idle capacity as well as upon the technological conditions and the structure of capital equipments. For the following discussion, it is sometimes more convenient to use $\frac{1}{\sigma}=c$ for σ. In this case,

$$K(t)=cY(t) \quad \cdots\cdots\cdots\cdots \quad (2)$$

is obtained and c is again called the capital coefficient.

Now, a part of the national products thus produced is consumed as national consumption, while the remaining part constitutes the new capital accumulation as national saving. Let the saving during the *t-th* period be $S(t)$. We suppose for simplicity that the saving is a linear homogeneous function of the national income. Then we get the equation

$$S(t)=sY(t), \quad \cdots\cdots\cdots\cdots \quad (3)$$

where s is called the saving ratio. Of course, the saving ratio is dependent upon the distributive structure of the national income as well as upon the social customs. But, such a complication will not be taken into consideration here in this paper.

Next, let the demand for new capital at the *t-th* period be $I(t)$. In equilibrium, it must be equal to the amount of saving at the same period. Thus, the equation

$$I(t)=S(t) \quad \cdots\cdots\cdots\cdots \quad (4)$$

expresses the macro-equilibrium condition of the national economy. Then, what does the behaviouristic equation about $I(t)$ look like? In case the capital equipments are fully at work with suitable idle reserve, we assume that $I(t)$ is proportional to the difference of $Y(t)$ and $Y(t\text{-}1)$, i. e. $\Delta Y(t)=Y(t)-Y(t\text{-}1)$. Let the coefficient of the proportion be v, then

$$I(t)=v\Delta Y(t) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

is the equation for investment, where the coefficient v is called the acceleration coefficient. As will be shown in what follows, there exists a definite relationship between v and c (or σ) with respect to the realization of developmental equilibrium. However, we have to be careful not to confuse these two notions. In particular, it seems to the author that many contemporary authors about macro-dynamics have failed to distinguish these two concepts.

Now, there remains finally the task to clarify the definition about the time unit. With this in view, let us define the unit period as

> "the interval from the time when the national product is produced as a result of the utilization of the national capital K up to the time when the national capital ΔK corresponding to the investment is again accumulated."

As a consequence of this definition, we obtain the following equation in equilibrium;

$$K(t)+I(t)\equiv K(t+1),$$

from which we can easily conclude

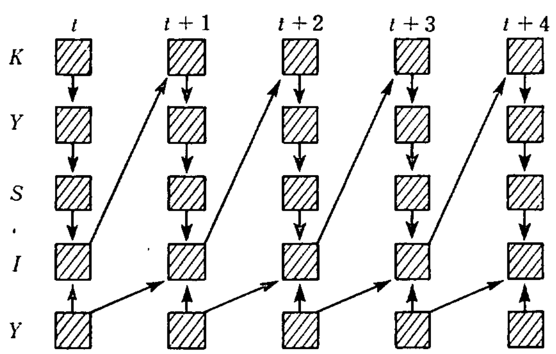
$$I(t)\equiv K(t+1)-K(t)\equiv\Delta K(t+1). \quad \cdots\cdots\cdots\cdots\cdots \quad (6)$$

Sometimes, the unit period is defined by means of

$$I(t)\equiv\Delta K(t) \qquad \therefore\ K(t-1)+I(t)\equiv K(t).$$

However, the proposed definition means that *the capital goods to be produced is calculated in advance in the original national capital necessary for produc-*

Fig. 1



ing it. The statement is clearly meaningless from the economic standpoint. Thus from the equation (2) and (6), we obtain

$$I(t)\equiv\Delta K(t+1)=c\Delta Y(t+1). \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (7)$$

As shown in Fig. 1, the above-mentioned relationships are represented by means of the so-called Tinbergen's arrow-schema. Each arrow shows the direction of the interaction which each variable exerts on another variable. From this schema, we can easily understand the economic meaning of the definition of $I(t)=\Delta K(t+1)$.

Another representation of the above relations are again shown in Fig. 2. In this figure, equation (1), (3) and (6) are respectively shown in the first, second and third quadrants. Each parallel dotted lines are located at the fourth quadrant with inclination of 45°, as both east and south axes indicate the same amount of the national capital. In this figure, the development of the national economy is well illustrated in its intertemporal order. For instance, a national economy which has started from $K(o)$ will develop in such a way as $K(o)\rightarrow Y(o)\rightarrow S(o)\rightarrow I(o)\rightarrow K(1)$— ········· maintaining equilibrium.

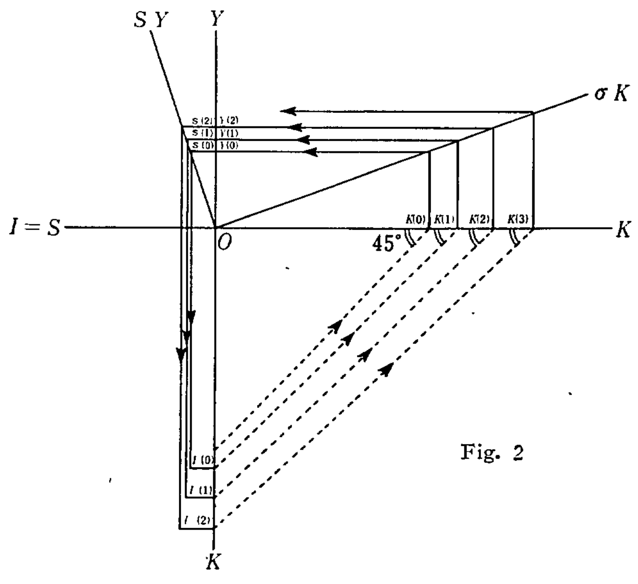


Fig. 2

III

Summing up the above-mentioned equations which determine the developmental equilibrium process, we have,

$$S(t)=sY(t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

$$I(t)=S(t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

$$I(t)=v\Delta Y(t) \quad \cdots\cdots (5)$$
$$I(t)=c\Delta Y(t+1). \quad \cdots\cdots (7)$$

As is easily seen, the variables of this system are three (i.e. Y, S and I), but the equations are four in number. Therefore, this system seems to be *over-determined.*

However, this over-determination can be easily avoided, when we observe that in developmental equilibrium the capital coefficient c and the acceleration coefficient v are linearly dependent upon each other, if the saving coefficient s is given.

In order to analyse the relation between the capital and acceleration coefficients, let us observe the following simultaneous equations;

$$(A) \begin{cases} S(t)=sY(t) & \cdots\cdots (3) \\ I(t)=S(t) & \cdots\cdots (4) \\ I(t)=c\Delta Y(t+1), & \cdots\cdots (7) \end{cases}$$

where three variables are uniquely determined by three equations. Solving with respect to Y, we have

$$c\Delta Y(t+1)=sY(t).$$

Let the national income at the *o-th* period be $Y(o)$. Then we have as general solution

$$Y(t)=\left(\frac{c+s}{c}\right)^t Y(o)=\left(1+\frac{s}{c}\right)^t Y(o), \quad \cdots\cdots (8)$$

Clearly, $s>o$ and $c>o$, from which we easily conclude that the economic system will expand at the rate of g to be defined as follows:

$$\frac{s}{c}=g. \quad \cdots\cdots (9)$$

This growth rate is in fact nothing but the so-called warranted rate of growth as defined by Harrod.

Next, we can present the following equation system,

$$(B) \begin{cases} S(t)=sY(t) & \cdots\cdots (3) \\ I(t)=S(t) & \cdots\cdots (4) \\ I(t)=v\Delta Y(t), & \cdots\cdots (5) \end{cases}$$

from which we obtain,

$$v\Delta Y(t)=sY(t). \quad \cdots\cdots (10)$$

From (10), we further obtain the following general solution of $Y(t)$ with respect to $Y(o)$ as initial condition,

$$Y(t)=\left(\frac{v}{v-s}\right)^t Y(o)=\left(1+\frac{s}{v-s}\right)^t Y(o). \quad \cdots\cdots (11)$$

As is easily seen, in case $v<s$, the national income level is respectively positive and negative at the period of even and odd order, as far as $Y(o)>o$.

This conclusion is seemingly very queer, if the developmental equilibrium should always have a positive accumulation of capital.

Inquiring into the relation between the capital and acceleration coefficients, we know that $v<s$ would never occur. From (8) and (11), we easily obtain the following relation,

$$\left(\frac{v}{v-s}\right)=\left(\frac{c+s}{c}\right)$$

which is further equivalent to the expression

$$v=c+s. \qquad (12)$$

It is clearly seen that, $v>s$ is true, i.e. the acceleration coefficient is larger than the saving ratio, as far as the capital coefficient c remains positive. Therefore, the possibility of the above-mentioned queer conclusion is completely ruled out. At any rate, the relation

acceleration coefficient=capital coefficient+saving ratio

holds true in the developmental equilibrium. It is to be remarked that v and c are the different notions from each other. In other words, the acceleration coefficient v is a parameter which determines the behavior of the demand for new capital, when the informations are given on the present and past movements of the national products. On the other hand, the capital coefficient c is another structural parameter which explains the productivity of the national capital.

Now, let c and s be given. In equilibrium, the equation (12) must hold. In terms of economics, this means that the effective supply restricted by the capital determines the development of the national economy, while the effective demand only follows the development thus determined. Next, let v and s be given. Again this means that the effective demand is responsible for determining the development of the national economy, while the effective supply is linearly dependent upon the course thus determined. In equilibrium —it is a characteristic of equilibrium—it does not make any difference which parameter is taken as linearly dependent upon another.

Up to now, we have been concerned with the analysis of the developmental equilibrium as represented by $I=S$. The national economy will keep its steady growth rate through the accumulation of capital as far as $I=S$ holds true. It should be remembered that such a long term growth rate is not given externally as in the Hicksian theory, but is accounted for within the system in terms of the saving, acceleration and capital coefficients.

Then what will happen when the national economy falls into $I\neq S$? If the equilibrium is instantaneously recovered, $I=S$ represents a stable equilibrium. Otherwise, the equilibrium would be unstable. In order to investigate the situation more in detail, it is necessary to analyse the property of the capital coefficient or the production coefficient of capital.

In general, the enterpriser reserves a certain amount of idle capacity in anticipation of unexpected increase of demand. Let the capacity reserve rate be $\lambda\%$ (for instance 40%) of the national capital K. Of course, this rate cannot be beyond unit, so

$$1 > \lambda > 0.$$

In case $\lambda=\bar{\lambda}=0$, *i.e.* when the capital equipments fully work within their physical limits, let the production coefficient σ be $\bar{\sigma}$. Of course, $\bar{\sigma}>\sigma>0$. In other words, the smaller the reserve rate, the larger the productivity. Thus, when a proper reserve rate is taken into consideration, we have about σ the following relation,

$$\sigma=\bar{\sigma}(1-\lambda). \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (13)$$

We remember that we have defined the production coefficient of capital as inverse of the capital coefficient. As is easily obtained from the equation (9), we get

$$g=\frac{s}{c}=\sigma s=\bar{\sigma}(1-\lambda)s. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (14)$$

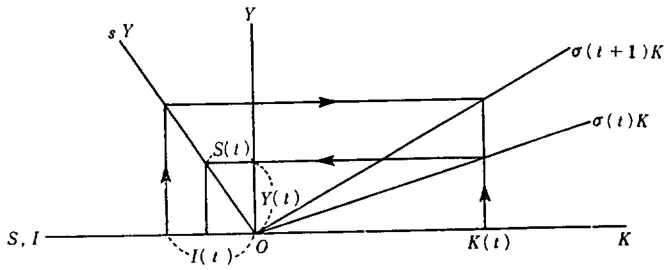
From this equation, we know that the smaller the proper reserve rate of capital λ, the larger the growth rate of the national income g, as far as other things remain constant.

Now, suppose the equation I=S remains maintained until up to the (t-1) *-th* period and I surpasses S at the *t-th* period for some reason (*e. g.* by the sudden innovation). We then make the following plausible assumption with respect to the behavior of the enterpriser;

> " In case there takes place a discrepancy between investment and saving at the *t-th* period, the working rate of capital at the $(t+1)$*-th* period will be the rate which produces the income level necessary for yielding the corresponding supply of saving."

The meaning of this assumption is well-illustrated in Fig. 3. Suppose $Y(t)$ is produced from $K(t)$ and $S(t)$ is supplied from $Y(t)$. By the assumption, $I(t)>S(t)$, which is shown in the left side of the figure. As is clear from the figure, the enterpriser should have had the higher working rate

Fig. 3



of capital in order to be $S(t)=I(t)$. Of course, it would be difficult to determine exactly the working rate of capital at the $(t+1)$-*th* period by means of this figure. However, the existence of such a tendency is highly plausible, when we remember that a part of the capital capacity is reserved in anticipation of such a situation.

Then, what kind of relationships exists between these two production coefficients? With this respect, we can prove the following equation,

$$\sigma(t+1)=\sigma(t)\frac{I(t)}{S(t)}, \quad \cdots\cdots (15)$$

from which we obtain $\sigma(t+1)>\sigma(t)$ when $I(t)>S(t)$

$\sigma(t+1)<\sigma(t)$ when $I(t)<S(t)$.

If the constancy is assumed about $\bar{\sigma}$ for the technical reason, these inequalities hold only by means of highering or lowering the working rate of capital.[1]

The lack of equilibrium between the investment and saving is thus responsible for the change of the working rate of capital to adjust the broken equilibrium. However, we show next that such an adjustment will in reali-

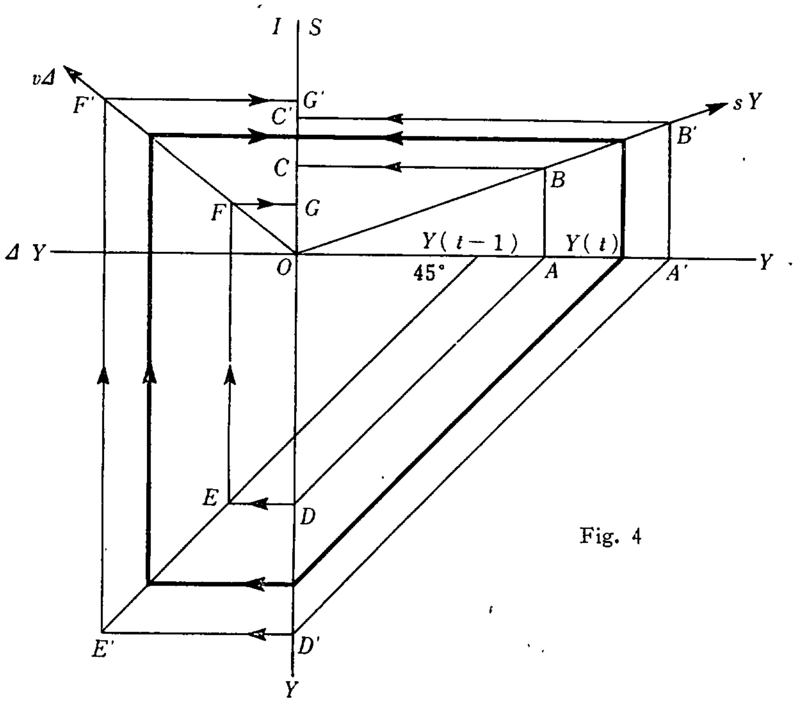


Fig. 4

ty have an effect of rather promoting the lack of equlibrium between the investment and saving.

In Fig. 4, the bold line indicates that the national income grows from $Y(t\text{-}1)$ to $Y(t)$ at the growth rate sufficient to realize $I(t)=S(t)$. If $Y(t)$ happens to be at the point A for some reason, the national saving is given by the point C passing through ABC, while the national investment is located at the point G going through $DEFG$. In other words, we have $I(t)<S(t)$. On the other hand, if $Y(t)$ happens to be at A', the national saving is given by C' passing through $A'B'C'$, while the national investment is at G' going through $D'E'F'G'$. In this case, we clearly have $I(t)>S(t)$.

From this figure, it is seen that $I(t)<S(t)$ or $I(t)>S(t)$ corresponds respectively to the smaller or larger growth rate of national income than that which can be obtained by solving $I(t)=S(t)$. This just corresponds to the case of the divergence of the actual growth rate from the warranted one as analysed by Harrod. As is easily seen from the above, the change of the working rate of capital to adjust $I\neq S$ brings about the higher or lower level of the national income than what would be brought in case there did not take place any change. For instance, let $I(t)>S(t)$. By the equation (12), we have

$$\frac{I(t+1)}{S(t+1)}=\frac{v}{s}\quad\frac{\Delta Y(t+1)}{Y(t+1)}>\frac{v}{s}\quad\frac{\Delta Y(t)}{Y(t)}=\frac{I(t)}{S(t)},\cdots\cdots\cdots \quad (16)$$

from which again by the equation (15), we obtain

$$\sigma(t+2)>\sigma(t+1)>\sigma(t). \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (17)$$

Under this assumption, the absolute divergence between the investment and saving becomes larger and larger. Paradoxically speaking, the increase of the working rate of capital for the maintenance of $I=S$ is helping to promote the failure of its maintenance and automatically strengthen the degree of utilization of capital. In short, it is clear that the process of the developmental equilibrium as expressed by $I=S$ is unstable under the above-mentioned assumptions.

V

We do not have any intention to develop a theory of business cycle here in this paper. We therefore wish to close the analysis with some supplementary comments on the upper bottle-neck, i.e. the ceiling which appears in the process of business cycles.

This problem has recently been made familiar to us by Hicks, Harrod and others. They conclude eventually that the full employment of labours constitutes the ceiling and checks the expansion of the national economy beyond a certain level.

We do not of course deny the possibility that the full employment of labours would constitute a ceiling to the further increase of national products. But it must be admitted that in most cases a plenty amount of unemployment is observed even amidst a boom. Rather it seems more plausible that the bottle-neck of capital constitutes the ceiling to be taken into consideration, in case there is not any external factor such as financial situations.

Making use of the above-observations, we are now in a position to make the following statement about this problem. "The technological restriction imposed upon the degree of utilization of capital, i.e. the existence of the condition $\bar{\sigma} > \sigma$ may check the further increase of national products, even when labours are unemployed." As was already observed, the self-cumulative expansion of the working rate of capital takes place accompanying the expanding process of disequilibrium. When such a self-cumulative expansion reaches to the technologically imposed limit, it can not go on any further and we reach to the bottle-neck of the capital.

Let's summerize what has been observed. We do not understand the process of business fluctuations as the process of moving equlibrium of I=S, but as the process of the disequilibrium between the effective demand and supply, i.e. the process of I≠S. The answer to the problem that a disequilibrium causes a further disequilibrium is found in the self-cumulative expansion of the working rate of capital. However, no systematic presentation of this problem has been attempted even by the advocates of disequilibrium theory of economic fluctuations. It seems to the author that such a lack of systematization is mainly originating from the confusion of the meaning of the acceleration coefficient and the capital coefficient as well as from the lack of detailed analysis on the working rate of capital. The present paper is a possible solution of this problem.[2]

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[1] Let

$$\sigma(t)K(t)=Y(t) \quad \cdots\cdots (1)$$

$$\sigma(t+1)K(t)=\bar{Y}(t) \quad \cdots\cdots (2)$$

where $\bar{Y}(t)$ is determined from $s\bar{Y}(t)=(I)t$ in accordance with the assumption. Clearly,

$$\sigma(t)=\frac{Y(t)}{K(t)}=\frac{1}{s}\,\frac{S(t)}{K(t)} \quad \cdots\cdots (3)$$

$$\sigma(t+1)=\frac{\bar{Y}(t)}{K(t)}=\frac{1}{s}\,\frac{I(t)}{K(t)}, \quad \cdots\cdots (4)$$

from which we obtain the equation (15) by (3)÷(4).

[2] There are many authors who restrict the notion of the equilibrium only to the stationary state. However, we must not forget the notion of the developmental equilibrium. Therefore, we have to consider not only the stability of the static equilibrium but also that of the developmental equilibrium. It is further to be remarked that the divergence of the solution of a dynamic equilibrium does not necessarily mean the instability of the divergent process of equilibrium, because some divergent processes may be stable.