Voluntary Participation Games in Public Good Mechanisms: Coalitional Deviations and Efficiency

by

Ryusuke Shinohara

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Koichi Tadenuma

Graduate School of Economics

Hitotsubashi University

Kunitachi, Tokyo, Japan

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CURRICULUM VITAE

Ryusuke Shinohara was born in Shizuoka, Japan, on October 31, 1977. In April 1996, he entered Tohoku University in Miyagi, Japan, from which he received the Bachelor of Arts in economics in March 2000. Subsequently he entered the Graduate School of Economics, Hitotsubashi University. He received the Master of Arts degree in economics in March 2002. He enrolled in the doctoral course of the Graduate School of Economics at Hitotsubashi University in April 2002. In the doctoral course, he specialized in game theory and applied micro economics. His thesis was supervised by Professor Koichi Tadenuma. In 2001-2004, he was a teaching assistant of the Graduate School of Economics at Hitotsubashi University. In 2003-2004, he was a research assistant for Professor Koichi Tadenuma.

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Contents

Chapter 1

Introduction

1.1 A participation problem in a public goods mechanism

The purpose of this dissertation is to examine a participation problem in a mechanism to produce a (pure) public good. The public good is one that satisfies non-excludability and non-rivalry: all agents can consume the same amount of a public good regardless of their contribution to it. Therefore, every agent has an incentive to free-ride the public good that is produced by other agents. As a result, the public good is provided at a low level. This situation is referred to as the "free-rider" problem.

A solution to the free-rider problem is the construction of an economic mechanism or system in which a socially efficient level of public goods is provided as a result of the strategic behavior of agents. The construction of such mechanisms has been studied in two distinct directions: one is *strategy-proofness*, and the other is *implementation theory*. In the theory of strategy-proofness, the mechanism designer, for example, the policy-implementation organization or the supplier of public goods, constructs a mechanism to elicit information about agents' preferences; this information is necessary for the provision of an efficient level of public goods as well as efficient cost distribution. However, preferences are usually only privately known. Therefore, it is possible that selfish agents try to misrepresent their preferences in order to manipulate the provision of the public good and their cost burdens. As a result of such strategic misrepresentation, the level of the public good may be socially inefficient. Thus, the construction of a procedure that gives all agents an incentive to announce their true preferences is an important issue in the provision of public goods. The importance of preference revelation was first pointed out by Samuelson (1954). His view on the possibility of truthful preference revelation is negative. He pointed out that there is room for strategic manipulation in the Lindahl mechanism, and he thought that the efficient provision of a public good could not be achieved through decentralized systems. Since the introduction of Samuelson 's viewpoint, there have been many studies on mechanisms in which the truthful revelation of preferences is a weakly dominant strategy for all agents; such mechanisms are referred to as *strategy-proof* mechanisms. Groves (1973) constructed a class of strategy-proof mechanisms, called the "Groves mechanism," which achieve the efficient provision of a public good on the domain of quasi-linear preferences. In the mechanisms, some amount of private goods must be discarded to give incentives to agents to reveal their true preferences. Thus, the allocations attained in the Groves mechanism are not Pareto-efficient. After the construction of the Groves mechanisms, Laffont and Maskin (1980) and Walker (1980) proved that only Groves mechanisms are strategy-proof mechanisms that can produce an efficient amount of the public good on a reasonable domain of preferences. Therefore, strategy-proofness and Pareto efficiency are incompatible in the public good economy.

Groves and Ledyard (1977) adopted a different approach from strategy-proofness, called *Nash implementation*. They gave up strategy-proofness and employed Nash equilibria as an equilibrium concept. They designed a mechanism that achieves a Pareto-efficient allocation at some Nash equilibrium. Therefore, in the theory of implementation, mechanisms can be constructed in such a way as to achieve Pareto efficiency, differently from the theory of strategy-proofness. Although the Groves and Ledyard mechanism does not satisfy the individual rationality condition[1], Hurwicz (1979) and Walker (1981) constructed mechanisms and succeeded in implementing the Lindahl allocation rule in Nash equilibria. Following these studies, many authors have constructed mechanisms that implement various desirable allocation rules under the concept of Nash equilibria.

However, implementation theory has limited the discussion to the construction of public goods mechanisms, and the participation problem in the mechanisms has not been sufficiently studied. In the implementation theory, all agents are assumed to participate in the mechanism. Hence, the results in this field indicate that a public good can be provided efficiently by constructing a public good mechanism under the assumption of the participation of all agents. Although the assumption is essential for the mechanism to fulfill its function in the economy, few studies have examined whether or not agents enter the mechanism voluntarily.

The participation problem is also important from a practical point of view. In the real world, there are many situations in which agents can decide whether or not to participate in the mechanism, and this participation decisions have serious effects on the effectiveness of the mechanism. For example, let us consider the case of international environmental agreements, such as the Kyoto Protocol. An environmental agreement can be regarded as a mechanism that provides a public good or eliminates harm to the

[1]The individual rationality condition requires that, for all agents, the allocation implemented by the mechanism should be at least as good as their initial endowments.

public in order to attain efficient allocations of resources. However, each country has the right to decide to participate in such an agreement. Since a sufficiently large number of countries must ratify such an agreement to put it into effect, the participation problem is very serious with regard to the implementation of the agreement. In fact, in the case of the Kyoto Protocol, the United States decided not to participate in the protocol, and the implementation of the protocol seemed improbable for a while. Another example is NHK, the public broadcasting network in Japan. NHK faces a serious participation problem these days. By law, every household must pay a fee to NHK. However, many individuals are refusing to pay this fee since they can watch NHK without paying the fee and the individuals that do not pay are not penalized.

As can be seen from these examples, the participation problem is important in a case in which an agent can decide whether or not to participate in a public good mechanism and non-participants can benefit from a public good provided by the participants at no cost. In such a case, every agent may have an incentive not to enter the mechanism, hoping that other agents will participate in the mechanism and provide a public good. This will generate another type of free-rider problem.

Palfrey and Rosenthal (1984) and Saijo and Yamato (1999) pointed out the importance of the strategic behavior of agents as they decide whether or not to participate in the mechanisms. Palfrey and Rosenthal (1984) formulated a participation game in a mechanism to implement a public project with identical agents. In this game, all agents simultaneously choose whether or not to participate. If they enter the mechanism, they contribute a fixed amount that is common to every participant. The public good is supplied only if the aggregate contribution of participants outweighs its production cost. Only the participants bear the cost of the public good, while non-participants can benefit from the public good at no cost because the public good is non-excludable. They characterized pure and symmetric mixed Nash equilibria and showed that the efficient provision of the public good is attained at a Nash equilibrium of this game but not all agents enter the mechanism.

Saijo and Yamato (1999) introduced a model of voluntary participation in a mechanism that implements the Lindahl allocation rule when the public good is perfectly divisible. Their model consists of two stages. In the first stage, agents decide simultaneously whether or not to participate in the public good mechanism. In the second stage, knowing the participation decisions of others, the agents that selected participation in the first stage choose their messages in the mechanism. As in Palfrey and Rosenthal (1984), only the participants bear the cost of the public good, but the non-participants can benefit from the public good at no cost in this model. In an economy in which all the agents have the same Cobb-Douglas utility function and the same initial endowments of a private good, Saijo and Yamato showed that the participation of all agents is not supported as a subgame perfect equilibrium of this game and the equilibrium allocation is not Pareto-efficient in many cases.

Okada (1993) examined the possibility of cooperation through the formation of an organization in an n-person prisoners' dilemma. In his game, agents face a similar participation problem in the organization. He also showed that participation of all agents is not necessarily achieved at the equilibrium of this game. The results presented in earlier literature lead to the conclusion that the free-rider problem with respect to the participation decision occurs and severely affects the resource allocations of the economy.

The existing literature has not considered the possibility that agents form a coalition and coordinate the participation decisions. Researchers characterized a set of participants that is stable against unilateral deviations of agents, focusing solely on subgame perfect Nash equilibria or Nash equilibria. However, in the theory of implementation, the mechanisms have been constructed not only under the concept of the Nash equilibrium but also under other equilibrium concepts, such as coalition-proof equilibria (Bernheim, Peleg, and Whinston, 1987) and strong equilibria (Aumann, 1959). If a mechanism is constructed under the assumption that agents form coalitions, then it is natural to consider that agents also coordinate participation decisions. Hence, in this case, it is important to analyze the participation decision, considering the possibility of the cooperative behavior of agents. In addition, behavior is based on various behavioral principles, and economists do not know which behavioral principles people will employ. Thus, considering the various possibilities is meaningful for understanding the consequences of strategic behavior.

In this dissertation, we consider the possibility that agents form a coalition in the participation decision stage. In addition to the unilateral deviations, we consider the following coordination behavior of agents:

- A subset of participants jointly switches to non-participation.
- Participants and non-participants form a coalition and coordinate the participation decisions with or without monetary transfers.
- A subset of non-participants jointly chooses participation.

Considering the possibility of coalitional behavior, we investigate (i) whether or not there is a set of participants that is stable against such coalitional behavior and (ii) which kinds of sets of participants are stable against the coalitional behavior.

We analyze the participation problem in a public good mechanism, as in Saijo and Yamato (1999). We examine coalition-proof equilibria and strong equilibria of the participation decision stage game. The notions of coalition-proof equilibria and strong equilibria are refinements of Nash equilibria that are immune to coalitional deviations. Therefore, even if a Nash equilibrium exists in this game, the participation game does not necessarily have a strong equilibrium and a coalition-proof equilibrium. One of the main purposes of this dissertation is to clarify whether there are coalition-proof equilibria and strong equilibria in this game and to characterize the two equilibria of this game, if such two equilibria exist.

1.2 Participation games in public good mechanisms

This section formally introduces the basic model in this dissertation. We consider the problem of providing a (pure) public good and distributing its cost. There are one private and one public goods in the economy. Let n be the number of agents. The set of agents is denoted by $N = \{1, \ldots, n\}$. Let Y be a given set of possible amounts of the public good. For example, $Y = \mathbb{R}_+$ when the public good is (perfectly) divisible, and $Y = \{0, 1\}$ when the public good is indivisible. Each agent i has a preference relation that is represented by the quasi-linear utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}$ such that $(y, x_i) \in Y \times \mathbb{R}_+ \mapsto v_i(y) - x_i \in \mathbb{R}$. The cost function of the public good is denoted by $c : Y \to \mathbb{R}_+$.

In this dissertation, we consider a situation in which there is a mechanism to provide a public good and distribute the cost of the public good, and each agent can simultaneously decide either participation or non-participation in the mechanism. A mechanism (or a game form) is a list of message spaces of all agents and an outcome function that associates an allocation with each profile of messages. The following two-stage game is considered: in the first stage, each agent simultaneously decides whether or not to participate in the mechanism. In the second stage, knowing the participation decisions of other agents, the agents who choose participation in the first stage select their messages from their message spaces of the mechanism. Only participants decide the quantity of the public good and the cost shares of each participant through the choice of messages. Denote by $(y^P, (x_j^P)_{j \in P})$ the allocation that is attained in the equilibria of the mechanism when $P \subseteq N$ is a set of participants. For example, if $Y = \mathbb{R}_+$, $v_i^{'}(y) > 0$ and $v_i^{''}(y) < 0$ for every $i \in P$ and every $y \in \mathbb{R}_+$, $c^{'}(y) > 0$ and $c^{''}(y) > 0$ for every $y \in \mathbb{R}_+$, and the *ratio allocation rule* introduced by Kaneko (1977a, 1977b) is achieved at the equilibrium of the mechanism, then, for every set of participants P,

$$\begin{aligned} y^P &\in \arg\max_{y \in \mathbb{R}_+} \sum_{j \in P} v_j(y) - c(y) \text{ and} \\ x_i^P &= \frac{v_i'(y^P)}{\sum_{j \in P} v_j'(y^P)} c(y^P) \text{ for all } i \in P. \end{aligned} \tag{1.1}$$

In this dissertation, we are not concerned with the implementation problem of the allocation rule. However, many researchers have constructed mechanisms that implement desirable allocation rules, such as the Lindahl allocation rule, under the various equilibrium concepts. Besides Groves and Ledyard (1977), Hurwicz (1979) and Walker (1981), discussed in this chapter, Peleg (1996) and Tian (2000) constructed mechanisms that doubly implement the Lindahl allocation rule in strong equilibria and Nash equilibria; Corchon and Wilkie (1996) constructed mechanisms that doubly implement the ratio equilibria in strong equilibria and Nash equilibria; Kalai, Postlewaite, and Roberts (1979) constructed a mechanism that implements the α core allocation rule in strong equilibria.[2]

In this dissertation, we assume that agents that selected non-participation can benefit from the public good at no cost because of the non-excludability of the public good.

[2]Okada (1993) modeled the stage of producing public goods and distributing these costs as a non-cooperative negotiation game among participants. An allocation that is Pareto efficient only within the participants is achieved in some subgame-perfect equilibria of this game.

Assumption 1.1 For every set of participants P and for every agent $i \notin P$, $x_i^P = 0$, and i consumes y^P.

Given the outcome of the second stage, the participation-decision stage can be reduced to the following simultaneous game. In the game, each agent i simultaneously chooses either $s_i = I$ (participation) or $s_i = O$ (non-participation), and then the set of participants is determined. Let P^s be the set of participants at an action profile $s = (s_1, \ldots, s_n)$. Then, each agent i obtains the utility $V_i(y^{P^s}, x_i^{P^s})$ at the action profile s. That is, the participants produce the public good, and they share the cost of the public good as above. Each non-participant can benefit from the public good at no cost. We call this reduced game a *participation game*, which is formally defined as follows.

Definition 1.1 (Participation game) A *participation game* is represented by $G = \left[N,\, S^n = \{I,\, O\}^n,\, (U_i)_{i \in N}\right]$, where U_i is the payoff function of i, which associates a real number $U_i(s)$ with each strategy profile $s \in S^n$: if P^s designates the set of participants at s, then $U_i(s) = V_i(y^{P^s}, x_i^{P^s})$ for all i.

We restrict our attention to the pure strategy profiles.

1.3 Equilibrium concepts

In this section, we introduce the notions of equilibria studied in this dissertation. The first notion is very basic.

Definition 1.2 (Nash equilibrium) A strategy profile $s^* \in S^n$ is a *Nash equilibrium* if, for all $i \in N$ and for all $\widehat{s}_i \in S$, $U_i(s_i^*, s_{-i}^*) \geq U_i(\widehat{s}_i, s_{-i}^*)$. A Nash equilibrium is *strict* if $U_i(s_i^*, s_{-i}^*) > U_i(\widehat{s}_i, s_{-i}^*)$ for all $i \in N$ and for all $\widehat{s}_i \in S \setminus \{s_i^*\}$.

The strict Nash equilibrium is an equilibrium concept that is strongly robust to unilateral deviations. Every strict Nash equilibrium is a *trembling-hand perfect Nash equilibrium* in normal form games. The trembling-hand perfect Nash equilibrium is the notion of non-cooperative equilibria that is robust to the possibility that players make mistakes with small probability.

Our second notion is the strong equilibria(Aumann, 1959). To define it, we use the following notation. For all $D \subseteq N$, we denote the complement of D by $-D$. For all coalitions D, $s_D \in S^{\#D}$ denotes a strategy profile for D. We simply write $s_N = s$.

Definition 1.3 (Strong equilibrium) A strategy profile $s^* \in S^n$ is a *strong equilibrium of* G if there exist no coalition $T \subseteq N$ and its strategy profile $\widetilde{s}_T \in S^{\#T}$ such that $U_i(\widetilde{s}_T, s^*_{-T}) \geq U_i(s^*)$ for all $i \in T$ with strict inequality for at least one $i \in T$.

A strong equilibrium is a strategy profile in which no subset of agents, taking the strategies of others as given, can jointly deviate in such a way that all members are at least as well off and at least one of its members is strictly better off.

Our third notion is the *coalition-proof equilibrium.* It was introduced by Bernheim, Peleg, and Whinston (1987) and is known as a refinement of Nash equilibria based on the stability against self-enforcing coalitional deviations. It is defined by using the notion of *restricted games.* A restricted game is a game in which a subset of agents play the game G, taking strategy profiles of agents outside the subset as given. We formally define it as follows. Let $T \subsetneq N$ and $t = \#T$. Let $\bar{s}_{N \setminus T} \in S^{n-t}$. A *restricted game* $G|\bar{s}_{N \setminus T}$ is a game in which the set of agents is T, the set of strategy profiles is S^t, and the payoff function for each $i \in T$ is the function $U_i(\cdot, \bar{s}_{N \setminus T})$ that associates a real value $U_i(s_T,\, \bar{s}_{N \setminus T})$ with each element s_T in S^t such that: $U_i(s_T,\, \bar{s}_{N \setminus T}) = V_i(y, x_i)$, where $(y, (x_j)_{j \in N})$ is the allocation when agents play $(s_T,\, \bar{s}_{N \setminus T})$ in G.

Definition 1.4 A *coalition-proof equilibrium* $(s_1^*, \ldots, s_n^*)$ is defined inductively with respect to the number of agents t:

- When $t = 1$, for all $i \in N$, s_i^* is a coalition-proof equilibrium for $G|s^*_{N\setminus\{i\}}$ if $s_i^* \in \arg\max\ U_i(s_i, s^*_{N\setminus\{i\}})$ s.t. $s_i \in S$.

- Let $T \subseteq N$ with $t = \#T \geq 2$. Assume that coalition-proof equilibria have been defined for all normal form games with fewer agents than t.

- Consider the restricted game $G|s^*_{N\setminus T}$ with t agents.

  ▷ A strategy profile $s_T^* \in S^t$ is called *self-enforcing* if, for all $Q \subsetneq T$, s_Q^* is a coalition-proof equilibrium of $G|s^*_{N\setminus Q}$.

  ▷ A strategy profile s_T^* is a coalition-proof equilibrium of $G|s^*_{N\setminus T}$ if it is a self-enforcing strategy profile and there is no other self-enforcing strategy profile $\widehat{s}_T \in S^t$ such that $U_i(\widehat{s}_T,\, s^*_{N\setminus T}) \geq U_i(s_T^*,\, s^*_{N\setminus T})$ for all $i \in T$ and $U_i(\widehat{s}_T,\, s^*_{N\setminus T}) > U_i(s_T^*,\, s^*_{N\setminus T})$ for some $i \in T$.

Coalition-proof equilibria are defined as the Pareto efficient frontier within the set of self-enforcing strategy profiles. The self-enforcing strategy profiles are recursively defined with respect to the number of agents in coalitions. At a self-enforcing strategy profile of N, no proper coalition of N can coordinate its members' strategies in such a way that all members of the coalition are at least as well off and at least one of them is strictly better off, and no proper subsets of the coalition further deviate in a self-enforcing way.

Remark 1.1 It is clear that both coalition-proof, strict, and strong equilibria are Nash equilibria. The set of strong equilibria is included in that of coalition-proof equilibria, because the coalition-proof equilibria are required to be stable only against *self-enforcing* coalitional deviations while the strong equilibria are defined to be stable against *all* *possible* coalitional deviations. The converse inclusion relation does not always hold. Note that the set of strict Nash equilibria and that of strong equilibria are not necessarily related by inclusion, nor are the set of strict Nash equilibria and that of coalition-proof equilibria.

1.4 The relation to the earlier literature

The participation game may include various models presented in the earlier literature. Using the framework of participation games, this section provides an overview of the models and the results of existing literature.

1.4.1 The case of a public project

A fixed-contribution game

Consider a case in which the level of the public good takes either zero or one: $Y = \{0, 1\}$. This kind of a public good is called a *public project*. Palfrey and Rosenthal (1984) first considered a game that is similar to the participation game and is called a *fixed-contribution game*. In this game, each player simultaneously decides whether or not to contribute a fixed amount that is common to every player. In addition, the public good is discrete and at most one unit of the public good is provided. If the aggregate contribution outweighs the cost of the public good, then one unit of the public good is provided. Otherwise, the public good is not produced. Let $\gamma > 0$ denote the contribution of every player. The preference relations of all players are represented by the same quasi-linear utility function. Let $\theta > 0$ be such that $v_i(y) = \theta y$ for all $i \in N$. Let $c > 0$ be the cost of producing one unit of the public good: that is, the cost function is $c(y) = cy$ for all $y \in Y$. We assume that there is a number of agents p^* such that $p^*\gamma \geq c$ and

$(p^* - 1)\gamma < c$. Palfrey and Rosenthal (1984) considered the two cases. One is a refund case, and the other is a non-refund case. If the number of contributors is less than p^*, then the contributions are refunded to the contributors in the refund case, but not in the non-refund case.

Interpreting the contribution of γ as the announcement of participation, we describe the fixed-contribution game by the participation game. Let P be a set of participants. Then, $x_i^P = 0$ for all $i \notin P$. If $\#P \geq p^*$, then $y^P = 1$ and $x_i^P = \gamma$ for all $i \in P$. Otherwise, $y^P = 0$ and $x_i^P = 0$ in the refund case, or $x_i^P = \gamma$ in the non-refund case.

Palfrey and Rosenthal (1984) compared the Nash equilibrium of the game under the refund case and that under the non-refund case, and presented the following results.

Proposition (Palfrey and Rosenthal, 1984) *(i) Under both the refund rule and the non-refund rule, p^* agents contribute γ in some pure-strategy Nash equilibrium. (ii) The set of pure-strategy Nash equilibria under the non-refund rule is included in that under the refund rule, and they may not coincide.*

From (ii) of the above proposition, the set of pure-strategy Nash equilibria varies under the different refund rules. Under the refund case, there may be a Nash equilibrium with the participation of fewer $p^* - 1$ players because every player i is indifferent between participation and non-participation when fewer than $p^* - 2$ players other than i choose I. On the other hand, under the non-refund rule, every player i clearly has an incentive to choose O in the same situation. This indicates that a strategy profile with the participation of fewer than $p^* - 1$ players is not supported as a Nash equilibrium of this game under the non-refund rule.

Palfrey and Rosenthal (1984) also examined the symmetric mixed Nash equilibria of this game. Since there is a possibility that the participants must pay γ, although the public good is not provided under the non-refund rule, the incentive for players to choose participation under the non-refund rule is intuitively less than that under the refund case. This intuition is reflected in the following proposition.

Proposition (Palfrey and Rosenthal, 1984) *The expected amount of contributions in symmetric mixed-strategy Nash equilibria under the refund rule is greater than that under the non-refund rule.*

Correlation and efficiency

Note that the symmetric mixed-strategy Nash equilibrium is not necessarily efficient. In the fixed-contribution game, pure-strategy Nash equilibria Pareto-dominates mixed Nash equilibria under both the refund and non-refund rules in many cases. The reason simply comes from the possibility that the public good is not provided with positive probability in mixed Nash equilibria. Cavaliere (2001) examined whether or not coordination behavior among agents leads to the efficiency of equilibria in the fixed-contribution game under the non-refund rule. He was motivated by the fact that the fixed-contribution game has efficient and inefficient Nash equilibria. One problem that was not considered by Palfrey and Rosenthal (1984) is why agents play one equilibrium rather than another. He considered that some sort of communication would solve the coordination problem and that agents could agree to play efficient strategy profiles through such communication. He investigated the relationship between communication and the coordination problem by using the concept of *correlated equilibrium* (Aumann, 1974, 1987), and showed the following.

Proposition (Cavaliere, 2001) *In the fixed-contribution game, there is an efficient and symmetric correlated equilibrium, which is a convex combination of efficient pure-*

strategy Nash equilibria.

This result is intuitive, because agents can play only the efficient pure-strategy Nash equilibria with positive probability by correlating their strategies. However, note that every Nash equilibrium is a correlated equilibrium by definition. Thus, if the fixed-contribution games have inefficient Nash equilibria, then there are inefficient correlated equilibria. We can not conclude from his result that the correlation of strategies solves the coordination problem in the fixed-contribution game.

A participation game in Coaseian bargaining

Dixit and Olson (2000) introduced a model of voluntary participation in a public project, which is similar to the model of Palfrey and Rosenthal (1984). Dixit and Olson (2000) considered the participation problem in Coaseian bargaining. In Coaseian bargaining, participants agree to achieve an allocation that satisfies Pareto efficiency only within the participants: the project is carried out, and the cost of the project is distributed in a budget-balancing way if the joint benefit from the project is greater than the cost of the public project; otherwise, the project is not undertaken, and the cost share of each participant is zero.

The formal model is as follows: the preferences of agents are represented by the same quasi-linear utility function. There is a unique number of agents $\widehat{p}$ such that $\widehat{p}\,\theta > c$ and $(\widehat{p}-1)\theta \leq c$. In their model, if $\widehat{p}$ or more agents choose I, then the project is undertaken, and the participants share its cost equally. Otherwise, the participants do not undertake the project and pay nothing. In our framework, the model of Dixit and Olson (2000) can be represented in the following way. Let $P \subseteq N$ be a set of participants. For every non-participant $i \notin P$, $x_i^P = 0$. If $\#P \geq \widehat{p}$, then $y^P = 1$ and $x_i^P = \dfrac{c}{\#P}$ for all $i \in P$. If not, $y^P = 0$ and $x_i^P = 0$ for all $i \in P$.

Dixit and Olson (2000) analyzed symmetric mixed Nash equilibria of this game and explained the properties of equilibrium strategies.

Proposition (Dixit and Olson, 2000) *In a symmetric mixed Nash equilibrium, the probability of choosing participation increases with respect to* $\widehat{p}$.

Since every agent receives positive payoffs only if $\widehat{p}$ or more agents choose I, he assigns a higher probability to participation as the minimal number of participants that is necessary to undertake the project gets higher. We confirm from this proposition that the probability of producing the public good is nearly zero when $\widehat{p}$ is small.[3] Therefore, in this game, there may be a Nash equilibrium at which the project is not done although undertaking the project is socially efficient.

Note that, in the model of Dixit and Olson (2000), there is ${}_nC_{\widehat{p}}$ pure-strategy Nash equilibria at which $\widehat{p}$ agents choose I and the efficient provision is achieved. Hence, the implication of voluntary participation to the equilibrium allocations are completely different, depending on pure or mixed-strategy equilibria.

1.4.2 The case of a perfectly divisible public good

Saijo and Yamato (1999) introduced a participation game in a mechanism that implements the Lindahl allocation rule. In their model, the public good is assumed to be perfectly divisible. They assumed that agents have the same initial endowments of the private good and preference relations that are represented by the same Cobb-Douglas

[3]By using many numerical examples, Dixit and Olson (2000) demonstrated that the project is undertaken with very low probability in every symmetric Nash equilibrium in a case in which $\widehat{p}$ is absolutely large but substantially smaller than n. This case corresponds to the case in which $\dfrac{\widehat{p}}{n}$ is close to zero. Hence, the case is equivalent to that of very small $\widehat{p}$ if n is fixed. Therefore, the result that is analogous to the observation in the numerical examples can be obtained from this proposition.

utility function. Let $\omega > 0$ be the initial endowments and let $V^\alpha(y,z) = y^{1-\alpha}z^\alpha$ for some $\alpha > 0$, where z denotes a private good consumption. One unit of the public good is produced from one unit of the private good. If the outcome of the second stage is given by the Lindahl allocation for the participants, the participation decision stage is described as follows: let P be a set of agents who choose participation in the first stage. Then, $y^P = (1-\alpha)\omega\#P$, $z_i^P = \alpha\omega$ for all $i \in P$, and $z_i^P = \omega$ for all $i \notin P$. Note that the level of the public good increases as the number of participants increases, differently from the case of a public project.

Saijo and Yamato (1999) characterized the number of participants at a pure-strategy subgame-perfect equilibrium for each value of α, and showed that the equilibrium number of participants becomes smaller as α gets larger. These results indicate that, although participation of all agents can be supported as an equilibrium for some α, some agents choose non-participation in equilibria for many other α. They also proved that all agents are less likely to choose I as the number of agents gets larger.

1.4.3 A participation problem in other economic models

Okada (1993) and Maruta and Okada (2001) introduced a model of group formation on the prisoners' dilemma game, which is called the *institutional arrangement game*. Their model consists of multiple stages, and the first stage is the participation decision stage in a group to achieve cooperative actions in the prisoners' dilemma, which is similar to the participation game. In the subgame-perfect equilibrium of their model, a sufficiently large number of agents choose participation in the group, but not all agents enter it. As a result, some agents choose cooperation, and others select defection in the prisoners' dilemma game in the equilibrium of this game. This situation corresponds to the case in which a positive but Pareto-inefficient amount of a public good can be produced by forming a group in the context of the provision of a public good.

Many other works have considered a participation model that has a similar structure to the participation game. The examples include participation in an international environmental treaty (Barrett, 1994; Carraro and Siniscalco, 1993, 1998; Ecchina and Mariotti, 1998; Hoel and Schneider, 1997) and the formation of cartels (d'Asprempnt et al., 1983; Thoron, 1998; Belleflamme, 2000)

1.5 Organization of this dissertation

In Part I of this dissertation, we deal with topics regarding the participation game when the public good is perfectly divisible as in Saijo and Yamato (1999). In Chapters 2 and 3, we examine a coalition-proof equilibrium of the participation game in a public good mechanism to implement the ratio equilibrium allocation rule, which is an extention of the Lindahl equilibrium allocation rule to the case of a convex cost function. We consider the case of identical agents in Chapter 2, and extend the analyses to a case of heterogeneous agents in Chapter 3. We mainly investigate the existence of coalition-proof equilibria and the number of participants at the equilibria. In Chapter 4, we consider a generalized game including the model of Chapter 2, and clarify the properties of the coalition-proof equilibria based on different notions of dominance relations.

In Part II, we consider the case of a discrete public good. Chapter 5 considers the case of a public project as in Palfrey and Rosenthal (1984). The participation game considered in this chapter corresponds to the game with refund rule introduced by Palfrey and Rosenthal (1984). However, the payments of participants depend on the set of participants in the participation game of this chapter, which is different from the case presented in their study. The mechanism achieves the allocations satisfy Pareto efficiency within the participants and individual rationality. We study strict Nash equilibria, coalition-proof equilibria, and the strong equilibria of the participation game as well as the relationships among the three equilibria. We show that there is a Pareto-efficient equilibrium allocation in this game. In Chapter 6, we first consider the participation game in the case of a public project. We consider the possibility that agents in a coalition coordinate their decisions through monetary transfers, and we examine the strong equilibria of this game. Second, we study the participation game in the case of a multi-unit public good; in this case, the public good is discrete and can have at most two units. We investigate the difference between the case of a public project and that of a multi-unit public good. We show that the equilibrium allocations are not Pareto efficient if the public good can be produced in multiple units, which is similar to the results of Saijo and Yamato (1999). Chapter 7 concludes this dissertation.

Part I

Participation Games with Perfectly Divisible Public Goods

Chapter 2

Coalition-proof Equilibria in Participation Games: Identical Agents

In this chapter, we consider the case in which the public good is perfectly divisible: $Y = \mathbb{R}_+$. We examine the participation game in a mechanism that implements the ratio allocation rule introduced by Kaneko (1977).[1] The purposes of this chapter are (i) to investigate whether there is a participation decision that is stable against coordination behavior of agents and (ii) to clarify which properties such participation decision satisfies if the decision exists. In this chapter, we consider a case in which agents have the identical preference. We show that the set of coalition-proof equilibria coincide with the Pareto efficient frontier of the set of Nash equilibria in this case. Thus, a coalition-proof equilibrium exists if there is a Nash equilibrium in this game.

[1]Note that the ratio allocation rule is equal to the Lindahl allocation rule when the cost function is linear.

2.1 Settings and examples

We assume that agents have an identical preference: $v(\cdot) = v_i(\cdot)$ for all $i \in N$. We assume that $v' > 0$, $v'' < 0$, and $v(0) = 0$. The cost function $c(\cdot)$ satisfies $c' > 0$, $c'' \geq 0$, and $c(0) = 0$. Moreover let us assume that $v'(0) > c'(0)$.

Example 2.1 Consider an example where $n = 3$, $v(y) = \alpha\sqrt{y}$, and $c(y) = y$, where $\alpha > 0$. When the set of participants is P with $p = \#P$, the public good provision y^P maximizes

$$p\left(\alpha\sqrt{y} - \frac{y}{p}\right).$$

Hence,

$$y^P = \left(\frac{\alpha p}{2}\right)^2.$$

The payoff to every agent $i \in P$ is

$$\frac{\alpha^2 p}{2} - \frac{1}{p}\left(\frac{\alpha p}{2}\right)^2 = \frac{\alpha^2 p}{4},$$

and the payoff to every $i \in N \backslash P$ is

$$\frac{\alpha^2 p}{2}.$$

Payoff matrix appears in Table 2.1, where agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There are two types of Nash equilibria in this game. One is the Nash equilibrium with participation of one agent, and the other is the Nash equilibrium with participation of two agents. Clearly, every Nash equilibrium with two participants is coalition-proof, and only the Nash equilibria are coalition-proof.

In this example, the set of coalition-proof equilibria coincides with that of Nash equilibria at which the number of participants is the greatest in the set of Nash equilibria.

	I	O
I	$\frac{3\alpha^2}{4}$, $\frac{3\alpha^2}{4}$, $\frac{3\alpha^2}{4}$	$\frac{\alpha_2}{2}$, α^2, $\frac{\alpha^2}{2}$
O	α^2, $\frac{\alpha^2}{2}$, $\frac{\alpha^2}{2}$	$\frac{\alpha^2}{2}$, $\frac{\alpha^2}{2}$, $\frac{\alpha^2}{4}$

I

	I	O
I	$\frac{\alpha^2}{2}$, $\frac{\alpha^2}{2}$, α^2	$\frac{\alpha^2}{4}$, $\frac{\alpha^2}{2}$, $\frac{\alpha^2}{2}$
O	$\frac{\alpha^2}{2}$, $\frac{\alpha^2}{4}$, $\frac{\alpha^2}{2}$	0, 0, 0

O

Table 2.1: Payoff matrix of Example 2.1

In the following section, we show that this statement generally holds in the case of identical agents.

2.2 Basic properties

Note that the payoffs to agents depend on the number of participants when agents are identical. Let us introduce the following notation for convenience.

Definition 2.1 Let $u_i : \{0, 1, \ldots, n-1\} \times \{I, O\} \to \mathbb{R}_+$ denote a payoff function of agent i that depends on the number of agents other than i and i's participation decision. If p designates the number of participants other than i and s_i designates i's participation decision, then i receives the payoff $u_i(p, s_i)$.

Let y^p be the level of the public good when p agents choose I for every $p \in \{0, \ldots, n\}$. Note that, for every $i \in N$ and for every p, $u_i(p, I) = v(y^{p+1}) - \dfrac{c(y^{p+1})}{p+1}$ and $u_i(p, O) = v(y^p)$. Since agents are identical, we have $u_i(p, I) = u_j(p, I)$ for all $i,\ j \in N$ and for all $p \leq n-1$ and $u_i(p, O) = u_j(p, O)$ for all $i,\ j \in N$ and for all $p \leq n$. Therefore, we can hereafter omit agents' indices of the payoff functions.

Lemma 2.1 For all numbers of participants $p,\ q \in \{0, \ldots, n\}$, if $p > q$, then $y^p > y^q$.

Lemma 2.1 is immediate. By Lemma 2.1, the level of public good gets higher as the number of participants increases.

Lemma 2.2 The payoff function of participants and that of non-participants are increasing with respect to the number of participants other than them: (i) for all $p,\ q \in \{0,\ldots,n-1\}$, if $p > q$, then $u(p,O) > u(q,O)$ and (ii) for all $p,\ q \in \{0,\ldots,n-1\}$, if $p > q$, then $u(p,I) > u(q,I)$.

Proof. Condition (i) is immediate from Lemma 2.1. We show condition (ii). Let $p,\ q \in \{0,\ldots,n-1\}$ such that $p > q$. The public good provision y^p satisfies

$$(p+1)v(y^{p+1}) - c(y^{p+1}) \geq (p+1)v(\widetilde{y}) - c(\widetilde{y})$$

for all $\widetilde{y}$. In particular, we have

$$(p+1)v(y^{p+1}) - c(y^{p+1}) \geq (p+1)v(y^{q+1}) - c(y^{q+1}).$$

Dividing the both sides of the above inequality by $p+1$ yields

$$v(y^{p+1}) - \frac{c(y^{p+1})}{p+1} \geq v(y^{q+1}) - \frac{c(y^{q+1})}{p+1}.$$

Since $p > q$, we have

$$v(y^{q+1}) - \frac{c(y^{q+1})}{p+1} > v(y^{q+1}) - \frac{c(y^{q+1})}{q+1}. \tag{2.1}$$

Therefore, we have $v(y^{p+1}) - \dfrac{c(y^{p+1})}{p+1} > v(y^{q+1}) - \dfrac{c(y^{q+1})}{q+1}$. Since the left hand side of (2.1) is equal to $u(p,I)$ and the right hand side is equal to $u(q,I)$, we have $u(p,I) > u(q,I)$.

■

Lemma 2.2 is a basic property of our model. Using this property, we show the main results of this chapter.

2.3 Results

2.3.1 Existence of coalition-proof equilibria and uniqueness of the number of participants at coalition-proof equilibria

In Example 2.1, there exist two types of Nash equilibria. Only the Nash equilibria with participation of two agents can be supported by coalition-proof equilibria: in other words, the number of participants in coalition-proof equilibria is uniquely determined by the largest number of participants in the set of Nash equilibria. Using only the conditions in Lemma 2.2 and the assumption of identical agents, we prove the following proposition.

Proposition 2.1 *Suppose that agents' preferences are identical. Let $p^{max} \leq n$ be the maximal number of participants in the set of Nash equilibria: $p^{max} = \max_{s \in NE(G)} \#\{i \in N \mid s_i = I\}$, in which $NE(G)$ is the set of Nash equilibria of G. In the participation game with a perfectly divisible public good, the set of coalition-proof equilibria coincides with the set of Nash equilibria at which p^{max} agents choose I.*

Let $s^{max} \in S^n$ be a Nash equilibrium at which p^{max} agents select participation. Let $D \subseteq N$ denote a coalition and $\overline{s}_D$ denote a strategy profile of D. The number of participants at $(\overline{s}_D,\, s^{max}_{-D})$ is denoted by $\overline{p}$. Note that, at the strategy profile s^{max}, agent i receives the payoff $u(p^{max} - 1, I)$ if i chooses I, and j obtains $u(p^{max}, O)$ if j chooses O. Similarly, at $(\overline{s}_D, s^{max}_{-D})$, the payoff to agent i choosing I and that to agent j selecting O are $u(\overline{p} - 1, I)$ and $u(\overline{p}, O)$, respectively. As preparations for proving Proposition 2.1, we show the following lemmas.

Lemma 2.3 If the coalition D deviates in a way in which $\overline{p} \leq p^{max}$, then some of its members are worse off. Moreover, for all agents $i \in D$, if $\overline{p} < p^{max}$, then the payoff to i

at s^{max} is greater than or equal to the i's payoff at $(\overline{s}_D, s^{max}_{-D})$.

Proof of Lemma 2.3. First, let us consider a case in which $\overline{p} = p^{max}$. In this case, there exists at least one member i of D with $s^{max}_i = O$ and $\overline{s}_i = I$.[2] By the definition of Nash equilibria, we have $u(p^{max}, O) \geq u(p^{max}, I)$. Since $p^{max} > \overline{p} - 1$, we obtain $u(p^{max}, I) > u(\overline{p}-1, I)$ by Lemma 2.2. Therefore, i's payoff decreases after the deviation.

Second, consider a case in which $\overline{p} < p^{max}$. We obtain from Lemma 2.2 that $u(p^{max} - 1, I) > u(\overline{p} - 1, I)$ for every member i of D with $s^{max}_i = \overline{s}_i = I$ and $u(p^{max}, O) > u(\overline{p}, O)$ for every $i \in D$ with $s^{max}_i = \overline{s}_i = O$. For every member i of D with $s^{max}_i = I$ and $\overline{s}_i = O$, the following inequalities are satisfied:

$$u(p^{max} - 1, I) \geq u(p^{max} - 1, O) \geq u(\overline{p}, O) \text{ with equality if } \overline{p} = p^{max} - 1. \tag{2.2}$$

The first inequality of (2.2) follows from the definition of Nash equilibria and the second inequality follows from Lemma 2.2. Hence, we have $u(p^{max} - 1, I) \geq u(\overline{p}, O)$ for every $i \in D$ with $s^{max}_i = I$ and $\overline{s}_i = O$. Similarly, for every member $i \in D$ with $s^{max}_i = O$ and $\overline{s}_i = I$, we have

$$u(p^{max}, O) \geq u(p^{max}, I) > u(\overline{p} - 1, I).$$

Therefore, the deviation by D does not improve the members' payoffs if $\overline{p} < p^{max}$.

■

Lemma 2.4 If $p^{max} < n$, then $u(t, O) - u(t, I) > 0$ for all $t \in \{p^{max}, \ldots, n - 1\}$.

Proof of Lemma 2.4. Suppose that $p^{max} < n$ and that there exist $t \in \{p^{max}, \ldots, n-1\}$ such that $u(t, O) - u(t, I) \leq 0$. Hence, $u(t, O) \leq u(t, I)$.

[2] In the case of $\overline{s}_D = s^{max}_D$, the payoffs to all the members trivially do not change.

If $t = n-1$, then $u(n-1, I) \geq u(n-1, O)$, which indicates that the participation of n agents is supported as a Nash equilibrium. This contradicts $p^{max} < n$. Consider a case in which $t < n-1$. Since $t+1 > p^{max}$, we have $u(t+1, O) < u(t+1, I)$. Otherwise, condition $u(t+1, O) \geq u(t+1, I)$, together with $u(t, O) \leq u(t, I)$, implies that the participation of $t+1$ agents can be supported as a Nash equilibrium and p^{max} is not the maximal number of participants in the set of Nash equilibria, which is a contradiction. By the same way, the following inequalities are obtained:

$$\begin{aligned} & u(t+2, O) < u(t+2, I), \\ & u(t+3, O) < u(t+3, I), \\ & \qquad \vdots \\ \text{and} \quad & u(n-1, O) < u(n-1, I). \end{aligned} \tag{2.3}$$

From (2.3), participation of n agents is a Nash equilibrium, which is a contradiction. Therefore, we have $u(t, O) - u(t, I) > 0$ for all $t \in \{p^{max}, \dots, n-1\}$.

■

Proof of Proposition 2.1. We first show that s^{max} is a coalition-proof equilibrium. If $p^{max} = n$, no coalitional deviations can improve their members' payoffs by Lemma 2.3. Then s^{max} is a strong equilibrium of G. Hence, it is a coalition-proof equilibrium. Let us consider a case in which $p^{max} = n-1$. By Lemma 2.3, no deviations after which p^{max} or less agents choose I improve their members' payoffs. Since $p^{max} = n-1$, the participation of n agents is not supported as a Nash equilibrium. Thus, we have $u(n-1, O) > u(n-1, I)$. It follows from this inequality that no deviations that achieve the participation of n agents are profitable. Thus, s^{max} is a strong equilibrium, which implies that it is also a coalition-proof equilibrium. Next, consider a case in which

$p^{max} \leq n-2$. Suppose, on the contrary, s^{max} is not coalition-proof. Then, there exist a coalition C and its self-enforcing strategy profile $\widehat{s}_C$ such that $U_i(\widehat{s}_C, s_{-C}) \geq U_i(s)$ for all $i \in C$ with strict inequality for at least one $i \in C$. Let $\widehat{p}$ be the number of participants at $(\widehat{s}_C, s^{max}_{-C})$.

Step 1. It follows that $\widehat{p} > p^{max}$.

Since all members of C are not worse off and at least one member of the coalition is better off after the deviation, we must have $\widehat{p} > p^{max}$ by Lemma 2.3. Hence, Step 1 holds.

Step 2. The strategy profile $\widehat{s}_C$ is not a Nash equilibrium of the restricted game $G|s^{max}_{-C}$.

Note that $\widehat{p} - 1 \geq p^{max}$ and that there is at least one member $i \in C$ with $s^{max}_i = O$ and $\widehat{s}_i = I$. Since $u(\widehat{p}-1, O) > u(\widehat{p}-1, I)$ by Lemma 2.4, agent i has an incentive to switch from I to O again after the deviation by C. It is straightforward that $\widehat{s}_C$ is not a Nash equilibrium of $G|s^{max}_{-C}$. Thus, Step 2 is true.

By Step 2, $\widehat{s}_C$ is not a self-enforcing strategy profile, which is a contradiction. Therefore, s^{max} is coalition-proof.

Secondly, we prove that p^{max} agents enter the mechanism in every coalition-proof equilibrium. We know from the proof of Lemma 2.3 that any strategy profile with the participation of less than p^{max} agents is Pareto dominated by s^{max}. Since s^{max} is a self-enforcing strategy profile, the participation of less than p^{max} agents is not supportable as a coalition-proof equilibrium. Every strategy profile with more than p^{max} participants also can not be coalition-proof because it is not a Nash equilibrium. Therefore, the number of participants in the set of coalition-proof equilibria is uniquely determined by p^{max}.

■

Corollary 2.1 A coalition-proof equilibrium exists in the participation game.

Proof. It is clear from Proposition 2.1 that the existence of a Nash equilibrium implies that of a coalition-proof equilibrium in the participation game. The Nash equilibrium is proved to exist in the same way as d'Aspremont et al. (1983). ■

Remark 2.1 Thoron (1998) obtained a similar result to Proposition 2.1 in the cartel formation problem. But Thoron (1998) used different conditions from ours. She used the following two conditions: (i) $u(p, O) > u(q, O)$ for all $p,\ q \in \{0, \dots, n-1\}$ such that $p > q$ and (ii) $u(p, O) > u(p-1, I)$ for all $p \in \{1, \dots, n-1\}$. Although conditions (i) and (ii) are satisfied in our model, we do not use condition (ii) in the proof; we use the condition $u(p, I) > u(q, I)$ for all $p,\ q \in \{0, \dots, n-1\}$ such that $p > q$, instead.

Remark 2.2 Note that Proposition 2.1 holds if agents have identical preferences and Lemma 2.2 is satisfied. Thus, the assumption that agents have a quasi-linear preference does not matter. Saijo and Yamato (1999) studied the *symmetric Cobb-Douglas economy*, in which all agents have the preference that is represented by the same Cobb-Douglas utility function $\alpha \ln z + (1-\alpha) \ln y$ for some $\alpha \in (0,\ 1)$, where z denotes the consumption of the private good. All agents are assumed to have the same initial endowments of the private good $\omega > 0$. They assumed that one unit of the private good yields one unit of the public good, and they considered the voluntary participation game in a public good mechanism to implement the Lindahl allocation rule. In this model, if P designates a set of participants and p denotes the number of agents in P, the provision of the public good is $y^p = (1-\alpha)\omega p$, the private good consumption of participants is $\alpha\omega$, and that of non-participants is ω. Hence, $u(p-1, I) = \alpha \ln \alpha\omega + (1-\alpha)\ln(\omega(1-\alpha))p$ and $u(p, O) = \alpha \ln \omega + (1-\alpha)\ln(1-\alpha)\omega p$. Since $u(\cdot, I)$ and $u(\cdot, O)$ are increasing functions in the first argument, Lemma 2.2 is satisfied. Therefore, the symmetric Cobb-Douglas economy has a coalition-proof equilibrium, and the number of participants at coalition-proof equilibria is unique.

2.3.2 Coalition-proof equilibria and the Pareto efficient frontier of the set of Nash equilibria

We provide another characterization of the set of coalition-proof equilibria in the participation game.

Definition 2.2 A strategy profile $s \in S^n$ is Pareto dominated by a strategy profile $\widetilde{s}$ if $U_i(\tilde{s}) \geq U_i(s)$ for all $i \in N$ and $U_i(\tilde{s}) > U_i(s)$ for some $i \in N$.

Proposition 2.2 *In the participation game, a strategy profile is a coalition-proof equilibrium if and only if it is a Nash equilibrium that is not Pareto dominated by any other Nash equilibrium.*

Proof. Let p^{max} be the maximal number of participants in the set of Nash equilibria. Take any coalition-proof equilibrium s^{cpe}. By Proposition 2.1, p^{max} agents choose I at s^{cpe}. By Lemma 2.3, s^{cpe} Pareto dominates any other Nash equilibrium with participation of less than p^{max} agents and it is undominated by any other Nash equilibrium with participation of p^{max} agents. Hence, s^{cpe} is a Nash equilibrium that is not Pareto dominated by any other Nash equilibrium. Conversely, let s^{pne} be a Nash equilibrium that is not dominated by any other Nash equilibrium. Then, p^{max} agents choose I in

s^{pne}. Therefore, it is coalition-proof. ■

It is clear from the definition of coalition-proof equilibria that the set of coalition-proof equilibria coincide with the Pareto efficient frontier of the set of Nash equilibria in every two-player game. However, the two sets do not necessarily coincide in games with more than two players. Bernheim, Peleg, and Whinston (1987) provided an example of three-player game in which there are two Nash equilibria, one is coalition-proof and the other is not, and the former is dominated by the latter. In the participation game, the two sets coincide regardless of the number of players.

Yi (1999) also investigated the equivalence between the set of coalition-proof equilibria and the Pareto efficient frontier of the set of Nash equilibria. Yi(1999) considered a game in which the strategy space of each player is a subset of real line and he showed that if a game satisfies *anonymity*[3], *monotone externality*[4], and *strategic substitutability*[5], then the set of coalition-proof equilibria and the Pareto efficient frontier of the set of Nash equilibria coincide. In the participation game, the anonymity condition is satisfied since agents are identical and the payoff function of every agent i depends on his participation decision and the number of agents other than i who choose I. The participation game satisfies the monotone externality condition because the utilities of participants and non-participants get higher as the number of participants increases. However, only a weaker

[3]A game satisfies anonymity if, for all $i \in N$, all $s_i \in S$, and all $s_{-i},\ \widehat{s}_{-i} \in S^{n-1}$, if $\sum_{j \neq i} s_j = \sum_{j \neq i} \widehat{s}_j$, then $U_i(s_i, s_{-i}) = U_i(s_i, \widehat{s}_{-i})$.

[4]A game satisfies monotone externality if, for all $i \in N$, all $s_i \in S$, and all s_{-i} and $\widehat{s}_{-i} \in S^{n-1}$, if $\sum_{j \neq i} s_j > \sum_{j \neq i} \widehat{s}_j$, then either $U_i(s_i, s_{-i}) \geq U_i(s_i, \widehat{s}_{-i})$ or $U_i(s_i, s_{-i}) \leq U_i(s_i, \widehat{s}_{-i})$ holds. If the former holds, the condition means *positive* externalities, and it represents *negative* externalities if the latter is satisfied.

[5]A game satisfies strategic substitutability if, for all $i \in N$, all $s_i,\ \widehat{s}_i \in S$, and all $s_{-i},\ \widehat{s}_{-i} \in S^{n-1}$, if $s_i > \widehat{s}_i$ and $\sum_{j \neq i} s_j > \sum_{j \neq i} \widehat{s}_j$, then $U_i(s_i, s_{-i}) - U_i(\widehat{s}_i, s_{-i}) < U_i(s_i, \widehat{s}_{-i}) - U_i(\widehat{s}_i, \widehat{s}_{-i})$.

condition than the strategic substitutability is satisfied in the participation game. Denoting strategies I and O by 1 and 0, respectively, we can describe the condition of strategic substitutability as follows: for all $i \in N$ and for all $p,\ \widehat{p} \in \{0, 1, \ldots, n-1\}$, if $p > \widehat{p}$, then $u_i(p, O) - u_i(p, I) > u_i(\widehat{p}, O) - u_i(\widehat{p}, I)$. Thus, the gain from switching from I to O is increasing with respect to the number of agents other than i that choose I. In the participation game, when agents' preferences are identical, Lemma 2.4 holds. The condition of Lemma 2.4 means that the gains of an agent from switching from I to O is positive if p^{max} or more agents other than the agent choose I. However, the gains is not necessarily increasing with respect to the number of participants. Thus, Proposition 2.2 provides different sufficient conditions from those of Yi (1999) for the set of coalition-proof equilibria to coincide with the Pareto efficient frontier of the set of Nash equilibria.

Finally, we mention that the coalition-proof equilibrium of our model is based on *weak domination*[6], while that of Yi (1999) is based on *strict domination*[7]. These two coalition-proof equilibria are not necessarily related by inclusion. (See Example 4.1 on page 57) Hence, it is not trivial and open that the set of coalition-proof equilibria under weak domination coincides with the strictly Pareto efficient frontier of the set of Nash equilibria under Yi (1999)'s conditions. We will examine the relationship between coalition-proof equilibria based on different dominance relations in Chapter 4.

---

[6]Strategy profile s weakly dominates strategy profile $\widehat{s}$ if there exists a coalition C such that all members of C are not worse off and at least one member of the coalition is better off by deviating from $\widehat{s}$ to s, holding the strategies of the others fixed.

[7]Strategy profile s weakly dominates strategy profile $\widehat{s}$ if there exists a coalition C such that all members of C can be better off by switching from y to x, taking the strategies of the players outside C as given.

2.3.3 Extension to the case of $v'(0) \leq c'(0)$

We extend the analysis to the case of $v'(0) \leq c'(0)$. We first consider the following case.

Case 1. There exists the number of participants $\widetilde{p} \leq n-1$ such that $v'(0) > c'(0)/(\widetilde{p}+1)$ and $v'(0) \leq c'(0)/\widetilde{p}$.

Example 2.2 Let $n = 3$ and $v(y) = -y\left(y - \frac{3}{4}\right)$. Let us assume that one unit of the public good can be provided from one unit of the private good. Note that $v'(0) = \frac{3}{4}$. The payoff matrix is shown in Table 2.2. In this example, $y^p = 0$ whenever $p < 2$. Hence,

	I	O
I	(0.04, 0.04, 0.04)	(0.02, 0.08, 0.02)
O	(0.08, 0.02, 0.02)	(0, 0, 0)

I

	I	O
I	(0.02, 0.02, 0.08)	(0, 0, 0)
O	(0, 0, 0)	(0, 0, 0)

O

Table 2.2: Payoff matrix of Example 2.2

the payoff to participants and non-participants are zero if the number of participants is less than two. There are two types of Nash equilibria: one is a Nash equilibrium with two participants and the other is a Nash equilibrium with zero participants. Clearly, the set of coalition-proof equilibria coincides with the set of strategy profiles in which two agents choose I.

In Case 1, the level of the public good is zero unless more than $\widetilde{p}$ agents choose I. The followings are analogous properties to Lemma 2.2.

Lemma 2.5 The payoffs to participants satisfy the following two conditions:

(P.1) For all $p \in \{0, \ldots, n\}$, if $p \leq \widetilde{p} - 1$, then $u(p, I) = 0$.

(P.2) For all p, $\widehat{p} \in \{\widetilde{p}, \ldots, n\}$, if $p > \widehat{p}$, then $u(p, I) > u(\widehat{p}, I)$.

The payoffs to non-participants satisfy conditions (N.1) and (N.2):

(N.1) For all $p \in \{0, \ldots, n\}$, if $p \leq \widetilde{p}$, then $u(p, O) = 0$.

(N.2) For all p, $\widehat{p} \in \{\widetilde{p}+1, \ldots, n\}$, if $p > \widehat{p}$, then $u(p, O) > u(\widehat{p}, O)$.

Conditions (P.1) and (N.1) are immediate. We can show (P.2) and (N.2) in a similar way to Lemma 2.2.

Lemma 2.6 Let s^* be a Nash equilibrium and let p^* be the number of agents that choose I in s^*. If $p^* > \widetilde{p}$, then s^* Pareto dominates every strategy profile in which the number of participants is $\widetilde{p}$ or less.

Lemma 2.6 can be shown in a way that is similar to Lemma 2.3.

Lemma 2.7 There is a Nash equilibrium in which more than $\widetilde{p}$ agents choose I.

Proof. If $u(n-1, I) \geq u(n-1, O)$, then n agents choose I in a Nash equilibrium. Otherwise, we have $u(n-1, I) < u(n-1, O)$. If $u(n-2, I) \geq u(n-2, O)$, then this inequality, together with $u(n-1, I) < u(n-1, O)$, implies that there is a Nash equilibrium in which $n-1$ agents choose I. Otherwise, we obtain $u(n-2, I) < u(n-2, O)$. Applying the same argument iteratively, if we have $u(\widetilde{p}+1, I) < u(\widetilde{p}+1, O)$, then this inequality, together with $u(\widetilde{p}, I) \geq u(\widetilde{p}, O) = 0$, indicates that $\widetilde{p}+1$ agents choose I in a Nash equilibrium. Hence, this game has Nash equilibria at which more than $\widetilde{p}$ choose I. ■

Proposition 2.3 *Let p^{max} denote the maximal number of participants in the set of Nash equilibria. Then, the set of coalition-proof equilibria coincides with the set of Nash equilibria in which p^{max} agents choose I in Case 1.*

Proof. By Lemma 2.7, we have $p^{max} \geq \widetilde{p}+1$. Using Lemmas 2.5 and 2.6, we can show the statement in a similar way to Proposition 2.1. ■

Case 2. $v'(0) \leq c'(0)/(\widetilde{p}+1)$ for all $\widetilde{p} \leq n-1$.

Finally, let us consider Case 2. In this case, no public goods are provided regardless of the number of participants. Thus, it is clear that all strategy profiles are coalition-proof. However, the various numbers of participants are supported as coalition-proof equilibria.

2.3.4 Non-existence of strong equilibria

In the participation game, there is not necessarily a strong equilibrium. The following is such an example.

Example 2.3 (Non-existence of strong equilibria) Consider a game with five agents. Each agent has the same preference relation as that in Example 2.1. There are two types of Nash equilibria in this game. Every Nash equilibrium with participation of one agent is not a strong equilibrium because it is Pareto dominated by Nash equilibria with participation of two agents. In every Nash equilibrium with two participants, non-participants receive the payoff α^2. When the number of participants is five, payoffs of all the agents are $5\alpha^2/4$. Hence, three non-participants in the Nash equilibrium can gain higher payoffs if all of the non-participants jointly deviate from O to I. Therefore, no Nash equilibria with two participants are strong equilibria, which indicates that there is no strong equilibrium in the game.

We characterize the set of strong equilibria in the participation game.

Proposition 2.4 *Let p^{max} be the number of participants that is attained in the set of Nash equilibria. A Nash equilibrium of the participation game is a strong equilibrium if* *and only if (i) it is a Nash equilibrium in which p^{max} agents choose I and (ii) no subset of agents can deviate from it in a way in which the number of participants is greater than p^{max} and all members of this coalition are not worse off and at least one member is better off.*

Proof. (*sufficiency*) Let s^{max} be a Nash equilibrium of the participation game in which p^{max} agents choose I. Suppose that s^{max} satisfies condition (ii). By Lemma 2.3, no deviation after which the number of participants is less than or equal to p^{max} improves the payoffs of its members. Therefore, s^{max} is a strong equilibrium in this game.
(*necessity*) Let s^* denote a strong equilibrium in this game. If s^* does not satisfy condition (i), then it is not coalition-proof. This is a contradiction. If s^* does not satisfy (ii), then a subset of agents can deviate profitably. This contradicts the idea that s^* is a strong equilibrium of this game. Hence, s^* must satisfy (i) and (ii). ■

We confirm from Propositions 2.1 and 2.4 that, in the participation game, a coalition-proof equilibrium is a strong equilibrium if and only if it satisfies condition (ii). Note that condition (ii) is not satisfied in many cases when the number of agents is large, since the payoff function of participants and that of non-participants are increasing with respect to the number of participants. In this game, it is less possible that a strong equilibrium exists as n gets larger, while a coalition-proof equilibrium exists regardless of the number of agents.

Chapter 3

Coalition-proof Equilibria in Participation Games: Heterogeneous Agents

We extend the analyses in Chapter 2 to a case of heterogeneous agents in this chapter. Let $v_i(\cdot) = \alpha_i v(\cdot)$ for all $i \in N$, where $\alpha_i > 0$, $v(0) = 0$, and $v(\cdot)$ is twice continuously differentiable and strictly concave. The values α_i may be different for each agent in this chapter. The cost function $c(\cdot)$ satisfies $c^{'} > 0$ and $c^{''} \geq 0$. Assume that $v^{'}(0) > c^{'}(0)$. The mechanism is assumed to implement the ratio allocation rule, as in Chapter 2.

Since the function $v(\cdot)$ is common to every agent, it can be interpreted that this setting of heterogeneity is close to the case of identical agents. However, the number of participants achieved at a coalition-proof equilibrium is not necessarily unique in this case.

Example 3.1 (The multiple numbers of participants in coalition-proof equilibria) Consider a game with three agents. Let α_1, α_2, and α_3 be such that $\alpha_1 > \alpha_2 = \alpha_3$

and $\alpha_1 < \alpha_2+\alpha_3$, say $\alpha_1 = 5$, and $\alpha_2 = \alpha_3 = 3$. In this example, we assume that one unit of the private good yields one unit of the public good and that the mechanism implements the Lindahl allocation rule. The payoff matrix of this game appears in Table 3.1. In this example, there are two coalition-proof equilibria. One is $s = (s_1, s_2, s_3) = (O, I, I)$ and the other is $s' = (s'_1, s'_2, s'_3) = (I, O, O)$. Two agents participate in the mechanism at s, while one agent enters the mechanism at s'. Thus, the number of participants attained at coalition-proof equilibria is not unique in this example.

	I	O
I	13.75, 8.25, 8.25	10, 12, 6
O	15, 4.5, 4.5	7.5, 4.5, 2.25

I

	I	O
I	10, 6, 12	6.25, 7.5, 7.5
O	7.5, 2.25, 4.5	0, 0, 0

O

Table 3.1: Payoff matrix of Example 3.1

In the following sections, we investigate sufficient conditions under which the number of participants is unique in coalition-proof equilibria.

3.1 Basic properties: Heterogeneous agents

3.1.1 Properties of payoff functions

We first introduce the payoff function that associates a real number with each set of participants.

Definition 3.1 A payoff function of $i \in N$, $u_i : 2^N \to \mathbb{R}_+$, is defined as follows:

$$\text{For any set of participants } P,\ u_i(P) = \begin{cases} \alpha_i v(y^P) - \dfrac{\alpha_i}{\sum_{j\in P}\alpha_j} c(y^P) & \text{if } i \in P, \\ \alpha_i v(y^P) & \text{otherwise.} \end{cases}$$

Lemma 3.1 proves that the level of the public good gets higher as the sum of the marginal willingness to pay of participants increases.

Lemma 3.1 For all sets of participants $P,\ Q \subseteq N$, if $\sum_{i\in P}\alpha_i > \sum_{i\in Q}\alpha_i$, then $y^P > y^Q$.

Proof. Let $P,\ Q \subseteq N$ be such that $\sum_{j\in P}\alpha_j > \sum_{j\in Q}\alpha_j$. Let y^P and y^Q be levels of the public good when the set of participants are P and Q, respectively. The public good provision y^P and y^Q satisfy the following conditions

$$v'(y^P) = \frac{c'(y^P)}{\sum_{j\in P}\alpha_j} \text{ and } v'(y^Q) = \frac{c'(y^Q)}{\sum_{j\in Q}\alpha_j}.$$

Suppose, on the contrary, $y^Q \geq y^P$. Then, the following inequalities are satisfied:

$$v'(y^P) = \frac{c'(y^P)}{\sum_{j\in P}\alpha_j} < \frac{c'(y^P)}{\sum_{j\in Q}\alpha_j} \leq \frac{c'(y^Q)}{\sum_{j\in Q}\alpha_j} = v'(y^Q).$$

Hence, we have $v'(y^P) < v'(y^Q)$. Since v' is strictly decreasing, $y^P > y^Q$. This is a contradiction. ■

Lemma 3.2 For all sets of participants $P,\ Q \subseteq N$, if $\sum_{i\in P}\alpha_i > \sum_{i\in Q}\alpha_i$, then conditions (3.1) and (3.2) are satisfied:

$$u_i(P) > u_i(Q) \text{ for all } i \notin P \cup Q, \quad \text{and} \tag{3.1}$$

$$u_i(P) > u_i(Q) \text{ for all } i \in P \cap Q. \tag{3.2}$$

Proof. It is immediate from Lemma 3.1 that (3.1) holds. We show (3.2). Let $P,\ Q \subseteq N$ be such that $\sum_{j\in P}\alpha_j > \sum_{j\in Q}\alpha_j$, and let $i \in P \cap Q$. Since y^P maximizes the sum of the utilities of agents in P,

$$\sum_{j\in P} u_j(P) = v(y^P)\sum_{j\in P}\alpha_j - c(y^P) \geq v(y^Q)\sum_{j\in P}\alpha_j - c(y^Q). \tag{3.3}$$

Multiplying the both sides of (3.3) by $\alpha_i/\sum_{j\in P}\alpha_j$, together with $\sum_{j\in P}\alpha_j > \sum_{j\in Q}\alpha_j$, yields

$$\begin{aligned}
\alpha_i\, v(y^P) - \frac{\alpha_i}{\sum_{j\in P}\alpha_j}c(y^P) \;&\geq\; \alpha_i\, v(y^Q) - \frac{\alpha_i}{\sum_{j\in P}\alpha_j}c(y^Q)\\
&>\; \alpha_i\, v(y^Q) - \frac{\alpha_i}{\sum_{j\in Q}\alpha_j}c(y^Q).
\end{aligned}$$

Hence, we obtain $u_i(P) > u_i(Q)$. ■

Lemma 3.2 is a basic property of our model. It shows that the payoff functions of both participants and non-participants are increasing with respect to the sum of the marginal willingness to pay for the public good of participants.

3.1.2 Nash equilibria and Pareto domination

Let $s \in S^n$ be a strategy profile and let P be the set of participants at s. Define $R(s)$ as the set of strategy profiles that can be reached from s by deviations of agents in P^s. Formally, we define this in the next definition.

Definition 3.2 Let s be a profile of strategies and let P be the set of agents that choose I. The subset of strategy profile $R(s)$ is defined as

$$\left\{\widehat{s} \in S^n | \text{there exists } D \in 2^{P^s}\backslash\{\emptyset\} \text{ such that } \widehat{s}_i = O \text{ for all } i \in D \text{ and } \widehat{s}_i = s_i \text{ for all } i \notin D\right\}.$$

For example, $R((I,\ldots,I))$ is equivalent to the set $S^n\backslash\{(I,\ldots,I)\}$ and $R((O,\ldots,O))$ is empty.

Lemma 3.3 Let $s \in S^n$ be a Nash equilibrium. Then, s Pareto dominates all the strategy profiles $\widehat{s} \in R(s)$.

Proof. Let s be a Nash equilibrium of G in which P is the set of participants, and let $\widehat{s} \in R(s)$ be a strategy profile in which $\widehat{P}$ is a set of participants. Note that $\widehat{P} \subsetneq P$ by the definition of $R(s)$. Thus, it follows that $\sum_{i\in P}\alpha_i > \sum_{i\in\widehat{P}}\alpha_i$. By $\sum_{i\in P}\alpha_i > \sum_{i\in\widehat{P}}\alpha_i$, (3.1), (3.2), and the definition of Nash equilibrium yields, we have the following three conditions:

$$u_i(P) > u_i(\widehat{P}) \text{ for all } i \in \widehat{P}, \tag{3.4}$$

$$u_i(P) > u_i(\widehat{P}) \text{ for all } i \in N\backslash P, \text{ and} \tag{3.5}$$

$$u_i(P) \geq u_i(P\backslash\{i\}) \geq u_i(\widehat{P}) \text{ for all } i \in P\backslash\widehat{P}. \tag{3.6}$$

Conditions (3.4) and (3.5) are immediate from $\sum_{i\in P}\alpha_i > \sum_{i\in\widehat{P}}\alpha_i$, (3.1) and (3.2). The first inequality of (3.6) follows from the definition of Nash equilibrium, and the second follows from (3.1) and holds with equality if $\widehat{P} = P\backslash\{i\}$. By (3.4), (3.5) and (3.6), $\widehat{s}$ is Pareto dominated by s. ■

3.2 Coalition-proof equilibria and the number of participants

We consider the situation in which the game G has multiple self-enforcing strategy profiles. Hence, coalition-proof equilibria may support various numbers of participants. The main purpose of this section is to establish a sufficient condition under which the number of participants supported as coalition-proof equilibria is unique.

Let p^{max} be the maximal number of participants in the set of self-enforcing strategy profiles of G. Let $s^{max} \in S^n$ be a self-enforcing strategy profile in which p^{max} agents choose I. Let us denote the set of participants at s^{max} by P^{max}.

Condition 3.1 $\alpha_i \geq \alpha_j$ for all $i \in P^{max}$ and all $j \in N \setminus P^{max}$.

Condition 3.1 means that all agents in P^{max} have at least as high marginal willingness to pay for the public good as the agents in $N \setminus P^{max}$. Note that Condition 3.1 is satisfied in the case of identical agents. The following proposition generalizes uniqueness of the number of participants in the set of coalition-proof equilibria in the case of identical agents.

Proposition 3.1 *Let p^{max} denote the maximal number of participants attained in the set of self-enforcing strategy profiles in G. Let P^{max} be a set of participants with $\#P^{max} = p^{max}$ that is supported as a self-enforcing strategy profile of G. If P^{max} satisfies Condition 3.1, then p^{max} is the unique number of participants that is achieved in the set of coalition-proof equilibria.*

Before proving Proposition 3.1, we show the following lemma.

Lemma 3.4 Let s^{max} be a self-enforcing strategy profile at which P^{max} is the set of participants and p^{max} agents choose I. Suppose that P^{max} satisfies Condition 3.1. Then, (i) s^{max} is not Pareto dominated by any strategy profile with participation of p^{max} agents and (ii) s^{max} Pareto dominates every strategy profile with participation of less than p^{max} agents.

Proof of Lemma 3.4. Let $\widehat{s} \in S^n$ be a profile of strategies. Let $\widehat{P}$ be a set of participants that is attained in $\widehat{s}$, and let $\widehat{p}$ be the number of agents in $\widehat{P}$. We consider the following two cases: one is the case of $p^{max} = \widehat{p}$ and the other is the case of $p^{max} > \widehat{p}$.

We first consider the first case and show the statement (i). If $\widehat{P} = P^{max}$, then $\widehat{s}$ does not Pareto dominate s^{max} trivially. Let us consider the case of $\widehat{P} \neq P^{max}$. Since $p^{max} = \widehat{p}$ and $\widehat{P} \neq P^{max}$, we have $\#[P^{max} \setminus \widehat{P}] = \#[\widehat{P} \setminus P^{max}] > 0$. For every agent

$i \in \widehat{P} \setminus P^{max}$, we have

$$u_i(P^{max}) \geq u_i(P^{max} \cup \{i\}) > u_i(\widehat{P}). \tag{3.7}$$

The first inequality follows from the definition of Nash equilibrium, and the second inequality holds since

$$\begin{aligned}
\sum_{j \in P^{max} \cup \{i\}} \alpha_j &= \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in P^{max} \setminus \widehat{P}} \alpha_j + \alpha_i \\
&\geq \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j + \alpha_i \\
&> \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j \\
&= \sum_{j \in \widehat{P}} \alpha_j.
\end{aligned} \tag{3.8}$$

Note that Condition 3.1 implies that $\alpha_k \geq \alpha_l$ for all $k \in P^{max} \setminus \widehat{P}$ and all $l \in \widehat{P} \setminus P^{max}$. Hence, we obtain $\sum_{j \in P^{max} \setminus \widehat{P}} \alpha_j \geq \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j$. The second inequality of (3.8) follows from this, and the third inequality of (3.8) follows from $\alpha_i > 0$. It follows from (3.7) that every agent $i \in \widehat{P} \setminus P^{max}$ is worse off by switching from s^{max} to $\widehat{s}$. This completes the proof of (i).

We second consider the case of $\widehat{p} < p^{max}$. Note that $\#[P^{max} \setminus \widehat{P}] > \#[\widehat{P} \setminus P^{max}]$ must be satisfied in this case. If $\widehat{s} \in R(s^{max})$, then s^{max} dominates $\widehat{s}$ by Lemma 3.3. If $\widehat{s} \notin R(s^{max})$, then the following claim is satisfied.

Claim 3.1 It follows that $\#[P^{max} \setminus \widehat{P}] > \#[\widehat{P} \setminus P^{max}] \geq 1$.

Proof of Claim 3.1. Since $\widehat{s} \notin R(s^{max})$, $\widehat{P} \setminus P^{max}$ is non-empty. Thus, $\#[\widehat{P} \setminus P^{max}] \geq 1$. We obtain $\#[P^{max} \setminus \widehat{P}] > \#[\widehat{P} \setminus P^{max}]$ because $p^{max} > \widehat{p}$. ■

Claim 3.2 Every agent that selects the same strategy at both s^{max} and $\widehat{s}$ is worse off:

$$\begin{aligned} &u_i(P^{max}) > u_i(\widehat{P}) \text{ for all } i \in P^{max} \cap \widehat{P}, \quad \text{and} \\ &u_i(P^{max}) > u_i(\widehat{P}) \text{ for all } i \in N \setminus (P^{max} \cup \widehat{P}). \end{aligned} \tag{3.9}$$

Proof of Claim 3.2. We first show $\sum_{j \in P^{max}} \alpha_j > \sum_{j \in \widehat{P}} \alpha_j$. Note that

$$\sum_{j \in P^{max}} \alpha_j = \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in P^{max} \setminus \widehat{P}} \alpha_j. \tag{3.10}$$

It follows from Condition 3.1 that $\alpha_k \geq \alpha_l$ for all $k \in P^{max} \setminus \widehat{P}$ and all $l \in \widehat{P} \setminus P^{max}$. By this condition and Claim 3.1, we have $\sum_{j \in P^{max} \setminus \widehat{P}} \alpha_j > \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j$. Hence,

$$\begin{aligned} (3.10) \quad &> \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j \\ &= \sum_{j \in \widehat{P}} \alpha_j. \end{aligned}$$

Therefore, we obtain $\sum_{j \in P^{max}} \alpha_j > \sum_{j \in \widehat{P} \setminus P^{max}}$. By Lemma 3.2 and this condition, we have (3.9).

■

Claim 3.3 Every agent that chooses I in s^{max} and O in $\widehat{s}$ is worse off: For all $i \in P^{max} \setminus \widehat{P}$, $u_i(P^{max}) \geq u_i(\widehat{P})$.

Proof of Claim 3.3. Let $i \in P^{max} \setminus \widehat{P}$. By the definition of Nash equilibrium, $u_i(P^{max}) \geq u_i(P^{max} \setminus \{i\})$. We first operate $\sum_{j \in P^{max} \setminus \{i\}} \alpha_j$ in the following way:

$$\begin{aligned} \sum_{j \in P^{max} \setminus \{i\}} \alpha_j &= \sum_{j \in P^{max}} \alpha_j - \alpha_i \\ &= \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in (P^{max} \setminus \widehat{P}) \setminus \{i\}} \alpha_j. \end{aligned} \tag{3.11}$$

By Condition 3.1 and $\#[P^{max} \setminus \widehat{P}] - 1 \geq \#[\widehat{P} \setminus P^{max}]$, we obtain $\sum_{j \in (P^{max} \setminus \widehat{P}) \setminus \{i\}} \alpha_j \geq \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j$. Therefore,

$$\begin{aligned}(3.11) &\geq \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j \\ &= \sum_{j \in \widehat{P}} \alpha_j.\end{aligned}$$

It is straightforward from Lemma 3.2 to show $u_i(P^{max}) \geq u_i(P^{max} \setminus \{i\})$.

$\blacksquare$

Claim 3.4 Every agent that chooses O in s^{max} and I in $\widehat{s}$ is worse off: For all $i \in \widehat{P} \setminus P^{max}$, $u_i(P^{max}) > u_i(\widehat{P})$.

Proof of Claim 3.4: Let $i \in \widehat{P} \setminus P^{max}$. By the definition of Nash equilibrium, $u_i(P^{max}) \geq u_i(P^{max} \cup \{i\})$. We show $\sum_{j \in P^{max} \cup \{i\}} \alpha_j > \sum_{j \in \widehat{P}} \alpha_j$ to prove $u_i(P^{max} \cup \{i\}) > u_i(\widehat{P})$. Note that

$$\sum_{j \in P^{max} \cup \{i\}} \alpha_j = \alpha_i + \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in P^{max} \setminus \widehat{P}} \alpha_j. \tag{3.12}$$

It follows from Condition 3.1 and Claim 3.1 that $\sum_{j \in P^{max} \setminus \widehat{P}} \alpha_j > \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j$. Thus,

$$\begin{aligned}(3.12) &> \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j + \alpha_i \\ &> \sum_{j \in P^{max} \cap \widehat{P}} \alpha_j + \sum_{j \in \widehat{P} \setminus P^{max}} \alpha_j \\ &= \sum_{j \in \widehat{P}} \alpha_j.\end{aligned}$$

Therefore, we have $u_i(P^{max}) \geq u_i(P^{max} \cup \{i\}) > u_i(\widehat{P})$. $\blacksquare$

By Claim 3.2, 3.3, and 3.4, s^{max} Pareto dominates every strategy profile with participation of less than p^{max} agents. Therefore, (ii) is satisfied. **(End of Proof of Lemma 3.4)**

Proof of Proposition 3.1. Since p^{max} is the maximal number of participants that is supportable as self-enforcing strategy profiles, participation of more than p^{max} agents is not achieved at coalition-proof equilibria. Profile s^{max} is self-enforcing and no strategy profiles with participation of less than p^{max} agents Pareto dominate s^{max} by Lemma 3.4. Thus, s^{max} is not Pareto dominated by any other self-enforcing strategy profile, which indicates that participation of p^{max} agents is supportable as a coalition-proof equilibrium. By Lemma 3.4, no strategy profile supporting participation of less than p^{max} agents are coalition-proof because such strategy profiles are Pareto dominated by s^{max}, which is self-enforcing.

■

3.3 Concluding Remarks

In Chapter 2 and Chapter 3, we have investigated the coalition-proof equilibria in the voluntary participation game in a public good mechanism. We have proved that the Nash equilibrium with the maximal number of participants is the unique coalition-proof equilibrium when agents' preferences are identical. In the case of heterogeneous agents, there may exist multiple coalition-proof equilibria and they support various numbers of participants. But if each of participants has at least as high marginal willingness to pay for the public good as non-participants in a self-enforcing strategy profile with the maximal number of participants, then the number of participants is unique. It follows from these results that coalition-proof equilibrium offers a much shaper prediction on the number of participants than Nash equilibrium in the cases of identical agents or heterogeneous agents under our sufficient condition. However, in this chapter, we did not provide the existence and characterizations of the set of coalition-proof equilibria when agents' preferences are heterogeneous because of the difficulty of them. For a future work, we characterize the set of coalition-proof equilibria in the case where agents are heterogeneous by assuming additional conditions.

We consider that agents collude only on participation in the mechanism. Hence, agents in a coalition do not offer side payments, or do not negotiate on the strategies in the mechanism. This form of coalition is weak. For another future research, we investigate the relationship between the set of participants in the public good mechanism and stronger forms of coalitions.

Chapter 4

Coalition-proofness and Dominance Relations

This chapter examines the relationship between coalition-proof equilibria based on different dominance relations. The notion of a coalition-proof equilibrium was introduced by Bernheim, Peleg, and Whinston (1987) and is known as a refinement of Nash equilibria based on the stability against credible coalitional deviations. However, there are two ways for a coalition to improve payoffs to its members. We consider the following two dominance relations:

(i) Strategy profile x *strictly dominates* strategy profile y if there exists a coalition S such that all members of S can be better off by switching y to x, taking the strategies of the players outside S as given.

(ii) Strategy profile x *weakly dominates* strategy profile y if there exists a coalition S such that all members are not worse off and at least one member of the coalition is better off by deviating from y to x, holding the strategies of the others fixed.

Under the notion of strict domination, all of the deviating players are better off, while, under that of weak domination, all members of a coalition are at least as well off, and at least one of them is better off. Thus, the set of equilibria under weak domination may be a subset of that under strict domination. This indeed applies to the strong Nash equilibrium and the core.

However, the set of coalition-proof equilibria under strict domination does not contain that under weak domination. Konishi, Le Breton, and Weber (1999) provided an example in which the set of coalition-proof equilibria under weak domination and that under strict domination are both non-empty and their intersection is empty. They also showed that, in the class of common agency games, any coalition-proof equilibria under weak domination is that under strict domination.

In this study, we consider the class of games with n players in which the strategy space of each player is a subset of the real line.[1] We show that, if a game satisfies the conditions of *anonymity*, *monotone externality*, and *strategic substitutability*, then the set of coalition-proof equilibria under weak domination is included in that under strict domination. It is interesting to point out that the same three conditions yield the equivalence of the set of coalition-proof equilibria under strict domination and the weakly Pareto efficient frontier of the set of Nash equilibria (Yi (1999)). The inclusion relation between the sets of coalition-proof equilibria under the two different dominance relations holds for the games that have interested economists, such as standard Cournot oligopoly games and voluntary participation games in a mechanism producing public goods.

[1]Note that common agency games do not belong to this class.

4.1 The Model

We consider a strategic game $G = [N, (X_i)_{i\in N}, (u_i)_{i\in N}]$, where N is a finite set of players, X_i is the set of pure strategies of player i that is a subset of real numbers, and $u_i \colon \prod_{j\in N} X_j \to \mathbb{R}$ is the payoff function of player i. In this dissertation, we focus solely on pure strategy equilibria.

The notions of coalition-proof equilibria are defined under strict domination and weak domination. In Definition 1.4 on page 16, the coalition-proof equilibria was defined under weak domination, which is called a *coalition-proof equilibrium under weak domination* in this chapter. The coalition-proof equilibria that are defined under strict domination are called a *coalition-proof equilibria under strict domination.*

The main difference between the two notions of coalition-proof equilibria lies in the notions of coalitional deviations. The idea behind the coalition-proof equilibria under weak domination is that a coalition deviates if all members in the coalition are at least as well off and at least one of them is better off. On the other hand, under strict domination, a coalition deviates only if every member of the coalition is better off. Note that, under either of the dominance relations, coalition-proof equilibria and self-enforcing strategy profiles are Nash equilibria. In games with two players, the set of self-enforcing strategy profiles coincides with that of Nash equilibria, since coalitions consist of only two or fewer players. Hence, the set of coalition-proof equilibria under weak domination coincides with the (strictly) Pareto efficient frontier of the set of Nash equilibria, and so does the set of coalition-proof equilibria under strict domination with the weakly Pareto efficient frontier of that of Nash equilibria. Therefore, the set of coalition-proof equilibria under weak domination is a subset of that under strict domination in two-player games. However, the inclusion relation between the sets of coalition-proof equilibria under the two different dominance relations does not necessarily hold in games with more than two players. Konishi, Le Breton, and Weber (1999) presented an example in which the sets of coalition-proof equilibria under strict and weak dominations are disjoint.

Example 4.1 (Konishi, Le Breton, and Weber, 1999) Consider the game with three players depicted in Table 4.1, in which agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There exist two pure strategy Nash equilibria, (a_1, b_1, c_2) and (a_2, b_2, c_1), where the former is a coalition-proof equilibrium under weak domination but not that under strict domination, and the latter is a coalition-proof equilibrium under strict domination but not that under weak domination. In this example, the set of coalition-proof equilibria under weak domination is not a subset of that under strict domination.

	b_1	b_2
a_1	1, 0, −5	−5, −5, 0
a_2	−5, −5, 0	0, 0, 10

c_1

	b_1	b_2
a_1	−1, −1, 5	5, −5, 0
a_2	−5, −5, 0	−2, −2, 0

c_2

Table 4.1: The payoff matrix of the example presented by Konishi, Le Breton, and Weber (1999). (Example 4.1)

4.2 The Main Result

In this section, we establish sufficient conditions under which the set of coalition-proof equilibria under weak domination is a subset of that of a coalition-proof equilibrium under strict domination.

The first condition is that of *anonymity.*

Anonymity. For all $i \in N$, all $x_i \in X_i$, and all $x_{-i},\ \widehat{x}_{-i} \in \prod_{j \neq i} X_j$, if $\sum_{j \neq i} x_j = \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) = u_i(x_i, \widehat{x}_{-i})$.

The anonymity condition means that the payoff function of every player depends on his strategy and on the aggregate strategy of all other players.

The next condition is that of *monotone externality.* The condition states that the payoffs to every player are either non-increasing or non-decreasing with respect to the sum of strategies of the other players.

Monotone externality. (i) For all $i \in N$, all $x_i \in X_i$, and all x_{-i} and $\widehat{x}_{-i} \in \prod_{j \neq i} X_j$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) \geq u_i(x_i, \widehat{x}_{-i})$ holds. (ii) For all $i \in N$, all $x_i \in X_i$, and all x_{-i} and $\widehat{x}_{-i} \in \prod_{j \neq i} X_j$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) \leq u_i(x_i, \widehat{x}_{-i})$ holds. If (i) holds, the condition means *positive* externalities, and it represents *negative* externalities if (ii) is satisfied. We define that a game satisfies the condition of monotone externality if either (i) or (ii) holds.

The third condition is that of *strategic substitutability.* Under this condition, the incentive of every player to reduce his strategy gets higher as the sum of the other players' strategies increases.

Strategic substitutability. For all $i \in N$, all $x_i,\ \widehat{x}_i \in X_i$, and all $x_{-i},\ \widehat{x}_{-i} \in \prod_{j \neq i} X_j$, if $x_i > \widehat{x}_i$ and $\sum_{j \neq i} x_j > \sum_{j \neq i} \widehat{x}_j$, then $u_i(x_i, x_{-i}) - u_i(\widehat{x}_i, x_{-i}) < u_i(x_i, \widehat{x}_{-i}) - u_i(\widehat{x}_i, \widehat{x}_{-i})$.

Proposition 4.1 *Suppose that a game satisfies anonymity, monotone externality, and* *strategic substitutability. Then, any coalition-proof equilibrium under weak domination is a coalition-proof equilibrium under strict domination.*

Proof. If the set of coalition-proof equilibria under weak domination is empty, then the statement of proposition is vacuously true. Hence, we consider the case in which there is a coalition-proof equilibria under weak domination in the game. Let us assume that a game satisfies anonymity, positive externality, and strategic substitutability.[2] We show by induction that the set of coalition-proof equilibria under weak domination is a subset of that under strict domination. Clearly, the statement is true for all games with a single player. In any two-player game, under either dominance relations, the set of self-enforcing strategy profiles coincides with that of Nash equilibria. Hence, the set of coalition-proof equilibria under weak domination coincides with the Pareto efficient frontier of the set of Nash equilibria, and so does the set of coalition-proof equilibria under strict domination with the weakly Pareto efficient frontier of that of Nash equilibria. As a result, the set of coalition-proof equilibria under strict domination contains that under weak domination in every two-player game.

Let $n \geq 3$, and suppose that any coalition-proof equilibrium under weak domination is a coalition-proof equilibria under strict domination for any game with fewer than n players as an induction hypothesis. Let x^* denote a coalition-proof equilibrium under weak domination of a game with n players. We need to show that x^* is a self-enforcing strategy profile under strict domination and that there is not other self-enforcing strategy profile under strict domination $\widetilde{x}$ where $u_i(\widetilde{x}) > u_i(x^*)$ for every $i \in N$.

Lemma 4.1 Any self-enforcing strategy profile under weak domination is that under strict domination.

[2] We can similarly show the statement in the case of negative externality.

The lemma above can be shown in the following way. Let x be a self-enforcing strategy profile under weak domination of G. Then, by definition, x_C is a coalition-proof equilibrium under weak domination in the restricted game $G|x_{-C}$ for every proper subset C of N. By the induction hypothesis, x_C is also a coalition-proof equilibrium under strict domination in $G|x_{-C}$. That is, for all proper subsets C of N, x_C is a coalition-proof equilibrium under strict domination of $G|x_{-C}$. Hence, x is a self-enforcing strategy profile under strict domination of G.

By Lemma 4.1, x^* is a self-enforcing strategy profile under strict domination.

Lemma 4.2 There is no other self-enforcing strategy profile under strict domination $\widetilde{x}$ such that $u_i(\widetilde{x}) > u_i(x^*)$ for all $i \in N$.

Proof of Lemma 4.2. Let us suppose, on the contrary, that there is a self-enforcing strategy profile under strict domination $\widetilde{x}$, at which $u_i(\widetilde{x}) > u_i(x^*)$ for all $i \in N$. Then, $\widetilde{x}$ must satisfy the following condition.

Claim 4.1 It follows that $\sum_{j \neq i} x_j^* < \sum_{j \neq i} \widetilde{x}_j$ for all $i \in N$.

Proof of Claim 4.1. Let us suppose, on the contrary, that there is player $i \in N$ such that $\sum_{j \neq i} x_j^* \geq \sum_{j \neq i} \widetilde{x}_j$. If $\sum_{j \neq i} x_j^* = \sum_{j \neq i} \widetilde{x}_j$, then we have $u_i(x^*) \geq u_i(\widetilde{x}_i, x_{-i}^*) = u_i(\widetilde{x})$ by the definition of Nash equilibrium and the condition of anonymity, which is a contradiction. If $\sum_{j \neq i} x_j^* > \sum_{j \neq i} \widetilde{x}_j$, then we have $u_i(x^*) \geq u_i(\widetilde{x}_i, x_{-i}^*)$ by the definition of Nash equilibrium and $u_i(\widetilde{x}_i, x_{-i}^*) \geq u_i(\widetilde{x})$ by positive externality. Therefore, we obtain $u_i(x^*) \geq u_i(\widetilde{x})$. This is a contradiction. $\|$

Claim 4.2 The strategy profile $\widetilde{x}$ is not a Nash equilibrium of G.

By Claim 4.1, it is satisfied that $\sum_{k\in N}\sum_{j\neq k} x_j^* < \sum_{k\in N}\sum_{j\neq k}\widetilde{x}_j$. Hence, $\sum_{k\in N} x_k^* < \sum_{k\in N}\widetilde{x}_k$. Therefore, $i \in N$ exists such that $\widetilde{x}_i > x_i^*$. By strategic substitutability, for player i, we have $u_i(x_i^*, \widetilde{x}_{-i}) - u_i(\widetilde{x}) > u_i(x^*) - u_i(\widetilde{x}_i, x_{-i}^*)$. Since x^* is a Nash equilibrium, $u_i(x^*) - u_i(\widetilde{x}_i, x_{-i}^*) \geq 0$. Therefore, $u_i(x_i^*, \widetilde{x}_{-i}) > u_i(\widetilde{x})$, which implies that $\widetilde{x}$ is not a Nash equilibrium of G. This contradicts the idea that $\widetilde{x}$ is a self-enforcing strategy profile under strict domination. Thus, there is no self-enforcing strategy profile under strict domination that dominates x^*, which implies that x^* is a coalition-proof equilibrium under strict domination in the n-person game. ■

Remark 4.1 We use none of three conditions in the proof of Lemma 4.1. Thus, Lemma 4.1 holds true in every game. Note that we use the conditions only when we prove Lemma 4.2.

Many interesting games in economics satisfy the conditions above. For instance, Cournot oligopoly games and the other games that have been studied as a part of industrial organization theory satisfy the conditions. For details, refer to Yi (1999). Here, we give an example in the context of the provision of pure public goods.

Example 4.2 Let us reconsider Example 2.1 on page 28. In this example, when p agents choose I, each of p agents receives $u(p-1, I) = \alpha^2 p/4$ and other agents obtain $u(p, O) = \alpha^2 p/2$. Table 4.2 is a payoff matrix that is reproduced from Table 2.1. In this example, the anonymity condition is satisfied, since payoffs to both participants and non-participants depend on the number of participants. The participation decision game satisfies the positive externality condition because the utilities of participants and non-participants get higher as the number of participants increases. The difference $u(p-1, I) - u(p-1, O) = \alpha^2(-p+2)/4$ is decreasing with respect to the number of participants p, which implies that the incentive to participate in the mechanism decreases as the number of participants increases. Therefore, the strategic substitutability condition holds in this example. From Proposition 4.1, the set of coalition-proof equilibria under strict domination contains that under weak domination. In fact, two agents choose participation in every coalition-proof equilibrium under weak domination, while one agent or two agents participate in the mechanism in coalition-proof equilibria under strict domination in this example.

	I	O
I	$\frac{3\alpha^2}{4}, \frac{3\alpha^2}{4}, \frac{3\alpha^2}{4}$	$\frac{\alpha^2}{2}, \alpha^2, \frac{\alpha^2}{2}$
O	$\alpha^2, \frac{\alpha^2}{2}, \frac{\alpha^2}{2}$	$\frac{\alpha^2}{2}, \frac{\alpha^2}{2}, \frac{\alpha^2}{4}$

I

	I	O
I	$\frac{\alpha^2}{2}, \frac{\alpha^2}{2}, \alpha^2$	$\frac{\alpha^2}{4}, \frac{\alpha^2}{2}, \frac{\alpha^2}{2}$
O	$\frac{\alpha^2}{2}, \frac{\alpha^2}{4}, \frac{\alpha^2}{2}$	0, 0, 0

O

Table 4.2: Payoff matrix of Example 4.2

Remark 4.2 The conditions of anonymity, monotone externality, and strategic substitutability do not necessarily guarantee equivalence between coalition-proof equilibria under the two different dominance relations. In fact, the participation decision game in Example 4.2 satisfies all of the conditions, but there is a coalition-proof equilibrium under strict domination that is not that under weak domination.

Remark 4.3 Let us reconsider Example 4.1. Let $a_1 = b_1 = c_1 = 0$ and $a_2 = b_2 = c_2 = 0$. Then, all of the three conditions do not hold in this example. Participation games with a public project considered in Chapters 5 and 6 do not necessarily satisfy the three conditions. However, the set of coalition-proof equilibria under weak domination is included in that under strict domination. The three conditions are not necessary conditions.

4.3 Concluding Remarks

In this chapter, the relationship between coalition-proof equilibria under strict and weak dominations was examined. In many equilibrium concepts, such as the core and strong Nash equilibria, the set of equilibria under weak domination is a subset of that under strict domination. However, the set of coalition-proof equilibria under strict domination and that under weak domination are not necessarily related by inclusion. We showed that, if a game satisfies the properties of anonymity, monotone externality, and strategic substitutability, then the set of coalition-proof equilibria under weak domination is a subset of that of coalition-proof equilibria under strict domination. This implies that the inclusion relation between the two sets of coalition-proof equilibria holds true in such interesting games studied in economics as the participation game in a public good mechanism and the standard Cournot oligopoly game.

The coalition-proof equilibrium is well known as a refinement of the Nash equilibrium. However, little is known about the structure of the equilibria. This chapter has focused on the relationship between coalition-proof equilibria and dominance relations. Although different strategies may be used in coalition-proof equilibria under the different dominance relations, their relationship had not been studied so far. The objective of this chapter was to give an answer to this problem. Clarifying other properties of this equilibrium concept may be left for future researches.

Part II

Participation Games with Discrete Public Goods

Chapter 5

Participation Problems in Public Projects

This chapter considers a participation game in a mechanism to implement a public project. We consider a mechanism that implements the following allocation rules: (i) the public project is undertaken if and only if the joint benefit of participants from it is more than its cost, (ii) the sum of payments from participants is equal to the cost of producing the public project, (iii) every participant bears a positive cost share, and (iv) the cost share of each participant is less than his willingness to pay for the public project. This kind of allocation rule includes many cost-sharing rules. A proportional cost-sharing rule is an example of such cost-sharing rules.

We first characterize the set of participants at strict Nash equilibria. We show that there exists a strict Nash equilibrium and that every strict Nash equilibrium supports an efficient allocation in the participation game. Secondly, we characterize strong equilibria and show that there is a strong equilibrium in the participation game. Our main result is that the set of strict Nash equilibria, that of strong equilibria, and that of coalition-proof equilibria coincide and that the sets of these three equilibria are not empty. Moreover, there are efficient allocations that are supportable as the three notions of equilibria, and all the equilibrium allocations are Pareto efficient.

We also extend our model to the case with a multi-unit public good and that with multiple projects. In these cases, the set of strict Nash equilibria, and that of strong equilibria, and that of coalition-proof equilibria do not necessarily coincide.

5.1 A participation game in a mechanism implementing a public project

We consider the case of $Y = \{0, 1\}$, $v_i(y) = \theta_i y$ where $\theta_i > 0$ for every $i \in N$, and $c(y) = cy$ where $c > 0$. Define $\theta_P := \sum_{i \in P} \theta_i$ for all $P \subseteq N$. We assume that there exists a mechanism that implements a Pareto efficient and individually rational allocation rule.

Assumption 5.1 For every set of participants P, the allocation to the participants $(y^P, (x_j^P)_{j \in P})$ satisfies

(i) $\theta_P > c$ if and only if $y^P = 1$,

(ii) if $y^P = 1$, then $\sum_{i \in P} x_i^P = c$,

(iii) $\theta_i > x_i^P$ for every $i \in P$, and

(iv) $x_i^P > 0$ for every $i \in P$ if and only if $y^P = 1$.

Condition (i) means that the public project is undertaken if and only if the sum that the participants are willing to pay for the project exceeds the project cost. Condition (ii) requires that the expenses paid by the participants be equal to the project cost when the project is undertaken. This is called the *budget balance* condition. Clearly, conditions (i) and (ii) imply that $(y^P, (x_j^P)_{j \in P})$ is a Pareto efficient allocation only for the preferences of agents in P. Item (iii) is the *individual rationality* condition, which means that the payoff of every participant after entering the mechanism is greater than 0, when the project is undertaken. Condition (iv) requires that every participant bear a positive cost share if and only if the public project is undertaken.

Several desirable allocation rules satisfy the conditions. The proportional cost-sharing rule under condition (i) is such an example: for all sets of participants P and for all i in P,

$$x_i^P = \begin{cases} \dfrac{\theta_i}{\theta_P}c & \text{if } y^P = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this model is a generalization of that of Dixit and Olson (2000).

In this dissertation, we are not concerned with the implementation problem of an allocation rule that satisfies (i), (ii), (iii), and (iv) in Assumption 5.1. However, there is a mechanism in which the above allocation rule is attainable in equilibria. For example, Jackson and Moulin (1992) constructed mechanisms which implement a class of cost-sharing rules satisfying all the above conditions in subgame perfect equilibria and undominated Nash equilibria.

The following is an example of the participation game in a mechanism that undertakes a public project.

Example 5.1 Let $N = \{1, 2, 3\}$, $\theta_1 = \theta_2 = \theta_3 = 3/4$, and $c = 1$. The cost is distributed among participants in proportion to their willingness to pay for the project. The payoff matrix of this example is depicted in Table 5.1, where agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There are two types of Nash equilibria. One is the Nash equilibrium with two participants, and the other is the Nash equilibrium with no participants. Only the Nash equilibria with participation of two agents are strict and strong.

	I	O
I	$\frac{5}{12}$, $\frac{5}{12}$, $\frac{5}{12}$	$\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{4}$
O	$\frac{3}{4}$, $\frac{1}{4}$, $\frac{1}{4}$	0, 0, 0

I

	I	O
I	$\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$	0, 0, 0
O	0, 0, 0	0, 0, 0

O

Table 5.1: Payoff matrix of Example 5.1

5.2 Nash equilibria of the participation game

In this section, we characterize the sets of participants attained at Nash equilibria. Since the payoffs to agents depend on the sets of participants, we introduce the following notations that are similar to those in Definition 3.1 for the sake of convenience.

Definition 5.1 A payoff function of i, $u_i \colon 2^N \to \mathbb{R}_+$, is defined as follows:

$$\text{For all sets of participants } P \in 2^N,\ u_i(P) = \begin{cases} (\theta_i - x_i^P)y^P & \text{if } i \in P, \\ \theta_i y^P & \text{otherwise.} \end{cases}$$

The set of feasible allocations of the economy is defined as A:

$$A = \left\{ (y, (x_j)_{j \in N}) \mid y \in \{0,1\},\ x_i \geq 0 \text{ for all } i \in N,\ \text{and } \sum_{i \in N} x_i \geq cy \right\}.$$

Assumption 5.2 *$\theta_N > c$.*

Definition 5.2 An allocation $(y, (x_j)_{j \in N})$ is called *Pareto efficient* if there is no allocation $(\widehat{y}, (\widehat{x}_j)_{j \in N}) \in A$ such that $V_i(\widehat{y}, \widehat{x}_i) \geq V_i(y, x_i)$ for all $i \in N$ and $V_i(\widehat{y}, \widehat{x}_i) > V_i(y, x_i)$ for some $i \in N$.

We, hereafter, consider a case in which Assumption 5.2 holds. By Assumption 5.2, the public project is undertaken at all Pareto efficient allocations. In the next Lemma, we characterize the sets of participants supported as strict Nash equilibria.

Lemma 5.1 A set of participants P is supported as a strict Nash equilibrium of the participation game if and only if $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$.

Proof. Let P be a set of participants that satisfies $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$, and let $(y^P, (x_j^P)_{j \in N})$ denote the allocation when P is the set of participants. Then, the following conditions are satisfied:

$$u_i(P) = \theta_i - x_i^P \ > \ 0 = u_i(P \setminus \{i\}) \text{ for all } i \in P, \text{ and}$$
$$u_i(P) = \theta_i \ > \ \theta_i - x_i^{P \cup \{i\}} = u_i(P \cup \{i\}) \text{ for all } i \notin P.$$

Therefore, P can be supported as a strict Nash equilibrium.

Secondly, we suppose that P is a set of participants at a strict Nash equilibrium. Then, we have $u_i(P) > u_i(P \setminus \{i\})$ for all $i \in P$ and $u_i(P) > u_i(P \cup \{i\})$ for all $i \notin P$. If $\theta_P \leq c$, then we have $u_i(P) = u_i(P \setminus \{i\}) = 0$ for all $i \in P$, which is a contradiction. Thus, it must be satisfied that $\theta_P > c$. Since $\theta_P > c$, $u_i(P) = \theta_i - x_i^P$ for all $i \in P$. If $\theta_P - \theta_j > c$ for some $j \in P$, then the agent j has an incentive to deviate from I to O because $u_j(P \setminus \{j\}) = \theta_j > \theta_j - x_i^P = u_j(P)$. This is a contradiction. Therefore, we must have $\theta_P - \theta_i \leq c$ for all $i \in P$. ■

In the following lemma, we verify that there is a strict Nash equilibrium in the participation game.

Lemma 5.2 There exists a strict Nash equilibrium in the game G under Assumption 5.2.

Proof. By Lemma 5.1, we show the existence of a set of participants $P \subseteq N$ that satisfies the condition

$$\theta_P > c \text{ and } \theta_P - \theta_i \leq c \text{ for all } i \in P, \tag{5.1}$$

in order to prove this statement. Let T be a set of participants such that:

$$T \in \arg\min_{Q \subseteq N} \ \theta_Q \text{ such that } \theta_Q > c. \tag{5.2}$$

Note that there is at least one set of participants R satisfying $\theta_R > c$ by Assumption 5.2. Now, suppose that $\theta_T - \theta_i > c$ for some $i \in T$. Since $\theta_T > \theta_{T \setminus \{i\}} > c$, θ_T is not the minimal number, which contradicts (5.2). Therefore, it holds true that $\theta_T - \theta_i \leq c$ for all $i \in T$. ■

In the participation game, there may be a non-strict Nash equilibrium. For example, a Nash equilibrium at which no agents choose I is obviously not strict in Example 5.1. Note that, if non-strict Nash equilibria exist, then the project is not done in the equilibrium, and the allocations supported as the non-strict Nash equilibria are Pareto-dominated by that attained at a strict Nash equilibrium. The following proposition shows that the set of strict Nash equilibria coincides with the set of Nash equilibria that support efficient allocations.

Proposition 5.1 *In the participation game, a strategy profile is a strict Nash equilibrium if and only if it is a Nash equilibrium at which an efficient allocation is attained.*

Proof. First, we prove that every strict Nash equilibrium is a Nash equilibrium that supports an efficient allocation. Obviously, every strict Nash equilibrium is a Nash equilibrium. Hence, we need to show that every allocation achieved at a strict Nash equilibrium is Pareto efficient. Assume that $(y^P, (x_j^P)_{j \in N})$ is the allocation attained at a strict Nash equilibrium. Note that $V_i(y^P, x_i^P) = \theta_i - x_i^P$ for all $i \in P$ and $V_i(y^P, x_i^P) = \theta_i$ for all $i \notin P$. Suppose, on the contrary, a feasible allocation $(\widehat{y}, (\widehat{x}_j)_{j \in N})$ Pareto dominates $(y^P, (x_j^P)_{j \in N})$. It must be satisfied that $V_i(\widehat{y}, \widehat{x}_i) = \theta_i$ for all $i \notin P$ because θ_i is the greatest payoff of agent i in A. Hence, there is at least one participant $j \in P$ such that $V_j(\widehat{y}, \widehat{x}_j) > V_j(y^P, x_j^P)$. Let $J \subseteq P$ be a set of such participants and let $\varepsilon_j = V_j(\widehat{y}, \widehat{x}_j) - V_j(y^P, x_j^P) > 0$ for all $j \in J$. Since $V_j(y^P, x_j^P) = \theta_j - x_j^P > 0$ for every $j \in J$, we must have $\widehat{y} = 1$: otherwise, $V_j(\widehat{y}, \widehat{x}_j) = 0$. Then, we learn that $V_j(\widehat{y}, \widehat{x}_j) = \theta_j - x_j^P + \varepsilon_j$ for all $j \in J$. By the argument above,

$$
\begin{aligned}
&\widehat{x}_j = 0 \text{ for all } j \notin P, \\
&\widehat{x}_j = x_j^P - \varepsilon_j \text{ for all } j \in J, \text{ and} \\
&\widehat{x}_j = x_j^P \text{ for all } j \in P \setminus J.
\end{aligned}
$$

Summing up $\widehat{x}_j$ for all $j \in N$ yields $\sum_{j \in N} \widehat{x}_j = \sum_{j \in P} x_j^P - \sum_{j \in J} \varepsilon_j = c - \sum_{j \in J} \varepsilon_j < c$, which contradicts the feasibility of $(\widehat{y}, (\widehat{x}_j)_{j \in N})$. Hence, $(y^P, (x_j^P)_{j \in N})$ is Pareto efficient.

Secondly, each Nash equilibrium that supports an efficient allocation is a strict Nash equilibrium. Let $s \in S^n$ be a Nash equilibrium that attains an efficient allocation. Denote the set of participants at s by P^s. Since the project is done at efficient allocations, we have $\theta_{P^s} > c$. Furthermore, it is satisfied that $\theta_{P^s} - \theta_i \leq c$ for all $i \in P^s$: if there is an agent $j \in P^s$ such that $\theta_{P^s} - \theta_j > c$, then agent j has an incentive to deviate from s because $u_j(P^s \setminus \{j\}) = \theta_j > \theta_j - x_j^{P^s} = u_j(P^s)$. This contradicts the idea that s is a Nash equilibrium. It follows from Lemma 5.1 that s is a strict Nash equilibrium. ■

5.3 Strong equilibria in the participation game

5.3.1 Equivalence between strict Nash equilibrium and strong equilibrium

First, we show that the set of strong equilibria coincides with that of strict Nash equilibria.

Proposition 5.2 *In the participation game with a public project, a strategy profile is a strong equilibrium if and only if it is a strict Nash equilibrium.*

Proof. ($\Leftarrow$) Let $s^* \in S^n$ denote a strict Nash equilibrium. Let P^* be the set of participants at s^*. Let $T \subseteq N$ be a coalition and $s_T \in S^{\#T}$ be a strategy profile of T. We show that some members of T are worse off by jointly deviating from s_T^* to s_T.

We take a partition of T consisting of four sets: $T_I^* \cap T_I$, $T_I^* \backslash T_I$, $T_I \backslash T_I^*$, and $T \backslash (T_I^* \cup T_I)$, where $T_I^* \equiv \{i \in T | s_i^* = I\}$ and $T_I \equiv \{i \in T | s_i = I\}$. The set of participants in (s_T, s_{-T}^*) is $(P^* \setminus (T_I^* \backslash T_I)) \cup (T_I \backslash T_I^*)$. We denote this set by $\widetilde{P}$. In the strict Nash equilibrium s^*,

$$u_i(P^*) = \theta_i - x_i^{P^*} > 0$$

for all $i \in P^*$, and

$$u_i(P^*) = \theta_i > 0$$

for all $i \notin P^*$. We calculate the payoffs of the members of T after the deviation. To do so, we need to consider the following two cases: $\theta_{\widetilde{P}} \leq c$, and $\theta_{\widetilde{P}} > c$.

First, consider the case in which $\theta_{\widetilde{P}} \leq c$. In this case, the public project is not undertaken at (s_T, s_{-T}^*). Since the payoffs of the members of T at (s_T, s_{-T}^*) are given by

$u_i(\widetilde{P}) = 0$ for all $i \in T$, we obtain the following four inequalities:

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T_I^* \cap T_I,$$

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T_I^* \backslash T_I,$$

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T_I \backslash T_I^*, \text{ and}$$

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T \backslash (T_I^* \cup T_I).$$

Therefore, the deviation cannot raise the members' payoffs.

Next, let us consider the case in which $\theta_{\widetilde{P}} > c$. Note that the public project is undertaken at (s_T, s^*_{-T}). If $T_I^* \backslash T_I$ is not empty, then it follows from Lemma 5.1 that $\theta_{P^*} - \theta_i \leq c$ for all $i \in T_I^* \setminus T_I$. Thus, we have $\theta_{P^*} - \theta_{T_I^* \backslash T_I} \leq c$. Because $\theta_{\widetilde{P}} = \theta_{P^*} - \theta_{T_I^* \backslash T_I} + \theta_{T_I \backslash T_I^*} > c$, we must obtain $\theta_{T_I \backslash T_I^*} > 0$. This implies that $T_I \backslash T_I^*$ is non-empty. It is satisfied that $u_i(P^*) > u_i(\widetilde{P})$ for all $i \in T_I \backslash T_I^*$ because $u_i(\widetilde{P}) = \theta_i - x_i^{\widetilde{P}}$ for every $i \in T_I \backslash T_I^*$. Therefore, if $T_I^* \backslash T_I$ is not empty, the deviation does not improve the members' payoffs. If $T_I^* \backslash T_I$ and $T_I \backslash T_I^*$ are empty sets, then $P^* = \widetilde{P}$ holds. Clearly, no member of T is better off by the deviation. If $T_I^* \backslash T_I$ is empty and $T_I \backslash T_I^*$ is non-empty, then none of the agents in $T_I \backslash T_I^*$ can improve their payoffs by the deviation since $u_i(P^*) = \theta_i > \theta_i - x_i^{\widetilde{P}} = u_i(\widetilde{P})$ for all $i \in T_I \backslash T_I^*$. Consequently, s^* is a strong equilibrium of G.

($\Rightarrow$) Let s^* be a strong equilibrium, and let P^* be the set of participants at s^*. If $\theta_{P^*} \leq c$ holds, then we have $u_i(P^*) = 0$ for all $i \in N$. When all agents jointly choose I, then every agent i has the payoff $u_i(N) = \theta_i - x_i^N$, which is positive by Assumption 5.1 and 5.2. This is a contradiction. Hence, we have $\theta_{P^*} > c$. It also holds that $\theta_{P^*} - \theta_i \leq c$ for all $i \in P^*$: if there exists an agent $j \in P^*$ such that $\theta_{P^*} - \theta_j > c$, then agent j has an incentive to deviate from s because $u_j(P^* \setminus \{j\}) = \theta_j > \theta_j - x_j^{P^*} = u_j(P^*)$. This contradicts the idea that s^* is a strong equilibrium. Therefore, s^* is a strict Nash equilibrium. ■

Although the sets of strict Nash equilibria and strong equilibria are subsets of that of Nash equilibria, it is not evident whether the two sets coincide. The two-player game depicted in Table 5.2 shows that the set of strong equilibria does not necessarily coincide with that of strict Nash equilibria. In this game, (B_1, B_2) is the only strict Nash equilibrium, and a strong equilibrium is uniquely determined by (A_1, A_2). Hence, the two sets have an empty intersection, and both of them exist. However, from Proposition 5.2, the set of strict Nash equilibria coincides with that of the strong equilibria in the participation game. An implication of Proposition 5.2 is that the two non-cooperative equilibrium concepts based on different types of stability coincide in the participation game with a public project.

Note that a weakly dominated strategy may be used at a strong equilibrium.[1] In the example in Table 5.2, A_1 is weakly dominated by B_1, and so is A_2 by B_2. However, (A_1, A_2) is a strong equilibrium of the game. In the participation game with a public project, every strong equilibrium is a strict Nash equilibrium, which implies that the strong equilibrium does not consist of weakly dominated strategies in the participation game.

$1\backslash 2$	A_2	B_2
A_1	2, 2	0, 2
B_1	2, 0	1, 1

Table 5.2: The set of strong equilibria and that of strict equilibria do not necessarily coincide.

[1] For every agent i, a strategy $s_i \in S$ is *weakly dominated* in the game G if there exists another strategy $s_i' \in S$ such that $U_i(s_i', s_{-i}) \geq U_i(s_i, s_{-i})$ for all s_{-i} with strict inequality for some s_{-i}.

By Lemma 5.2, Proposition 5.1, and Proposition 5.2, the set of strong equilibria and the set of Nash equilibria that support efficient allocations coincide, and a strong equilibrium exists in the participation game.

Corollary 5.1 The set of strong equilibria coincides with the set of Nash equilibria that support an efficient allocation in the participation game.

Corollary 5.2 The participation game has a strong equilibrium.

These results contrast with those of a participation game with a perfectly divisible public good. Saijo and Yamato (1999) introduced a model of voluntary participation in a mechanism to provide a perfectly divisible public good. We find from their results that the Nash equilibria of the game are not always Pareto efficient. Hence, if agents have the right to decide either participation or non-participation in the mechanism, then efficient allocations are not necessarily attained even if the mechanism is constructed to implement efficient allocations in its equilibrium. It is also proven in Chapter 2 that the game does not always have a strong equilibrium. In contrast, in a participation game with a public project, there exist strong equilibria, and an efficient allocation of the economy can be supported as the equilibrium.

5.3.2 Coalition-proof equilibria and strong equilibria

Proposition 5.3 *In the participation game with a public project, a strategy profile is a strong equilibrium if and only if it is a coalition-proof equilibrium.*

Proof. By the definitions of coalition-proof equilibria and strong equilibria, every strong equilibrium is a coalition-proof equilibrium. We show that a coalition-proof equilibrium $s \in S^n$ is a strong equilibrium. Suppose, on the contrary, that s is not a strong equilibrium. Since the profile s is a coalition-proof equilibrium, it must be a Nash equilibrium. If s is a strict Nash equilibrium, then it is also a strong equilibrium by Proposition 5.2. Therefore, s must be a non-strict Nash equilibrium. By Proposition 5.1, s does not support an efficient allocation. Because of this, we have $U_i(s) = 0$ for all $i \in N$. By Lemma 5.2, there is at least one strict Nash equilibrium in this game. Denote a strict Nash equilibrium by s^*. Note that s^* must be a coalition-proof equilibrium; hence, it must also be a self-enforcing strategy profile. By Proposition 5.2, s^* is a strong equilibrium, and we have $U_i(s^*) > 0$ for every $i \in N$. Since s is Pareto-dominated by the self-enforcing strategy profile s^*, s is not coalition-proof, which is a contradiction. Therefore, s is a strong equilibrium. ■

Konishi et al. (1997a, 1997b, 1997c) studied the *no-spillover game*, in which the strategy spaces of all players are common. In the no-spillover game, for each player i, his payoff is not affected by the choices of those players who choose strategies different from i.[2] These authors established sufficient conditions for the existence of strong equilibria and the equivalence between coalition-proof equilibria and strong equilibria in the game. One of the sufficient conditions is the condition of *positive population monotonicity*: the payoff of every player i increases if more players choose the same strategy as players i.[3] Konishi et al. (1997a) proved that, if the population monotonicity condition is satisfied, the set of coalition-proof equilibria coincides with that of strong equilibria in every no-spillover game. Konishi et al. (1997b) also showed that strong equilibria exist in games in which the set of pure strategies for each player consists of two alternatives. Although the

[2]The no-spillover game is formally defined as follows: a game is called a *no-spillover game* if, for all pairs of agents $i,\ j \in N$, for all strategy profiles $s \in S^n$, and for all strategies for i, $\widehat{s}_i$, if $s_j \neq s_i$ and $s_j \neq \widehat{s}_i$, then $U_j\left(s_i,\, s_j,\, s_{N\setminus\{i,j\}}\right) = U_j\left(\widehat{s}_i,\, s_j,\, s_{N\setminus\{i,j\}}\right)$.

[3]The game satisfies *positive population monotonicity* if, for all $i, j \in N$ and for all $s \in S^n$, if $s_i \neq s_j$, then $U_j(s_i, s_j, s_{N\setminus\{i,j\}}) \leq U_j(s_j, s_j, s_{N\setminus\{i,j\}})$.

participation game is a no-spillover game, it does not satisfy positive population monotonicity because the payoffs of non-participants decrease when a participant switches to non-participation and the project is then not undertaken. It was also proven by Konishi et al. (1997c) that, if a no-spillover game satisfies *negative population monotonicity*[4] and *anonymity*[5], then the game has a strong equilibrium. The participation game with a public project does not satisfy negative population monotonicity. Furthermore, the participation game is not anonymous because agents are heterogeneous and the payoffs of participants depend not on the number of participants but on their composition in our model. Although the conditions of Konishi et al. are not sufficiently met in our game, the set of strong equilibria coincides with that of coalition-proof equilibria and is not empty. The existence of strong equilibria has also been studied in the context of *congestion games*, which can be interpreted as a sort of a participation game in mechanisms providing local public goods with congestion effects. The congestion games satisfying some conditions have a strong equilibrium (Holzman and Yone, 1997). Although the participation game studied in this chapter is not a congestion game, it has a strong equilibrium.

The following theorem summarizes the results that have been obtained so far.

Theorem 5.1 *In the participation game, the set of strict Nash equilibria, that of strong equilibria, that of coalition-proof equilibria, and the set of Nash equilibria that support efficient allocations coincide.*

[4]The game satisfies *negative population monotonicity* if, for all $i, j \in N$, for all $s \in S^n$, if $s_i \neq s_j$, then $U_j(s_i, s_j, s_{N\setminus\{i,j\}}) \geq U_j(s_j, s_j, s_{N\setminus\{i,j\}})$.

[5]The condition of anonymity requires that the payoff of a player depend only on the number of players who choose the same strategy. The formal definition is as follows: a game is *anonymous* if, for all s, $\widehat{s} \in S^n$ and all $i \in N$, if $s_i = \widehat{s}_i$ and $\#\{j \in N | s_j = \bar{s}\} = \#\{j \in N | \widehat{s}_j = \bar{s}\}$ for all $\bar{s} \in S$, then $U_i(s) = U_i(\widehat{s})$.

Remark 5.1 Let us consider an allocation rule that satisfies (ii), (iv), and the following conditions:

(i)$'$ For all sets of participants P, $\theta_P \geq c$ if and only if $y^P = 1$.

(iii)$'$ For all $P \subseteq N$ and for all $i \in P$, $\theta_i \geq x_i$. (weakly individual rationality)

In the participation game in a mechanism to implement this allocation rule, the set of strong equilibria contains that of strict Nash equilibria, and they do not always coincide. Furthermore, a strict Nash equilibrium does not necessarily exist in the game. However, the game has a Nash equilibrium at which efficient allocations are attained, and every set of participants at Nash equilibria that support efficient allocations is characterized as $P \subseteq N$ with $\theta_P \geq c$ and $\theta_P - \theta_i < c$ for all $i \in P$. We can show that the set of Nash equilibria that support efficient allocations, that of strong equilibria, and that of coalition-proof equilibria coincide in a similar way to Propositions 5.2 and 5.3. Therefore, the equivalence between a strong equilibrium and a coalition-proof equilibrium can be obtained in a case in which the allocation rule satisfies (i)$'$ and (iii)$'$ instead of (i) and (iii).

5.4 More general participation games: examples

In Section 5.3, we prove that the set of strong, strict Nash, and coalition-proof equilibria coincide in the participation game with a public project. In this section, we consider two natural generalizations of the participation game with a public project: *participation games with a multi-unit public good* and *participation games with multiple public projects.* The purpose of this section is to investigate whether or not the results in Section 5.3 can be extended to the more general participation games.

5.4.1 Participation games with a multi-unit public good

There is one private and one public good. We assume that the public good is produced in the units of integers only. Let $l > 1$ be a natural number. Let Y be a subset of $\mathbb{R}^l_+$ such that $Y = \{(y_1, \dots, y_l) \in \{0,1\}^l | \, y_1 \geq y_2 \geq \cdots \geq y_l\}$: in this model, at most l units of the public good can be produced. Let $c > 0$ be the cost of producing one unit of the public good. Each agent i has a preference relation that is represented by the utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}_+$, which associates a real value $V_i(y, x_i) = \sum_{k \in \{1,2,\dots,l\}} \theta_i^k y_k - x_i$ with each element (y, x_i) in $Y \times \mathbb{R}_+$, where $\theta_i^k > 0$ denotes agent i's willingness to pay for the k-th unit of the public good.

Example 5.2 Let $N = \{1, 2, 3, 4\}$. Let $l = 2$. Suppose that $\theta_i^1 = 2$ and $\theta_i^2 = 0.8$ for all $i \in N$ and $c = 1$. Assume that a mechanism implements the equal cost-sharing rule. Let P be a set of participants. Note that one unit of the public good is produced if $\#P = 1$, and two units of the public good are provided if $\#P \geq 2$. Table 5.3 shows the payoffs to participants and non-participants in this example. From the table, we can easily find that one and only one agent enters the mechanism at all strict Nash and coalition-proof equilibria. However, these Nash equilibria are not strong equilibria, since three non-participants at the Nash equilibrium can gain higher payoffs if all of them jointly deviate from non-participation to participation; thus, a strong equilibrium does not exist in this example. Therefore, the set of strict Nash equilibria and that of strong equilibria do not necessarily coincide in the participation game with a multi-unit public good.

The number of participants	Payoffs to participants	Payoffs to non-participants
0	-	0
1	1	2
2	1.8	2.8
3	$\frac{32}{15}$	2.8
4	2.3	-

Table 5.3: Payoffs of Example 5.2

5.4.2 Participation games with multiple public projects

Let us consider an economy with two public projects (A and B) and their corresponding mechanisms. The set of strategies of every agent is denoted by $S = \{A, B, O\}$: A means participation in the mechanism undertaking the public project A, B designates participation in the mechanism implementing the public project B, and O represents participation in neither mechanism. The public project A is produced from c units of the private good, and B is produced from αc units of the private good, where $\alpha > 0$. The production costs of public projects A and B are shared by participants equally. Every agent i has a preference relation that is represented by the quasi-linear utility function $\theta_i^A y_A + \theta_i^B y_B - x_i$, where $y_A \in \{0, 1\}$ and $y_B \in \{0, 1\}$ represent the public projects A and B, and θ_i^A and θ_i^B denote the willingness to pay for public projects A and B, respectively.

Example 5.3 Assume that $\theta_1^A = \theta_1^B = \theta_2^A = \theta_2^B = \theta > 0$, $2\theta > \alpha c > \theta > c$, and $1 < \alpha < 2$, say $\alpha = 1.5$, $c = 1$, and $\theta = 1.25$. The payoff matrix is depicted in Table 5.4. In this example, the cost of project B is higher than that of project A. Project A is undertaken if one or two agents choose A, and project B is undertaken only if two agents choose B. Thus, it is a Nash equilibrium for the two agents to select B. This strategy profile is also coalition-proof, because (A, A) is the only deviation that improves payoffs of the two agents, but the deviation is not self-enforcing. However, strategy profile (B, B) is not strong since the deviation from (B, B) to (A, A) is profitable. Hence, in the participation game with two projects, there may be a coalitional deviation that increases payoffs of its members but is not self-enforcing. Therefore, the set of strong equilibria does not always coincide with that of coalition-proof equilibria.

$1\backslash 2$	A	B	O
A	0.75, 0.75	0.25, 1.25	0.25, 1.25
B	1.25, 0.25	0.5, 0.5	0, 0
O	1.25, 0.25	0, 0	0, 0

Table 5.4: A participation game with two public projects

The above examples indicate that the equivalence among the three sets of equilibria does not always hold in the games with a discrete public good and multiple public projects. Therefore, it is an essential assumption to the equivalence result that there is one and only one public project in the economy.

Remark 5.2 Konishi et al. (1997a) showed that the set of coalition-proof equilibria and that of strong equilibria coincide in many games of the provision of local public goods. (Refer to Greenberg and Weber (1993) and Konishi et al. (1998) for games of the provision of local public goods.) However, in games of the provision of non-excludable public goods, the equivalence rarely holds. The above results show that the two equilibrium sets coincide in the participation game with a public project, while they may fail to coincide if the public good can be provided in multiple units or if there are multiple projects.

5.5 Concluding Remarks

We have investigated a participation game in a mechanism providing a public project. We characterized the strict Nash, strong, and coalition-proof equilibria of the participation game. We showed that the set of strict Nash, strong, and coalition-proof equilibria coincide and that all of the equilibria exist. We find from the result that the participation in a public project is in a class of games in which the three different non-cooperative equilibria coincide. Furthermore, an efficient allocation of the economy can be achieved as various notions of equilibria, and only the efficient allocations are supportable as the equilibria. These results are contrasted with those in the models of providing a perfectly divisible public good, such as a participation game with a perfectly divisible public good in Chapter 2 and the voluntary contribution of a perfectly divisible public good. The equivalence between the sets of coalition-proof and strong equilibria is established, although the conditions of the earlier literature have not been sufficiently satisfied in our model. This chapter clarified the conditions that the set of coalition-proof equilibria and that of strong equilibria coincide in the game of the provision of non-excludable public goods.

Although efficient allocations are attained at the equilibria, the allocations are less desirable from the viewpoint of equity. In Example 5.1 on page 67, there exist strict Nash equilibria at which two agents enter the mechanism. Obviously, these Nash equilibria support efficient allocations. However, in these equilibria, only two agents bear the cost for the public project, and the other agent enjoys the project at no cost. To achieve more equitable allocations, it is desirable that all agents participate in the mechanism. It is left for future researches to study the possibility of constructing the mechanism, in which all agents participate at equilibria.

Chapter 6

Participation Games with Multiple-choice Public Goods

In this chapter, a participation game in a mechanism to implement a proportional cost-sharing rule is examined. We first consider the participation game in the case of a public project, which is similar to the case of Chapter 5. However, unlike in Chapter 5, we assume that agents in a coalition can freely transfer their utilities among the members of the coalition. We prove that a strong equilibrium exists in this game under the assumption and that the set of strict Nash equilibria contains that of strong equiibria. Second, we extend our analysis to the case of a multi-unit public good. In particular, we focus on the participation game in which the public good is discrete and at most two units of the public good. In this case, Nash equilibria of the game do not necessarily support efficient allocations. We show that no Nash equilibrium supports an efficient allocation if agents are identical and some conditions hold.

6.1 A participation game in a mechanism implementing the proportional cost-sharing rule

In this chapter, we assume that a public good mechanism implements the proportional cost-sharing rule, which appears in the following assumption.

Assumption 6.1 Let P denote a set of participants. The allocation to participants $(y^P, (x_j^P)_{j\in P})$ that satisfies the following conditions is attained in the equilibrium of the mechanism:

$$\text{if } \theta_P > c, \text{ then } x_i^P = \frac{\theta_i}{\theta_P} c \text{ for all } i \in P \text{ and } y^P = 1, \text{ and}$$
$$\text{if } \theta_P \leq c, \text{ then } x_i^P = 0 \text{ for all } i \in P \text{ and } y^P = 0.$$

Definition 6.1 (Strong equilibrium) A strategy profile $s^* \in S^n$ is a *strong equilibrium of* G if there exist no coalition $T \subseteq N$ and its strategy profile$\widetilde{s}_T \in S^{\#T}$ such that $\sum_{i\in T} U_i(\widetilde{s}_T, s^*_{-T}) > \sum_{i\in T} U_i(s^*)$ for all $i \in T$.

In this chapter, we assume that members of a coalition can coordinate their participation decision through monetary transfers. A strong equilibrium is a strategy profile in which no coalitions can jointly deviate in a way that increases the sum of the payoffs to the members in the coalitions. Note that the set of strong equilibria without monetary transfers contains the set of strong equilibria with monetary transfers. But the converse is not always true. Therefore, existence of a strong equilibrium without monetary transfers does not necessarily imply that of a strong equilibrium with monetary transfers.

Finally, we assume that $\theta_N > c$.

6.2 Strong equilibria of the participation game

Note that the proportional cost-sharing allocation rule in Assumption 6.1 is included in the class defined in Assumption 5.1. Thus, from Lemma 5.1, Lemma 5.2 and Proposition 5.1, a strict Nash equilibrium exists in this participation game and the set of strict Nash equilibria coincides with that of Nash equilibria that support efficient allocations.

From Theorem 5.1, the set of strong equilibria and the set of strict Nash equilibria exist and coincide in the absence of monetary transfers. However, it is unclear that the strong equilibrium exists and this equivalence between the two equilibria holds when members in a coalition can transfer their utilities because the set of strong equilibria with monetary transfers is included in that without monetary transfers. The following example shows that not every strict Nash equilibrium is a strong equilibrium in the presence of monetary transfers.

Example 6.1 Let $N = \{1, 2, 3\}$ and let $\theta_1 = \theta_2 = 8$, $\theta_3 = 4$, and $c = 10$. Table 6.1 shows the payoff matrix of this example. This game has three strict Nash equilibria: $(s_1, s_2, s_3) = (O, I, I)$, (I, O, I) and (I, I, O). All the strict Nash equilibria support efficient allocations. We now focus on the strategy profile $s^* = (I, I, O)$. The payoffs at s^* are $U_1(s^*) = U_2(s^*) = 3$, and $U_3(s^*) = 4$. Suppose a coalition $C = \{2, 3\}$ is formed and deviate from s^*_C to $\widetilde{s}_C = (O, I)$. Note that the public project is undertaken at $(s^*_1, \widetilde{s}_C)$. The payoffs of agent 2 and 3 at $(s^*_1, \widetilde{s}_C)$ are $U_2(s^*_1, \widetilde{s}_C) = 8$ and $U_3(s^*_1, \widetilde{s}_C) = 2/3$. Hence, the aggregate payoff for C at $(s^*_1, \widetilde{s}_C)$ is 26/3, which is greater than the sum of payoffs of C at s^*. Therefore, the strategy profile s^* is not a strong equilibrium, while the other strict Nash equilibria are strong equilibria.

In Example 6.1, the sum of the benefits that participants receive from the project is 12 in all strong equilibria, which is the smallest sum of the benefits of participants

	I	O
I	4, 4, 2	$\frac{4}{3}$, 8, $\frac{2}{3}$
O	8, $\frac{4}{3}$, $\frac{2}{3}$	0, 0, 0

I

	I	O
I	3, 3, 4	0, 0, 0
O	0, 0, 0	0, 0, 0

O

Table 6.1: Payoff matrix of Example 6.1

that can be attained in the set of strict Nash equilibria. In the following subsection, we identify which strict Nash equilibrium is a strong equilibrium in the participation game.

6.2.1 A characterization of strong equilibria

Proposition 6.1 *Let $s^* \in S^n$ denote a strict Nash equilibrium and let P^* be the set of participants at s^*. The strict Nash equilibrium s^* is a strong equilibrium of G if and only if there is not coalition T and its strategy profile $\widehat{s}_T \in S^{\#T}$ such that*

$$T_I^* \subsetneq P^*,\ \theta_{T_I^* \setminus \widehat{T}_I} > \theta_{\widehat{T}_I \setminus T_I^*} > 0, \textit{and}\ \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*} > c, \tag{6.1}$$

where $T_I^* = \{i \in T | s_i^* = I\}$ and $\widehat{T}_I = \{i \in T | \widehat{s}_i = I\}$.

Proof. Let s^* denote a strict Nash equilibrium. Denote the set of participants by P^*. Let T denote a coalition and let $\widehat{s}_T$ denote a profile of strategies for T. The set of participants at $(\widehat{s}_T, s^*_{-T})$ is denoted by $\widehat{P}$. If we define $T_I^* = P^* \cap T$ and $\widehat{T}_I = \widehat{P} \cap T$, then $\widehat{P} = (P^* \setminus (T_I^* \setminus \widehat{T}_I)) \cup (\widehat{T}_I \setminus T_I^*)$. Note that $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T_I^* \setminus \widehat{T}_I} + \theta_{\widehat{T}_I \setminus T_I^*}$.

We first show the following lemma.

Lemma 6.1 Only the deviations that satisfy (6.1) improve the sum of the payoffs that members of T obtain.

Proof of Lemma 6.1.

Claim 6.1 If $\theta_{\widehat{P}} \geq \theta_{P^*}$, then the sum of the payoffs that agents in T obtain before the deviation is greater than or equal to the sum of the payoffs that members of T receive before the deviation.

Proof of Claim 6.1. The sum of the payoffs of agents in T at s^* is

$$\theta_T - \frac{\theta_{T_I^*}}{\theta_{P^*}} c, \tag{6.2}$$

and that at $(\widehat{s}_T, s^*_{-T})$ is

$$\theta_T - \frac{\theta_{\widehat{T}_I}}{\theta_{\widehat{P}}} c. \tag{6.3}$$

Subtracting (6.3) from (6.2) yields

$$\begin{aligned}
& -\frac{\theta_{T_I^*}}{\theta_{P^*}} c + \frac{\theta_{\widehat{T}_I}}{\theta_{\widehat{P}}} c \\
=& \frac{c}{\theta_{P^*}\theta_{\widehat{P}}} (\theta_{P^*}\theta_{\widehat{T}_I} - \theta_{\widehat{P}}\theta_{T_I^*}) \\
=& \frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*}\theta_{\widehat{T}_I} - \theta_{T_I^*}\left(\theta_{P^*} - \theta_{T_I^*\setminus\widehat{T}_I} + \theta_{\widehat{T}_I\setminus T_I^*}\right)\right) \\
=& \frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*}\left(\theta_{\widehat{T}_I} - \theta_{T_I^*}\right) - \theta_{T_I^*}\left(\theta_{\widehat{T}_I\setminus T_I^*} - \theta_{T_I^*\setminus\widehat{T}_I}\right)\right).
\end{aligned}$$

Using the equation $\theta_{\widehat{T}_I} - \theta_{T_I^*} = \theta_{\widehat{T}_I\setminus T_I^*} - \theta_{T_I^*\setminus\widehat{T}_I}$, we obtain

$$\frac{c}{\theta_{P^*}\theta_{\widehat{P}}} \left(\theta_{P^*} - \theta_{T_I^*}\right)\left(\theta_{\widehat{T}_I\setminus T_I^*} - \theta_{T_I^*\setminus\widehat{T}_I}\right). \tag{6.4}$$

We have $\theta_{P^*} - \theta_{T_I^*} \geq 0$ because $T_I^* \subseteq P^*$. Since $\theta_{\widehat{P}} \geq \theta_{P^*}$, we obtain $\theta_{\widehat{T}_I\setminus T_I^*} \geq \theta_{T_I^*\setminus\widehat{T}_I}$. Therefore, (6.4) is greater than or equal to zero. **(End of Proof of Claim 6.1)**

By Claim 6.1, the deviations by T satisfies $\theta_{P^*} > \theta_{\widehat{P}}$ only if the deviations result in improvements. Since $\theta_{P^*} > \theta_{\widehat{P}}$, we obtain $\theta_{T_I^*\setminus\widehat{T}_I} > \theta_{\widehat{T}_I\setminus T_I^*}$.

Claim 6.2 If $\theta_{\widehat{P}} \leq c$, the deviation does not increase the sum of payoffs of agents in T.

Proof of Claim 6.2. Note that the project is not undertaken at $(\widehat{s}_T, s^*_{-T})$; thus, the sum of the payoffs that members of T receive after the deviation is zero. Since (6.2) is more than zero, the deviations after which $\theta_{\widehat{P}} \leq c$ is satisfies are not profitable. **(End of Proof of Claim 6.2)**

Combining Claim 6.1 and Claim 6.2 gives $\theta_{P^*} > \theta_{\widehat{P}} > c$. By Lemma 5.1 on page 69, $\theta_{P^*} - \theta_i \leq c$ for all $i \in P^*$. Therefore, $\theta_{P^*} - \theta_{T_I^* \backslash \widehat{T}_I} \leq c$. By Claim 6.2, $\theta_{\widehat{P}} = \theta_{P^*} - \theta_{T_I^* \backslash \widehat{T}_I} + \theta_{\widehat{T}_I \backslash T_I^*} > c$. Thus, we have $\theta_{\widehat{T}_I \backslash T_I^*} > 0$. Accordingly, it follows that $\theta_{P^*} > \theta_{\widetilde{P}} > c$ and $\theta_{T_I^* \backslash \widehat{T}_I} > \theta_{\widehat{T}_I \backslash T_I^*} > 0$.

Claim 6.3 If $T_I^* = P^*$, then the total payoff of T at s^* is equal to that at $(\widehat{s}_T, s^*_{-T})$.

Proof of Claim 6.3. Note that the difference between the total payoff of T at s^* and that at $(\widehat{s}_T, s^*_{-T})$ is equal to (6.4). Therefore, if $T_I^* = P^*$, then (6.4) is equal to zero. **(End of Proof of Claim 6.3)**

By Claims 6.1, 6.2, and 6.3, the statement of Lemma 6.1 is proven. **(End of Proof of Lemma 6.1)**

It is clear from Lemma 6.1 that a strict Nash equilibrium is a strong equilibrium in the participation game if and only if there are no coalitional deviations that satisfies (6.1). ■

Proposition 6.1 says that a deviation from a strict Nash equilibrium results in improvements if and only if there exists the following situation: at the strict Nash equilibrium, a proper subset of the set of participants and non-participants form a coalition and they can coordinate in a way in which the sum of the benefits from the project of participants decreases and the project is undertaken. In this situation, members of the coalition changing their strategies I to O get benefits, and those who alter O to I suffer losses. However, by transferring part of the benefits to the agents altering O to I, the members switching I to O can make up for the losses. As a result, all members of the coalition can improve their payoffs after this deviation.

From Proposition 6.1, we confirm that no deviations after which the total benefits from the project of participants increases are profitable. Therefore, it is not profitable that participants at a strict Nash equilibrium commit themselves to choose participation and induce non-participants at the equilibrium to select participation by transferring money to the non-participants. Since we can interpret that allocations are more equitable as the number of participants increases, we conclude that the coalitional deviations from a strict Nash equilibrium to attain more equitable allocations are not profitable.

The following corollary shows that every strict Nash equilibrium at which only one agent chooses I is a strong equilibrium.

Corollary 6.1 If there is an agent $i \in N$ such that $\theta_i > c$, then $\{i\}$ is a set of participants at a strong equilibrium.

Proof. Suppose that there is an agent $i \in N$ be such that $\theta_i > c$. Then, the set $\{i\}$ is supportable as a strict Nash equilibrium in the game. Let $s^* \in S^n$ be the strict Nash equilibrium at which $\{i\}$ is the set of participants. By Proposition 6.1, s^* is a strong equilibrium if and only if no coalitions deviate from s^* in a way that satisfies (6.1). Because the proper subset of $\{i\}$ is empty, it is clear that no deviations from s^* satisfy (6.1). Therefore, $\{i\}$ is attained at a strict Nash equilibrium. ■

Finally, we mention multiplicity of strong equilibria. In Example 6.1, the sum of the benefits that participants receive at strong equilibria is unique. However, this does not hold true in some cases. By Corollary 6.1, every strict Nash equilibrium with only one participant is a strong equilibrium even if the benefit of the participant is more than $\theta_{P^{min}}$. For example, consider an example where $n = 3$, $c = 10$, $\theta_1 = 5$, $\theta_2 = 6$, and $\theta_3 = 12$. Then, $\{1, 2\}$ and $\{3\}$ are the sets of participants that are supportable as a strong equilibrium. Hence, strong equilibria may support multiple sums of the benefits of participants.

6.2.2 Existence of a strong equilibrium

Proposition 6.2 *A strong equilibrium exists in the participation game.*

Proof. Let P^{min} denote a set of participants such that $\theta_{P^{min}}$ is the smallest sum of the benefits that participants receive in the set of strict Nash equilibria. Let $s^{min} \in S^n$ be a strict Nash equilibrium at which P^{min} is the set of participants. We show that s^{min} is a strong equilibrium. By Proposition 6.1, it is sufficient to show that there is no deviation that satisfies (6.1). Suppose, on the contrary, that there is a coalition T and its strategy profile s_T such that $T_I^{min} \subsetneq P^{min}$, $\theta_{T_I^{min} \backslash T_I} > \theta_{T_I \backslash T_I^{min}} > 0$, and $\theta_{P^{min}} - \theta_{T_I^{min} \backslash T_I} + \theta_{T_I \backslash T_I^{min}} > c$, where $T_I^{min} = \{i \in T | s_i^{min} = I\}$ and $T_I = \{i \in T | s_i = I\}$. Note that the set of participants at (s_T, s_{-T}^{min}) is $(P^{min} \cup (T_I \backslash T_I^{min})) \backslash (T_I^{min} \backslash T_I)$. Let us describe this set of participants by $\widetilde{P}$.

First of all, note that $\theta_{P^{min}} > \theta_{\widetilde{P}}$. Since $\theta_{P^{min}}$ is the smallest sum of the benefits that participants receive from the project at a strict Nash equilibrium, $\widetilde{P}$ can not be supported as a strict Nash equilibrium. Thus, by Lemma 5.1, there is at least one agent $i \in \widetilde{P}$ such that $\theta_{\widetilde{P}} - \theta_i > c$.

Claim 6.4 Let $i \in \widetilde{P}$ be such that $\theta_{\widetilde{P}} - \theta_i > c$. Then, $i \in \widetilde{P} \backslash P^{min}$.

Proof of Claim 6.4. Let $i \in \widetilde{P}$ be such that $\theta_{\widetilde{P}} - \theta_i > c$. Suppose, on the contrary, $i \in \widetilde{P} \cap P^{min}$. Then, $\theta_{P^{min}} - \theta_i \leq c$ holds. From this condition and $\theta_{T_I^{min} \setminus T_I} > \theta_{T_I \setminus T_I^{min}}$, we obtain $\theta_{P^{min}} - \theta_i - \theta_{T_I^{min} \setminus T_I} + \theta_{T_I \setminus T_I^{min}} < c$. As $\theta_{\widetilde{P}} = \theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} + \theta_{T_I \setminus T_I^{min}}$, we obtain that $\theta_{\widetilde{P}} - \theta_i < c$, which contradicts $\theta_{\widetilde{P}} - \theta_i > c$. Therefore, we have $i \in \widetilde{P} \setminus P^{min}$. **(End of Proof of Claim 6.4)**

Note that $\widetilde{P} \setminus P^{min} = T_I \setminus T_I^{min}$. From the conditions $\theta_{\widetilde{P}} = \theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} + \theta_{T_I \setminus T_I^{min}} > c$ and $\theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} \leq c$, we obtain $\theta_{T_I \setminus T_I^{min}} > 0$. Thus, $T_I \setminus T_I^{min}$ is not empty. Suppose, without loss of generality, that the set $T_I \setminus T_I^{min}$ consists of h agents. Denote this set by $\{j_1, \ldots, j_h\}$. In addition, let us assume that $\theta_{j_1} \leq \theta_{j_2} \leq \cdots \leq \theta_{j_h}$. First, consider the set $\widetilde{P} \setminus \{j_1\}$. Since $\widetilde{P}$ is not supported as strict Nash equilibria, we have $\theta_{\widetilde{P} \setminus \{j_1\}} > c$: otherwise, we have

$$c \geq \theta_{\widetilde{P}} - \theta_{j_1} \geq \theta_{\widetilde{P}} - \theta_k \text{ for all } k \in T_I \setminus T_I^{min}, \text{ and}$$
$$c \geq \theta_{P^{min}} - \theta_k > \theta_{\widetilde{P}} - \theta_k \text{ for all } k \in \widetilde{P} \cap P^{min},$$

which means that $\widetilde{P}$ is supportable as a strict Nash equilibrium. This is a contradiction. If $\theta_{\widetilde{P} \setminus \{j_1\}} - \theta_{j_2} \leq c$, then $\widetilde{P} \setminus \{j_1\}$ is supportable as a strict Nash equilibrium since $\theta_{\widetilde{P} \setminus \{j_1\}} - \theta_j \leq c$ for all $j \in \widetilde{P} \setminus \{j_1\}$. This contradicts the idea that $\theta_{\widetilde{P}}$ is the smallest sum of the benefits of participants that is attained at strict Nash equilibria. If $\theta_{\widetilde{P} \setminus \{j_1\}} - \theta_{j_2} > c$, then consider the set of participants $\widetilde{P} \setminus \{j_1, j_2\}$. If $\theta_{\widetilde{P} \setminus \{j_1, j_2\}} - \theta_{j_3} \leq c$, then the set $\widetilde{P} \setminus \{j_1, j_2\}$ is supportable as a strict Nash equilibrium, which is a contradiction by the same reason above. If else, consider the set $\widetilde{P} \setminus \{j_1, j_2, j_3\}$. Applying the same argument and using the facts that $\theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} \leq c$ and $\theta_{P^{min}} - \theta_{T_I^{min} \setminus T_I} + \theta_{T_I \setminus T_I^{min}} > c$, we can find the set $K \subseteq T_I \setminus T_I^{min}$ such that $\theta_{\widetilde{P} \setminus K} > c$ and $\theta_{\widetilde{P} \setminus K} - \theta_j \leq c$ for all $j \in \widetilde{P} \setminus K$. This implies that $\widetilde{P} \setminus K$ is supportable as a strict Nash equilibrium. This is a contradiction. Therefore, coalition T can not deviate in a way that satisfies (6.1). ■

From Proposition 6.1, the set of strict Nash equilibria contains that of strong equilibria, but the converse is not always true. However, in the case of identical agents, every strict Nash equilibrium is a strong equilibrium.

Corollary 6.2 Suppose that agents are identical: $\theta_i = \theta_j$ for all pairs of agents $\{i, j\}$. Then, all strict Nash equilibria are strong equilibria in the participation game.

Proof. Let $\theta = \theta_i$ for all $i \in N$ and let P be a set of participants that is supported as a strict Nash equilibrium. By Lemma 5.1, P satisfies $\#P\,\theta > c$ and $(\#P - 1)\theta \leq c$, or $c/\theta < \#P \leq (c/\theta) + 1$. Since $\#P$ is a natural number, we find from these inequalities that $\#P$ is unique. Therefore, $\#P\,\theta$ is the smallest sum of the benefits that participants receive from the project in the set of strict Nash equilibria. In the proof of Proposition 6.2, we show that a strict Nash equilibrium in which the sum of the benefits of the participants is the smallest in the set of strict Nash equilibria is strong. Thus, P is attained at a strong equilibrium of the game. ■

In the participation game, strict Nash and strong equilibria are both non-empty and the set of strong equilibria is included in that of strict Nash equilibria. This is an interesting respect of our model, because strict Nash equilibria and strong equilibria are based on different stability and there is not always the inclusion relation between the two sets of equilibria. It follows from Lemma 5.2 that there exists an efficient allocation which is supportable as a Nash equilibrium. Moreover, some of the efficient allocations are also supported as a strong equilibrium of the participation game.

The following theorem summarizes the results that have been obtained so far.

Theorem 6.1 *In the participation game with a public project, (i) there is a Nash equilibrium at which the efficiency of an allocation is achieved, (ii) the set of Nash equilibria* *that supports efficient allocations coincides with the set of strict Nash equilibria, (iii) a strong equilibrium exists, and (iv) the set of strict Nash equilibria includes that of strong equilibria, but the converse inclusion relation does not necessarily hold.*

6.3 Participation games with a multi-unit public good

6.3.1 A participation game in which at most two units of the public good can be produced

In this section, we consider a participation game in a mechanism that implements the proportional cost-sharing rule in which at most two units of the public good can be provided. Let Y be a *public good space* such that $Y = \{(y_1, y_2) \in \{0,1\}^2 | y_1 \geq y_2\}$: if $y_1 = y_2 = 1$, then two units of the public good are produced; if $y_1 = 1$ and $y_2 = 0$, then one unit of the public good is produced; if $y_1 = y_2 = 0$, then zero units of the public good are produced. Let $c > 0$ be the cost of producing one unit of the public good. Each agent i has a preference relation that is represented by the utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}_+$, which associates a real value $V_i(y, x_i) = \sum_{k \in \{1,2\}} \theta_i^k y_k - x_i$ with each element (y, x_i) in $Y \times \mathbb{R}_+$, where $\theta_i^k > 0$ denotes agent i's marginal benefit from the k-th unit of the public good. We denote $\theta_P^k = \sum_{j \in P} \theta_j^k$ for all $k \in \{1, 2\}$ and for all $P \subseteq N$. Let us assume that $\theta_i^1 > \theta_i^2$ for all $i \in N$ and $\theta_N^2 > c$. Thus, at every Pareto efficient allocation, two units of the public good are produced.

Assumption 6.2 There exists a mechanism that implements the following allocation rule. Let P denote a set of participants and $(y^P, (x_j^P)_{j \in P})$ be the allocation that is implemented by the mechanism. Then,

$$y^P = \max\{k \in \{0,1,2\} \mid \theta_P^k - c > 0\}, \text{ and}$$

$$\text{for all } i \in P,\ x_i^P = \begin{cases} 0 & \text{if } y^P = 0, \\ \dfrac{\sum_{k=1}^{y^P} \theta_i^k}{\sum_{k=1}^{y^P} \theta_P^k} y^P c & \text{otherwise.} \end{cases}$$

Example 5.2 on page 79 indicates that a Nash equilibrium does not always support an efficient allocation and strong equilibria do not necessarily exist in the participation game with a multi-unit public good. In Example 5.2, two units of the public good are provided at every efficient allocation. However, one unit of the public good is produced at every Nash equilibrium and no strong equilibrium exists. Therefore, no efficient allocations are supportable as Nash equilibria. This is a remarkable difference between the participation game with a public project and that with a multi-unit public good.

6.3.2 Non-existence of equilibria that support efficient allocations

In this subsection, we investigate whether or not a Nash equilibrium supports an Pareto efficient allocation in the participation game in which at most two units of the public good can be produced. For this, we first characterize the set of Nash equilibria at which two units of the public good are produced.

Proposition 6.3 *Two units of the public good are produced at a Nash equilibrium in the participation game if and only if there is a set of participants $P \subseteq N$ that satisfies (i) $\theta_P^2 > c$, (ii) $\theta_P^2 - \theta_i^2 \leq c$ for all $i \in P$, and (iii) if there is an agent $i \in P$ such $\theta_P^1 - \theta_i^1 > c$, then $\theta_i^2 \geq \dfrac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$.*

Proof. (sufficiency) Suppose that there is a set P that satisfies the conditions (i), (ii), and (iii). By (i), two units of the public good are produced if P is a set of participant. By conditions (ii) and (iii), no agent $i \in P$ have an incentive to switch I to O. Clearly, no agents $i \notin P$ do not have an incentive to participate in the mechanism, given the participation of P. Hence, P is a set of participants that is supportable as a Nash equilibrium.

(necessity) Let us assume that there exists a Nash equilibrium at which two units of the public good are produced. Let P be the set of participant attained at the Nash equilibrium. Since two units of the public good are provided, condition (i) must be satisfied. If P do not satisfy (ii), then there exists agent i such that $\theta_P^2 - \theta_i^2 > c$. Hence, agent i has an incentive to deviate from I to O, which is a contradiction. Suppose that there is an agent $i \in P$ such that $\theta_P^1 - \theta_i^1 > c$ and $\theta_i^2 < \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$. Then, he obtains the payoff θ_i^1 if he chooses O, and, in the case of participation, he receives the payoff $\sum_{k=1}^2 \theta_i^k - \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$. Since $\theta_i^2 < \frac{\sum_{k=1}^2 \theta_i^k}{\sum_{k=1}^2 \theta_P^k}(2c)$, he has an incentive to switch from I to O. This is a contradiction. Therefore, P satisfies (i), (ii), and (iii). ■

We examine whether two units of the public good are produced at a Nash equilibrium or not. First consider the following case.

Case 1. For all $P \subseteq N$, if P satisfies $\theta_P^2 > c$, then $\theta_{P\setminus\{i\}}^1 > c$ for some $i \in P$.

The following is the result in this case:

Proposition 6.4 *Suppose that, for all $P \subseteq N$, if P satisfies $\theta_P^2 > c$, then $\theta_{P\setminus\{i\}}^1 > c$ for some $i \in P$. Suppose that agents are identical. Then, Nash equilibria do not support efficient allocations at almost all values θ^2 in Case 1.*

Proof. Let P^* be a set of participants such that $\theta_{P^*}^2 > c$ and $\theta_{P^*\setminus\{i\}}^1 > c$ for some

$i \in P^*$. If all agents in P^* participate in the mechanism, then two units of the public good are produced. But, one unit of the public good is provided if agent $i \in P^*$ deviates from participation to non-participation. Since $\theta^1_{P^*\setminus\{i\}} > c$, we have $\#P \geq 2$. We focus on a case of identical agents: let $\theta^1 = \theta^1_i$ and $\theta^2 = \theta^2_i$ for all $i \in N$. By Proposition 6.3, the set P^* is supportable as a Nash equilibrium if and only if it satisfies (i), (ii), and (iii). By condition (i),

$$\#P^*\theta^2 > c. \tag{6.5}$$

By condition (ii), we have $(\#P^* - 1)\theta^2 \leq c$. Therefore,

$$\theta^2 \leq \frac{c}{\#P^* - 1}. \tag{6.6}$$

By condition (iii),

$$\theta^2 \geq \frac{2c}{\#P^*}. \tag{6.7}$$

Note that it is sufficient to focus solely on equations (6.6) and (6.7). Subtracting $\frac{2c}{\#P^*}$ from $\frac{c}{\#P^* - 1}$ yields

$$\frac{c}{\#P^*(\#P^* - 1)}(2 - \#P^*). \tag{6.8}$$

Since $\#P^* \geq 2$, we have (6.8) ≤ 0 with equality if $\#P^* = 2$. Therefore, it is impossible for a Nash equilibrium to support the provision of two units of the public good if $\#P^* > 2$. When $\#P^* = 2$, two units of the public good are produced only in the case of $\theta^2 = c$. Therefore, in this case, two units of the public good are hardly provided when agents are identical. ■

Proposition 6.4 confirms that the strategic behavior on the participation decisions often leads to the inefficiency of the allocations, even though a mechanism is constructed in a way that implements an efficient allocation rule. Hence, the implication that is similar to Saijo and Yamato (1999) can be obtained even in the participation game in which the public good is discrete and at most two units of the public good can be produced.

The following example indicates that Nash equilibria may support the efficient allocation if agents' preferences are heterogeneous.

Example 6.2 Let $N = \{1, 2\}$ and let $\theta_1^1 = 21$, $\theta_1^2 = 9$, $\theta_2^1 = 7$, $\theta_2^2 = 6.9$, and $c = 10$. In this example, two units of the public good are produced only if two agents choose I, and one unit of the public good is provided only when agent 1 chooses I and agent 2 chooses O. Table 6.2 is the payoff matrix of this example. In this example, there is a Nash equilibrium at which two agents enter the mechanism and two units of the public good are provided.

1\2	I	O
I	16.33, 7.29	11, 7
O	0, 0	0, 0

Table 6.2: The payoff matrix of Example 6.2

Proposition 6.5 *In the participation game in which at most two units of the public good, if a strategy profile is a strong equilibrium, then it is a Nash equilibrium that supports an efficient allocation.*

Proof. Suppose, on the contrary, that there is a strong equilibrium $s \in S^n$ that supports an inefficient allocation. If zero units of the public good is produced at s, then every agent receives the payoff zero, and if one units of the public good is produced, then every participant i obtains $\theta_i^1 - \dfrac{\theta_i^1}{\theta_P^1}c$ and every non-participant j receives θ_j^1. Note that the sum of the payoffs of all agents is zero, when no public good is provided; the sum of the payoffs to all agents is $\theta_N^1 - c > 0$, when one unit of the public good is supplied. On the other hand, when all agents choose I, the sum of the payoffs to all agents is $\sum_{k=1}^{2} \theta_N^k - 2c$, which is greater than $\theta_N^1 - c$. Therefore, if the grand coalition forms and every member chooses I, then all members of N are better off, which is a contradiction. ■

Obviously, every strong equilibrium is Pareto efficient within the set of strategy profiles. However, in this model, it is not clear that a strong equilibrium supports an efficient allocation, because the strategy sets of all agents consists of two alternatives. Proposition 6.5 shows that two units of the public good is provided at every strong equilibrium of this game. It follows from Propositions 6.4 and 6.5 that a strong equilibrium does not necessarily exist in the participation game with a multiunit public good.

Finally, we briefly mention the case in which Case 1 is not satisfied: there exists a set of participants P such that $\theta_P^1 - \theta_i^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$. Note that, for all $D \subseteq N$, if D is not empty, then $\theta_D^1 > \theta_D^2$. Thus, the above P satisfies $\theta_{P\setminus\{i\}}^2 < \theta_{P\setminus\{i\}}^1 \leq c$ for all $i \in P$ and $\theta_P^2 > c$. If all agents in P choose participation, then two units of the public good is provided. By Proposition 6.3, P can be supported as a Nash equilibrium. Thus, two units of the public good are produced at a Nash equilibrium in this case.

6.4 Concluding Remarks

We have investigated the participation game in the mechanism implementing the proportional cost-sharing rule. First, we considered the case of a public project. We have shown that, in this case, strict Nash equilibria exist, the set of strict Nash equilibria and the set of Nash equilibria that supports an efficient allocation coincide, there are strong equilibria, and the set of strict Nash equilibria contains that of strong equilibria. Secondly, we have considered the case in which at most two units of the public good can be provided. In this case, there is not always a Nash equilibrium that supports an efficient allocation and there is not necessarily a strong equilibrium. We have found from these results that the assumption that only one unit of the public good can be produced is essential to the existence of a Nash equilibrium that supports the efficient allocation and that of a strong equilibrium. We also found that the strategic behavior on the participation decisions leads to the inefficiency of the allocations even in the participation game in which the public good is discrete and at most two units of the public good can be produced.

Chapter 7

Conclusion

We have examined the participation game under various equilibrium concepts and various forms of the provision of a public good. In the real world, the participation problem in a public good mechanism is one of the most important issues, as is shown by many examples. However, the participation problem has not been studied adequately so far. In this dissertation, considering the possibility that agents form a coalition and coordinate their actions, we studied the participation problem in a public good mechanism.

In Part I, we considered the participation game in a mechanism to produce a perfectly divisible public good. In Chapter 2, we considered the case in which agents' preferences are identical and examined the coalition-proof equilibria of this game, which are stable against the self-enforcing deviations. We provided an example in which the participation game has multiple Nash equilibria that support the different numbers of participants. However, we showed that the coalition-proof equilibria exist and only the maximal number of participants in the set of Nash equilibria is supportable as a coalition-proof equilibrium. We further showed that the set of coalition-proof equilibria coincides with the Pareto-efficient frontier of the set of Nash equilibria. Therefore, the possibility of coalition deviations guarantees the improvement of the allocative efficiency in the economy. We also considered a strong equilibrium of this game, which is stable against all possible coalitional deviations. Since the coalition-proof equilibrium is stable only against the self-enforcing deviations, the notion of strong equilibria is stronger than that of coalition-proof equilibria. However, a strong equilibrium does not necessarily exist in this game. We characterized the set of strong equilibria and proved that this equilibrium is less likely to exist as the number of agents increases. In the participation game, all members of a coalition can be better off only if the coalitional deviation increases the number of participants, but this type of deviation is not self-enforcing. Thus, coalition-proof equilibria exist, but strong equilibria do not. This is the difference between the two equilibrium concepts in the participation game. Since the definition of coalition-proof equilibria is very complicated, little is known about the properties of this equilibrium. Furthermore, it is difficult to examine the coalition-proof equilibria under general circumstances. The contribution of this chapter is to show the existence of coalition-proof equilibria and to clarify the properties of this equilibrium in the participation game under the assumption of identical agents.

In Chapter 3, we extended the analysis in Chapter 2 to the case of heterogeneous agents. In this chapter, we investigated the number of participants that is attained at coalition-proof equilibria. When agents' preferences are heterogeneous, the number of participants in coalition-proof equilibria may be multiple, differently from the case of identical agents. The main result of this chapter was to provide a sufficient condition under which the number of participants in the coalition-proof equilibria is unique. As a result, we confirmed that a unique number of participants can be achieved not only in the case of identical agents but also in some cases of heterogeneous agents. Although we generalized the uniqueness result regarding the number of participants to the case of heterogeneous agents, we did not show the existence of coalition-proof equilibria when agents' preferences are heterogeneous. The coalition-proof equilibria are defined with recursion of the number of agents in a coalition, and it is quite complicated to show the existence of the coalition-proof equilibria in the case of heterogeneous agents. This task will be undertaken in future research.

In Chapter 4, we studied coalition-proof equilibria based on two different dominance relations: strict dominations and weak dominations. A coalition deviates only if all members of the coalition can be better off by switching their strategies under the notion of strict domination, while a coalition deviates if all members of the coalition are not worse off and at least one of the members is better off by changing their strategies under the notion of weak domination. In equilibrium concepts based on coalition deviations such as the core and strong equilibria, a set of equilibria under weak domination is included in that under strict domination. However, the set of coalition-proof equilibria under strict domination and that under weak domination are not necessarily related by inclusion. We showed that, if a game satisfies the conditions of anonymity, monotone externality, and strategic substitutability, then the set of coalition-proof equilibria under strict domination contains that under weak domination. Since this class of games contains many interesting games, such as the participation games in the case of identical agents, which were studied in Chapter 2, and the Cournot oligopoly game, the inclusion relation holds in many games studied in economics. However, there is a game that does not satisfy the three conditions but in which the inclusion relation between the two coalition-proof equilibria holds. For example, in Example 3.1 on page 42, the two sets of coalition-proof equilibria coincide. This game does not satisfy the condition of anonymity because agents have heterogeneous preferences. The inclusion relation between the two coalition-proof equilibria probably holds when agents' preferences are heterogeneous. A further direction of this study will be to find other classes of games in which the inclusion relation holds and to generalize a sufficient condition.

In Part II, we considered the participation game in a mechanism to produce a discrete public good. First, we considered the participation game in a mechanism to implement a public project in Chapter 5. The mechanism implements the allocation rule that satisfies the following requirements: (i) the public project is undertaken if and only if the joint benefit of participants from it is greater than its cost; (ii) the sum of payments from participants is equal to the cost of the project; (iii) every participant bears a positive cost burden; and (iv) the cost burden of every participant is less than his willingness to pay for the project. There may be multiple Nash equilibria, and both efficient and inefficient allocations are supportable as the equilibria in this game.

We first examined strict Nash equilibria of the participation game. The strict Nash equilibria are those in which every agent is worse off if he deviates unilaterally. We showed that this game has a strict Nash equilibrium, and the set of strict Nash equilibria coincides with the set of Nash equilibria to support efficient allocations. This result is in contrast with that of Saijo and Yamato (1999), who showed that there are no Nash equilibria to support the efficient allocations when the public good is perfectly divisible in many cases. However, in this chapter, we showed that the efficient allocation is achieved in a Nash equilibrium if the level of the public good is fixed.

Next, we proved that, in this game, both coalition-proof and strong equilibria exist, and the set of coalition-proof equilibria and that of strong equilibria coincide. We further showed that the two sets of equilibria coincide with the set of strict Nash equilibria. Combining these results with those above, we determined that the three different sets of equilibria are not empty and coincide. These are new findings in the two following respects. First, in Chapter 2, we pointed out that a strong equilibrium does not necessarily exist in the participation game if the public good is perfectly divisible. In contrast, the strong equilibrium is shown to exist if the level of the public good is of a fixed size. The existence of strong equilibria has been studied in the cases of the provision of local public goods. However, this dissertation provided sufficient conditions for a strong equilibrium to exist in the case of non-excludable public goods. Second, we establish sufficient conditions for the sets of coalition-proof and strong equilibria to coincide. As in the case of the existence of strong equilibria, the equivalence between the two equilibria has been studied in the field of local public good provision and not in the case of non-excludable public goods.

Although efficient allocations are achieved in a Nash equilibrium, not all agents participate in the mechanism. Therefore, there may be agents that benefit from the public project at no cost. It is desirable from the viewpoint of equity that all beneficiaries share the cost of the project. A solution to this free-ride problem is to construct mechanisms in which all agents voluntarily participate. It is left for future studies to determine the possibilities for constructing such mechanisms.

In Chapter 6, we studied the relationship between the allocative efficiency and the forms of public good provision. We considered the participation game in a mechanism to implement the proportional cost-sharing rule and the following two multiple-choice public goods: one is a public project, and the other is a discrete public good that is produced in either one or two units. As we proved in Chapter 6, there are Nash equilibria in which the efficient allocations are achieved, and such Nash equilibria are strong equilibria in the participation game in the public project. In addition to the previous chapter, we considered the possibility that members of a coalition transfer their utilities among the members, and we examined the strong equilibria of this game. We showed that there is a strong equilibrium and some efficient allocations are supportable as a strong equilibrium in this game. In contrast, there is not necessarily a Nash equilibrium that supports an efficient allocation in the participation game to produce a discrete public good. We showed that no Nash equilibria support efficient allocations if agents are identical and a mild condition is satisfied. These findings indicate that the results in the case of a multi-unit public good differ greatly from those in the case of a public project. We concluded from these results that the inefficiency stemming from the agents' strategic behavior with respect to the participation decision often occurs when the level of public good takes more than one positive value.

Table 7.1 on page 107 is a summary of the main results of this dissertation regarding the existence of equilibria. In this table, each column represents a form of the provision of a public good, and each row represents the notion of equilibria. We confirmed from Table 7.1 that Nash equilibria supporting efficient allocations and strong equilibria do not exist in the case of a perfectly divisible public good and that of a discrete and multi-unit public good. The results in the two cases indicate a similar tendency. On the other hand, both equilibria exist in the case of a public project. In conclusion, the inefficiency of Nash equilibria and the non-existence of strong equilibria are due to the setting that the level of public good can take multiple positive values.

In the cases of a perfectly divisible public good and a discrete and multi-unit public good, the coalition-proof equilibria exist, although there is no strong equilibrium. From the results of Chapter 2, the set of coalition-proof equilibria coincides with the Pareto-efficient frontier of the set of Nash equilibria. This indicates that agents have an incentive to coordinate their participation decisions at an inefficient Nash equilibrium when agents can form a coalition. As a result, the coordination leads to the Pareto-efficient frontier of the set of Nash equilibria, and the allocative efficiency is improved. The same applies to the cases of a perfectly divisible public good and a discrete and multi-unit public good. Thus, we conclude that the possibility of coalition deviations improves the efficiency of equilibrium allocations in the class of participation games. Note that the improvement of payoffs to members of coalitions does not necessarily imply the improvement of efficiency in allocations. It will, therefore, be worthwhile to emphasize this point.

Finally, we hope that the findings in this dissertation will serve as the foundations for future studies of the participation problem in public good mechanisms and the solution to the participation problem in the real world.

	$\mathbb{R}_+$ (identical agents)	$\{0,1\}$	$\{0,1,2\}$ (identical agents)
Nash equilibrium achieving the efficiency	$-$	$+$	$-$
Coalition-proof equilibrium	$+$	$+$	$+$ $(*1)$
Strong equilibrium	$-$	$+$	$-$

Table 7.1: Existence of equilibria. Symbol $+$ means the existence of the equilibrium, and $-$ indicates non-existence of the equilibrium. $(*1)$ Although we have not provide a proof of existence of coalition-proof equilibria in the case of a discrete public good, the existence can be shown in a way that is similar to the proof of Proposition 2.1.

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