JSPS Grants-in-Aid for Creative Scientific Research

Understanding Inflation Dynamics of the Japanese Economy

Working Paper Series No.3

Optimal Monetary Policy at the Zero Interest Rate Bound: The Case of Endogenous Capital Formation

Tamon Takamura
Tsutomu Watanabe
And
Takeshi Kudo

November 06

Research Center for Price Dynamics
Institute of Economic Research, Hitotsubashi University
Naka 2-1, Kunitachi-city, Tokyo 186-8603, JAPAN
Tel/Fax: +81-42-580-9138
E-mail: sousei-sec@ier.hit-u.ac.jp
http://www.ier.hit-u.ac.jp/~ifd/

Optimal Monetary Policy at the Zero Interest Rate Bound: The Case of Endogenous Capital Formation

Tamon Takamura* Tsutomu Watanabe† Takeshi Kudo‡

First Version: December 1, 2005

Current Version: April 22, 2006

Abstract

This paper characterizes optimal monetary policy in an economy with the zero interest rate bound and endogenous capital formation. First, we show that, given an adverse shock to productivity growth, the natural rate of interest is less likely to fall below zero in an economy with endogenous capital than the one with fixed capital. However, our numerical exercises show that, unless investment adjustment costs are very close to zero, we still have a negative natural rate of interest for large shocks to productivity growth. Second, the optimal commitment solution is characterized by a negative interest rate gap (i.e., real interest rate is lower than its natural rate counterpart) before and after the shock periods during which the natural rate of interest falls below zero. The negative interest rate gap *after* the shock periods represents the history dependence property, while the negative interest rate gap *before* the shock periods emerges because the central bank seeks to increase capital stock before the shock periods, so as to avoid a decline in capital stock after the shock periods, which would otherwise occur due to a substantial decline in investment during the shock periods. The latter property may be seen as central bank's preemptive action against future binding shocks, which is entirely absent in fixed capital models. We also show that the targeting rule to implement the commitment solution is characterized by history-dependent inflation-forecast targeting. Third, a central bank governor without sophisticated commitment technology tends to resort to preemptive action more than the one with it. The governor without commitment technology controls natural rates of consumption, output, and so on in the future periods, by changing capital stock today through monetary policy.

JEL Classification Numbers: E31; E52; E58; E61
Keywords: Deflation; zero lower bound for interest rates; liquidity trap; endogenous capital formation

*Graduate School of Economics, Hitotsubashi University, ed051007@srv.cc.hit-u.ac.jp
†Institute of Economic Research, Hitotsubashi University, tsutomu.w@srv.cc.hit-u.ac.jp
‡Department of Economics, Nagasaki University, tkudo@net.nagasaki-u.ac.jp

1 Introduction

Recent literature on optimal monetary policy with the zero interest rate bound has assumed that capital stock is exogenously given. This assumption of fixed capital stock has some important implications. First, the natural rate of interest is exogenously determined simply due to the lack of endogenous state variables: namely, it is affected by exogenous factors such as changes in technology and preference, but not by changes in endogenous variables. For example, Jung et al. (2005) and Eggertsson and Woodford (2003a, b) among others, start their analysis by assuming that the natural rate of interest is an exogenous process, which is a deterministic or a two-state Markov process. More recent researches such as Adam and Billi (2004a, b) and Nakov (2005) extend analysis to a fully stochastic environment, but continue to assume that the natural rate process is exogenously given. These existing researches typically consider a situation in which the natural rate of interest, whether it is a deterministic or a stochastic process, declines to a negative level entirely due to exogenous shocks, and conduct an exercise of characterizing optimal monetary policy responses to the shock, as well as monetary policy rules to implement the optimal outcome.

Second, no serious attention has been paid to the channel through which interest rate adjustments conducted by a central bank would affect an equilibrium through a change in capital stock. The existing researches have found that it would contribute to consumption smoothing and consequently to an improvement in welfare if a central bank lowers short-term interest rates before and/or after the periods during which the natural rate of interest is below zero. Specifically, Jung et al. (2005) and Eggertsson and Woodford (2003a, b) emphasize the role of history dependent monetary policy by showing that central bank's credible commitment about monetary easing *after* the periods with a negative natural rate would contribute to consumption smoothing. On the other hand, Adam and Billi (2004a, b) and Nakov (2005) stress the role of central bank's preemptive action by showing that an interest cut *before* the periods with a negative natural rate would mitigate a downward pressure upon consumption and inflation, thereby improving economic welfare. However, the existing papers are entirely silent about the possible effects of an interest rate change, whether it is before or after the shock periods, upon a change in capital stock, which should be closely related to consumption smoothing and therefore economic welfare.

The purpose of the present paper is to see how analysis would change if we remove the assumption of fixed capital stock and instead employ a more realistic assumption of endogenous capital formation. Specifically, we introduce the zero bound constraint into the variable capital model in which rental market for capital stock exists, and characterize optimal monetary policy responses to shocks that lead to a decline in the natural rate of interest to a negative level, as well as monetary policy rules to implement the optimal outcome.

To illustrate the role of endogenous capital stock in an economy with the zero interest rate bound, let us consider a situation in which a substantial decline in productivity growth in period T leads to a decline in the natural rate of interest to a negative level. The best action that a central bank can take in period T is to lower nominal interest rate to its lower bound; however, because of a negative natural rate in period T, the interest rate gap, defined as the discrepancy between the real interest rate and its natural rate counterpart, takes a positive value in period T. Then the consumption Euler equation implies that, other things being equal, the positive interest rate gap in period T leads to a decline in consumption in period T. Then we ask ourselves how one can fix this problem.

Let us first think about a discretionary solution under the assumption of fixed capital stock. Consumption in the period just after the shock, c_{T+1} must coincide with its natural rate counterpart, c^n_{T+1}, simply because a central bank under discretionary policy chooses to close a interest rate gap whenever it is possible. Moreover, the natural level of consumption is exogenously given under the assumption of fixed capital. Therefore c_{T+1} is determined entirely by exogenous sources, and there is no way to avoid the decline in consumption in period T. However, if the central bank can make a credible commitment about future monetary policy, there is a room to improve the situation. Specifically, as shown by Jung et al (2005) among others, the central bank's commitment to monetary easing in period $T+1$ leads to an increase in c_{T+1}, thereby contributing to an increase in c_T, even if the interest rate gap in period T remains unchanged.

In fact, this kind of history dependent policy is the sole way to fix the problem as far as we stick to the assumption of fixed capital. If we relax this assumption, however, we have another channel to fix it. That is, if the central bank lowers short-term interest rate in period $T-1$, firm investment in period $T-1$ would increase, and thus capital stock at the end of period $T-1$ would increase as well.[*1] Other things being equal, this increase in capital stock at the end of period $T-1$ leads to an increase in capital stock at the end of period T as well, which contributes to an expansion of production capacity in period $T+1$, thereby successfully increasing c^n_{T+1}. An important thing to note is that even a central bank without commitment technology is able to increase c_T making use of this channel; it is still possible for such a central bank to increase c_T even if the bank sticks to closing the gap after the shock ($c_{T+1} = c^n_{T+1}$). This mechanism can be seen as a new channel to realize consumption smoothing through capital stock adjustments, which is entirely missing in the fixed capital model that has been used in the analysis of optimal monetary policy with the zero interest rate bound.

This paper characterizes optimal monetary policy in an economy with the zero interest rate bound and

[*1] Note that, in period T, labor supply and production increases in response to monetary easing.

endogenous capital formation. First, we show that, given an adverse shock to productivity growth, the natural rate of interest is less likely to fall below zero in an economy with endogenous capital than the one with fixed capital. However, our numerical exercises show that, unless investment adjustment costs are very close to zero, we still have a negative natural rate of interest for large shocks to productivity growth. Second, the optimal commitment solution is characterized by a negative interest rate gap (i.e., real interest rate is lower than its natural rate counterpart) before and after the shock periods during which the natural rate of interest falls below zero. The negative interest rate gap *after* the shock periods represents the history dependence property, while the negative interest rate gap *before* the shock periods emerges because the central bank seeks to increase capital stock before the shock periods, so as to avoid a decline in capital stock after the shock periods, which would otherwise occur due to a substantial decline in investment during the shock periods.

The main findings of this paper are as follows. First, we show that, given an adverse shock to productivity growth, the natural rate of interest is less likely to fall below zero in an economy with endogenous capital model than the one with fixed capital. This is a direct reflection of consumption smoothing through capital adjustments in an economy with perfectly flexible prices. However, our numerical exercises show that, unless investment adjustment costs are very close to zero, we still have a negative natural rate of interest for large and persistent shocks to productivity growth.

Second, the optimal commitment solution is characterized by a negative interest rate gap (i.e., real interest rate is lower than its natural rate counterpart) before and after the shock periods with negative natural rate of interest. The negative interest rate gap *after* the shock periods represents the history dependence property, which was emphasized by the existing studies such as Jung et al (2005) and Eggertsson and Woodford (2003). On the other hand, the negative interest rate gap *before* the shock periods emerges because the central bank seeks to increase capital stock just before the shock periods, so as to avoid a decline in capital stock after the shock periods, which would otherwise occur due to a substantial decline in investment during the shock periods.

The latter property may be seen as a central bank's preemptive action against future binding shocks, which is unique to endogenous capital model. It should be emphasized that such a preemptive action is completely different from the central bank's pre-shock behavior extensively studied by Adam and Billi (2004a, b) and Nakov (2005) among others in the setting of fixed-capital model. The preemptive action these papers have focused on is nothing but a central bank's policy response to a decline in the *current* inflation rate, which occurs because private agents anticipate the possibility of hitting the zero lower bound in the future and thereby adjust their inflation expectations and their spending. We also show that the targeting rule to implement the commitment solution is characterized by history-dependent inflation-forecast targeting.

Third, a central bank governor without sophisticated commitment technology tends to resort to preemptive action more than the one with it. The governor without commitment technology controls natural rates of consumption, output, and so on in the future periods, by changing capital stock today through monetary policy.

The rest of the paper is organized as follows. Section 2 presents a model with endogenous capital formation, and characterizes its steady state, natural rates, log-linearized system, and utility-based loss function. Section 3 discusses when and how frequently the zero bound constraint is binding. Section 4 characterizes solutions under commitment, as well as under discretion. Section 5 concludes the paper.

2 The model

2.1 The optimal decisions of economic agents

We use a New Keynesian dynamic general equilibrium model with capital stock accumulation. For simplicity, we assume the rental market for capital stock, which Woodford(2005) describes in his paper in comparison with his model of firm-specific capital stock. We assume that there is one firm that specializes in accumulating capital stock of the entire economy (type I firm) and there are other firms which rent capital stock through the rental market for producing goods (type II firms). The optimality conditions for the representative household are as below.

$$u_c\left(C_t;\xi_t\right)=\lambda_t \tag{2.1}$$

$$v_h\left(h_t(i);\zeta_t\right)=w_t(i)\lambda_t,\ \left(i\in[0,1]\right), \tag{2.2}$$

$$\lambda_t Q_{t,t+1}\frac{P_{t+1}}{P_t}=\beta\lambda_{t+1}, \tag{2.3}$$

where λ_t is the marginal utility of real income, C_t is the aggregate consumption, $u_c(\cdot)$ is the marginal utility of consumption, $h_t(i)$ is the supply of labor to firm i of type II firms, $v_h(\cdot)$ is the marginal disutility of labor, w_t is the real wage rate, ξ_t and ζ_t are the preference shocks affecting the utility functions, $Q_{t,t+1}$ is the nominal stochastic discount factor, P_t is the price level, β is the subjective discount factor. The risk-free one-period (gross) nominal interest rate, R_t, must satisfy

$$\left(R_t\right)^{-1}=E_t\left\{Q_{t,t+1}\right\}. \tag{2.4}$$

The type I firm is endowed with the capital stock at time 0, K_0, and makes an investment decision every period in order to maximize the following objective function.

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}}\quad E_0\sum_{t=0}^{\infty}Q_{0,t}\frac{P_t}{P_0}\left[\rho_t K_t-I_t\right],$$

$$s.t. \quad I_t = I\left(\frac{K_{t+1}}{K_t}\right)K_t. \tag{2.5}$$

$I\left(\frac{K_{t+1}}{K_t}\right)$ is a convex function representing costs of investment, adjustment and depreciation altogether per unit of the existing capital stock when the firm chooses K_{t+1} to be the level of capital stock next period. As in Woodford(2005), this function satisfies $I(1) = \delta$, $I'(1) = 1$ and $I''(1) = \epsilon_\psi$, where δ is the depreciation rate of capital stock and $\epsilon_\psi > 0$. This firm's first-order condition is

$$I'\left(\frac{K_{t+1}}{K_t}\right) = E_t Q_{t,t+1}\Pi_{t+1}\left[\rho_{t+1} - I\left(\frac{K_{t+2}}{K_{t+1}}\right) + I'\left(\frac{K_{t+2}}{K_{t+1}}\right)\frac{K_{t+2}}{K_{t+1}}\right], \tag{2.6}$$

where ρ_t is the real rental rate of capital stock, and Π_t is the gross inflation rate from time t to $t+1$.

Type II firms, which produce intermediate goods for consumption and investment in monopolistic competition, exist continuously along the $[0,1]$ line. Whenever possible, each individual firm makes a pricing decision for its product to maximize the discounted sum of profits in the future states in which it is unable to reoptimize its price. The specification of price stickiness is assumed to be the Calvo type as in Rotemberg and Woodford(1997). We also assume that government subsidizes the type II firms at a rate $\tau = 1/(\theta-1)$ per unit of production in order to remove the distortion of monopolistic competition[*2]. We first characterize real marginal cost function for firm $j \in [0,1]$, which can be derived by solving the following cost minimization problem.

$$\begin{aligned} &\min_{\{h_t^d(j),\, k_t(j)\}} \quad w_t(j)h_t^d(j) + \rho_t k_t(j) \\ &s.t. \quad y_t(j) = f\left(A_t \frac{h_t^d(j)}{k_t(j)}\right)k_t(j), \end{aligned} \tag{2.7}$$

where $h_t^d(j)$ and $k_t(j)$ are the demand for labor and capital stock, respectively, A_t is the productivity level and $f(\cdot)$ is the strictly increasing and convave production function. Since the Lagrange multiplier associated with (2.7) is the real marginal cost function, denoted by $S_t(j)$, it can be expressed as

$$S_t(j) = \frac{w_t(j)}{A_t f'\left(A_t \frac{h_t^d(j)}{k_t(j)}\right)} = \frac{\rho_t}{f\left(A_t \frac{h_t^d(j)}{k_t(j)}\right) - A_t \frac{h_t^d(j)}{k_t(j)} f'\left(A_t \frac{h_t^d(j)}{k_t(j)}\right)}. \tag{2.8}$$

We assume constant-returns-to-scale production technology so that the profit maximization problem for firm j is to set a price, $p_t(j)$, to maximize the following objective function.

$$\max_{\{p_t(j)\}} \quad E_t \sum_{k=0}^{\infty} \alpha^k Q_{t,t+k}\left\{p_t(j)(1+\tau)\left(\frac{p_t(j)}{P_{t+k}}\right)^{-\theta} Y_{t+k} - P_{t+k}S_{t+k}(j)\left(\frac{p_t(j)}{P_{t+k}}\right)^{-\theta} Y_{t+k}\right\}$$

Here, we assume that every firm can optimally choose its price every period with probability $1-\alpha$, independent of its history, and $Q_{t,t+k}$ is the nominal stochastic discount factor from time t to $t+k$. We have also used

[*2] θ is the elasticity of demand derived from the Dixit-Stiglitz aggregator for consumption goods.

the demand condition, $y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\theta} Y_t$, to substitute $y_t(j)$ in the objective function, where $y_t(j)$ and Y_t are firm j's output the aggregate production, respectively. It is then straightforward to derive the first-order condition,

$$p_t(j) = \frac{E_t \sum_{k=0}^{\infty} (\alpha\beta)^k u_c (C_{t+k}; \xi_{t+k}) P_{t+k}^{\theta} Y_{t+k} S_{t+k}(j)}{E_t \sum_{k=0}^{\infty} (\alpha\beta)^k u_c (C_{t+k}; \xi_{t+k}) P_{t+k}^{\theta} Y_{t+k} P_{t+k}^{-1}}. \tag{2.9}$$

Finally, we specify the following market-clearing conditions.

$$Y_t = C_t + I_t + G_t, \tag{2.10}$$

$$K_t = \int_0^1 K_t(j) dj \tag{2.11}$$

$$h_t(i) = h_t^d(j) \quad \text{for } i = j \in [0, 1]. \tag{2.12}$$

2.2 The steady state

We characterize the zero-inflation steady state around which we shall log-linearize the optimality conditions. We denote steady-state variables in characters without time subscripts. Assuming that $A = 1$ and $\xi = \zeta = G = 0$ in the steady state, the optimality conditions obtained in the previous section can be written as below.

$$\frac{v_h (h(i); 0)}{u_c (C; 0)} = w(i), \tag{2.13}$$

$$1 = \beta (\rho + 1 - \delta), \tag{2.14}$$

$$R = \beta^{-1},$$

$$P - PS(j) = 0 \tag{2.15}$$

$$S(j) = \frac{w(j)}{f'\left(\frac{h^d(j)}{K(j)}\right)} = \frac{\rho}{f\left(\frac{h^d(j)}{K(j)}\right) - \frac{h^d(j)}{K(j)} f'\left(\frac{h^d(j)}{K(j)}\right)}, \tag{2.16}$$

Equation (2.13) represents the steady-state consumption-leisure choice and equation (2.14) is the corresponding no-arbitrage condition between risk-free nominal interest rate and the rental rate of capital. Equation (2.15) states that price is set at the marginal cost level reflecting the fact that government subsidy induces production of goods at the efficient level. The market-clearing conditions and constraints for optimization problems in the steady state are

$$\begin{aligned} &Y = C + \delta K, \\ &h(i) = h^d(j) \quad \text{for} \quad i = j, \qquad (2.17) \\ &K = \int_0^1 K(j) dj, \\ &y(j) = f\left(\frac{h^d(j)}{K(j)}\right) K(j), \end{aligned}$$

where Y is defined as $Y \equiv \left(\int_0^1 y(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}$.

From the set of equations above, we can observe some properties of the steady state. First, labor-capital ratio is identical across firms. To see this, combine equation (2.15), (2.16) and (2.17) to obtain

$$\rho = f\left(\frac{h(j)}{K(j)}\right) - \frac{h(j)}{K(j)} f'\left(\frac{h(j)}{K(j)}\right). \tag{2.18}$$

Since the right-hand side of equation (2.18) is positive and strictly increasing in $\frac{h(j)}{K(j)}$, there exists a unique positive $\frac{h(j)}{K(j)}$ that satisfies this equation. This in turn implies that the steady-state wages across firms are identical. Consequently, from equation (2.13), the level of employment is the same across firms and hence the level of capital used in each firm must be equivalent.

Accordingly, the steady state can be characterized by the following equations.

$$\frac{v_h(h;0)}{u_c(C;0)} = w, \tag{2.19}$$

$$1 = \beta(\rho + 1 - \delta), \tag{2.20}$$

$$R = \beta^{-1},$$

$$w = f'\left(\frac{h}{K}\right), \tag{2.21}$$

$$\rho = f\left(\frac{h}{K}\right) - \frac{h}{K} f'\left(\frac{h}{K}\right), \tag{2.22}$$

$$Y = C + \delta K$$

Thus, given the initial level of capital stock, we can identify the steady-state values of all the variables.

Finally, for the later purpose of approximating welfare function, we derive some convenient expressions. First, denoting that $\phi_h \equiv f(y/k) / \left(f'\left(f^{-1}(y/k)\right) f^{-1}(y/k)\right)$, (2.19) and (2.21) yield

$$\frac{Yu_c}{hv_h} = \frac{f\left(f^{-1}(y/k)\right)}{f'(f^{-1}(y/k))\, f^{-1}(y/k)} = \phi_h. \tag{2.23}$$

Second, (2.20) and (2.22) imply that

$$1 - \phi_h^{-1} = 1 - \frac{\frac{h}{K} f'\left(\frac{h}{K}\right)}{f\left(\frac{h}{K}\right)} = \frac{\rho}{f\left(\frac{h}{K}\right)} = \frac{\rho K}{f\left(\frac{h}{K}\right) K} = \rho k = k\left(\beta^{-1} - (1-\delta)\right). \tag{2.24}$$

2.3 Log-linearized system

In this section, we log-linearize the structural equations derived in section 2 around the zero-inflation steady state.

2.3.1 Market-clearing conditions

The market-clearing conditions (2.10) and (2.11) and the identity (2.5) are linearized in the following way[*3].

$$\hat{Y}_t = \hat{C}_t + \hat{I}_t + \hat{G}_t, \tag{2.25}$$

$$\hat{K}_t = \int_0^1 \hat{K}_t(j)dj,$$

$$\hat{I}_t = k\left\{\hat{K}_{t+1} - (1-\delta)\hat{K}_t\right\}, \tag{2.26}$$

where $\hat{Y}_t \equiv (Y_t - Y)/Y$, $\hat{C}_t \equiv (C_t - C)/Y$, $\hat{I}_t \equiv (I_t - I)/Y$, $\hat{K}_t \equiv (K_t - K)/K$ and $\hat{G}_t \equiv G_t/Y$[*4].

2.3.2 Household behavior

Log-linearizing equations (2.1) to (2.4) and noticing the market-clearing condition for labor, (2.12), we obtain the following approximation.

$$-\sigma^{-1}\left(\hat{C}_t - \bar{c}_t\right) = \hat{\lambda}_t, \tag{2.27}$$

$$\nu\hat{h}_t(j) - \nu\bar{h}_t = \hat{w}_t(j) + \hat{\lambda}_t,$$

$$\hat{Q}_{t,t+1} + \hat{\Pi}_{t+1} = \hat{\lambda}_{t+1} - \hat{\lambda}_t,$$

$$\hat{\lambda}_t = \hat{R}_t - E_t\Pi_{t+1} + E_t\hat{\lambda}_{t+1}, \tag{2.28}$$

where $\sigma^{-1} \equiv -\frac{u_{cc}Y}{u_c}$, $\bar{c}_t \equiv -\frac{u_{c\xi}}{u_{cc}Y}\xi_t$, $\nu \equiv \frac{v_{hh}h}{v_h}$, $\bar{h}_t \equiv -\frac{v_{h\zeta}}{v_{hh}h}\zeta_t$. From (2.25),(2.26), (2.27) and (2.28), we obtain the following IS relation.

$$\hat{Y}_t - k(2-\delta)\hat{K}_{t+1} + k(1-\delta)\hat{K}_t + kE_t\hat{K}_{t+2} - E_t\hat{Y}_{t+1} + \sigma\hat{R}_t - \sigma E_t\hat{\Pi}_{t+1} - g_t + E_t g_{t+1} = 0, \tag{2.29}$$

where $g_t \equiv \hat{G}_t + \bar{c}_t$.

2.3.3 Firm behavior

For type II firms, we have (2.8) from the cost minimization problem and the first-order condition (2.9). Two log-linearized expressions can be obtained from (2.8), the former being the real marginal cost function and the latter being the relation between marginal products of labor and capital.

$$\hat{S}_t(j) = \omega\left(\hat{y}_t(j) - \hat{k}_t(j)\right) + \nu\hat{k}_t(j) - \hat{\lambda}_t - \omega q_t, \tag{2.30}$$

$$\hat{\rho}_t = \rho_y\hat{y}_t(j) - \rho_k\hat{k}_t(j) - \hat{\lambda}_t - \omega q_t, \tag{2.31}$$

[*3] To simplify notation, we hereafter do not distinguish labor supply and demand.
[*4] We follow this notation in Woodford(2005).

where $\omega q_t \equiv (1+\nu)a_t + \nu \overline{h_t}$[*5]. Defining the average of log-linearized variables in the production sector as $\hat{X}_t \equiv \int_0^1 \hat{X}_t(j)dj$ for a variable X,[*6] the respective averages of (2.30) and (2.31) are simply

$$\hat{S}_t = \omega\left(\hat{Y}_t - \hat{K}_t\right) + \nu\hat{K}_t - \hat{\lambda}_t - \omega q_t, \tag{2.32}$$
$$\hat{\rho}_t = \rho_y\hat{Y}_t - \rho_k\hat{K}_t - \hat{\lambda}_t - \omega q_t. \tag{2.33}$$

Then, subtracting the averages from their original equations, we obtain

$$\begin{aligned}
&\hat{S}_t(j) = \hat{S}_t + \omega\left(\hat{y}_t(j) - \hat{Y}_t\right) - (\omega - \nu)\left(\hat{k}_t(j) - \hat{K}_t\right), \\
&\rho_y\left(\hat{y}_t(j) - \hat{Y}_t\right) = \rho_k\left(\hat{k}_t(j) - \hat{K}_t\right).
\end{aligned} \tag{2.34}$$

Hence,

$$\hat{S}_t(j) = \hat{S}_t + \frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\left(\hat{y}_t(j) - \hat{Y}_t\right). \tag{2.35}$$

Since the demand for firm j is $y_t(j) = (p_t(j)/P_t)^{-\theta}\, Y_t$, we can derive the following relationship between price and output deviations from average.

$$\hat{y}_t(j) - \hat{Y}_t = -\theta\left(\hat{p}_t(j) - \hat{P}_t\right). \tag{2.36}$$

Turning to the first-order condition (2.9) and log-linearizing, we obtain

$$\sum_{k=0}^{\infty} E_t\,(\alpha\beta)^k\left(\hat{p}_t(j) - \hat{P}_{t+k} - \hat{S}_{t+k}(j)\right) = 0.$$

Then, substitute (2.35) and (2.36) into the above equation to obtain

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty} E_t\,(\alpha\beta)^k\left(\hat{p}_t(j) - \hat{P}_{t+k} - \hat{S}_{t+k} + \theta\frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\left(\hat{p}_t(j) - \hat{P}_{t+k}\right)\right) \\
&= \sum_{k=0}^{\infty} E_t\,(\alpha\beta)^k\left(\psi\tilde{p}_t(j) - \hat{S}_{t+k} - \psi\sum_{h=1}^{k}\hat{\Pi}_{t+h}\right),
\end{aligned}$$

where $\psi = 1 + \theta\frac{(\rho_y - \omega)\nu}{\rho_k}$ and $\tilde{p}_t(j) \equiv \hat{p}_t(j) - \hat{P}_t$. After some manipulation, we can solve out the optimal pricing rule of firm j as follows.

$$\tilde{p}_t^* = \frac{1-\alpha\beta}{\psi}\hat{S}_t + \alpha\beta E_t\hat{\Pi}_{t+1} + \alpha\beta E_t\tilde{p}_{t+1}^*,$$

where $\tilde{p}_t^*$ denotes the optimal price in period t for all firms revising their prices. But because of the type of price rigidity that we assume, $\tilde{p}_t^* = \frac{\alpha}{(1-\alpha)}\hat{\Pi}_t$. Substituting this relationship into the optimal pricing formula, we obtain the standard New Keynesian Phillips curve.

$$\hat{\Pi}_t = \kappa\hat{S}_t + \beta E_t\hat{\Pi}_{t+1}, \tag{2.37}$$

[*5] $a_t \equiv \log A_t$.

[*6] Note that $Y_t = \left\{\int_0^1 \hat{y}_t^{\frac{\theta-1}{\theta}}(j)dj\right\}^{\frac{\theta}{\theta-1}}$ and its first-order approximation is $Y_t = \int_0^1 \hat{y}_t(j)dj$.

where $\kappa = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha\psi}$. Combining (2.25),(2.26), (2.27), (2.32) and (2.37), we can express the Phillips curve in the following way.

$$\hat{\Pi}_t = \kappa\left(\omega + \sigma^{-1}\right)\hat{Y}_t - \kappa\sigma^{-1}k\hat{K}_{t+1} + \kappa\left(\sigma^{-1}k\left(1-\delta\right) - \left(\omega - \nu\right)\right)\hat{K}_t - \kappa\sigma^{-1}g_t - \kappa\omega q_t + \beta E_t\hat{\Pi}_{t+1} \quad (2.38)$$

For the type I firm, the capital-stock dynamics is given by log-linearizing (2.6).

$$\hat{\lambda}_t + \epsilon_\psi\left(\hat{K}_{t+1} - \hat{K}_t\right) = E_t\hat{\lambda}_{t+1} + \left[1 - \beta\left(1-\delta\right)\right]E_t\hat{\rho}_{t+1} + \beta\epsilon_\psi\left(E_t\hat{K}_{t+2} - \hat{K}_{t+1}\right). \quad (2.39)$$

Repeating the same procedures to substitute out λ and using (2.33), we can write (2.39) as below.

$$\begin{aligned}0 =& \hat{Y}_t + \left[k\left(1-\delta\right) + \sigma\epsilon_\psi\right]\hat{K}_t - \left[k + \sigma\epsilon_\psi\left(1+\beta\right) + \beta k\left(1-\delta\right)^2 + \sigma\rho_k\left\{1 - \beta\left(1-\delta\right)\right\}\right]\hat{K}_{t+1} \\ &+ \left[\sigma\rho_y\left\{1-\beta\left(1-\delta\right)\right\} - \beta\left(1-\delta\right)\right]E_t\hat{Y}_{t+1} + \left[\beta k\left(1-\delta\right) + \beta\sigma\epsilon_\psi\right]E_t\hat{K}_{t+2} \\ &- g_t + \beta\left(1-\delta\right)E_t g_{t+1} - \sigma\left[1-\beta\left(1-\delta\right)\right]\omega E_t q_{t+1}. \end{aligned} \quad (2.40)$$

2.4 Natural variables

In this section, we explain how the natural variables are defined and determined in the model with endogenous capital formation. Consider the following system of equations that are consistent with a hypothetical flexible-price economy that begins at time t.[*7] We denote any variable determined in the system of equations at time t as $\hat{z}^f_{t+j|t}$ for $\forall j > 0$. The time-t flexible-price economy is characterized by the following equations.

$$-\sigma^{-1}\hat{Y}^f_{t+j|t} + \sigma^{-1}k\hat{K}^f_{t+1+j|t} - \sigma^{-1}k\left(1-\delta\right)\hat{K}^f_{t+j|t} + \sigma^{-1}g_{t+j} = \hat{\lambda}^f_{t+j|t}, \quad (2.41)$$

$$\hat{\lambda}^f_{t+j|t} = \mathrm{E}_t\hat{\lambda}^f_{t+1+j|t} + \hat{r}^f_{t+j|t}, \quad (2.42)$$

$$\hat{\lambda}^f_{t+j|t} + \epsilon_\psi\left(\hat{K}^f_{t+1+j|t} - \hat{K}^f_{t+j|t}\right) = \mathrm{E}_t\hat{\lambda}^f_{t+1+j|t} + \left[1-\beta\left(1-\delta\right)\right]\mathrm{E}_t\hat{\rho}^f_{t+1+j|t} + \beta\epsilon_\psi\left(\hat{K}^f_{t+2+j|t} - \hat{K}^f_{t+1+j|t}\right), \quad (2.43)$$

$$0 = \omega\hat{Y}^f_{t+j|t} - \left(\omega - \nu\right)\hat{K}^f_{t+j|t} - \omega q_{t+j} - \lambda^f_{t+j|t}, \quad (2.44)$$

$$\hat{\rho}^f_{t+j|t} = \rho_y\hat{Y}^f_{t+j|t} - \rho_k\hat{K}^f_{t+j|t} - \hat{\lambda}^f_{t+j|t} - \omega q_{t+j}. \quad (2.45)$$

Equation (2.41) shows the determinants of the marginal utility of consumption. (2.42) comes from the Euler equation for the household and (2.43) is the optimality condition for capital stock accumulation. (2.44) is derived from the real marginal cost function in the flexible-price economy and (2.45) results from the cost minimization problem of the firm. Note that the determination of variables in the flexible-price economy starting at time t depends on the level of capital stock in the sticky-price economy, K_t. Thus, if the path of capital stock in the sticky-price economy after time t does not coincide with the path of $\left\{\hat{K}^f_{t+1+j|t}\right\}_{j=0}^{\infty}$, the flexible-price economy equilibrium starting from a period later than t is different from the path of time-t-flexible-price-economy equilibrium.

[*7] We call the flexible-price economy that starts at time t "time-t flexible-price economy".

The natural variables defined in Woodford(2003), $\hat{z}^n_{t+j}$, is equivalent to $\hat{z}^f_{t+j|t+j}$ ($\forall j \geq 0$) in the notation that we use. That is, it conditions each natural variable on the capital stock determined in the sticky-price economy in each period. On the other hand, Neiss and Nelson(2003) consider the flexible-price-economy equilibrium starting at a particular fixed-date as natural variables. If we consider natural levels of variables to be the central bank's targets to achieve, both definitions have their own merits. In the case of discretionary monetary policy, Woodford's definition is suitable since the central bank reoptimizes every period by taking the existing level of capital stock as given. In the case of commitment, on the other hand, the flexible-price-economy-equilibrium paths starting at the time of commitment is the appropriate central bank's target.

Whether the difference matters depends on the situation considered. As we can see from the shape of the loss function for the central bank in the next section, the distinction between the two is unnecessary in the present model when the zero-lower-bound of nominal rate of interest (hereafter referred to as the ZLB) is not an issue of concern. This is because the central bank is able to completely offset the effects of shocks represented as the natural rate of interest by controlling the nominal rate of interest. As a result, the equilibrium in the sticky-price economy will be identical to that of the flexible-price economy every period. If there is a positive probability that the ZLB will bind, however, the distinction will be important[*8].

For the convenience of discussion in later sections, we denote the gap between the sticky-price-economy equilibrim and the time-t flexible-price-economy equilibrium for any variable $\hat{z}_{t+j}$ as below.

$$\tilde{z}^*_{t+j|t} \equiv \hat{z}_{t+j} - \hat{z}^f_{t+j|t}$$

2.5 Utility-based loss function

In order to derive optimal monetary policies, an appropriate criterion for the central bank's optimization problem should be established. In this section, following the methods of Edge(2003) and Onatski and Williams(2004), we approximate the households' welfare function around the steady state up to second order. Households' welfare in this model can be defined as

$$W_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u\left(C_t; \xi_t\right) - \int_0^1 v\left(h_t\left(i\right); \zeta_t\right) di \right\}.$$

As a result of approximation, we obtain the central bank's loss function.[*9]

$$L_0 = E_0 \sum_{t=0}^{\infty} \beta^t \hat{L}_t + \text{t.i.p.} + O(3),$$

[*8] If there exists a cost-push shock or any other type of disturbance whose impact the central bank cannot offset, the distinction will also be important.

[*9] The derivation of loss function is discussed in Appendix A.

where

$$
\begin{aligned}
\hat{L}_t =& \left(\sigma^{-1} + \omega\right) \hat{Y}_t^2 + \sigma^{-1} \hat{I}_t^2 + \epsilon_\psi k \left(\hat{K}_{t+1} - \hat{K}_t\right)^2 + \rho_k k \left(\beta^{-1} - (1-\delta)\right) \hat{K}_t^2 - 2\sigma^{-1} \hat{Y}_t \hat{I}_t \\
& - 2\hat{Y}_t \left(\sigma^{-1} g_t + \omega q_t\right) - 2(\omega - \nu) \hat{Y}_t \hat{K}_t - 2k(1-\delta)\left(\sigma^{-1} g_t + \omega q_t\right) \hat{K}_t \\
& + 2k\left(\sigma^{-1} g_t \hat{K}_{t+1} + \beta^{-1} \omega q_t \hat{K}_t\right) + \left(1 + \frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\theta\right) \frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)} \hat{\Pi}_t^2,
\end{aligned} \tag{2.46}
$$

and t.i.p. are the terms independent of policy. The loss function can be further expressed in terms of deviation of each variable from time-0 flexible-price economy equilibrium path. That is,

$$
\begin{aligned}
\hat{L}_t =& \left(\sigma^{-1} + \omega\right)\left(\hat{Y}_t - \hat{Y}_{t|0}^f\right)^2 + \sigma^{-1}\left(\hat{I}_t - \hat{I}_{t|0}^f\right)^2 + \rho_k k\left(\beta^{-1} - (1-\delta)\right)\left(\hat{K}_t - \hat{K}_{t|0}^f\right)^2 \\
& + \epsilon_\psi k\left(\left(\hat{K}_{t+1} - \hat{K}_{t+1|0}^f\right) - \left(\hat{K}_t - \hat{K}_{t|0}^f\right)\right)^2 - 2\sigma^{-1}\left(\hat{Y}_t - \hat{Y}_{t|0}^f\right)\left(\hat{I}_t - \hat{I}_{t|0}^f\right) \\
& - 2(\omega - \nu)\left(\hat{Y}_t - \hat{Y}_{t|0}^f\right)\left(\hat{K}_t - \hat{K}_{t|0}^f\right) + \left(1 + \frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)}\hat{\Pi}_t^2 \\
=& \left(\sigma^{-1} + \omega\right)\tilde{Y}_{t|0}^{*2} + \sigma^{-1}\left(k\tilde{K}_{t+1|0}^* - k(1-\delta)\tilde{K}_{t|0}^*\right)^2 + \rho_k k\left(\beta^{-1} - (1-\delta)\right)\tilde{K}_{t|0}^{*2} \\
& + \epsilon_\psi k\left(\tilde{K}_{t+1|0}^* - \tilde{K}_{t|0}^*\right)^2 - 2\sigma^{-1}\tilde{Y}_{t|0}^*\left(k\tilde{K}_{t+1|0}^* - k(1-\delta)\tilde{K}_{t|0}^*\right) \\
& - 2(\omega - \nu)\tilde{Y}_{t|0}^*\tilde{K}_{t|0}^* + \left(1 + \frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)}\hat{\Pi}_t^2,
\end{aligned}
$$

Hence, the first-best outcome for the economy is to follow exactly the same paths as the time-0 flexible price economy achieves in equilibrium. If we can ignore the ZLB, this is actually what the optimal commitment and discretionary policies must achieve in the present model. To see this intuitively, transform the structural equations in the gap form:

$$
0 = \tilde{Y}_{t|0}^* - k(2-\delta)\tilde{K}_{t+1|0}^* + k(1-\delta)\tilde{K}_{t|0}^* + kE_t\tilde{K}_{t+2|0}^* - E_t\tilde{Y}_{t+1|0}^* + \sigma\left(\hat{R}_t - E_t\hat{\Pi}_{t+1} - \hat{r}_{t|0}^f\right), \tag{2.47}
$$

$$
\hat{\Pi}_t = \beta E_t\hat{\Pi}_{t+1} + \omega\left(\tilde{Y}_{t|0}^* - \tilde{K}_{t|0}^*\right) + \nu\tilde{K}_{t|0}^* + \sigma^{-1}\left(\tilde{Y}_{t|0}^* - k\tilde{K}_{t+1|0}^* + k(1-\delta)\tilde{K}_{t|0}^*\right),
$$

$$
\begin{aligned}
0 =& \tilde{Y}_{t|0}^* + \left[k(1-\delta) + \sigma\epsilon_\psi\right]\tilde{K}_{t|0}^* - \left[k + \sigma\epsilon_\psi(1+\beta) + \beta k(1-\delta)^2 + \sigma\rho_k\{1 - \beta(1-\delta)\}\right]\tilde{K}_{t+1|0}^* \\
& + \left[\sigma\rho_y\{1 - \beta(1-\delta)\} - \beta(1-\delta)\right]E_t\tilde{Y}_{t+1|0}^* + \left[\beta k(1-\delta) + \beta\sigma\epsilon_\psi\right]E_t\tilde{K}_{t+2|0}^*.
\end{aligned}
$$

Suppose that the central bank commits to set the policy rate at $\hat{R}_t = \hat{r}_{t|0}^f$ in (2.47) each period. This is consistent with all the gap variables and inflation rate being zero all the time, which is the first-best outcome. Thus the optimal commitment policy demands the central bank to attain the flexible-price-economy equilibrium starting at the initial period in the absence of ZLB. Discretionary central bank reoptimizes every period but will achieve the same outcome because the central bank governor in the initial period knows that leaving the capital stock at the level equivalent to $K_{1|0}^f$ is consistent with all the governors in the following periods setting the policy rates at the appropriate levels that realize the capital stock level at $\left\{\hat{K}_{t|0}^f\right\}_{t=2}^\infty$. The same argument applies to all the following periods. Of course, this argument no longer holds if the ZLB binds, which is the topic we address in this paper.

3 When does the ZLB bind?

In section 2.5, we observed that the optimal policies can achieve the first-best solution if the nominal interest rates never hit the ZLB. As Rogoff(1998) points out, however, there are reasons to question the possibility of negative natural rate of interest when it is endogenously determined. In the context of our model, the question is related to the variability of the capital stock. For a representative household with consumption smoothing motive, investment contributes to diminishing the volatility in the real interest rate through the consumption Euler equation. Here, we attempt to illustrate whether the ZLB matters for the central bank through a simple numerical exercise[*10]. Suppose there is an unexpected 3 percent rise in productivity at time 0, a_0, which follows an AR(1) process with the persistence parameter denoted by ρ_a[*11]. This shock generates negative expected productivity *growth* which lowers current and expected future natural rates of interest. Figure 1(a) presents impulse responses of the natural capital stock and the natural rate of interest in percentage deviation from the steady state for various values of ϵ_ψ when $\rho_a = 0.5$. It is observed that the larger the size of ϵ_ψ, the greater the responses of natural rate of interest. However, in this case, the natural rate of interest is lower than the ZLB only for high values of ϵ_ψ. Figure 1(b) presents the case when $\rho_a = 0$ for the same unexpected shock in productivity at time 0. In contrast to the previous case, the natural rate of interest hits the ZLB for lower values of ϵ_ψ. These two examples suggest that as long as a change in productivity stems from an unexpected shock in productivity, given that the magnitude of the shock is large enough, the size of ϵ_ψ and the statistical property of the shock play an important role in determining whether the natural rate of interest hits the ZLB. The optimal monetary policy when the natural rate of interest unexpectedly falls below the ZLB is discussed in Jung et al (2005) and Eggertsson and Woodford (2003).

The recent interests in the practical discussion of monetary policy include how the central bank should act against imminent danger of liquidity trap not necessarily implied by the current level of shocks. The simple numerical exercise above implies that if a sharp drop in productivity is expected to occur in the near future, then the central bank may not be free from the ZLB unless the adjustment cost of capital stock is very small. This calls for the central bank to act pre-emptively to minimize the damage from a liquidity trap, which we shall discuss in the following sections.

[*10] For all the numerical exercises in this paper, we employ the following parameter values that were taken from Woodford(2003, 2005): $\alpha = 0.66$, $\beta = 0.99$, $\sigma^{-1} = 1$, $\nu = 0.11$, $\phi_h^{-1} = 0.75$, $\omega_p = 0.33$, $(\theta - 1)^{-1} = 0.15$, $\delta = 0.12/4 = 0.03$.

[*11] In the numerical exercises of this paper, we only consider productivity shocks and hold other shocks constant. Therefore, $g_t = 0$ for all t and $\omega q_t = (1 + \nu)\, a_t$.

4 Optimal monetary policy with the ZLB

4.1 Optimal policy under commitment

Commitment solution can be obtained by minimizing (2.46) subject to (2.29), (2.38), (2.40) and $\hat{R}_t \geq -\frac{1-\beta}{\beta}$. By taking derivatives with respect to $\hat{Y}_t$, $\hat{\Pi}_t$ and $\hat{K}_{t+1}$, first-order conditions can be obtained as follows.

$$
\begin{aligned}
0 &= \left(\sigma^{-1} + \omega\right)\hat{Y}_t - \left(\sigma^{-1}g_t + \omega q_t\right) - (\omega - \nu)\hat{K}_t - \sigma^{-1}k\left(\hat{K}_{t+1} - (1-\delta)\hat{K}_t\right) \\
&\quad + \phi_{1t} - \beta^{-1}\phi_{1,t-1} - \kappa\left(\omega + \sigma^{-1}\right)\phi_{2t} + \phi_{3t} + \beta^{-1}\left(\sigma\rho_y\left(1 - \beta(1-\delta)\right) - \beta(1-\delta)\right)\phi_{3,t-1} \qquad (4.1) \\
0 &= \Theta\hat{\Pi}_t - \beta^{-1}\sigma\phi_{1,t-1} + \phi_{2t} - \phi_{2,t-1} \qquad (4.2) \\
0 &= \sigma^{-1}k^2\left(\hat{K}_{t+1} - (1-\delta)\hat{K}_t\right) - \beta\sigma^{-1}k^2(1-\delta)\left(E_t\hat{K}_{t+2} - (1-\delta)\hat{K}_{t+1}\right) + \epsilon_\psi k\left(\hat{K}_{t+1} - \hat{K}_t\right) \\
&\quad - \beta\epsilon_\psi k\left(E_t\hat{K}_{t+2} - \hat{K}_{t+1}\right) + \rho_k k\left(1 - \beta(1-\delta)\right)\hat{K}_{t+1} - \sigma^{-1}k\hat{Y}_t + \beta\sigma^{-1}k(1-\delta)E_t\hat{Y}_{t+1} - \beta(\omega-\nu)E_t\hat{Y}_{t+1} \\
&\quad - \beta k(1-\delta)\left(\sigma^{-1}E_t g_{t+1} + \omega E_t q_{t+1}\right) + \sigma^{-1}kg_t + k\omega E_t q_{t+1} + \beta^{-1}k\phi_{1,t-1} - k(2-\delta)\phi_{1t} + \beta k(1-\delta)E_t\phi_{1,t+1} \\
&\quad + \kappa\sigma^{-1}k\phi_{2t} - \beta\kappa\left[\sigma^{-1}k(1-\delta) - \omega + \nu\right]E_t\phi_{2,t+1} + \left[k(1-\delta) + \sigma\epsilon_\psi\right]\phi_{3,t-1} \\
&\quad - \left[k + \sigma\epsilon_\psi(1+\beta) + \beta k(1-\delta)^2 + \sigma\rho_k\left\{1 - \beta(1-\delta)\right\}\right]\phi_{3t} + \beta\left[k(1-\delta) + \sigma\epsilon_\psi\right]E_t\phi_{3,t+1} \qquad (4.3) \\
0 &= \phi_{1t}\left(\hat{R}_t + \frac{1-\beta}{\beta}\right) \qquad (4.4)
\end{aligned}
$$

where ϕ_{1t}, ϕ_{2t} and ϕ_{3t} are Lagrange multipliers associated with (2.29), (2.38) and (2.40), respectively, and $\Theta \equiv \left(1 + \frac{(\rho_y - \omega)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)}$. The Kuhn-Tucker condition requires that $\phi_{1t} > 0$ if and only if $\hat{R}_t > -\frac{1-\beta}{\beta}$. We rearrange (4.1), (4.2) and (4.3) in order to eliminate ϕ_{2t} and ϕ_{3t}, and obtain a first-order condition of the form

$$
\begin{aligned}
G(L)E_t\phi_{1,t} &= -H(L)\left(E_t\hat{\Pi}_t + \mu E_t\Delta\tilde{Y}^*_{t|0}\right) \\
&\equiv -H(L)\tilde{\Pi}_t, \qquad (4.5)
\end{aligned}
$$

where the lag polynomials are given by

$$
\begin{aligned}
G(L) &= 5.3411(1 - 0.5561L^{-1})(1 - 1.0748L)(1 - 0.9380L)(1 - 0.6316L), \\
H(L) &= 62.7678(1 - 0.9286L^{-1})(1 - 0.9380L), \\
\mu &= 0.1343,
\end{aligned}
$$

under the parameter values used in section 3 and $\epsilon_\psi = 3$.[*12] Similar to the fixed-capital-stock model of Jung et al. (2005) and Eggertsson and Woodford (2003a, b), the first-order condition (4.5) presents policy inertia responding to past economic performances. In addition, the optimal commitment policy responds to expectations of future deviation in endogenous variables, due to the presence of channel to affect future states via capital stock. But notice that capital-stock gaps do not appear in (4.5). This presumably reflects

[*12] Note that in deriving (4.5), we replace exogenous variables with the time-0-flexible-price-economy equilibrium path. The same computational procedure is used to transform the loss function into the gap form. See Appendix A for the details.

the fact that the central bank is consistent in its policy through time under commitment and there is no concern about possible future shifts in its target, which can be characterized as the deviation of capital stock from the first-best path.

4.2 Targeting rules to implement the commitment solution

To know more about central bank's behavior in the commitment solution, we derive targeting rules to implement it. First of all, we rewrite the first-order condition (4.5) in three different ways.*[13]

$$(1-1.7983L)(1-0.6316L)\phi_{1t} = -19.6637\left[\frac{1-0.9286L^{-1}}{1-0.9304L^{-1}}E_t\tilde{\Pi}_t\right] \tag{4.6}$$

$$(1-1.0748L)(1-0.6316L)\phi_{1t} = -11.7519\left[\frac{1-0.9286L^{-1}}{1-0.5561L^{-1}}E_t\tilde{\Pi}_t\right] \tag{4.7}$$

$$(1-0.6316L)\phi_{1t} = 10.9343\left[\frac{(1-0.9286L^{-1})L^{-1}}{(1-0.9304L^{-1})(1-0.5561L^{-1})}E_t\tilde{\Pi}_t\right] \tag{4.8}$$

These three equations have a similar structure in that terms with lag operators are all gathered on the left hand side, while terms with lead operators are all put together on the right hand side. Lag terms represent monetary policy inertia, while lead terms represent central bank's reaction to their inflation-forecasts, in which inflation is measured by $\tilde{\Pi}$. Generally speaking, as pointed out by Giannoni and Woodford (2002a, b), multiple targeting rules could be consistent with a single first-order condition. In our case, the above three equations are all consistent with the first-order condition (4.5), and it could be possible to obtain multiple targeting rules, corresponding to each of the above three equations, to implement the commitment solution.

An important criterion to choose one among these multiple targeting rules is that it should include a targeting rule for the fixed-capital model as a special case. This criterion is particularly important if one wants to compare a targeting rule derived in the variable-capital model with those obtained in the fixed-capital model, such as "history dependent price level targeting" in Eggertsson and Woodford (2003a, b) and "history dependent inflation targeting" in Iwamura et al. (2005). To see whether these three equations include a targeting rule for the fixed-capital model as a special case, we compute (4.6)-(4.8) again, but now under the assumption that investment adjustment costs are extremely large ($\epsilon_\psi = 100$).

$$(1-1.5119L)(1-0.6706L)\phi_{1t} = -11.5703\left[\frac{1-0.9789L^{-1}}{1-0.9789L^{-1}}E_t\tilde{\Pi}_t\right]$$

*[13] We divide the both sides of equation (4.5) by $1-0.9380L$ before obtaining these equations. In that sense, these equations should be seen as an approximation of equation (4.5).

$$(1 - 1.0216L)(1 - 0.6706L)\phi_{1t} = -7.8180\left[\frac{1 - 0.9789L^{-1}}{1 - 0.6614L^{-1}}E_t\tilde{\Pi}_t\right]$$

$$(1 - 0.6706L)\phi_{1t} = 7.6528\left[\frac{(1 - 0.9789L^{-1})L^{-1}}{(1 - 0.9789L^{-1})(1 - 0.6614L^{-1})}E_t\tilde{\Pi}_t\right]$$

It should be noted that the denominator and the numerator in the squared bracket on the right hand side of the first equation are canceled out, so that the expression in the squared bracket equals to $\tilde{\Pi}_t$, implying that the central bank does not pay any attention to inflation forecasts in this limiting case. This is exactly the property of a targeting rule for the fixed-capital model: the central bank never takes inflation forecasts into consideration, simply because there is no endogenous state variable that could be controlled by the central bank through monetary policy. In this sense, equation (4.6) includes a targeting rule for the fixed-capital model as a special case. In contrast, the second and third equations, each of which corresponds to (4.7) and (4.8), contain inflation forecasts even in this limiting case, implying that neither of equations (4.7) and (4.8) does *not* include a targeting rule for the fixed-capital model as a special case.

Given the above argument, we now concentrate on equation (4.6), and characterize a targeting rule corresponding to it. We first define central bank's inflation forecasts as of period t as

$$F_t(\tilde{\Pi}) \equiv \sum_{j=0}^{\infty}\Psi_j E_t\tilde{\Pi}_{t+j} = \frac{1 - 0.9286L^{-1}}{1 - 0.9304L^{-1}}E_t\tilde{\Pi}_t, \tag{4.9}$$

where $F_t(z)$ denotes a linear combination of forecasts of the variable z at various future horizons, with weights $\{\Psi_j\}$ being normalized by $\Psi_0 = 1$. Closely following a procedure adopted by Eggertsson and Woodford (2003a, b) and Iwamura et al. (2005), we denote a target for inflation forecast $F_t(\tilde{\Pi})$ by Π_t^{TAR}. We also denote the target shortfall by Δ_t^{Π} ($\Delta_t^{\Pi} \equiv \Pi_t^{TAR} - F_t(\tilde{\Pi})$). Given these definitions, we substitute $\phi_{1t} = 19.6637\Delta_t^{\Pi}$ and (4.9) into (4.6) to obtain a target updating rule of the form

$$\Pi_{t+1}^{TAR} = 2.4299\Delta_t^{\Pi} - 1.1358\Delta_{t-1}^{\Pi}. \tag{4.10}$$

Consider the following targeting rule. The inflation target for period 0 is set at zero ($\Pi_0^{TAR} = 0$), and the targets for the subsequent periods are determined by equation (4.10). The central bank chooses, in each period t, the level of the overnight interest rate in each period so as to achieve the target criterion

$$F_t(\tilde{\Pi}) = \Pi_t^{TAR}. \tag{4.11}$$

if it is possible. If it is not possible because of the zero lower bound on nominal interest rates, the central bank simply sets the overnight interest rate at zero. If the central bank successfully shoots the target in each period starting from period 0, then Δ_t^{Π} is always zero, therefore the target in each period never deviates from zero. This can be seen as a standard inflation targeting. However, if the central bank fails to achieve the target due to the zero bound constraint, Δ_t^{Π} takes a positive value, and consequently the predetermined target for the next period becomes higher than zero. Given a higher target, the central bank must adopt an easier monetary policy (probably, a zero interest rate policy) in the near future. This raises inflation expectations further in the liquidity trap, which is exactly needed to escape from the trap.[*14]

The targeting rule characterized by (4.10) and (4.11) can be interpreted as "history-dependent inflation-forecast targeting". First, it has the feature of history dependence represented by lagged target shortfalls, Δ_t^{Π} and Δ_{t-1}^{Π}, in equation (4.10). This is exactly the same property observed by Eggertsson and Woodford (2003a, b) and Iwamura et al. (2005) in the setting of fixed-capital model. Second, not only the inflation rate in the current period, but also forecasts of the inflation rate in the future periods enter the target criterion (4.12). This is in sharp contrast with the targeting rules derived in the fixed capital model, in which only current inflation (or current price-level) comes in. Adjustments in current overnight interest rates have no influences on future inflation rates in the setting of fixed capital model, so that the central bank needs not pay any attention to future inflation. In the setting of variable capital, however, since changes in current overnight interest rates could affect future inflation rates through changes in capital stock, the central bank needs to take into account future developments in inflation.

Inflation-forecast targeting could be interpreted as central bank's preemptive actions against future inflation or deflation. It should be emphasized that such a preemptive action is completely different from the central bank's pre-shock behavior extensively studied by Adam and Billi (2004a, b) and Nakov (2005) among others in the setting of fixed-capital model. The preemptive action these papers have focused on is nothing but a central bank's policy response to a decline in the *current* inflation rate, which occurs because private agents anticipate the possibility of hitting the zero lower bound in the future and thereby adjust their inflation expectations and spending. To see how the preemptive action in the variable capital model differs from the one studied by the previous papers, it must be sufficient to recall that the target criterion (4.11) converges to

$$\tilde{\Pi}_t = \Pi_t^{TAR}$$

if investment adjustment costs are extremely large (ϵ_ψ takes an extremely large value); namely, inflation forecasts disappear when investment adjustment costs are extremely large. In the fixed capital model, including the one studied by Adam and Billi (2004a, b) and Nakov (2005), private agents behave in a forward-looking manner but the central bank does not do so. The central bank's forward-looking behavior

[*14] Price-level targeting to implement the commitment solution can be derived in a similar way. We denote a price-level forecast as $F_t(\tilde{P}) \equiv \frac{1-0.9286L^{-1}}{1-0.9304L^{-1}} E_t \tilde{P}_t$, where $\tilde{P}_t$ is defined by $\tilde{P}_t \equiv \hat{P}_t + 0.1343\tilde{Y}^*_{t|0}$. Denote the target shortfall as $\Delta_t^P \equiv P_t^{TAR} - F_t(\tilde{P})$. Then, substituting $\phi_{1t} = 19.6637\Delta_t^P$ into (4.5) yields a target updating rule: $P_t^{TAR} = P_{t-1}^{TAR} + 2.4299\Delta_{t-1}^P - 1.1358\Delta_{t-2}^P$. Note that the target criterion, $F_t(\tilde{P}) = P_t^{TAR}$, involves price-level forecasts rather than inflation forecasts. This could be seen as an extended version of the history-dependent price-level targeting, derived by Eggertsson and Woodford (2003a, b), to the setting of variable capital.

is entirely absent, simply because the current central bank governor has no weapon to fight against future inflation/deflation.

To evaluate a quantitative importance of central bank's preemptive action, Figure 2 shows the relative importance of forecasts for $\tilde{\Pi}_{t+j}$ at future horizons, for different values of ϵ_ψ. Under the standard value of ϵ_ψ ($\epsilon_\psi = 3$), the relative weight on one-period and two-period ahead forecasts are 0.0019 and 0.0017, respectively, indicating that the weights are extremely small with the exception of very small ϵ_ψ (i.e., when investment adjustment costs are very small). Indeed, the mean future horizon of these forecasts, which is defined by

$$\sum_{j\geq 1} \Psi_j j / \sum_{j\geq 1} \Psi_j,$$

is equal to 1.3725 quarters in the case of $\epsilon_\psi = 3$. These results indicate that central bank's preemptive action indeed exists in the targeting rule of the commitment solution, but it does not play an important role at least quantitatively.

4.3 A Non-inertial Policy

The analysis in the previous subsection indicates that a central bank governor with sophisticated commitment technology is able to reduce losses by making use of policy inertia. This might be a reason why a preemptive action does not play an important role, at least quantitatively, in the targeting rule to implement the commitment solution. If this is the case, we might be able to observe a stronger preemptive action in the behavior of a central bank without such a sophisticated commitment technology. Based on this idea, this subsection derives a targeting rule under the assumption that the central bank conducts policy in a discretionary manner, and compares it with the targeting rule obtained in the previous subsection.

Discretionary policymaking is defined as a process that presumes period-by-period reoptimization involving each period's start-up conditions. Specifically, we consider a central bank that reoptimizes in each period independently of past events. To express this idea of policy "resetting", we assume that a central bank governor takes a restriction of the form

$$0 = \phi_{1,t-1} = \phi_{2,t-1} = \phi_{3,t-1} \tag{4.12}$$

into consideration when he reoptimizes in period t. As we saw in the previous subsection, it is possible for a central bank governor in period t to improve economic welfare by manipulating private-sector's expectations about future inflation and outputs. However, such a management of expectations is worthless to thegovernor we are now considering, because he has a perception that it never affects economic welfare. The governor perceives in period t, for example, that the lagged Lagrange multiplier associated with the IS equation, $\phi_{1,t-1}$, is zero; therefore, the IS equation is not binding in period $t-1$; consequently, any change in $E_{t-1}\hat{Y}_t$

has no impacts on central bank losses. In this sense, the restriction (4.12) can be interpreted as representing the governor's perception of "bygones-be-bygones".

McCallum and Nelson (2000) studies a targeting rule in a model in which the AS equation with inflation persistence is assumed, and thus π_{t-1} plays a role of endogenous state variable in decision making in period t. In their discussion of discretionary optimization, they make an interesting statement that there are two different concepts in discretion: the one obtained by a dynamic programming approach (i.e., optimal discretionary solution) and the other obtained under the restriction that the Lagrange multiplier in period $t-1$ equals to zero, as in (4.12).

Discretionary policymaking with (4.12) has the following features. First, the central bank governor does not pay attention to the channel through which a change in overnight interest rate in period t affects inflation and outputs in period $t+1$ through a change in capital stock at the end of period t. In particular, the central governor ignores the channel through which private-sector's expectations in period t, including $E_t\hat{Y}_{t+1}$, $E_t\hat{\pi}_{t+1}$ and $E_t\hat{K}_{t+2}$, are altered through a change in capital stock at the end of period t, $\hat{K}_{t+1}$. Ignoring this channel obviously deteriorates economic welfare (although it is perceived to be costless to the governor), implying that the non-inertial solution (i.e., discretionary policymaking with (4.12)) leads to a lower economic welfare as compared with the optimal discretionary solution, not to speak of the commitment solution.[*15] Second, the central bank governor does his best in each period so as to close gaps, i.e., the deviation of each variable from its natural rate counterpart that is conditioned on the current level of capital stock (Woodford (2003)), including $\tilde{Y}_t^*$, $\tilde{K}_{t+2}^*$, and $\hat{\Pi}_t$. To see this, we simply rewrite period $t+j$ loss in the gap form

$$\begin{aligned}\hat{L}_{t+j} = &\left(\sigma^{-1}+\omega\right)\tilde{Y}^{*2}_{t+j|t} + \sigma^{-1}\left(k\tilde{K}^*_{t+j+1|t} - k\left(1-\delta\right)\tilde{K}^*_{t+j|t}\right)^2 + \rho_k k\left(\beta^{-1}-\left(1-\delta\right)\right)\tilde{K}^{*2}_{t+j|t}\\ &+\epsilon_\psi k\left(\tilde{K}^*_{t+j+1|t} - \tilde{K}^*_{t+j|t}\right)^2 - 2\sigma^{-1}\tilde{Y}^*_{t+j|t}\left(k\tilde{K}^*_{t+j+1|t} - k\left(1-\delta\right)\tilde{K}^*_{t+j|t}\right)\\ &-2\left(\omega-\nu\right)\tilde{Y}^*_{t+j|t}\tilde{K}^*_{t+j|t} + \left(1+\frac{\left(\rho_y-\omega\right)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{\left(1-\alpha\right)\left(1-\alpha\beta\right)}\hat{\Pi}^2_{t+j}\end{aligned}$$

and the three structural equations again in the gap form.

$$0 = \tilde{Y}^*_{t+j|t} - k\left(2-\delta\right)\tilde{K}^*_{t+j+1|t} + k\left(1-\delta\right)\tilde{K}^*_{t+j|t} + kE_t\tilde{K}^*_{t+j+2|t} - E_t\tilde{Y}^*_{t+j+1|t} + \sigma\left(\hat{R}_{t+j} - E_t\hat{\Pi}_{t+j+1} - \hat{r}^f_{t+j|t}\right)$$

$$\hat{\Pi}_{t+j} = \beta E_t\hat{\Pi}_{t+j+1} + \omega\left(\tilde{Y}^*_{t+j|t} - \tilde{K}^*_{t+j|t}\right) + \nu\tilde{K}^*_{t+j|t} + \sigma^{-1}\left(\tilde{Y}^*_{t+j|t} - k\tilde{K}^*_{t+j+1|t} + k\left(1-\delta\right)\tilde{K}^*_{t+j|t}\right)$$

$$\begin{aligned}0 = &\tilde{Y}^*_{t+j|t} + \left[k\left(1-\delta\right)+\sigma\epsilon_\psi\right]\tilde{K}^*_{t+j|t} - \left[k+\sigma\epsilon_\phi\left(1+\beta\right)+\beta k\left(1-\delta\right)^2 + \sigma\rho_k\left\{1-\beta\left(1-\delta\right)\right\}\right]\tilde{K}^*_{t+j+1|t}\\ &+\left[\sigma\rho_y\left\{1-\beta\left(1-\delta\right)\right\} - \beta\left(1-\delta\right)\right]E_t\tilde{Y}^*_{t+j+1|t} + \left[\beta k\left(1-\delta\right)+\beta\sigma\epsilon_\psi\right]E_t\tilde{K}^*_{t+j+2|t}\end{aligned}$$

[*15] Note that the optimal discretionary solution has a feature of policy inertia since a central bank manipulates private-sector's expectations through a change in capital stock. The main purpose of our analysis in this subsection is to derive a targeting rule for policy making without inertia, and compare it with the one obtained in the previous subsection. The non-inertial solution is more suitable to that purpose than the optimal discretionary solution.

These two features of the non-inertial solution imply its suboptimality, but a casual observation seems to indicate that central banking practice is more close to, and well approximated by the non-inertial solution: there are many central banks that do not pay much attention to the policy transmission channel via capital stock; there are many central banks that mainly focus on closing inflation and output gaps, which are conditioned on the current level of capital stock.

The targeting rule for the non-inertial solution can be characterized as follows. We first substitute (4.12) into (4.1), (4.2), and (4.3), and then eliminate ϕ_2 and ϕ_3 using those equations to obtain a difference equation with respect to ϕ_{1t}.

$$\begin{aligned}
&\left[k(1-\delta) - \sigma\epsilon_\psi(1+\beta) - \beta k(1-\delta)^2 - \sigma\rho_k\{1-\beta(1-\delta)\}\right]\phi_{1,t} + \beta\sigma\epsilon_\psi E_t\phi_{1,t+1} \\
&= \kappa\Theta\left[\left(\omega+\sigma^{-1}\right)\left(k+\sigma\epsilon_\psi(1+\beta)+\beta k(1-\delta)^2+\sigma\rho_k\{1-\beta(1-\delta)\}\right) - \sigma^{-1}k\right]\left(\hat{\Pi}_t + (\kappa\Theta)^{-1}\hat{Y}_t\right) \\
&\quad - \beta\left[k(1-\delta)+\sigma\epsilon_\psi\right]\left[\sigma^{-1}k(1-\delta)-(\omega-\nu)\right]\hat{K}_{t+1} \\
&\quad + \left[\left(k+\sigma\epsilon_\psi(1+\beta)+\beta k(1-\delta)^2+\sigma\rho_k\{1-\beta(1-\delta)\}\right)\left(\sigma^{-1}k(1-\delta)-(\omega-\nu)\right) - \sigma^{-1}k^2(1-\delta) - \epsilon_\psi k\right]\hat{K}_t \\
&\quad + \beta\kappa\Theta\left[\sigma^{-1}k(1-\delta)-(\omega-\nu)-\left(\omega+\sigma^{-1}\right)\left\{k(1-\delta)+\sigma\epsilon_\psi\right\}\right]\left(E_t\hat{\Pi}_{t+1} + (\kappa\Theta)^{-1}E_t\hat{Y}_{t+1}\right) \\
&\quad + k\left(\sigma^{-1}g_t - \sigma^{-1}E_t g_{t+1}\right) - \left[k+\sigma\epsilon_\psi(1+\beta)+\beta k(1-\delta)^2+\sigma\rho_k\{1-\beta(1-\delta)\}\right]\left(\sigma^{-1}g_t+\omega q_t\right) \\
&\quad + \left[k+\beta\sigma\epsilon_\psi\right]\left(\sigma^{-1}E_t g_{t+1} + \omega E_t q_{t+1}\right). \qquad (4.13)
\end{aligned}$$

Furthermore, using the following relationship, we replace the exogenous shocks with the variables determined in the time-t flexible-price economy.

$$\sigma^{-1}g_t + \omega q_t = \left(\omega+\sigma^{-1}\right)\hat{Y}^f_{t|t} - \sigma^{-1}k\hat{K}^f_{t+1|t} + \sigma^{-1}k(1-\delta)\hat{K}_t - (\omega-\nu)\hat{K}_t \qquad (4.14)$$

$$\sigma^{-1}E_t g_{t+1} + \omega E_t q_{t+1} = \left(\omega+\sigma^{-1}\right)E_t\hat{Y}^f_{t+1|t} - \sigma^{-1}kE_t\hat{K}^f_{t+2|t} + \sigma^{-1}k(1-\delta)\hat{K}^f_{t+1|t} - (\omega-\nu)\hat{K}^f_{t+1|t} \qquad (4.15)$$

$$\begin{aligned}
\sigma^{-1}g_t - \sigma^{-1}E_t g_{t+1} &= \sigma^{-1}\hat{Y}^f_{t|t} - \sigma^{-1}k\left(\hat{K}^f_{t+1|t} - (1-\delta)\hat{K}_t\right) - \epsilon_\psi\left(\hat{K}^f_{t+1|t} - \hat{K}_t\right) + \beta\epsilon_\psi\left(E_t\hat{K}^f_{t+2|t} - \hat{K}^f_{t+1|t}\right) \\
&\quad + \left[(1-\beta(1-\delta))\left(\rho_y - \omega - \sigma^{-1}\right) - \sigma^{-1}\beta(1-\delta)\right]E_t\hat{Y}^f_{t+1|t} \\
&\quad + \sigma^{-1}k\left(E_t\hat{K}^f_{t+2|t} - (1-\delta)\hat{K}^f_{t+1|t}\right) + (1-\beta(1-\delta))(\omega-\nu-\rho_k)\hat{K}^f_{t+1|t} \qquad (4.16)
\end{aligned}$$

Finally, by substituting (4.14), (4.15) and (4.16) into (4.13), we can express the first-order condition in a much simpler way.

$$\Gamma\phi_{1t} + \beta\sigma\epsilon_\psi E_t\phi_{1,t+1} = \Lambda_1\left(\hat{\Pi}_t + (\kappa\Theta)^{-1}\tilde{Y}^*_{t|t}\right) + \Lambda_2\left(E_t\hat{\Pi}_{t+1} + (\kappa\Theta)^{-1}E_t\tilde{Y}^*_{t+1|t}\right) + \Lambda_3\tilde{K}^*_{t+1|t}, \qquad (4.17)$$

where

$$\begin{aligned}
\Gamma &\equiv k(1-\delta) - \sigma\epsilon_\psi(1+\beta) - \beta k(1-\delta)^2 - \sigma\rho_k\{1-\beta(1-\delta)\}, \\
\Lambda_1 &\equiv \kappa\Theta\left[\left(\omega+\sigma^{-1}\right)\left(k+\sigma\epsilon_\psi(1+\beta)+\beta k(1-\delta)^2+\sigma\rho_k\{1-\beta(1-\delta)\}\right) - \sigma^{-1}k\right], \\
\Lambda_2 &\equiv \beta\kappa\Theta\left[\sigma^{-1}k(1-\delta)-(\omega-\nu)-\left(\omega+\sigma^{-1}\right)\left\{k(1-\delta)+\sigma\epsilon_\psi\right\}\right], \\
\Lambda_3 &\equiv -\beta\left\{k(1-\delta)+\sigma\epsilon_\phi\right\}\left\{\sigma^{-1}k(1-\delta)-(\omega-\nu)\right\}.
\end{aligned}$$

A simple comparison between (4.17) and the corresponding condition in the commitment solution, (4.5), shows the following features of the non-inertial solution. First, lagged Lagrange multipliers, $\phi_{1,t-1}$, $\phi_{1,t-2}$ and so on, are entirely absent in (4.17). Second, variables appearing in (4.17) are all expressed in the gap form, in which the corresponding natural rates are all conditioned on the current level of capital stock. This reflects the bygones-be-bygones property of the non-inertial solution. Third, we compute a numerical expression of (4.17) to evaluate quantitative importance of central bank's preemptive action in the non-inertial solution. The equation corresponding to (4.6) in the case of the commitment solution is

$$\phi_{1t} = -27.0970\frac{1-0.3719L^{-1}}{1-0.5135L^{-1}}E_t\left(\hat{\Pi}_t + 0.1343\tilde{Y}^*_{t|t}\right) + 8.9455\frac{1}{1-0.5135L^{-1}}E_t\tilde{K}^*_{t+1|t},$$

where the relative importance of forecasts of inflation and the output gap at various future horizons is represented by

$$\frac{1-0.3719L^{-1}}{1-0.5135L^{-1}}E_t\left(\hat{\Pi}_t + 0.1343\tilde{Y}^*_{t|t}\right).$$

Figure 3 shows the relative weights on forecasts at different future horizons for $\hat{\Pi}_t$ and $\tilde{Y}^*_{t|t}$, with the weight on the current period being normalized to unity. If we compare them with the corresponding figures in the case of the commitment solution, which is shown in Figure 2, we see that the relative weights on one-period and two-period ahead forecasts are now 0.1416 and 0.0727, respectively, under the standard value of ϵ_ψ ($\epsilon_\psi = 3$), and are much greater than the corresponding figures in Figure 2. Also, the mean future horizon of these forecasts is now 1.4634 quarters, indicating that preemptive action plays an important role, not only qualitatively but also quantitatively.

5 Numerical analysis

In this section, we have two purposes to present numerical exercises. First, we would like to see the equilibrium paths of endogenous variables under the optimal commitment, optimal discretionary and non-inertial policies for the baseline parameter specification in order to confirm the basic characteristics of these solutions discussed in the previous section. For simplicity, we take up the perfect foresight equilibrium in each case. The baseline setting assumes -3 percent change in the productivity from its steady-state level in periods 5,6 and 7, which are correctly anticipated in period 0.[*16] It is also assumed that the economy has been in the steady state before the news of the shock is revealed in period 0, and that other types of shocks do not disturb the economy. Under the baseline setting, the anticipated change in productivity from period 4 to 5 makes the natural rate of interest in period 4 to fall below the ZLB. The reason for focusing on the anticipated shock is to see how pre-emptive monetary easing may differ under different policy regiemes.

[*16] That is, $\hat{a}_5 = \hat{a}_6 = \hat{a}_7 = -0.03$.

The responses of various variables under the respective policies are shown in Figures 4 to 6. In each figure, the red lines in the panels for output, capital stock, consumption, inflation and nominal interest rate show paths of the hat variables in the log-linearized system. The blue and green lines superimposed in each panel present the movements of the associated time-0 flexible price equilibrium and natural levels in Woodford's definition, respectively.

Second, we alter the value of ϵ_ψ to analyze how the equilibrium nominal interest rate paths under the optimal commitment and discretionary policies would change. The discussion in section 3 indicates that a larger value of ϵ_ψ brings about a greater fall in the natural rate of interest. Our interest lies in how the length of periods with binding ZLB varies depending on the value of ϵ_ψ.

5.1 Solutions for the baseline case

Figures 4 to 6 show the numerical solutions of endogenous variables under the optimal commitment, optimal discretionary, and non-inertial policies, respectively. The baseline parameter values are taken from Woodford (2005) with $\epsilon_\psi = 3$.

Figure 4 confirms the analysis in section 3.1. The commitment policy reacts to the positive real interest rate gap in period 4 by the pre-emptive monetary easing in period 3, followed by the zero-interest-rate policy in period 4 and the inertial monetary easing in period 5.[*17] As a result, the equilibrium inflation is positive in period 5, which mitigates the deflationary pressure in the previous periods. Recalling the discussion in section 3.2 that the forward-lookingness of the target criterion is not very large under the parameter values we assume in this paper, monetary easing in period 3 is considered to be a central bank's response to potential deflationary pressure in that period, which results from the private sector's anticipation of the binding shock in period 4, rather than its own response to the binding shock in period 4. The figure also indicates the target paths for the central bank with commitment technology by the blue line; that is, the time-0 flexible price economy equilibrium. Overall, the commitment solutions closely track the paths that the central bank would wish to attain in the initial period if the ZLB constraint on the nominal interest rate did not exist.

The optimal discretionary solution is shown in Figure 5. With the lack of commitment technology, the central bank in period 5 will never wish to create positive inflation to counteract a substantial decline in the natural rate of interest in period 4. For the central bank after period 5 will set the nominal rate of interest equal to the natural rate of interest (in Woodford's definition) every period. This makes the

[*17] The numerical algorithm that we use is an extension of Jung et al. (2005), which can be applied for solving the perfect foresight equilibrium.

optimal discretionary solution inferior to the optimal commitment solution in terms of economic welfare evaluated at the initial-period. Therefore, the discretionary optimizer in the variable capital stock model must be forward-looking in deciding optimally the level of capital stock upon which the paths of endogenous variables and their target paths, which can be characterized as the time-t flexible price equilibrium for the period-t central bank, are conditioned. If $\hat{K}_{t+1}$ deviates from its natural level, $\hat{K}^{f}_{t+1|t}$, the central bank in the next period will have the target paths different from the current ones even when the shocks are deterministic. Because the central bank would wish the future target paths to be kept as close as possible to their own, the decision of the central bank in each period involves passing on the capital stock to the next generation, such that it helps minimize the discounted sum of future expected losses, $\sum_{j=1}^{\infty} \beta^j \hat{L}_{t+j}$. To illustrate how this works, let us compare the discretionary solution with the outcome obtained under a simple-minded Taylor rule of the form, $\hat{R}_t = \max\left\{\hat{r}^n_t + E_t\hat{\Pi}_{t+1}, 1 - 1/\beta\right\}$. This is the policy which attempts to set the real rate of interest at the natural rate whenever possible. Obviously, this policy is able to achieve full stabilization after (and including) period 5 so that the endogenous variables are always equivalent to the time-5 flexible-price equilibrium. However, as Figure 7 shows, the suboptimality of this policy not only leads to unfavorable outcomes prior to period 5 but also renders the paths of endogenous variables distant from the time-0 flexible-price-equilibrium paths even after period 5. A simple comparison between Figures 5 and 7 clearly shows that central bank's deliberate decision making about capital stock transferred to future generations significantly improves economic welfare. Note that this feature of the discretionary policy is entirely absent in the fixed-capital stock model where the future value function is independent from the current decision, or in the model with inflation inertia where the natural variables (in Woodford's definition) do not depend on the state variables determined in the sticky-price economy.

Finally, we make a comparison between the equilibrium paths under the non-inertial policy and the optimal discretionary policy. A comparison between Figures 5 and 6 shows that there are only minor differences between the two, which might suggest that the targeting rule to implement the non-inertial solution, which was derived in section 4, be a good approximation to the one to implement the optimal discretionary solution, at least under the baseline parameter values and the type of shock that we assume here.

5.2 Policy regime and the length of ZLB binding

In this section, we make alterations to the baseline setting in the previous analysis by changing the parameter values of ϵ_ψ in order to compare how the length of periods in which the central bank must set the nominal rate of interest at zero changes in equilibrium under the optimal commitment and discretionary solutions.

Figures 8 and 9 show the equilibrium paths for $\epsilon_\psi = 10$ and 30, respectively. The results indicate that, with the commitment technology to manage expectations, the central bank needs to confront with the ZLB only in period 4 even for a larger size of natural-rate-of-interest shock. In constrast, for the discretionary central bank, the number of periods with binding ZLB increases with the value of ϵ_ψ. When $\epsilon_\psi = 30$, the discretionary central bank begins the zero-interest-rate policy as early as period 2. A larger ϵ_ψ, which increases the size of shock in terms of the natural rate of interest, induces the discretionary central bank to fight against a greater expected deflation and thus to take pre-emptive action earlier to make its future expected loss as small as possible. This result confirms the strong stabilization property of the commitment policy analyzed in section 4.

6 Conclusion

This paper has characterized optimal monetary policy in an economy with the zero interest rate bound and endogenous capital formation, and obtained the following findings. First, we have shown that, given an adverse shock to productivity growth, the natural rate of interest is less likely to fall below zero in an economy with endogenous capital model than the one with fixed capital. This is a direct reflection of consumption smoothing through capital adjustments in an economy with perfectly flexible prices. However, our numerical exercises show that, unless investment adjustment costs are very close to zero, we still have a negative natural rate of interest for large and persistent shocks to productivity growth.

Second, the optimal commitment solution is characterized by a negative interest rate gap (i.e., real interest rate is lower than its natural rate counterpart) before and after the shock periods with negative natural rate of interest. The negative interest rate gap *after* the shock periods represents the history dependence property, which was emphasized by the existing studies such as Jung et al (2005) and Eggertsson and Woodford (2003). On the other hand, the negative interest rate gap *before* the shock periods emerges because the central bank seeks to increase capital stock just before the shock periods, so as to avoid a decline in capital stock after the shock periods, which would otherwise occur due to a substantial decline in investment during the shock periods.

The latter property may be seen as a central bank's preemptive action against future binding shocks, which is unique to endogenous capital model. It should be emphasized that such a preemptive action is completely different from the central bank's pre-shock behavior extensively studied by Adam and Billi (2004a, b) and Nakov (2005) among others in the setting of fixed-capital model. The preemptive action these papers have focused on is nothing but a central bank's policy response to a decline in the *current* inflation rate, which occurs because private agents anticipate the possibility of hitting the zero lower bound in the future and thereby adjust their inflation expectations and their spending. We also show that the targeting rule to implement the commitment solution is characterized by history-dependent inflation-forecast targeting.

Third, a central bank governor without sophisticated commitment technology tends to resort to preemptive action more than the one with it. The governor without commitment technology controls natural rates of consumption, output, and so on in the future periods, by changing capital stock today through monetary policy.

References

[1] Adam, Klaus, and Roberto Billi (2004a). "Optimal monetary policy under commitment with a zero bound on nominal interest rates." Mimeo.

[2] Adam, Klaus, and Roberto Billi (2004b). "Optimal monetary policy under discretion with a zero bound on nominal interest rates." Mimeo.

[3] Benigno, Pierpaolo and Michael Woodford (forthcoming). "Optimal Taxation in an RBC Model: A Linear-Quadratic Approximation." *Journal of Economic Dynamics and Control*.

[4] Edge, Rochelle (2003). "A Utility-Based Welfare Criterion in a Model with Endogenous Capital Accumulation." Mimeo.

[5] Eggertsson, Gauti, and Michael Woodford (2003a). "The Zero Interest-rate Bound and Optimal Monetary Policy." *Brookings Papers on Economic Activity* 1, 139-211.

[6] Eggertsson, Gauti, and Michael Woodford (2003b). "Optimal Monetary Policy in a Liquidity Trap." NBER Working Papers 9968.

[7] Iwamura, Mitsuru, Kudo, Takeshi, Watanabe, Tstutomu, 2005. Monetary and Fiscal Policy in a Liquidity Trap: The Japanese Experience 1999-2004, NBER Working Paper No. 11151

[8] Jung, Taehun, Yuki Teranishi, and Tsutomu Watanabe (2005). "Optimal Monetary Policy at the Zero-Interest-Rate Bound." *Journal of Money, Credit, and Banking* 37(5), 813-835.

[9] McCallum, Bennett and Edward Nelson (2000). "Timeless Perspective vs. Discretionary Monetary Policy in Forward-looking Models." NBER Working Paper 7915.

[10] Nakov, Anton (2005). "Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate." Mimeo.

[11] Neiss, Katherine and Edward Nelson (2003). "The Real Interest rate Gap as an Inflation Indicator." *Macroeconomic Dynamics* 7(2), 239-62.

[12] Onatski, Alexei and Noah Williams (2004). "Empirical and Policy Performance of a Forward-Looking Monetary Model." Mimeo.

[13] Rogoff, Kenneth (1998). "It's Baaack: Japan's Slump and the Return of the Liquidity Trap: Comment." *Brookings Papers on Economic Activity* 2, 194-99.

[14] Rotemberg, Julio and Michael Woodford (1997). "An Optimization-based Econometric Framework for the Evaluation of Monetary Policy." *NBER Macroeconomics Annual* 12, 297-346.

[15] Woodford, Michael(2005). "Firm-Specific Capital and the New-Keynesian Phillips Curve." *International Journal of Central Banking* 1(2), 1-46.

[16] Woodford, Michael(2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

Appendix A: Deriving the loss function

The derivation of loss function in this paper follows a method similar to Edge(2003) and Onatski and Williams(2004). Suppose $f(x_t;\eta_t)$ is the function to be approximated where x_t is an endogenous variable and η_t is a vector of exogenous variables. Let $x_t = g(z_t,\mu_t)$ be the function relating x_t to vectors of endogenous variables z_t and exogenous variables μ_t. A second-order Taylor approximation to $f(\cdot)$ with respect to z_t and η_t is

$$f(x_t;\epsilon_t) = f(x;0) + \left[f_z^T, f_\mu^T, f_\epsilon^T\right]\begin{bmatrix} dz_t \\ d\mu_t \\ d\epsilon_t \end{bmatrix} + \frac{1}{2}\left[dz_t^T, d\mu_t, d\epsilon_t^T\right]\begin{bmatrix} f_{zz^T} & f_{z\mu^T} & f_{z\epsilon^T} \\ f_{\mu z^T} & f_{\mu\mu^T} & f_{\mu\epsilon^T} \\ f_{\epsilon z^T} & f_{\epsilon\mu^T} & f_{\epsilon\epsilon^T} \end{bmatrix}\begin{bmatrix} dz_t \\ d\mu_t \\ d\epsilon_t \end{bmatrix} + O\left(\|\eta\|^3\right),$$

where T is transpose of a matrix and the subscripts, z and ϵ, represents first and second derivatives of f with respect to these vectors evaluated at the steady state. Letting $\Lambda(z)$ denote a diagonal matrix formed from a vector z and $\hat{z}_t$ be a vector whose elements are percentage deviation of elements of z_t from their steady state values,

$$dz_t = \Lambda(z)\left(\hat{z}_t + \frac{1}{2}\Lambda(\hat{z}_t)\hat{z}_t\right) + O(3).$$

Substituting this into the above equation and rearranging, we obtain

$$\begin{aligned} & f(x_t;\epsilon_t) \\ &= \left[f_x g_z^T \Lambda(z), f_\mu^T, f_\epsilon^T\right]\begin{bmatrix} \hat{z}_t \\ d\mu_t \\ d\epsilon_t \end{bmatrix} \\ &+ \frac{1}{2}\left[\hat{z}_t^T, d\mu_t^T, d\epsilon_t^T\right]\begin{bmatrix} \Lambda(z)\left(f_x g_{zz^T} + f_{xx} g_z g_z^T\right)\Lambda(z) + \Lambda(z)\Lambda(f_x g_z) & \Lambda(z) f_x g_\mu & \Lambda(z) f_{x\epsilon} g_z \\ f_x g_\mu^T \Lambda(z) & f_{\mu\mu^T} & f_{\mu\epsilon^T} \\ f_{x\epsilon} g_z^T \Lambda(z) & f_{\epsilon\mu^T} & f_{\epsilon\epsilon^T} \end{bmatrix}\begin{bmatrix} \hat{z}_t \\ d\mu_t \\ d\epsilon_t \end{bmatrix} \\ &+ \text{t.i.p.} + O\left(\|\epsilon\|^3\right). \end{aligned} \tag{A.1}$$

To approximate utility of consumption, let $f = u(C_t;\xi_t)$, $g = Y_t - I\left(\frac{\hat{K}_{t+1}}{\hat{K}_t}\right)\hat{K}_t - G_t$, $z_t = [Y_t, K_{t+1}, K_t]^T$, $\mu_t = G_t$, $\epsilon_t = \xi_t$, $\hat{z}_t = \left[\hat{Y}_t, \hat{K}_{t+1}, \hat{K}_t\right]^T$ and $z = [Y, K, K]^T$.[*18] We can compute the matrices of coefficients as

[*18] Note that we defined $\hat{G}_t \equiv \frac{G_t}{Y}$ rather than in percentage deviation from the steady state of itself. Hence, $dG_t = Y\hat{G}_t$. The formula needs an adjustment such that appropriate elements of the second-order coefficient matrix are multiplied by Y or Y^2.

below.

$$\Lambda(z)\, f_x g_z = Yu_c \begin{bmatrix} 1 \\ -k \\ (1-\delta)\,k \end{bmatrix}, \quad \Lambda(z)\, f_x g_{zz^T} \Lambda(z) = Yu_c \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\epsilon_\psi k & \epsilon_\psi k \\ 0 & \epsilon_\psi k & -\epsilon_\psi k \end{bmatrix},$$

$$\Lambda(z)\, f_{xx} g_z g_z^T \Lambda(z) = Yu_c \begin{bmatrix} -\sigma^{-1} & \sigma^{-1}k & -\sigma^{-1}k\,(1-\delta) \\ \sigma^{-1}k & -\sigma^{-1}k^2 & \sigma^{-1}k^2\,(1-\delta) \\ -\sigma^{-1}k\,(1-\delta) & \sigma^{-1}k^2\,(1-\delta) & -\sigma^{-1}k^2\,(1-\delta) \end{bmatrix},$$

$$\Lambda(z)\,\Lambda(f_x g_z) = Yu_c \begin{bmatrix} 1 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & (1-\delta)\,k \end{bmatrix}$$

If we substitute these matrices in (A.1), it follows that

$$\begin{aligned} u(C_t;\xi_t) =& Yu_c\left[\hat{Y}_t - k\hat{K}_{t+1} + k(1-\delta)\hat{K}_t\right] \\ &+ \frac{u_c Y}{2}\Big[-\left(\sigma^{-1}-1\right)\hat{Y}_t^2 - k\left(\sigma^{-1}k + 1 + \epsilon_\psi\right)\hat{K}_{t+1}^2 - k\left(\sigma^{-1}k(1-\delta)^2 - (1-\delta) + \epsilon_\psi\right)\hat{K}_t^2 \\ &+ 2\sigma^{-1}k\hat{Y}_t\hat{K}_{t+1} - 2\sigma^{-1}k(1-\delta)\hat{Y}_t\hat{K}_t + 2k\left(\sigma^{-1}k(1-\delta) + \epsilon_\psi\right)\hat{K}_{t+1}\hat{K}_t + 2\sigma^{-1}g_t\hat{Y}_t \\ &- 2\sigma^{-1}kg_t\hat{K}_{t+1} + 2\sigma^{-1}k(1-\delta)g_t\hat{K}_t\Big] + \text{t.i.p.} + O\left(\|\xi\|^3\right). \end{aligned} \tag{A.2}$$

Similarly, disutility of labor can be computed by letting $f = v(h_t(j);\zeta_t)$, $g = f^{-1}(y_t(j)/k_t(j))\,k_t(j)/A_t$, $z_t = [y_t(j), k_t(j), A_t]^T$ and $\epsilon_t = \zeta_t$. The coefficient matrices are

$$\Lambda(z)\, f_x g_z = hv_h \begin{bmatrix} \phi_h \\ (1-\phi_h) \\ -1 \end{bmatrix}, \quad \Lambda(z)\, f_x g_{zz^T} \Lambda(z) = hv_h \begin{bmatrix} \omega_p\phi_h & -\omega_p\phi_h & -\phi_h \\ -\omega_p\phi_h & \omega_p\phi_h & \phi_h - 1 \\ -\phi_h & \phi_h - 1 & 2 \end{bmatrix},$$

$$\Lambda(z)\, f_{xx} g_z g_z^T \Lambda(z) = h^2 v_{hh} \begin{bmatrix} \phi_h^2 & \phi_h(1-\phi_h) & -\phi \\ \phi_h(1-\phi_h) & (1-\phi_h)^2 & -(1-\phi_h) \\ -\phi_h & -(1-\phi_h) & 1 \end{bmatrix},$$

$$\Lambda(z)\,\Lambda(f_x g_z) = hv_h \begin{bmatrix} \phi_h & 0 & 0 \\ 0 & (1-\phi_h) & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Invoking the relationship between Yu_c and hv_h in equation (2.23),

$$\begin{aligned} \int_0^1 v(h_t(j);\zeta_t)\,dj =& Yu_c \int_0^1 \left[\hat{y}_t(j) + \frac{1-\phi_h}{\phi_h}\hat{k}_t(j)\right] dj \\ &+ \frac{Yu_c}{2}\int_0^1 \Big[(1+\omega)\hat{y}_t^2(j) + \frac{1-\phi_h}{\phi_h}(1-\rho_k)\hat{k}_t^2(j) - 2(\omega - \nu)\hat{y}_t(j)\hat{k}_t(j) \\ &- 2\omega q_t\left(\hat{y}_t(j) + \frac{1-\phi_h}{\phi_h}\hat{k}_t(j)\right)\Big] dj + \text{t.i.p.} + O\left(\|\zeta\|^3\right). \end{aligned} \tag{A.3}$$

Recall that we defined $Y_t \equiv \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}$ and $K_t \equiv \int_0^1 k_t(j)dj$. Second-oder approximation to these aggregators are

$$\hat{Y}_t = \mathrm{E}_j \hat{y}_t(j) + \frac{1-\theta^{-1}}{2} \mathrm{var}_j \hat{y}_t(j) + O(3),$$
$$\hat{K}_t = \mathrm{E}_j \hat{k}_t(j) + \frac{1}{2} \mathrm{var}_j \hat{k}_t(j) + O(3).$$

From (2.34), we have

$$\mathrm{var}_j \hat{k}_t(j) = \left(\frac{\rho_y}{\rho_k}\right)^2 \mathrm{var}_j \hat{y}_t(j),$$
$$\mathrm{cov}_j \left\{\hat{y}_t(j), \hat{k}_t(j)\right\} = \frac{\rho_y}{\rho_k} \mathrm{var}_j \hat{y}_t(j).$$

Substituting these expressions into (A.3) yields

$$\begin{aligned}
\int_0^1 v\left(h_t(j); \zeta_t\right) dj &= Yu_c \left[\hat{Y}_t + \frac{1-\phi_h}{\phi_h} \hat{K}_t\right] \\
&\quad + \frac{u_c Y}{2} \Bigg[(1+\omega)\hat{Y}_t^2 + \frac{1-\phi_h}{\phi_h}(1-\rho_k)\hat{K}_t^2 - 2(\omega-\nu)\hat{Y}_t\hat{K}_t \\
&\quad - 2\omega q_t \left(\hat{Y}_t + \frac{1-\phi_h}{\phi_h}\hat{K}_t\right) + \left(\omega + \theta^{-1}\right) \mathrm{var}_j \hat{y}_t(j) \\
&\quad + \frac{\phi_h - 1}{\phi_h} \rho_k \mathrm{var}_j \hat{k}_t(j) - 2(\omega-\nu)\, \mathrm{cov}_j \left\{\hat{y}_t(j), \hat{k}_t(j)\right\}\Bigg] \\
&= Yu_c \left[\hat{Y}_t + \frac{1-\phi_h}{\phi_h} \hat{K}_t\right] \\
&\quad + \frac{u_c Y}{2} \Bigg[(1+\omega)\hat{Y}_t^2 + \frac{1-\phi_h}{\phi_h}(1-\rho_k)\hat{K}_t^2 - 2(\omega-\nu)\hat{Y}_t\hat{K}_t \\
&\quad - 2\omega q_t \left(\hat{Y}_t + \frac{1-\phi_h}{\phi_h}\hat{K}_t\right) + \left(\omega + \theta^{-1} + \frac{\phi_h - 1}{\phi_h}\frac{\rho_y{}^2}{\rho_k} - 2(\omega-\nu)\frac{\rho_y}{\rho_k}\right) \mathrm{var}_j \hat{Y}_t\Bigg] \\
&= Yu_c \left[\hat{Y}_t + \frac{1-\phi_h}{\phi_h} \hat{K}_t\right] \\
&\quad + \frac{u_c Y}{2} \Bigg[(1+\omega)\hat{Y}_t^2 + \frac{1-\phi_h}{\phi_h}(1-\rho_k)\hat{K}_t^2 - 2(\omega-\nu)\hat{Y}_t\hat{K}_t \\
&\quad - 2\omega q_t \left(\hat{Y}_t + \frac{1-\phi_h}{\phi_h}\hat{K}_t\right) + \theta^{-1}\left(1 + \frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\theta\right) \mathrm{var}_j \hat{y}_t(j)\Bigg].
\end{aligned} \tag{A.4}$$

The final line in (A.4) used the fact that

$$\begin{aligned}
\omega + \theta^{-1} + \frac{\phi_h - 1}{\phi_h}\frac{\rho_y^2}{\rho_k} - 2(\omega-\nu)\frac{\rho_y}{\rho_k} &= \omega + \theta^{-1} + \frac{\rho_y}{\rho_k}\left(\frac{\phi_h - 1}{\phi_h}\rho_y - 2(\omega-\nu)\right) \\
&= \omega + \theta^{-1} + \frac{\rho_y}{\rho_k}\left(\frac{\phi_h - 1}{\phi_h}\left(\phi_h \nu + \frac{\phi_h}{\phi_h - 1}\omega_p\right) - 2(\omega-\nu)\right) \\
&= \omega + \theta^{-1} - \frac{\rho_y}{\rho_k}(\omega-\nu) \\
&= \theta^{-1}\left(1 + \frac{\left(\rho_y - \omega\right)\nu}{\rho_k}\theta\right) > 0.
\end{aligned}$$

This term is positive since $\rho_y - \omega = \phi_h \nu + \frac{\phi_h}{\phi_h - 1}\omega_p - \omega = \frac{\omega_p}{\phi_h - 1} > 0$. The output dispersion term in (A.4), $\mathrm{var}_j \hat{y}_t(j)$, can be replaced by inflation. To show this, log-linearize the demand function for an individual firm.

$$\hat{y}_t(j) - \hat{Y}_t = -\theta\left(\hat{p}_t(j) - \hat{P}_t\right) + O(2).$$

Squaring both sides and taking expections over j,

$$\mathrm{var}_j \hat{y}_t(j) = \theta^2 \mathrm{var}_j \hat{p}_t(j) + O(3).$$

As in Rotemberg and Woodford(1997), the price dispersion on the right-hand side can be further transformed as below.

$$\begin{aligned}
\mathrm{var}_j \hat{p}_t(j) &= \mathrm{var}_j \log p_t(j) \\
&= \mathrm{var}_j \left\{\log p_t(j) - \overline{p}_{t-1}\right\} \quad \left(\text{where } \overline{p}_t \equiv \mathrm{E}_j \left\{\log p_t(j)\right\}\right) \\
&= \mathrm{E}_j \left\{\log p_t(j) - \overline{p}_{t-1}\right\}^2 - \left(\mathrm{E}_j \log p_t(j) - \overline{p}_{t-1}\right)^2 \\
&= \alpha \mathrm{E}_j \left(\log p_{t-1}(j) - \overline{p}_{t-1}\right)^2 + (1-\alpha)\left(\log p_t^* - \overline{p}_{t-1}\right)^2 - \left(\Delta \overline{p}_t\right)^2, \qquad \text{(A.5)}
\end{aligned}$$

where $\log p_t^*$ is the optimal price for firms that are allowed to reoptimize at time t. Note that in this model, firms are able to hire capital input for production every period depending on the level of production in the same period. Thus, firms that are free from price rigidity must choose to set the same price. This also implies that

$$\mathrm{E}_j \log p_t(j) = \alpha \mathrm{E}_j \log p_{t-1}(j) + (1-\alpha)\log p_t^*.$$

Therefore, $\log p_t^* - \overline{p}_{t-1} = \frac{1}{1-\alpha}\Delta \overline{p}_t$ and (A.5) reduces to

$$\begin{aligned}
\mathrm{var}_j \hat{p}_t(j) &= \alpha \mathrm{var}_j \hat{p}_{t-1}(j) + \frac{\alpha}{1-\alpha}\left(\Delta \overline{p}_t\right)^2 \\
&= \alpha^{t+1} \mathrm{var}_j \hat{p}_{-1}(j) + \frac{\alpha}{1-\alpha}\sum_{h=0}^{t} \alpha^h \hat{\Pi}_{t-h}^2.
\end{aligned}$$

If we integrate this forward,

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \mathrm{var}_j \hat{p}_t(j) &= \frac{\alpha}{1-\alpha}\sum_{t=0}^{\infty}\sum_{h=0}^{t} \beta^t \alpha^h \hat{\Pi}_{t-h}^2 + \text{t.i.p.} \\
&= \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\sum_{t=0}^{\infty} \beta^t \hat{\Pi}_t^2 + \text{t.i.p.}
\end{aligned}$$

Hence, the output dispersion term can be replaced by

$$\mathrm{var}_j \hat{y}_t(j) = \frac{\alpha\theta^2}{(1-\alpha)(1-\alpha\beta)}\hat{\Pi}_t^2 + \text{t.i.p.} + O(3). \qquad \text{(A.6)}$$

As a result, (A.2), (A.4) and (A.6) constitute the approximation of households' utility function.

However, for the reasons explained in Benigno and Woodford(forthcoming), we need to show that the first-order terms in (A.2), and (A.3) cancel out. If we collect the first-order terms and recalling (2.24),

$$\hat{Y}_t - k\hat{K}_{t+1} + k(1-\delta)\hat{K}_t - \left[\hat{Y}_t + \frac{1-\phi_h}{\phi_h}\hat{K}_t\right] = -k\hat{k}_{t+1} + k(1-\delta)\hat{K}_t - \frac{1-\phi_h}{\phi_h}\hat{K}_t$$
$$= -k\left[\hat{K}_{t+1} - \beta^{-1}\hat{K}_t\right]. \tag{A.7}$$

Integrating this expression forward,

$$-\sum_{t=0}^{\infty}\beta^t k\left[\hat{K}_{t+1} - \beta^{-1}\hat{K}_t\right] = \frac{K}{\beta}\hat{K}_0 = \text{t.i.p.} \tag{A.8}$$

Thus, the welfare function consists of second-order terms only. Collecting the remaining terms from (A.2), (A.3) and (A.8), the loss function of central bank can be expressed as $L_0 = \frac{Yu_c}{2}\hat{L}_0 + \text{t.i.p.} + O(3)$, where $\hat{L}_0$ is given below

$$\hat{L}_0 = \sum_{t=0}^{\infty}\beta^t\Bigg\{\left(\sigma^{-1}+\omega\right)\hat{Y}_t^2 + \sigma^{-1}\hat{I}_t^2 + \epsilon_\psi k\left(\hat{K}_{t+1}-\hat{K}_t\right)^2 + \rho_k k\left(\beta^{-1}-(1-\delta)\right)\hat{K}_t^2 - 2\sigma^{-1}\hat{Y}_t\hat{I}_t$$
$$- 2\hat{Y}_t\left(\sigma^{-1}g_t + \omega q_t\right) - 2(\omega-\nu)\hat{Y}_t\hat{K}_t - 2k(1-\delta)\left(\sigma^{-1}g_t+\omega q_t\right)\hat{K}_t$$
$$+ 2k\left(\sigma^{-1}g_t\hat{K}_{t+1} + \beta^{-1}\omega q_t\hat{K}_t\right) + \left(1 + \frac{\left(\rho_y-\omega\right)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)}\hat{\Pi}_t^2\Bigg\} \tag{A.9}$$

Finally, we shall change (A.9) into a form using gap variables. This requires replacing the exogenous terms that appear in (A.9) by flexible-price equilibrium values. From the discussion in §5, variables in time-0 flexible price equilibrium have the following relationship.

$$\sigma^{-1}g_t + \omega q_t = \left(\omega+\sigma^{-1}\right)\hat{Y}^f_{t|0} - \sigma^{-1}k\hat{K}^f_{t+1|0} + \left[\sigma^{-1}k(1-\delta) - \omega + \nu\right]\hat{K}^f_{t|0} \tag{A.10}$$

$$-\sigma^{-1}\hat{Y}^f_{t|0} + \sigma^{-1}\hat{I}^f_{t,0} + \sigma^{-1}g_t + \epsilon_\psi\left(\hat{K}^f_{t+1|0} - \hat{K}^f_{t|0}\right)$$
$$= \beta(1-\delta)E_t\left(-\sigma^{-1}\hat{Y}^f_{t+1|0} + \sigma^{-1}\hat{I}^f_{t+1|0} + \sigma^{-1}g_{t+1}\right) + \beta\epsilon_\psi\left(E_t\hat{K}^f_{t+2|0} - \hat{K}^f_{t+1|0}\right)$$
$$+ \left[1-\beta(1-\delta)\right]E_t\left(\rho_y\hat{Y}^f_{t+1|0} - \rho_k\hat{K}^f_{t+1|0} - \omega q_{t+1}\right) \tag{A.11}$$

(A.11) can be further transformed to

$$\sigma^{-1}g_t\hat{K}_{t+1} = \sigma^{-1}E_t g_{t+1}\hat{K}_{t+1} + \hat{K}_{t+1}\Big[\sigma^{-1}\hat{Y}^f_{t|0} - \sigma^{-1}\hat{I}^f_{t|0} - \epsilon_\psi\left(\hat{K}^f_{t+1|0} - \hat{K}^f_{t|0}\right) + \beta\epsilon_\psi\left(E_t\hat{K}^f_{t+2|0} - \hat{K}^f_{t+1|0}\right)$$
$$+ \left\{(1-\beta(1-\delta))\left(\rho_y - \omega - \sigma^{-1}\right) - \sigma^{-1}\beta(1-\delta)\right\}E_t\hat{Y}^f_{t+1|0} + \sigma^{-1}E_t\hat{I}^f_{t+1|0}$$
$$+ (1-\beta(1-\delta))(\omega - \nu - \rho_k)\hat{K}^f_{t+1|0}\Big] \tag{A.12}$$

Substitute (A.10) and (A.12) into (A.9). After a bit of algebra, we obtain

$$\begin{aligned}
\hat{L}_0 = \sum_{t=0}^{\infty}\beta^t\Big\{ & \left(\sigma^{-1}+\omega\right)\left(\hat{Y}_t^2 - 2\hat{Y}_{t|0}^f\hat{Y}_t\right) + \sigma^{-1}\left(\hat{I}_t^2 - 2\hat{I}_{t|0}^f\hat{I}_t\right) + \rho_k k\left(\beta^{-1}-(1-\delta)\right)\left(\hat{K}_t^2 - 2\hat{K}_{t|0}^f\hat{K}_t\right) \\
& + \epsilon_\psi k\left(\hat{K}_{t+1}^2 - 2\hat{K}_{t+1}\hat{K}_t + \hat{K}_t^2 - 2\hat{K}_{t+1|0}^f\hat{K}_{t+1} + 2\hat{K}_{t|0}^f\hat{K}_{t+1} + 2\hat{K}_{t+1|0}^f\hat{K}_t - 2\hat{K}_{t|0}^f\hat{K}_t\right) \\
& - 2\sigma^{-1}\left(\hat{Y}_t\hat{I}_t - \hat{Y}_{t|0}^f\hat{I}_t - \hat{I}_{t|0}^f\hat{Y}_t\right) - 2\left(\omega-\nu\right)\left(\hat{Y}_t\hat{K}_t - \hat{Y}_{t|0}^f\hat{K}_t - \hat{K}_{t|0}^f\hat{Y}_t\right) \\
& + \left(1 + \frac{\left(\rho_y-\omega\right)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)}\hat{\Pi}_t^2\Big\} \\
= \sum_{t=0}^{\infty}\beta^t\Big\{ & \left(\sigma^{-1}+\omega\right)\left(\hat{Y}_t - \hat{Y}_{t|0}^f\right)^2 + \sigma^{-1}\left(\hat{I}_t - \hat{I}_{t|0}^f\right)^2 + \rho_k k\left(\beta^{-1}-(1-\delta)\right)\left(\hat{K}_t - \hat{K}_{t|0}^f\right)^2 \\
& + \epsilon_\psi k\left(\left(\hat{K}_{t+1} - \hat{K}_{t+1|0}^f\right) - \left(\hat{K}_t - \hat{K}_{t|0}^f\right)\right)^2 - 2\sigma^{-1}\left(\hat{Y}_t - \hat{Y}_{t|0}^f\right)\left(\hat{I}_t - \hat{I}_{t|0}^f\right) \\
& - 2\left(\omega-\nu\right)\left(\hat{Y}_t - \hat{Y}_{t|0}^f\right)\left(\hat{K}_t - \hat{K}_{t|0}^f\right) + \left(1 + \frac{\left(\rho_y-\omega\right)\nu}{\rho_k}\theta\right)\frac{\alpha\theta}{(1-\alpha)(1-\alpha\beta)}\hat{\Pi}_t^2\Big\} \qquad \text{(A.13)}
\end{aligned}$$

Appendix B: Computational method for the optimal discretionary solution

We present the method for computing the optimal discretionary solution in section 5, where the productivity shock is assumed to follow a particular deterministic process. In this particular case, the optimal discretionary solution can be computed by solving the problem backwards, starting from period 8. More specifically, we first compute the value and policy functions under the optimal discretionary policy for period 8. Since it is assumed that no shock is expected to occur after period 8, this problem can be solved in the linear-quadratic framework using the algorithm of Söderlind(1999). Then, for periods 0 to 7, we create scenarios for the central bank, describing in which periods it conducts the zero-interest-rate policy. The number of possible scenarios is 2^8 in total. For each scenario, we compute value and policy functions backwards and numerical values of solutions in each period recursively using the initial condition on the capital stock, $\hat{K}_0 = 0$. Finally, we check whether these results do not violate non-negativity of the nominal interest rate and whether the central bank taking the zero-interest-rate policy in any period has an incentive to deviate. The details of each computational procedure are described below.

Step1: Dynamic programming specification

Write the Bellman equation for the problem in period 0 to 7 as below.

$$\left[\hat{K}_t, \mathbf{s}_t\right] P_t \begin{bmatrix} \hat{K}_t \\ \mathbf{s}_t \end{bmatrix} = \min_{\{\mathbf{x}_t\}} \left\{ \left[\mathbf{x}_t^T, \hat{K}_t, \omega q_t\right] Q \begin{bmatrix} \mathbf{x}_t \\ \hat{K}_t \\ \omega q_t \end{bmatrix} + \beta \left[\hat{K}_{t+1}, \mathbf{s}_{t+1}^T\right] P_{t+1} \begin{bmatrix} \hat{K}_{t+1} \\ \mathbf{s}_{t+1} \end{bmatrix} \right\} \tag{B.1}$$

$$\text{s.t.} \begin{cases} M_1\mathbf{x}_{t+1} + M_2\mathbf{x}_t + M_3\hat{K}_t + M_4\mathbf{q}_t + M_5\left(1 - 1/\beta\right) = 0 & \text{if} \quad \hat{R}_t = 1 - 1/\beta \\ N_1\mathbf{x}_{t+1} + N_2\mathbf{x}_t + N_3\hat{K}_t + M_4\mathbf{q}_t = 0 & \text{if} \quad \hat{R}_t > 1 - 1/\beta \end{cases}, \tag{B.2}$$

where

$$\mathbf{x}_t = \left[\hat{\Pi}_t, \hat{Y}_t, \hat{K}_{t+1}\right]^T, \quad \mathbf{q}_t = \left[\omega q_t, \omega q_{t+1}\right]^T, \quad \mathbf{s}_t = \left[\omega q_t, \omega q_{t+1}, \ldots, \omega q_7, (1 - 1/\beta)\right]^T.$$

Q and P are the matrices representing coefficients in (2.46) and the value function in quadratic form, respectively. In periods where the ZLB bind, $\mathbf{x}_t$ is determined by (2.29), (2.38) and (2.40) only. The matrices M_1 to M_5 contain the coefficients of these structural equations.[*19] On the other hand, when the ZLB does not bind, the optimization problem to solve $\mathbf{x}_t$ in that period is considered subject to (2.38) and (2.40). The nominal interest rate can be backed out from (2.29). N_1 to N_4 are the coefficient matrices for (2.38) and (2.40).

[*19] The value function can be obtained by inserting the values of endogenous variables into (B.1).

Step2: Value and policy functions for $t = 8$

Using the algorithm in Söderlind(1999), the value and policy functions in period 8 can be computed. In this period, the problem is to minimize (2.46) subject to (2.38) and (2.40). In addition, $\omega q_t = 0$ and $\mathbf{s}_t = 0\ \forall t$. Since the policy function is known to take the form of $\mathbf{x}_{t+1} = C_{t+1}\hat{K}_{t+1}$,[*20] it can replace the expectation term in the constraints, and the optimization problem is equivalent to[*21]

$$\hat{K}_t P_t \hat{K}_t = \min_{\{\mathbf{x}_t\}} \left\{ \left[\mathbf{x}_t^T, \hat{K}_t\right] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \hat{K}_t \end{bmatrix} + \beta \mathbf{x}_t^T R^T P_{t+1} R \mathbf{x}_t \right\} \tag{B.3}$$

$$\text{s.t. } \left(N_2 + N_1 C_{t+1} R\right) \mathbf{x}_t + N_3 \hat{K}_t = 0. \tag{B.4}$$

After taking the first-order conditions and rearranging, we finally obtain the following recursive expressions.

$$\mathbf{x}_t = C_t \hat{K}_t$$

$$P_t = C_t^T Q_{11} C_t + 2C_t^T Q_{12} + Q_{22} + \beta C_t^T R^T P_{t+1} R C_t \tag{B.5}$$

$$C_t \equiv -\left(Q_{11} + \beta R^T P_{t+1} R\right)^{-1} \left\{Q_{12} + \left(N_2 + N_1 C_{t+1} R\right)^T \Gamma_t\right\} \tag{B.6}$$

$$\Gamma_t \equiv \left\{ \left(N_2 + N_1 C_{t+1} R\right) \left(Q_{11} + \beta R^T P_{t+1} R\right)^{-1} \left(N_2 + N_1 C_{t+1} R\right)^T \right\}^{-1} \left\{ N_3 - \left(N_2 + N_1 C_{t+1} R\right) \left(Q_{11} + \beta R^T P_{t+1} R\right)^{-1} Q_{12} \right\}.$$

If P and C converge, starting from some appropriate initial values for P_{t+1} and C_{t+1}, those are the value and policy functions to be used in the next step.

Step3: Solving backwards

For periods $t = 0, 1, 2, ..., 7$, the problem can be solved backwards, given the value and policy functions in period 8. Note that we guess scenarios for the timing of taking the zero-interest-rate policy and check whether each scenario satisfies the optimality conditions later. In our problem, policy funciton is linear and thus can be expressed in the form of $\mathbf{x}_t = X_{k,t}\hat{K}_t + X_{s,t}\mathbf{s}_t$.

If the ZLB binds

Substitute $\mathbf{x}_{t+1} = X_{k,t+1}\hat{K}_{t+1} + X_{s,t+1}\mathbf{s}_{t+1}$ in the first set of constraints in (B.2) to obtain

$$X_{k,t} = -\left(M_1 X_{k,t+1} R + M_2\right)^{-1} M_3$$

$$X_{s,t} = -\left(M_1 X_{k,t+1} R + M_2\right)^{-1} \left(M_1 X_{s,t+1} S^s + M_4 S^q + M_5 S^r\right)$$

$$S^s \equiv \begin{bmatrix} \mathbf{0}_{(8-t)\times 1} & \mathrm{I}_{(8-t)} \end{bmatrix},\ S^q \equiv \begin{bmatrix} \mathrm{I}_2 & \mathbf{0}_{2\times(7-t)} \end{bmatrix},\ S^r \equiv \begin{bmatrix} \mathbf{0}_{1\times(8-t)} & 1 \end{bmatrix}$$

The value function for period t can be computed by substituting the above results in (B.1). If we divide the coefficient matrix for the value function into four appropriately-sized blocks of matrices, it can be expressed

[*20] C_{t+1} is a 3×1 matrix whose elements are to be determined

[*21] $\hat{K}_{t+1} = R\mathbf{x}_t$, where $R = [0,\ 0,\ 1]$.

as

$$P_t = \begin{bmatrix} P_t(11) & P_t(12) \\ P_t(12)^T & P_t(22) \end{bmatrix} \tag{B.7}$$

$$P_t(11) = X_{k,t}^T Q_{11} X_{k,t} + 2X_{k,t}^T Q_{12} + Q_{22} + \beta X_{k,t}^T R^T P_{t+1}(11) R X_{k,t} \tag{B.8}$$

$$\begin{aligned} P_t(12) =& X_{k,t}^T Q_{11} X_{s,t} + Q_{12}^T X_{s,t} + X_{k,t}^T Q_{13} S^{\omega} + Q_{23} S^{\omega} \\ &+ \beta \left(X_{k,t}^T R^T P_{t+1}(11) R X_{s,t} + X_{k,t}^T R^T P_{t+1}(12) S^{s} \right) \end{aligned} \tag{B.9}$$

$$\begin{aligned} P_t(22) =& X_{s,t}^T Q_{11} X_{s,t} + 2X_{s,t} Q_{13} S^{\omega} + S^{\omega T} Q_{33} S^{\omega} \\ &+ \beta \left(X_{s,t}^T R^T P_{t+1}(11) R X_{s,t} + 2X_{s,t}^T R^T P_{t+1}(12) S^{s} + S^{sT} P_{t+1}(22) S^{s} \right), \end{aligned} \tag{B.10}$$

where $S^{\omega} \equiv [1, 0, \ldots, 0]$ is a $1 \times (8 - t)$ vector.

<u>If the ZLB does not bind</u>

In a period where the ZLB does not bind, minimize (B.1) subject to the second set of constraints in (B.2) to obtain

$$X_{k,t} = -\left(Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} \left(Q_{12} + A^T \Gamma_{1,t} \right) \tag{B.11}$$

$$X_{s,t} = -\left(Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} \left(Q_{13} S^{\omega} + \beta R^T P_{t+1}(12) S^{s} + A^T \Gamma_{2,t} \right) \tag{B.12}$$

$$\begin{aligned} A &\equiv N_2 + N_1 X_{k,t+1} R \\ B &\equiv N_3 \\ C &\equiv N_1 X_{s,t+1} S^{s} + N_4 S^{q} \\ \Gamma_{1,t} &\equiv \left\{ A \left(Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} A^T \right\}^{-1} \left\{ B - A \left(Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} Q_{12} \right\} \\ \Gamma_{2,t} &\equiv \left\{ A \left(Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} A^T \right\}^{-1} \left\{ C - A \left(Q_{11} + \beta R^T P_{t+1}(11) R \right)^{-1} \left(Q_{13} S^{\omega} + \beta R^T P_{t+1}(12) S^{s} \right) \right\}. \end{aligned}$$

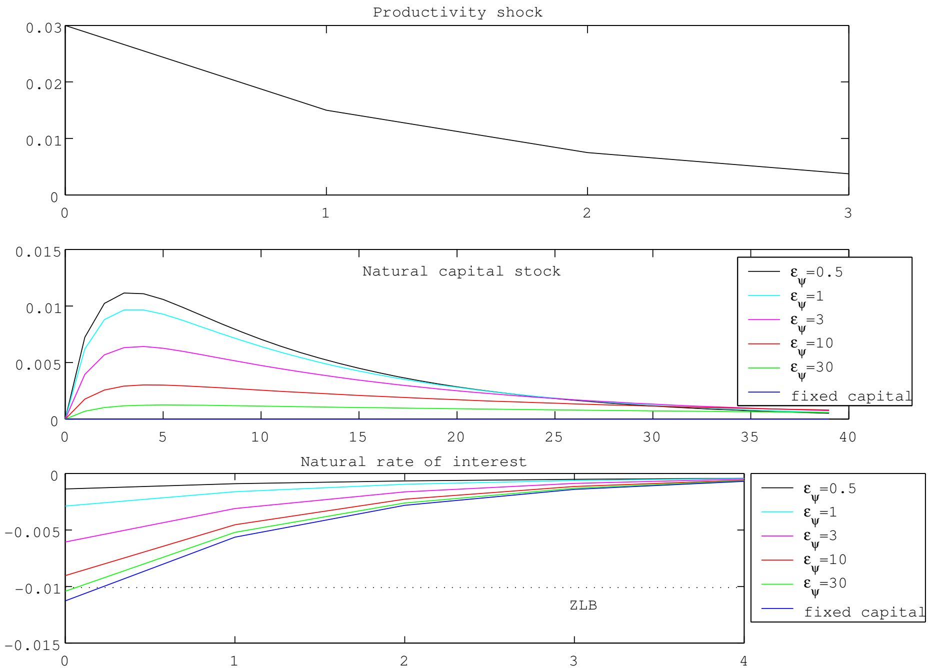
The coefficient matrix for the value function are the same as (B.7) to (B.10), given (B.11) and (B.12).

Step4: Consistency with the optimality conditions

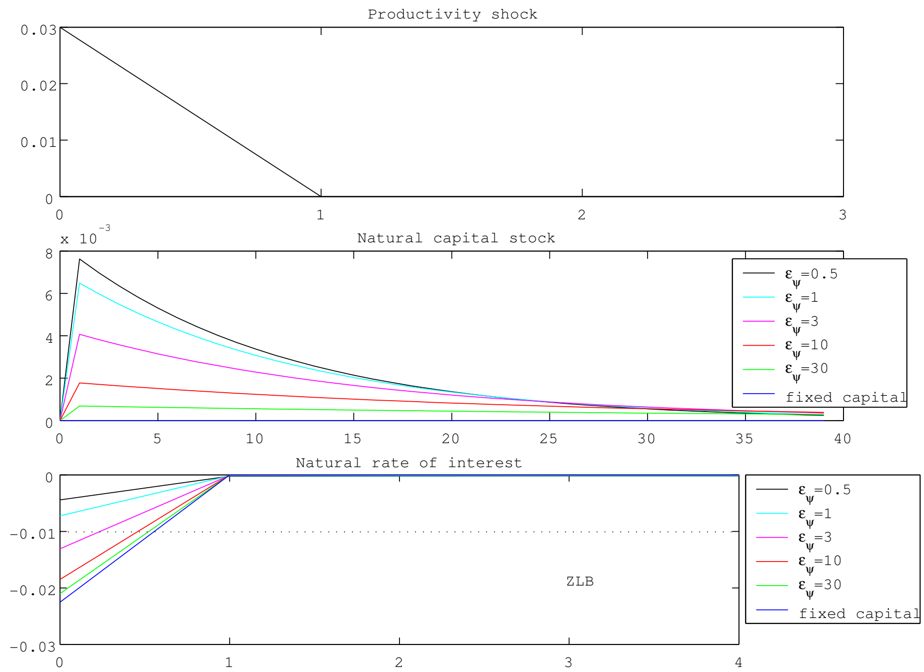
For each scenario regarding the choice of taking the zero-interest-rate policy in periods 0 to 7, we can compute the values of $\{\mathbf{x}_t\}_{t=0}^{7}$ and $\left\{\hat{R}_t\right\}_{t=0}^{7}$ by applying the methods in step 1 to 3 and using the initial condition on the capital stock. If $\hat{R}_t < 1 - 1/\beta$ in any period, that scenario violates the non-negativity of the nominal interest rate. We also need to check whether the central bank is optimally taking the zero-interest-rate policy. Given the capital stock in each period and the next period's value function, the central bank must have no incentive to deviate from the zero-interest-rate policy. If choosing a positive level of interest rate turns out to improve the economic welfare under a certain scenario, that should be excluded from the set of possible solutions. Scenarios which survived this consistency-check process are the solutions.

Fig. 1 Impulse responses of capital stock and nominal interest rate to an unexpected productivity shock at $t = 0$.

(a) $\rho_a = 0.5$



(b) $\rho_a = 0$



Note: Fixed-capital stock model used in this numerical example is the one in which capital stock at the individual firm level is fixed. Thus, it is not exactly the same as the case in which $\epsilon_\psi \to \infty$ in the variable-capital stock model that we use.

Fig. 2 Relative weights on forecasts at different horizons in the optimal commitment policy.

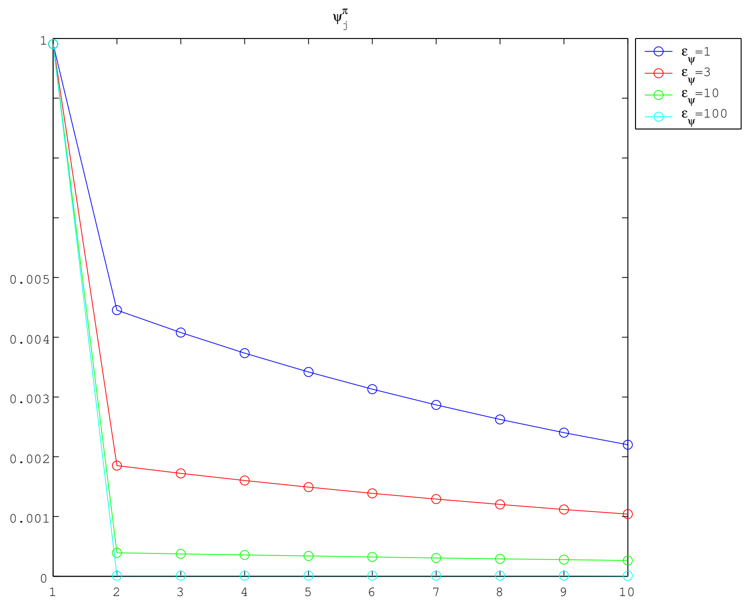


Fig. 3 Relative weights on forecasts at different horizons in the non-inertial policy.

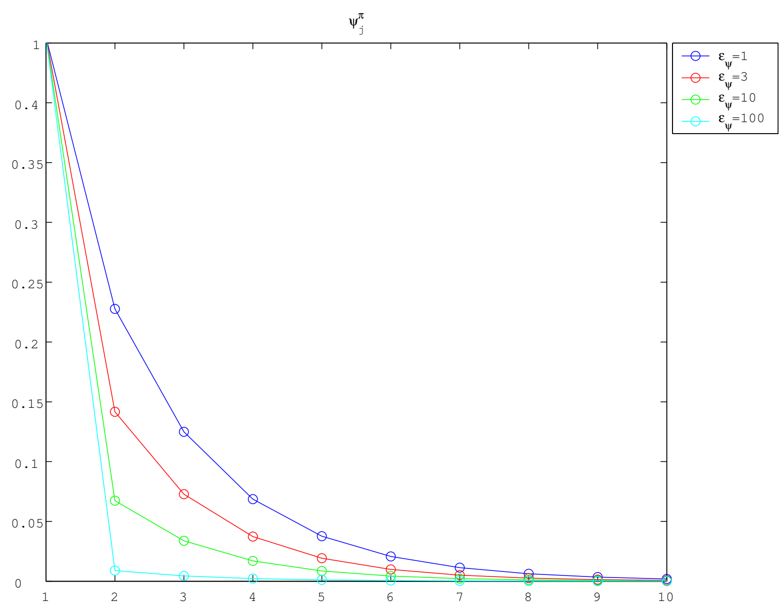


Fig. 4 The optimal commitment solution ($\epsilon_\psi = 3$)

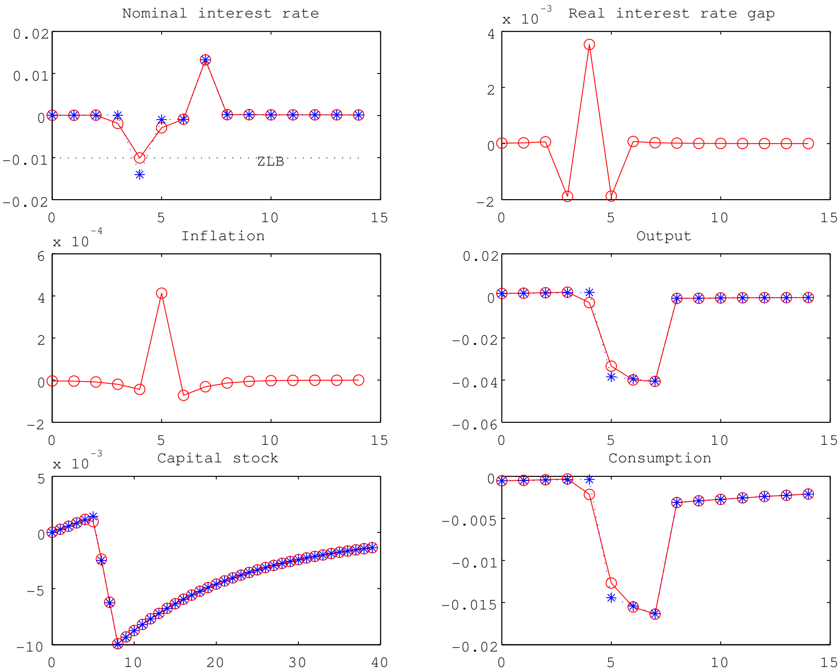
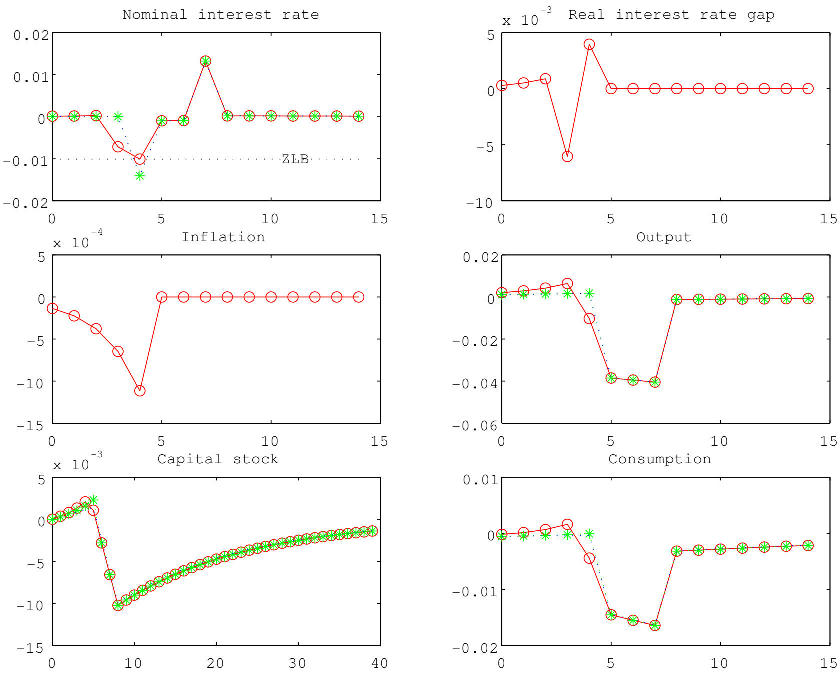


Fig. 5 The optimal discretionary solution ($\epsilon_\psi = 3$)



Note: In each panel, a red line indicates the level of sticky-price equilibrium path (hat variables in the log-linearized system). Green and blue lines show the natural levels (rates) in Woodford's definition and the time-0 flexible-price equilibrium paths, respectively.

Fig. 6 The non-inertial policy ($\epsilon_\psi = 3$)

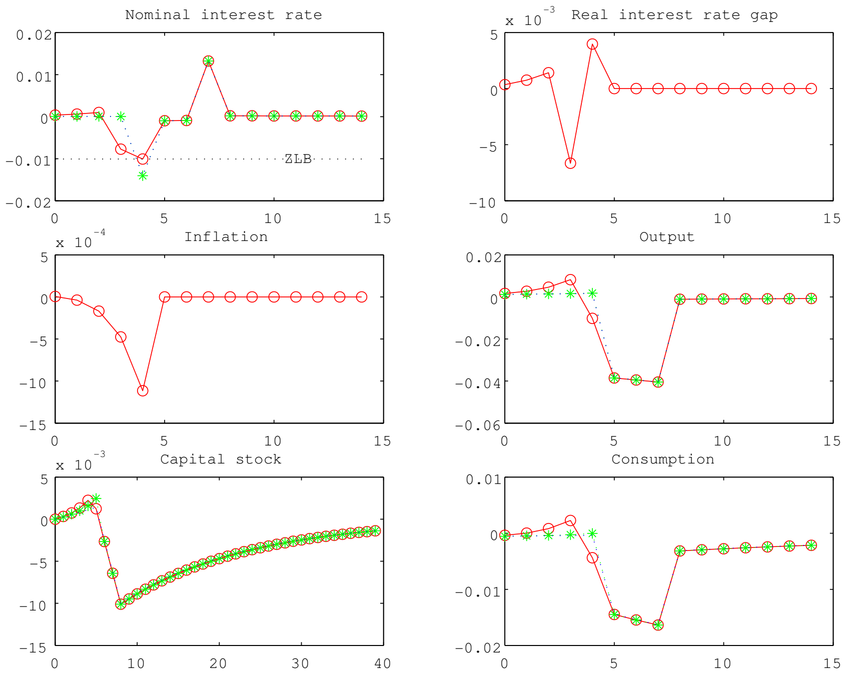
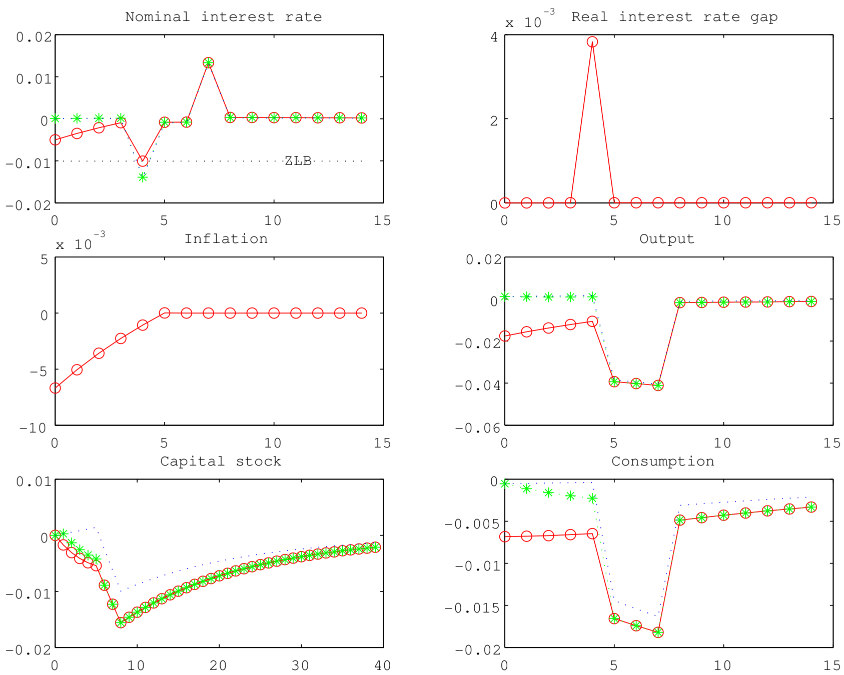


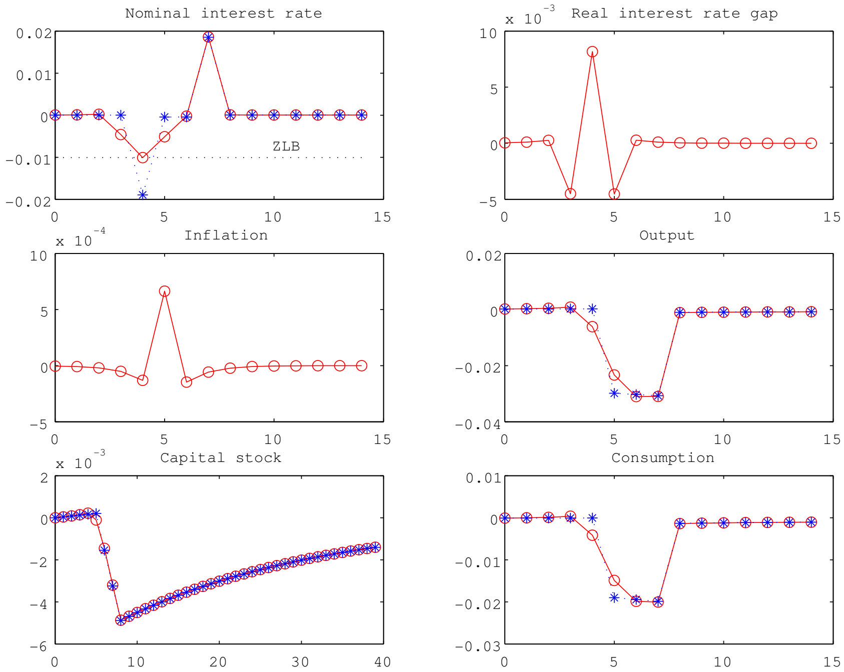
Fig. 7 $\hat{R}_t = \max\left\{\hat{r}^n_t + E_t\hat{\Pi}_{t+1}, 1 - 1/\beta\right\}$, ($\epsilon_\psi = 3$)



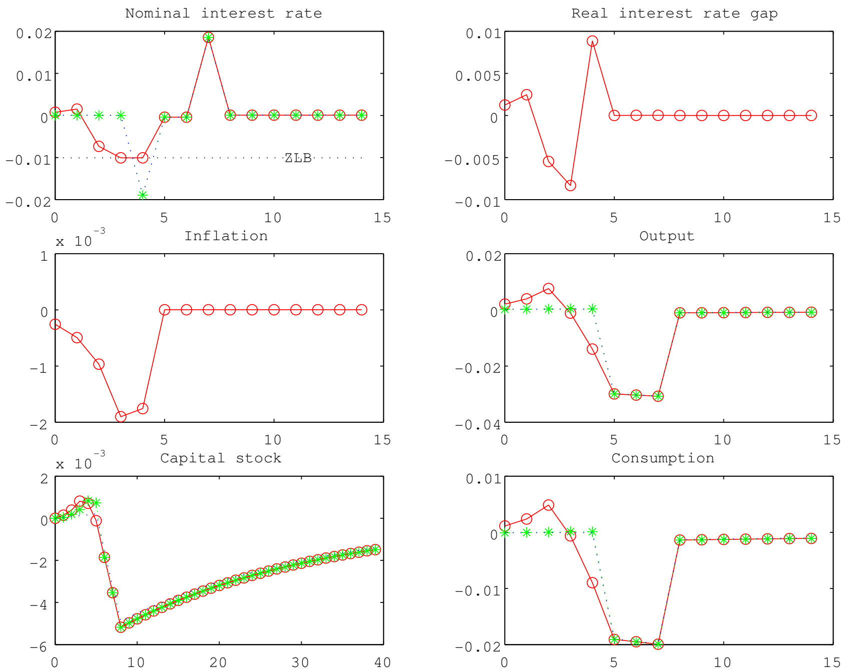
Note: In each panel, a red line indicates the level of sticky-price equilibrium path (hat variables in the log-linearized system). Green and blue lines show the natural levels (rates) in Woodford's definition and the time-0 flexible-price equilibrium paths, respectively.

Fig. 8 The length of periods with binding ZLB ($\epsilon_\psi = 10$)

(a) The optimal commitment solution



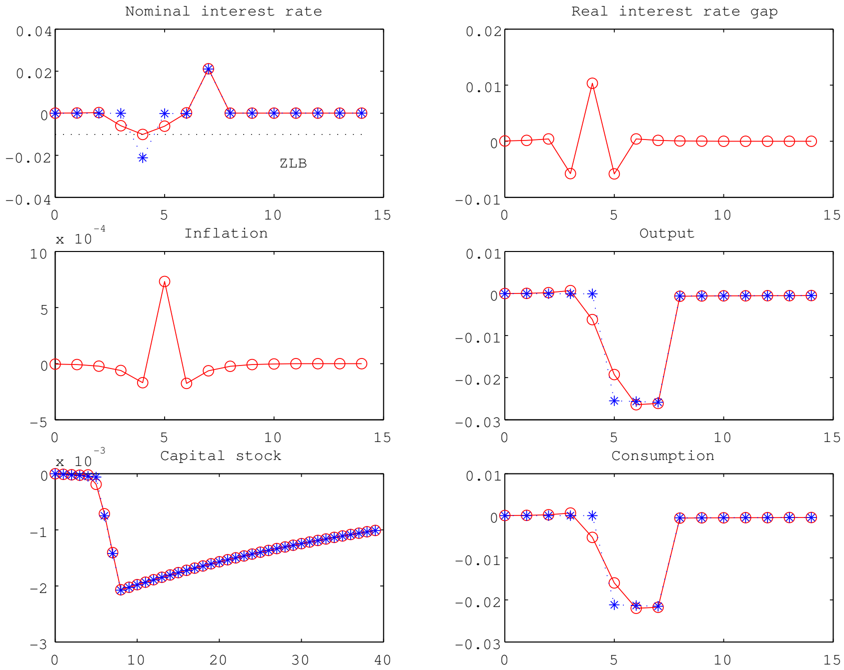
(b) The optimal discretionary solution



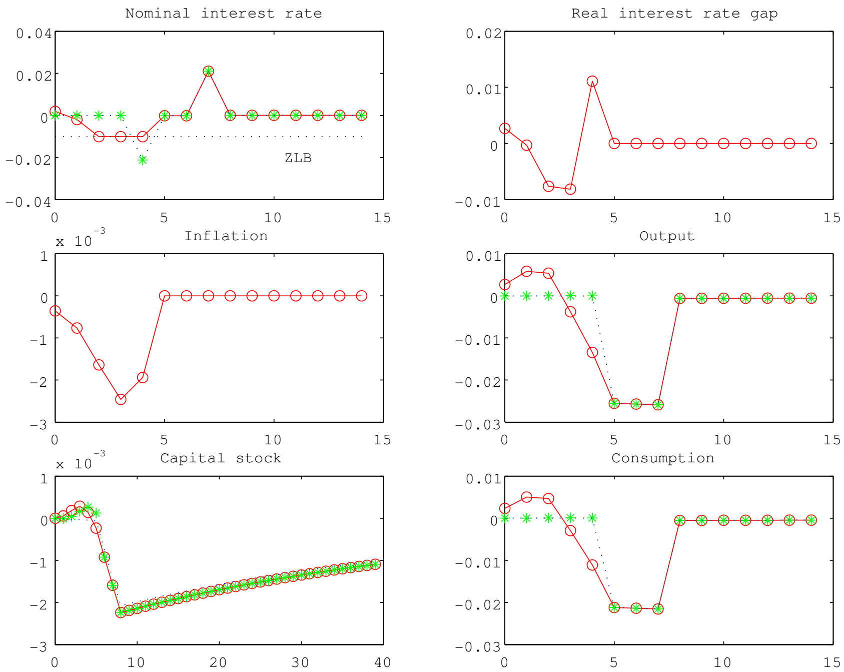
Note: In each panel, a red line indicates the level of sticky-price equilibrium path (hat variables in the log-linearized system). Green and blue lines show the natural levels (rates) in Woodford's definition and the time-0 flexible-price equilibrium paths, respectively.

Fig. 9 The length of periods with binding ZLB ($\epsilon_\psi = 30$)

(a) The optimal commitment solution



(b) The optimal discretionary solution



Note: In each panel, a red line indicates the level of sticky-price equilibrium path (hat variables in the log-linearized system). Green and blue lines show the natural levels (rates) in Woodford's definition and the time-0 flexible-price equilibrium paths, respectively.