ESSAYS ON TESTING FOR STATIONARITY POSSIBLY WITH SEASONALITY AND A STRUCTURAL CHANGE

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PREFACE

This dissertation investigates three testing problems, testing for stationarity against seasonal unit roots, testing for periodic stationarity against periodic integration, and testing for stationarity with a break against a unit root. Since the tests for stationarity and a unit root complement each other, Chapter 1 briefly reviews the three testing problems in a reversed direction, that is, testing for a (seasonal, periodic) unit root against stationarity. Chapters 2 to 4 develop three testing procedures for the above testing problems. In each chapter, we derive the LM test statistics, their limiting distributions, and their characteristic functions. Power functions are drawn and the test statistics are investigated under the alternatives as well as under the null hypothesis.

I am so much obliged to Prof. Taku Yamamoto, my principal advisor, and Prof. Katsuto Tanaka. Prof. Yamamoto first stimulated my interest in time series analysis, especially nonstationary time series analysis. He always encourages me and has given me thoughtful comments in my study. Prof. Tanaka stimulated my interest in nonstationary problems, and taught me mathematical and statistical methods to study such problems. I believe that, according to their teaching, I got the wide knowledge about the time series analysis as well as econometrics.

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Contents

Chapter 1.

Introduction: Brief Reviews of Testing for Seasonal Unit Roots, Testing for a Periodic Unit Root, and Testing for a Unit Root against Stationarity with a Break

This Chapter briefly reviews three testing problems: Testing for seasonal unit roots, testing for a periodic unit root, and testing for a unit root with a trend break. We will suppose in later chapters the null hypothesis in reversed direction compared with the above problems, that is, we will fundamentally assume the null of stationarity. Since our tests in later chapters and the above three tests compliment each other, it is necessary to understand the above three testing problems.

1. Introduction

It has been great concern to economists whether there is persistence of a unit root in the time series or not, and testing for a unit root has an important role in the practical analysis. From many empirical studies following the seminal work of Nelson and Plosser (1982), it is believed that many macroeconomic time series have a unit root and, more or less, persistence is found in such data.

In addition, seasonality and possibility of a structural change as well as nonstationarity have been the much discussed problems and studied theoretically and in a practical analysis. In this chapter, we briefly review the three testing problems concerning with nonstationarity, seasonality and a structural change, that is, testing for seasonal unit roots, testing for a periodic unit root, and testing for a unit root against stationarity with a break. Though our main concern in this dissertation is testing for stationarity and the null hypothesis we are interested in is *not* a unit root *but* stationarity, we review the above testing procedures since testing for a unit root is strongly related to testing for stationarity. For example, suppose that the researcher is interested in a unit root and the Dickey-Fuller (DF) test rejects the null hypothesis of a unit root. Since the DF test has the power against more general alternatives than a stationary one, a rejection of the null hypothesis does not necessarily indicate stationarity of the series. Thus once the test rejects the null of a unit root, the primary interest in turn becomes the null of stationarity, possibly with seasonality and a break. In this sense, both tests for a unit root and stationarity should compliment each other.

The plan of this chapter is as follows. Section 2 reviews the tests for seasonal unit roots and the tests for stationarity against seasonal unit roots. As in the case of the non-seasonal unit root tests, testing procedures for both the null of seasonal unit roots and that of stationarity have been developed in the literature. Section 3 reviews the tests for a periodic unit root, and Section 4 discusses the unit root test against trend stationarity with a break.

2. Seasonal Unit Roots

2.1. Testing for Seasonal Unit Roots

Many macroeconomic time series are observed quarterly or monthly and seasonality is one of the important characteristics of the data. The economic variable is sometimes considered to be generated from an integrated process and a seasonal difference operator such as $(1-B^s)$ seems to be appropriate to make the data stationary, where B is a backshift operator and s denotes a seasonal period. A seasonal autoregressive integrated moving average (SARIMA) model is one of the useful models to express such a seasonally nonstationary behavior.

Adequacy of such a seasonal difference operator is investigated in Dickey, Hasza and Fuller (1984). They considered the following model:

$$y_t = \sum_{j=1}^{s} \delta_j D_{jt} + \alpha y_{t-s} + e_t, \quad t = 1, \cdots, T, \qquad (1-1)$$

where $\{e_t\} \sim i.i.d.(0, \sigma^2)$ and $D_{jt} = 1$ if $t = j$ (mod s) and 0 otherwise. A seasonal difference operator is adequate when $\alpha = 1$ and then they investigated two test statistics, $T(\hat{\alpha}-1)$ and the t-statistic for testing $\alpha = 1$ where $\hat{\alpha}$ is a least squares (LS) estimator of α. The model (1-1) is extended to have non-zero mean and a linear trend and also to the case when $\{e_t\}$ is dependent. Tanaka (1996) considered the locally best invariant (LBI) test, the locally best invariant and unbiased (LBIU) test and the point optimal invariant (POI) test for the above hypothesis.

Moreover, since a seasonal difference operator $(1-B^s)$ has s roots on the unit circle, each root except for 1 is called a seasonal unit root and whether the time series has some of seasonal unit roots as well as a non-seasonal unit root becomes interesting to the researcher. For example, a quarterly seasonal difference, $(1-B^4)$, can be decomposed into $(1-B^4) = (1-B)(1+B)(1-iB)(1+iB)$ where $i = \sqrt{-1}$ and then it has four roots, 1, -1 and $\pm i$ on the unit circle, which correspond to frequencies 0, π and $\pi/2$. We can consider the following seven lag polynomials with roots on the unit circle.

$$\begin{aligned} A_1(B) &= (1-B). \\ A_2(B) &= (1+B). \\ A_3(B) &= (1+B^2). \\ A_4(B) &= (1-B)(1+B) = (1-B^2). \qquad (1\text{-}2) \end{aligned}$$

$$\begin{aligned} A_5(B) &= (1-B)(1+B^2) = (1-B+B^2-B^3). \\ A_6(B) &= (1+B)(1+B^2) = (1+B+B^2+B^3). \\ A_7(B) &= (1-B)(1+B)(1+B^2) = (1-B^4). \end{aligned}$$

In the above quarterly model, we call the roots of -1 and $\pm i$ seasonal unit roots and especially -1 a negative unit root. Then, testing for adequacy of a quarterly seasonal difference is equivalent to testing for the joint hypothesis that $\{y_t\}$ has four roots on the unit circle. Then, even if the null hypothesis is rejected, it is still possible for the process to have some of unit roots. For example, $(1+B)y_t = e_t$ does not have all the unit roots and the null of four unit roots will be rejected, but it still has a negative unit root and is nonstationary. Neither a usual difference $(1-B)$ nor a seasonal difference $(1-B^4)$ is adequate to be applied to $\{y_t\}$. In this sense, it becomes important to test for each seasonal unit root as well as a non-seasonal unit root.

The limiting distributions of the LS estimators of autoregressive (AR) parameters when the process has several unit roots are investigated in Ahtola and Tiao (1987) and Chan and Wei (1988). When the process has complex unit roots at frequency θ, the model can be expressed as

$$\left(1-e^{i\theta}B\right)\left(1-e^{-i\theta}B\right)y_t = e_t.$$

Expanding the left hand side, we have

$$y_t = 2\cos\theta y_{t-1} + y_{t-2} + e_t.$$

Ahtola and Tiao (1987) considered the above complex unit roots model and investigated the limiting distributions of $T(\hat{\phi}_1-\phi_1)$ and $T(\hat{\phi}_2+1)$, the LS estimators of the AR parameters of the regression,

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + e_t.$$

Chan and Wei (1988) considered a more general model. Let us consider the p-th order AR model,

$$\phi(B)y_t = e_t, \quad \phi(B) = 1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p, \tag{1-3}$$

and express the characteristic polynomial $\phi(z)$ as

$$\phi(z) = (1-z)^a(1+z)^b \prod_{j=1}^{l}(1-2\cos\theta_j z + z^2)^{d_j}\psi(z),$$

where a, b, l and d_j are nonnegative integers, $\theta_j \in (0,\pi)$ and $\psi(z)$ is a polynomial of order $q = p - (a+b+2d_1+\cdots+2d_l)$ which has all roots outside the unit circle. Denoting the LS estimator of $\phi = (\phi_1, \cdots, \phi_p)'$ as $\hat{\phi}$, Chan and Wei (1988) derived the limiting distribution of $(\hat{\phi} - \phi)$ multiplied by the appropriate weighting matrix.

Testing for seasonal unit roots are developed in Hylleberg, Engle, Granger and Yoo (1990) (HEGY) using the above results. They extend the model (1-3) to that with a deterministic component,

$$\phi(B)y_t = \mu_t + e_t.$$

As a deterministic component, they considered five cases: i) no deterministic, ii) an intercept, iii) an intercept and a seasonal dummy, iv) an intercept and a trend, and v) an intercept, a sasonal dummy and a trend. According to HEGY, the above model can be expressed as

$$y_{4t} = \pi_1 y_{1t-1} + \pi_2 y_{2t-1} + \pi_3 y_{3t-2} + \pi_4 y_{3t-1} + \sum_{j=1}^{k}\phi_j^* y_{4t-j} + \mu_t + e_t, \qquad (1-4)$$

where

$$\begin{aligned} y_{1t} &= (1+B+B^2+B^3)y_t, \\ y_{2t} &= -(1-B+B^2-B^3)y_t, \\ y_{3t} &= -(1-B^2)y_t, \\ y_{4t} &= (1-B^4)y_t, \end{aligned}$$

and k is defined appropriately. It can be shown that $\pi_1 = 0$, $\pi_2 = 0$ and $\pi_3 = \pi_4 = 0$ under the null hypothesis of a unit root at frequency 0, π and $\pi/2$, respectively. The t-statistics of π_1 and π_2 are used for the test for a unit root and a negative unit root with a lower area as a critical region and the F-statistic testing for $\pi_3 = \pi_4 = 0$ is for the test for complex unit roots at frequency $\pi/2$ as rejecting the null when it takes a large value. The limiting distributions of these test statistics can be derived using the result of Chan and Wei (1988).

HEGY's tests are extended in several directions. Joint tests for seasonal unit roots were developed in Ghysels, Lee and Noh (1994), and Smith and Taylor (1988) considered the same testing problem as HEGY with a different assumption of a deterministic component. Breitung and Franses (1998) investigated the testing procedure for seasonal unit roots which uses the nonparametric correction for nuisance parameters. According to their Monte Carlo simulation, their test may be more powerful than HEGY's test, whereas it suffers from severe size distortions in some situations. All of these tests are for the quarterly observed data, and seasonal unit roots tests are extended to the monthly time series in such as Beaulieu and Miron (1993) and Taylor (1998).

2.2. Testing for Stationarity against Seasonal Unit Roots

All of the above tests consider the null hypothesis of seasonal unit roots. But once the null hypothesis is rejected, our next concern in turn becomes the null of stationarity. Note that, since the tests for seasonal unit roots have the non-trivial power against more general alternatives compared with those assumed when the test statistics are constructed, rejection of the seasonal unit roots does not imply stationarity of the time series.

Tests for the null of stationarity against a unit root are developed in Kwiatkowski, Phillips, Schmidt and Shin (1992) (KPSS) and Leybourne and McCabe (1994). KPSS considered the following error component model,

$$y_t = \xi t + r_t + e_t, \quad r_t = r_{t-1} + u_t,$$

where $\{u_t\} \sim i.i.d.(0, \sigma_u^2)$ and independent of $\{e_t\}$. For $\{e_t\}$, it is assumed that

$$\lim_{T\to\infty} T^{-1} E\left[\left(\sum_{t=1}^{T} e_t\right)^2\right] = \sigma^2 < \infty.$$

The null hypothesis is $\sigma_u^2 = 0$, in which case $\{y_t\}$ becomes trend stationary whereas when $\sigma_u^2 > 0$ $\{y_t\}$ has a unit root component. The Lagrange Multiplier (LM) statistic, which is equivalent to the locally best invariant (LBI) test under the assumption of normality, is

$$\eta_T = \frac{1}{s^2(l)T^2} \sum_{t=1}^{T} S_t^2, \quad S_t = \sum_{j=1}^{t} \hat{e}_t, \quad s^2(l) = \frac{1}{T}\sum_{t=1}^{T} \hat{e}_t^2 + \frac{2}{T}\sum_{j=1}^{l} w(j,l) \sum_{t=1}^{T-j} \hat{e}_t \hat{e}_{t+j},$$

where $\hat{e}_t$ are regression residuals of y_t on a constant and t, $w(j,l) = 1 - |j|/(l+1)$ is the Bartlett window, the lag truncation parameter $l \to \infty$ as $T \to \infty$ and $l = O(T^{1/2})$. It can be shown that

$$\eta_T \xrightarrow{d} \int_0^1 \left\{ W(r) + (2r - 3r^2)W(1) + (-6r + 6r^2) \int_0^1 W(s)ds \right\}^2 dr,$$

where $W(\cdot)$ is a standard Brownian motion and $\xrightarrow{d}$ denotes convergence in distribution. They also showed that $\eta_T = O_p(T/l)$ under the alternative of $\sigma_u^2 > 0$, that is, the LM statistic η_T is consistent.

The problem of the KPSS test is that it may suffer from a size distortion in some cases under the null hypothesis as shown by the Monte Carlo simulation in KPSS. Instead, Leybourne and McCabe (1994) considered the following local level model,

$$\phi(B)y_t = \xi t + r_t + e_t, \quad \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p.$$

Firstly we have to construct the series $y_t^* = y_t - \sum_{i=1}^p \phi_i^* y_{t-i}$, where ϕ_i^*'s are the maximum likelihood estimators of ϕ_i's from the fitted ARIMA(p,1,1) model, $\triangle y_t = \xi + \sum_{i=1}^p \phi_i \triangle y_{t-i} + \zeta_t - \theta\zeta_{t-1}$ with $\triangle = (1-B)$. Next we calculate residuals $\hat{e}_t^*$ from the LS regression of y_t^* on an intercept and a linear trend. Then, it is shown that the test statistic

$$\hat{\eta}_T = \frac{1}{\hat{\sigma}_e^2 T^2} \sum_{t=1}^T \hat{S}_t^{*2}, \quad S_t^* = \sum_{j=1}^t \hat{e}_t^*,$$

where $\hat{\sigma}_e^{*2} = T^{-1}\sum_{t=1}^T \hat{e}_t^{*2}$, has the same limiting distribution as the KPSS test. According to their Monte Carlo simulation, the size distortion of $\hat{\eta}_T$ is not so severe as that of η_T in some cases.

The above testing procedures are extended to the test for the null of stationarity against the alternative of seasonal unit roots in Canova and Hansen (1995) and Caner (1998). It may be seen that the former extends the KPSS test to the seasonal model and the latter the Leybourne-McCabe test.

Canova and Hansen (1995) considered the following model with s seasons,

$$y_t = \mu + x_t'\beta + f_t'\gamma_t + e_t, \quad \gamma_t = \gamma_{t-1} + u_t, \qquad (1-5)$$

where $\{u_t\}$ is $i.i.d.(0, \sigma_u^2)$, $\{e_t\} \sim (0, \sigma_e^2)$ satisfies mixing-type conditions as in Phillips (1987) and is uncorrelated with $\{x_t\}$, $\{x_t\}$ is any nontrending variables that satisfy standard weak dependence conditions, γ_0 ($\neq 0$) is a fixed $q \times 1$ vector with $q = s/2$, and $f_t = [f_{1t}', \cdots, f_{qt}']'$ with $f_{jt} = [\cos((j/q)\pi t), \sin((j/q)\pi t)]'$ for $j < q$ and $f_{qt} = \cos(\pi t) = (-1)^t$. It can be shown that $\mu + f_t'\gamma$ in the model (1-5) is equivalent to the dummy variables $\sum_{j=1}^{s} \delta_j D_{jt}$. They also considered that, to test for seasonal unit roots at only a subset of the seasonal frequencies, the formulation of γ is modified as

$$A'\gamma_t = A'\gamma_{t-1} + u_t,$$

where A is a full rank $(s-1) \times a$ matrix which selects the a elements of γ_t that we wish to test for nonstationarity. For example, to test whether the entire vector γ is stable, set $A = I_{s-1}$ and to test for unit roots only at frequency θ ($< \pi$), set $A = (0, I_2, 0)'$ and for frequency π, set $A = (0, 1)'$. Note that this model does not include a non-seasonal unit root.

The Canova-Hansen Test is constructed as

$$L = \frac{1}{T^2} \sum_{t=1}^{T} \hat{F}_t' A (A' \hat{\Omega}^f A)^{-1} A' \hat{F}_t, \quad \hat{\Omega}^f = \sum_{j=-l}^{l} w(j,l) \frac{1}{T} \sum_{t=1}^{T-j} f_{t+j} \hat{e}_{t+j} f_t' \hat{e}_t,$$

where $\hat{e}_t$ is the regression residuals of y_t on 1, x_t and f_{it}, $\hat{F}_t = \sum_{j=1}^{t} f_t \hat{e}_t$ and $w(j,l)$ is the Bartlett window. Note that $\hat{\Omega}^f$ is a consistent estimator of the long-run covariance matrix of $f_t e_t$, $\Omega^f = \lim_{T\to\infty} T^{-1} E[F_T F_T']$ with $F_T = \sum_{j=1}^{T} f_t e_t$. It is shown that

$$L \xrightarrow{d} \int_0^1 W_a(r)' W_a(r) dr \equiv VM(a),$$

where $W_a(\cdot)$ is an a-dimensional standard Brownian bridge.

On the other hand, Caner (1998) extended the model of Leybourne and McCabe (1994) to

$$\phi(B) y_t = \mu + f_t' \gamma_t + e_t, \quad \gamma_t = \gamma_{t-1} + u_t, \quad \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p. \qquad (1-6)$$

They proposed the following test statistic:

$$D = \frac{1}{\hat{\sigma}_e^2 T^2} \sum_{t=1}^{T} \hat{F}_t^{*\prime} G \hat{F}_t^*,$$

where $\hat{\sigma}_e^{*2} = T^{-1}\sum_{j=1}^T \hat{e}_j^{*2}$, $\hat{F}_t^* = \sum_{j=1}^t f_j \hat{e}_j^*$ and $G = diag\{2, \cdots, 2, 1\}$. $\hat{e}_t^*$ is regression residuals of y_t^* on a constant and seasonal dummies, where $y_t^* = y_t - \sum_{j=1}^p \phi_j^* y_{t-j}$ with ϕ_j^*'s the maximum likelihood estimators of ϕ's from the fitted model,

$$\tilde{y}_t = \tilde{\mu} + \sum_{j=1}^p \phi_j \tilde{y}_{t-j} + \Theta(B)\zeta_t,$$

where $\tilde{y}_t = (1 + L + \cdots + L^{s-1})y_t$ and $\Theta(B)$ an MA$(s-1)$ polynomial. They showed that D has the same limiting distribution as L, $D \xrightarrow{d} VM(s-1)$. The test for a subset of seasonal unit roots is also conducted as in Canova and Hansen's method.

On the other hand, Tam and Reinsel (1997, 1998) and Tanaka (1996) developed testing procedures for moving average (MA) seasonal unit roots with $(1 - B^4)$, but the model considered in these papers are a little different from (1-5) and (1-6).

The above two models do not have a linear trend as well as a non-seasonal unit root. However, many macroeconomic time series are trending variables. Then, in Chapter 2, we will consider the Canova-Hansen type model with a linear trend and investigate the limiting power properties of the test statistics. We will also consider the joint test for seasonal and non-seasonal unit roots as well as for each seasonal unit root.

3. Testing for a Periodic Unit Root

One of the other time series models which are useful to represent a seasonal behavior is a periodic autoregressive (PAR) model investigated by Gladyshev (1961), Pagano (1978), Tiao and Grupe (1980) and Troutman (1979), among others. A typical example is a quarterly PAR(1) model,

$$y_t = \sum_{s=1}^4 \phi_s D_{st} y_{t-1} + e_t. \qquad (1-7)$$

The important property of the PAR model is that the AR parameters vary with seasons. For example, when t corresponds to the first quarter, $\{y_t\}$ is generated from the AR(1) process, $y_t = \phi_1 y_{t-1} + e_t$, while for the second quarter, $y_t = \phi_2 y_{t-1} + e_t$ is an appropriate model. Then, the effect of one quarter to the other quarters is different with seasons. This assumption of seasonally varying parameters may be appropriate for the time series which have a seasonal pattern and whose seasonal behavior can not be well approximated by such as a seasonal deterministic term or a SARIMA model. Then, the seasonal properties of the observed data would be made clearer using the PAR model. This model can also be extended to have a periodic moving average disturbance.

Though the PAR model is not stationary, whether the process is stable or not depends on the PAR parameters. Stacking each variable to the 4×4 annualized vector, the model (1-7) can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_2 & 1 & 0 & 0 \\ 0 & -\phi_3 & 1 & 0 \\ 0 & 0 & -\phi_4 & 1 \end{bmatrix} \begin{bmatrix} y_{4j-3} \\ y_{4j-2} \\ y_{4j-1} \\ y_{4j} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\phi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{4(j-1)-3} \\ y_{4(j-1)-2} \\ y_{4(j-1)-1} \\ y_{4(j-1)} \end{bmatrix} + \begin{bmatrix} e_{4j-3} \\ e_{4j-2} \\ e_{4j-1} \\ e_{4j} \end{bmatrix},$$

or

$$\Phi_0 Y_j = \Phi_1 Y_{j-1} + E_j,$$

where, e.g., $Y_j = [y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}]'$. Then, the model (1-7) can be seen to be the four dimensional vector autoregressive (VAR) process of order one, so that whether $\{Y_j\}$ is stationary, or whether $\{y_t\}$ is stable, depends on the root of $|\Phi_0 - \Phi_1 z| = |1 - \phi_1\phi_2\phi_3\phi_4 z| = 0$. If $|\phi_1\phi_2\phi_3\phi_4| < 1$, $\{y_t\}$ is called periodically stationary and if $\phi_1\phi_2\phi_3\phi_4 = 1$, periodically integrated of order one, PI(1), and in this case, $\{y_t\}$ is said to have a periodic unit root. This definition is extended to the more general PAR(p) model in Chapter 3.

Using the PAR model, we can express the nonstationary aspect of the economic time series with a concept of periodic integration. Intuitively, a periodically integrated AR (PIAR) model has a single stochastic trend just like a unit root process and seasonally varying AR parameters. Then, the PIAR model may be suitable for expressing the time series data which seems to be nonstationary and has a seasonal variation.

Testing for a periodic unit root of the PAR(1) model is proposed in Boswijk and Franses (1995), and for the PAR(p) model, the testing procedure is developed in Boswijk and Franses (1996) and Boswijk, Franses and Haldrup (1997). Let us consider the following PAR(p) model,

$$y_t = \sum_{s=1}^{4} \phi_{1s} D_{st} y_{t-1} + \cdots + \sum_{s=1}^{4} \phi_{ps} D_{st} y_{t-p} + e_t. \qquad (1-8)$$

According to Boswijk and Franses (1996), if $\{y_t\}$ has a periodic unit root, (1-8) can be rewritten as

$$y_t = \sum_{s=1}^{4} \delta_s D_{st} y_{t-1} + \sum_{i=1}^{p-1} \psi_{is} D_{st} (y_{t-i} - \delta_{s-i} y_{t-i-1}) + e_t, \qquad (1-9)$$

where δ_s's satisfy the condition $\delta_1\delta_2\delta_3\delta_4 = 1$ and we use the convention $\delta_{s-4k} = \delta_s$ for $k \in \mathbf{N}$. Denoting the log likelihood from the model (1-9) as $\tilde{\mathcal{L}}$ (the restricted maximized log likelihood) and that from the model (1-8) as $\hat{\mathcal{L}}$ (the unrestricted maximized log likelihood), the likelihood ratio (LR) statistic is defined by $LR = -2(\tilde{\mathcal{L}} - \hat{\mathcal{L}})$. Boswijk and Franses (1996) showed that

$$LR \xrightarrow{d} \left(\int_0^1 W^2(r)dr \right)^{-1} \left(\int_0^1 W(r)dW(r) \right)^2 .$$

Similar results are obtained for the model with a constant and a linear trend.

The above test is constructed for the null hypothesis of a periodic unit root. From the same reason as the seasonal unit roots test, the test for the null of periodic stationarity against a periodic unit root is also important, but such a testing procedure is not proposed in the literature. In Chapter 3, we will derive the LM test statistic for the null of periodic stationarity and investigate its limiting properties as well as finite sample properties.

4. Testing for a Unit Root with a Break

Testing for a unit root against stationarity with a break has been investigated from the work of Perron (1989) and it has been discussed whether there is persistence in the macroeconomic time series or not. By Perron (1989), it is shown that the DF test can not reject the null hypothesis of a unit root under the alternative of trend stationarity with a structural change, that is, the DF test does not have the non-trivial power against stationarity with a break. Perron (1989) considered the following three models of a break:

$$\begin{aligned}
\text{Model } A^{AO} &: \; y_t = \mu_1 + \beta t + \mu_2 DU_t + x_t, \\
\text{Model } B^{AO} &: \; y_t = \mu + \beta_1 t + \beta_2 DT_t + x_t, \\
\text{Model } C^{AO} &: \; y_t = \mu_1 + \beta_1 t + \mu_2 DU_t + \beta_2 DT_t + x_t,
\end{aligned}$$

where $DU_t = 1(t > T_B)$ with $1(\cdot)$ an indicator function and T_B a date of a break and $DT_t = 1(t > T_B) \times (t - T_B)$. For $\{x_t\}$, it is assumed that $\phi(B)x_t = e_t$ where $\phi(B) = (1 - \alpha B)\phi^*(B)$

is a lag polynomial of order $p+1$ with $\phi^*(B)$ invertible and $\{e_t\} \sim i.i.d.(0, \sigma_e^2)$. It is assumed that $\alpha = 1$ under the null hypothesis and $|\alpha| < 1$ under the alternative. The above model is said to be the additive outlier (AO) model, that is, a shock has an effect on y_t only at one time. On the other hand, the following model is called the innovational outlier (IO) model, in which the effect of a structural change pervades the variables with lags.

$$\begin{aligned}
\text{Model } A^{IO} &: \phi(B)y_t = \mu_1 + \beta t + \mu_2 DU_t + e_t, \\
\text{Model } B^{IO} &: \phi(B)y_t = \mu + \beta_1 t + \beta_2 DT_t + e_t, \\
\text{Model } C^{IO} &: \phi(B)y_t = \mu_1 + \beta_1 t + \mu_2 DU_t + \beta_2 DT_t + e_t.
\end{aligned}$$

Under the assumption that the break date T_B is known, the testing procedures for a unit root in the above models are proposed and critical points are tabulated.

However, the exogeneity of a break point is sometimes criticized, since the break date is chosen by visual inspection of the data and the date chosen as a break point also depends on the data. Then, it is insisted that the break date should be treated as an unknown parameter. Banerjee et al. (1992), Perron (1997) and Zivot and Andrews (1992) developed the testing procedures for a unit root with an unknown break point. In these papers, a structural change is not assumed under the null of a unit root. On the other hand, Vogelsang and Perron (1998) considered unit root tests that allow a shift in a trend at an unknown time under the null.

The testing procedures are different between the AO model and the IO model. Testing for a unit root in the AO model consists of two steps. The first step involves detrending the series by the following regressions,

$$\begin{aligned}
\text{Model } A^{AO} &: y_t = \mu_1 + \beta t + \mu_2 DU_t + \tilde{x}_t, \\
\text{Model } B^{AO} &: y_t = \mu + \beta_1 t + \beta_2 DT_t + \tilde{x}_t, \\
\text{Model } C^{AO} &: y_t = \mu_1 + \beta_1 t + \mu_2 DU_t + \beta_2 DT_t + \tilde{x}_t.
\end{aligned}$$

The next step is that the unit root hypothesis is tested using the t-statistic for testing $\alpha = 1$ in the following regressions,

$$\text{Models } A^{AO},\ C^{AO} \quad : \quad \tilde{x}_t = \sum_{i=1}^{p} \omega_i D(T_B)_{t-i} + \alpha \tilde{x}_{t-1} + \sum_{i=1}^{p} c_i \Delta \tilde{x}_{t-i} + e_t,$$

$$\text{Model } B^{AO} \quad : \quad \tilde{x}_t = \alpha \tilde{x}_{t-1} + \sum_{i=1}^{p} c_i \Delta \tilde{x}_{t-i} + e_t,$$

where $D(T_B)_t = 1(t = T_B + 1)$.

For the IO model, the t-statistics for testing $\alpha = 1$ are used in the following model.

$$\text{Model } A^{IO} \quad : \quad y_t = \mu_1 + \beta t + dD(T_B)_t + \mu_2 DU_t + \alpha y_{t-1} + \sum_{i=1}^{p} c_i \Delta y_{t-i} + e_t,$$

$$\text{Model } B^{IO} \quad : \quad y_t = \mu + \beta_1 t + \beta_2 DT_t + \alpha y_{t-1} + \sum_{i=1}^{p} c_i \Delta y_{t-i} + e_t,$$

$$\text{Model } C^{IO} \quad : \quad y_t = \mu_1 + \beta_1 t + dD(T_B)_t + \mu_2 DU_t + \beta_2 DT_t + \alpha y_{t-1} + \sum_{i=1}^{p} c_i \Delta y_{t-i} + e_t.$$

To construct the t-statistics in the above regressions, we need to determine the break date T_B. One selecting method is to choose T_B such that the t-statistic for testing $\alpha = 1$ is minimized. This strategy is based on an idea that the selection of a break point is the outcome of an estimation procedure designed to fit $\{y_t\}$ to a certain trend stationary representation. Then, this strategy gives the most weight to the trend stationarity alternative to choose the break date. The other method is that T_B is chosen so that the absolute value of the t-statistic for testing $\mu_2 = 0$ or $\beta_2 = 0$, or the F-statistic for testing $[\mu_2, \beta_2] = [0, 0]$, is maximized.

The limiting distributions of the test statistics depend on which selecting methods of T_B to be used as well as whether a structural change is assumed under the null hypothesis or not. For each cases, critical points are collected from the above papers and tabulated in Vogelsang and Perron (1998).

In Chapter 4, we will investigate the test for the null of stationarity with a break against a unit root. In our testing procedure, a structural change is always assumed under the null hypothesis.

Chapter 2.

The Limiting Properties of Seasonal and/or Non-Seasonal Unit Roots Tests

In this chapter we investigate unit roots tests with the quarterly seasonal model, extending the model of Kwiatkowski, Phillips, Schmidt and Shin (1992). We derive the LM test statistic, which is slightly different from that of Canova and Hansen (1995) and Caner (1998), for the null hypothesis of stationarity against the alternative hypothesis of nonstationarity with seasonal and/or non-seasonal unit roots. We develop the asymptotic theory of this statistic under both the null and the alternative. We also investigate the test against the alternative of nonstationarity, not specifying particular unit roots.

1. Introduction

In this chapter, we consider both seasonal and non-seasonal unit roots under the alternative with a deterministic linear trend and develop the testing procedures. Though some of our procedures are the same as Canova and Hansen (1995) and Caner (1998), we investigate the properties of the tests by deriving the explicit local power functions using the Fredholm approach, which is extensively developed in such as Nabeya and Tanaka (1988) and Tanaka (1990a, 1990b, 1996), and these functions would help us to understand the properties of the tests more clearly. We also consider the test for the null of stationarity against the alternative of nonstationarity without specifying the particular type of roots, since we are sometimes interested not in the particular type of roots but only in whether the process is stationary or not. Then we need to include a non-seasonal unit root as well as seasonal unit roots in the model, unlike the model considered in Canova and Hansen (1995) and Caner (1998). Using the Fredholm approach again, we will find the consistent test against the alternative of unknown roots.

This chapter proceeds as follows. In Section 2 we see the model and notations. In Section 3 we derive the LM test for the null of stationarity against the alternative of the existence of particular unit roots. The limiting distribution and its characteristic function will be derived both under the null hypothesis and under a sequence of local alternatives. The explicit power functions will be derived by the numerical integration and compared each other. Section 4 develops the test for the null of stationarity against nonstationarity, not assuming the particular type of roots. Section 5 concludes this chapter. All proofs are given in Appendix 2.

2. The Model and Notations

Consider the following quarterly model.

$$y_t = x_t'\beta + w_t, \quad w_t = r_t + u_t, \quad A_i(B)r_t = \sqrt{\kappa_i}\epsilon_t, \quad (i = 1, \cdots, 7, \ t = 1, \cdots, T) \qquad (2-1)$$

where x_t is a deterministic component, $\{u_t\}$ and $\{\epsilon_t\}$ are independent and $NID(0, \sigma_u^2)$ with $\sigma_u^2 > 0$, $NID(0, \sigma_\epsilon^2)$ with $\sigma_\epsilon^2 \geq 0$, respectively, $\kappa_1 = \kappa_2 = 1$, $\kappa_3 = \kappa_4 = \kappa_5 = \kappa_6 = 2$, $\kappa_7 = 4$, and we set $r_0 = r_{-1} = r_{-2} = r_{-3} = 0$. $A_i(B)$ denotes one of the lag polynomials defined in (1-2), and we assume that $N = T/4$ is an integer. κ_i has a role for r_T to have the same variance, $T\sigma_\epsilon^2$, for each i. Though it does not affect the properties of the test itself, we compare each test in the later section and then, for fair comparisons, we need to have the same variance of r_T for each $A_i(B)$. Note that when $\sigma_\epsilon^2 > 0$, the variance of $\{r_t\}$ tends to infinity as the process evolves. Then, in this chapter, we say $\{y_t\}$ is nonstationary when $\sigma_\epsilon^2 > 0$, that is, when it has the nonstationary stochastic component.

Our model (2-1) is slightly different from that of Canova and Hansen (1995) and Caner (1998). In their model, the seasonal stochastic component r_t is in an additive form,

$$r_t = f_t'\gamma_t, \quad \gamma_t = \gamma_{t-1} + e_t, \tag{2-2}$$

where $f_t = (\cos(\pi t/2), \sin(\pi t/2), \cos(\pi t))'$ and $e_t \sim i.i.d.(0, \sigma_\epsilon^2\Omega_e)$ with $\Omega_e = diag(2,2,1)$, while the corresponding component in our model is in a multiplicative form as in (2-1). We use the model (2-1) only for notational convenience. In fact, we can investigate the same testing problem with the model (2-2) using the Fredholm approach, but such investigations become notationally more complicated. Moreover, we can show that the same limiting distributions in this chapter are derived using the model (2-2) with $\Omega_e = diag(1,1,1)$, and then we may interpret our results as derived by slight modification of the Canova-Hansen model.

Note also that we allow (2-1) to have a deterministic linear trend and a non-seasonal unit root, and then, in this sense, our model may be seen as an extension of the Canova-Hansen model. Inclusion of a non-seasonal unit root enables us to test against the existence of seasonal and non-seasonal unit roots jointly and also against the alternative of nonstationarity with unknown unit roots.

Stacking each variable from $j = 1$ to T, we have

$$y = X\beta + w, \quad w = r + u, \quad r = \sqrt{\kappa_i}L_i\epsilon,$$

where, e.g., $y' = [y_1, \cdots, y_T]$, and

$$L_i = \begin{bmatrix} V_{i0} & & & \mathbf{0} \\ V_{i1} & V_{i0} & & \\ \vdots & \ddots & \ddots & \\ V_{i1} & \cdots & V_{i1} & V_{i0} \end{bmatrix},$$

for $i = 1, \cdots, 7$ with

$$V_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad V_{11} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$V_{20} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \quad V_{21} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

$$V_{30} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad V_{31} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$V_{40} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad V_{41} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$V_{50} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad V_{51} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$V_{60} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad V_{61} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$V_{70} = V_{71} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4,$$

where I_j denotes the $j \times j$ identity matrix. Note that L_i^{-1} corresponds to $A_i(B)$ for $i = 1, \cdots, 7$.

We specify the deterministic term X as follows.

Case A : A seasonal constant and a linear trend.

$$X = [d_1, d_2] \text{ with } d_1' = [I_4, \cdots, I_4], \text{ and } d_2' = [1, 2, \cdots, T].$$

Case B : A seasonal constant.

$$X = [d_1] \text{ with } d_1' = [I_4, \cdots, I_4].$$

Case C : No deterministic term.

It follows that the dimension of β varies according to the definition of X.

Here we define the following functions, which will be used in the later sections.

$$\begin{aligned} K_1(s,t) &= \min(s,t) - 4st + 3st(s+t) - 3s^2t^2. \\ K_2(s,t) &= \min(s,t) - st. \\ K_3(s,t) &= 1 - \max(s,t). \end{aligned}$$

$$\begin{aligned} D_1(\lambda) &= \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right). \\ D_2(\lambda) &= \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}. \\ D_3(\lambda) &= \cos\sqrt{\lambda}. \end{aligned}$$

Note that $D_i(\lambda)$ is the Fredholm determinant associated with the integral equation of the second kind,

$$f(t) = \lambda \int_0^1 K_i(s,t) f(s) ds,$$

for $i = 1, 2, 3$. We denote a sequence of eigenvalues associated with the above integral equation as $\{\lambda_{in}\}$.

3. Testing for Stationarity against Nonstationarity with Particular Roots

Let us consider the LM test for the testing problem

$$H_0 : \rho = 0 \quad v.s. \quad H_1^i : \rho = \frac{c^2}{N^2} \qquad (2-3)$$

where $\rho = \sigma_\epsilon^2/\sigma_u^2$, c is a constant, and H_1^i $(i = 1, \cdots, 7)$ denotes a particular alternative with $A_i(B)$. For example, H_1^1 denotes the alternative hypothesis that $\{r_t\}$ has a non-seasonal unit root while H_1^2 denotes the alternative of a negative unit root. So (2-3) signifies the testing problem, the null hypothesis of no unit roots against the local alternatives of particular roots.

From the assumption, we have

$$y \sim N\left(X\beta, \sigma_u^2(\kappa_i \rho L_i L_i' + I_T)\right).$$

Then the log likelihood except for a constant term is give by

$$\mathcal{L} = -\frac{1}{2}\ln|\Omega| - \frac{1}{2}(y - X\beta)'\Omega^{-1}(y - X\beta),$$

where $\Omega = \sigma_u^2(\kappa_i \rho L_i L_i' + I_T)$, and the first derivative of $\mathcal{L}$ with respect to ρ is given by

$$\frac{\partial \mathcal{L}}{\partial \rho} = -\frac{\kappa_i \sigma_u^2}{2} trace\left(\Omega L_i L_i'\right) + \frac{\kappa_i \sigma_u^2}{2}(y - X\beta)'\Omega^{-1} L_i L_i' \Omega^{-1}(y - X\beta).$$

Then the LM test for (2-3) is given by

$$\left.\frac{\partial \mathcal{L}}{\partial \rho}\right|_{H_0} = \frac{\kappa_i}{\tilde{\sigma}_u^2}(y - X\tilde{\beta})' L_i L_i'(y - X\tilde{\beta}) + \ a\ constant, \qquad (2-4)$$

as rejecting H_0 when (2-4) takes large values, where $\tilde{\beta}$ and $\tilde{\sigma}_u^2$ are the maximum likelihood estimators of β and σ_u^2 under H_0 and given by

$$\tilde{\beta} = (X'X)^{-1}X'y, \quad \tilde{\sigma}_u^2 = \frac{1}{T}\tilde{w}'\tilde{w},$$

where $\tilde{w} = My$ with $M = I_T - X(X'X)^{-1}X'$. See also Kwiatkowski, Phillips and Schmidt (1992) for its derivation. Then we use the following statistic,

$$S_i^j = \frac{\kappa_i}{N^2 \tilde{\sigma}_u^2}(y - X\tilde{\beta})' L_i L_i'(y - X\tilde{\beta}),$$

for $i = 1, \cdots, 7$ and $j = A, B, C$. If there seems to be no confusion, we omit the superscript j and abbreviate S_i^j as S_i. Note that the LM test is equivalent to the locally best invariant (LBI) test as discussed in such as King and Hillier (1985), KPSS (1992) and Tanaka (1996).

3.1. The Limiting Properties under H_0

In this section, we derive the limiting distribution of S_i^j for $j = A, B, C$ under H_0 and compare their distributions.

Let us consider the case A. Since $y \sim N(X\beta, \sigma_u^2 I_T)$ under H_0, we can see that

$$\begin{aligned} S_i &\stackrel{d}{=} \frac{\kappa_i \sigma_u^2}{N^2 \tilde{\sigma}_u^2} z' M L_i L_i' M z \\ &\stackrel{d}{=} \frac{\kappa_i \sigma_u^2}{N^2 \tilde{\sigma}_u^2} z' L_i' M L_i z \end{aligned}$$

where $z \sim N(0, I_T)$ and $\stackrel{d}{=}$ denotes equality in distribution. Since we can easily see that $\tilde{\sigma}_u^2$ converges to σ_u^2 in probability, it is enough to consider the following statistic.

$$\overline{S}_i = \frac{\kappa_i}{N^2} z' L_i' M L_i z. \qquad (2-5)$$

To derive the limiting distribution and the characteristic function of $\overline{S}_i$, we use the next lemma, which is due to Theorem 1 of Nabeya and Tanaka (1988).

Lemma 2.1 *Consider the following statistic.*

$$S_N = \frac{1}{N} \sum_{j,k=1}^{N} K\left(\frac{j}{N}, \frac{k}{N}\right) z_j' z_k, \qquad (2-6)$$

where $K(s,t)$ is a symmetric, continuous and positive definite function and $\{z_j\} \sim NID(0, I_q)$ with some fixed integer q.

Let $D(\lambda)$ be the Fredholm determinant of $K(s,t)$ and $\{\lambda_n\}$ a sequence of eigenvalues of $K(s,t)$. Then the limiting distribution of S_N is given by

$$S_N \stackrel{d}{\longrightarrow} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} Z_n' Z_n, \qquad (2-7)$$

where $Z_n \sim NID(0, I_q)$. The characteristic function of (2-7) is given by

$$\lim_{N\to\infty} E\left[e^{i\theta S_N}\right] = \left[\prod_{n=1}^{\infty} \left(1 - \frac{2i\theta}{\lambda_n}\right)^{-1/2}\right]^q = (D(2i\theta))^{-q/2}.$$

Proof: See Nabeya and Tanaka (1988).

Let us apply Lemma 2.1 to deal with $\overline{S}_i$ in (2-5). For this purpose, partition $L_i' M L_i$ into $N \times N$ blocks with each block a 4×4 matrix and denote the (j,k) block of $L_i' M L_i$ as $(L_i' M L_i)(j,k)$ for $j,k = 1, \cdots, N$. Then we can write

$$(L_i' M L_i)(j,k) = (L_i' L_i)(j,k) - \left(L_i' X (X'X)^{-1} X' L_i\right)(j,k).$$

Note that

$$(L_i' L_i)(j,k) = (N - \max(j,k))\, V_{i1}' V_{i1} + O(1), \qquad (2-8)$$

$$(X'X)^{-1} = \begin{bmatrix} \frac{1}{N} I_4 + \frac{3}{4N} e_4 e_4' & -\frac{3}{8N^2} e_4 \\ -\frac{3}{8N^2} e_4' & \frac{3}{16N^3} \end{bmatrix} + \begin{bmatrix} O(N^{-2}) & O(N^{-3}) \\ O(N^{-3}) & O(N^{-4}) \end{bmatrix}$$

and the j-th row block of $L_i'X$ is expressed as

$$L_i'X(j,\cdot) = \left[(N-j)V_{i1}', 2(N^2-j^2)V_{i1}'e_4\right] + [O(1), O(N)], \qquad (2-9)$$

where $e_4' = [1111]$. Then the (j,k) block of $L_i'ML_i/N$ can be expressed as

$$\frac{1}{N}(L_i'ML_i)(j,k) = F_i\left(\frac{j}{N}, \frac{k}{N}\right) + O(N^{-1}), \qquad (2-10)$$

for all j, k, where

$$\begin{aligned} F_i\left(\frac{j}{N}, \frac{k}{N}\right) &= \left\{1 - \max\left(\frac{j}{N}, \frac{k}{N}\right) - \left(1-\frac{j}{N}\right)\left(1-\frac{k}{N}\right)\right\} V_{i1}'V_{i1} \\ &\quad -\frac{1}{4}\left\{3\left(1-\frac{j}{N}\right)\left(1-\frac{k}{N}\right) - 3\left(1-\frac{j^2}{N^2}\right)\left(1-\frac{k}{N}\right)\right. \\ &\quad \left. -3\left(1-\frac{j}{N}\right)\left(1-\frac{k^2}{N^2}\right) + 3\left(1-\frac{j^2}{N^2}\right)\left(1-\frac{k^2}{N^2}\right)\right\} V_{i1}'e_4e_4'V_{i1} \\ &= K_2\left(\frac{j}{N}, \frac{k}{N}\right) V_{i1}'V_{i1} - \frac{1}{4}\left\{K_2\left(\frac{j}{N}, \frac{k}{N}\right) - K_1\left(\frac{j}{N}, \frac{k}{N}\right)\right\} V_{i1}'e_4e_4'V_{i1}. \end{aligned}$$

Note that the order of the second term of (2-10) is N^{-1} for all j, k. By (2-10) and Lemma 3 of Nabeya and Tanaka (1988), we have

$$\begin{aligned} E\left|\overline{S}_i - \frac{\kappa_i}{N}\sum_{j,k=1}^{N} z_j' F_i\left(\frac{j}{N}, \frac{k}{N}\right) z_k\right| &= E\left|\frac{\kappa_i}{N}\sum_{j,k=1}^{N} z_j'\left\{\frac{1}{N}(L_i'ML_i)(j,k) - F_i\left(\frac{j}{N}, \frac{k}{N}\right)\right\} z_k\right| \\ &\to 0, \end{aligned}$$

so that

$$\overline{S}_i - \frac{\kappa_i}{N}\sum_{j,k=1}^{N} z_j' F_i\left(\frac{j}{N}, \frac{k}{N}\right) z_k$$

converges to zero in probability from Markov's inequality. Then it is enough to consider the limiting distribution of

$$\tilde{S}_i = \frac{\kappa_i}{N}\sum_{j,k=1}^{N} z_j' F_i\left(\frac{j}{N}, \frac{k}{N}\right) z_k. \qquad (2-11)$$

Next let us consider to diagonalize $V_{i1}'V_{i1}$ and $V_{i1}'e_4e_4'V_{i1}$ to apply Lemma 2.1. Define a 4×4 orthogonal matrix P as

$$P = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & 1 & \sqrt{2} & 0 \end{bmatrix}. \qquad (2-12)$$

A little algebra reveals that $V_{i1}'V_{i1}$ and $V_{i1}'e_4e_4'V_{i1}$ are diagonalized at the same time using the matrix P.

Lemma 2.2 *$V_{i1}'V_{i1}$ and $V_{i1}'e_4e_4'V_{i1}$ $(i = 1, \cdots, 7)$ are diagonalized using the matrix P.*

$$
\begin{aligned}
P'V_{11}'V_{11}P &= diag(16,0,0,0). & P'V_{11}'e_4e_4'V_{11}P &= 4 \times diag(16,0,0,0).\\
P'V_{21}'V_{21}P &= diag(0,16,0,0). & P'V_{21}'e_4e_4'V_{21}P &= 0.\\
P'V_{31}'V_{31}P &= diag(0,0,4,4). & P'V_{31}'e_4e_4'V_{31}P &= 0.\\
P'V_{41}'V_{41}P &= diag(4,4,0,0). & P'V_{41}'e_4e_4'V_{41}P &= 4 \times diag(4,0,0,0).\\
P'V_{51}'V_{51}P &= diag(4,0,2,2). & P'V_{51}'e_4e_4'V_{51}P &= 4 \times diag(4,0,0,0).\\
P'V_{61}'V_{61}P &= diag(0,4,2,2). & P'V_{61}'e_4e_4'V_{61}P &= 0.\\
P'V_{71}'V_{71}P &= diag(1,1,1,1). & P'V_{71}'e_4e_4'V_{71}P &= 4 \times diag(1,0,0,0).
\end{aligned}
$$

Note that $P'V_{i1}'e_4e_4'V_{i1}P$ for $i = 2,3,6$ are zero since $V_{i1}'e_4 = 0$. By this relation and the equation (2-9), the j-th row block of $L_i'X$ is expressed as

$$L_i'X(j,\cdot) = [(N-j)V_{i1}', 0] + [O(1), O(N)],$$

for $i = 2,3,6$, so that a linear trend plays no role on the limiting distribution of S_i and then, as will be shown in the later, each limiting distribution in the case A is the same as that in the case B for $i = 2,3,6$.

Now applying Lemma 2.1 to $\tilde{S}_i$ using Lemma 2.2, we can obtain the limiting distribution of S_i and its characteristic function.

Theorem 2.1 *The LM test statistic S_i^A converges in distribution under H_0 and its characteristic function, $\phi_i^A(\theta; H_0)$, is given by*

$$
\begin{aligned}
\phi_1^A(\theta; H_0) &= [D_1(32i\theta)]^{-1/2}.\\
\phi_2^A(\theta; H_0) &= [D_2(32i\theta)]^{-1/2}.\\
\phi_3^A(\theta; H_0) &= [D_2(16i\theta)]^{-1}.\\
\phi_4^A(\theta; H_0) &= [D_1(16i\theta)]^{-1/2} \times [D_2(16i\theta)]^{-1/2}.
\end{aligned}
$$

$$\begin{aligned}
\phi_5^A(\theta; H_0) &= [D_1(16i\theta)]^{-1/2} \times [D_2(8i\theta)]^{-1}. \\
\phi_6^A(\theta; H_0) &= [D_2(16i\theta)]^{-1/2} \times [D_2(8i\theta)]^{-1}. \\
\phi_7^A(\theta; H_0) &= [D_1(8i\theta)]^{-1/2} \times [D_2(8i\theta)]^{-3/2}.
\end{aligned}$$

Next we derive the limiting distribution in the case B. As in the case A, note that

$$(X'X)^{-1} = \frac{1}{N} I_4, \qquad (2-13)$$

and the j-th row block of $L_i'X$ is expressed as

$$(L_i'X)(j,1) = (N-j)V_{i1}' + O(1). \qquad (2-14)$$

Then from (2-8), (2-13), and (2-14), the (j,k) block of $L_i'ML_i/N$ is given by

$$\frac{1}{N}(L_i'ML_i)(j,k) = K_2\left(\frac{j}{N}, \frac{k}{N}\right) V_{i1}'V_{i1} + O(N^{-1})$$

for all j, k. Then, from the same discussion as in the case A, it is enough to consider the limiting distribution of

$$\tilde{S}_i = \frac{\kappa_i}{N} \sum_{j,k=1}^{N} z_j' \left(K_2\left(\frac{j}{N}, \frac{k}{N}\right) V_{i1}'V_{i1}\right) z_k. \qquad (2-15)$$

Using the result of Lemma 2.2, we have the following theorem.

Theorem 2.2 *The LM test statistic S_i^B converges in distribution under H_0 and its characteristic function, $\phi_i^B(\theta; H_0)$, is given by*

$$\begin{aligned}
\phi_1^B(\theta; H_0) = \phi_2^B(\theta; H_0) &= [D_2(32i\theta)]^{-1/2}. \\
\phi_3^B(\theta; H_0) = \phi_4^B(\theta; H_0) &= [D_2(16i\theta)]^{-1}. \\
\phi_5^B(\theta; H_0) = \phi_6^B(\theta; H_0) &= [D_2(16i\theta)]^{-1/2} \times [D_2(8i\theta)]^{-1}. \\
\phi_7^B(\theta; H_0) &= [D_2(8i\theta)]^{-2}.
\end{aligned}$$

As is mentioned in the case A, we can see that $\phi_2^B(\theta; H_0)$, $\phi_3^B(\theta; H_0)$, and $\phi_6^B(\theta; H_0)$ are the same as $\phi_2^A(\theta; H_0)$, $\phi_3^A(\theta; H_0)$, and $\phi_6^A(\theta; H_0)$, respectively. We will also see that, under

H_1^i for $i = 2, 3, 6$, the limiting distribution of S_i^A is the same as S_i^B. Then a linear trend has no effect on the limiting distribution not only under the null hypothesis but also under the alternative hypothesis, for $i = 2, 3, 6$. Thus, even if we do not know whether a linear trend should be included or not in the model, we can test (2-3) using the model with a linear trend, having the same limiting power as the model with no linear trend.

Note that, as far as the test for a non-seasonal unit root is concerned (the case when $A_i(B) = A_1(B)$), the limiting distribution of the test statistic with a seasonal constant (and a linear trend) is the same as that with a non-seasonal constant (and a linear trend), which can be proved completely in the same way as Theorems 2.1 and 2.2. See also Section 9.10 of Tanaka (1996). Then our test statistic S_1 has the same limiting properties as those of the KPSS test. The only difference between them is that our test statistic is normalized by N^2 while the KPSS test is by T^2. Since $N = T/4$, we can see the relation '$S_1 = 16\times$ (the KPSS statistic)'. Thus our test for a non-seasonal unit root, or equivalently the KPSS test, is applicable to the model both with a seasonal constant and with a non-seasonal constant.

In the case C, we have no deterministic term and then the (j, k) block of $L_i'ML_i/N$ is given by

$$\frac{1}{N}(L_i'ML_i)(j,k) = \frac{1}{N}(L_i'L_i)(j,k) = K_3\left(\frac{j}{N}, \frac{k}{N}\right) V_{i1}'V_{i1} + O(N^{-1}).$$

Since we can derive the characteristic function in the case C as in Theorems 2.1 and 2.2, we omit a proof of the following theorem.

Theorem 2.3 *The LM test statistic S_i^C converges in distribution under H_0 and its characteristic function, $\phi_i^C(\theta; H_0)$, is given by*

$$\begin{aligned}
\phi_1^C(\theta; H_0) = \phi_2^C(\theta; H_0) &= [D_3(32i\theta)]^{-1/2}. \\
\phi_3^C(\theta; H_0) = \phi_4^C(\theta; H_0) &= [D_3(16i\theta)]^{-1}. \\
\phi_5^C(\theta; H_0) = \phi_6^C(\theta; H_0) &= [D_3(16i\theta)]^{-1/2} \times [D_3(8i\theta)]^{-1}. \\
\phi_7^C(\theta; H_0) &= [D_3(8i\theta)]^{-2}.
\end{aligned}$$

From the above three theorems, we can obtain the distribution functions, $G_i^A(x; H_0)$, $G_i^B(x; H_0)$, and $G_i^C(x; H_0)$, by inverting the characteristic functions. Since each limiting distribution is nonnegative, we have, using Lévy's inversion formula,

$$G_i^j(x;H_0) = \frac{1}{\pi}\int_0^\infty Re\left[\frac{1-e^{-i\theta x}}{i\theta}\phi_i^j(\theta;H_0)\right]d\theta,$$

for $i = 1, \cdots, 7$ and $j = A, B, C$.

Tables 1, 2, and 3 show percent points of the limiting distributions for the cases A, B, and C, respectively. Note that, as shown in Theorems 2.1 and 2.2, we have $G_2^A = G_2^B$, $G_3^A = G_3^B$ and $G_6^A = G_6^B$, so that the rows corresponding to these cases in Tables 1 and 2 are completely the same. From each table, we can see that, for a fixed i, $G_i^B(x;H_0)$ is located to the right compared with $G_i^A(x;H_0)$ and so is $G_i^C(x;H_0)$ compared with $G_i^B(x;H_0)$. For example, the 95% points of $G_1^j(x;H_0)$ are about 2.4, 7.4, 26.5 for $j = A, B, C$, respectively. That is, the less complicated the deterministic term is, the further is the limiting distribution shifted to the right.

On the other hand, we can see that, for $i < j$, the limiting distribution of S_i has fatter tails than S_j except for the case A. In other words, for the cases B and C, the less the number of roots is, the fatter is the both tails of the limiting distribution.

3.2. The Limiting Properties under H_1^i

As in the previous section, we derive the limiting distribution of S_i in each case under H_1^i.

Firstly we consider the case A. Since $y \sim N(X\beta, \sigma_u^2(\kappa_i\rho L_iL_i' + I_T))$ under H_1^i, we can see that

$$\begin{aligned} S_i &\overset{d}{=} \frac{\kappa_i\sigma_u^2}{N^2\tilde{\sigma}_u^2}z'(\kappa_i\rho L_iL_i' + I_T)^{1/2}ML_iL_i'M(\kappa_i\rho L_iL_i' + I_T)^{1/2}z \\ &\overset{d}{=} \frac{\kappa_i\sigma_u^2}{N^2\tilde{\sigma}_u^2}z'L_i'M(\kappa_i\rho L_iL_i' + I_T)ML_iz. \end{aligned}$$

As in the previous section, we can see that $\tilde{\sigma}_u^2$ converges to σ_u^2 in probability under H_1^i, so that it is enough to consider the following statistic.

$$\begin{aligned} \overline{S}_i &= \frac{\kappa_i}{N^2}z'L_i'M(\kappa_i\rho L_iL_i' + I_T)ML_iz \\ &= \frac{\kappa_i}{N}z'\left\{\frac{1}{N}L_i'ML_i + \frac{\kappa_ic^2}{N^3}(L_i'ML_i)^2\right\}z \\ &= \frac{\kappa_i}{N}\sum_{j,k=1}^N z_j'\left\{\frac{1}{N}(L_i'ML_i)(j,k) + \frac{\kappa_ic^2}{N}\sum_{l=1}^N\frac{1}{N}(L_i'ML_i)(j,l)\frac{1}{N}(L_i'ML_i)(l,k)\right\}z_k. \end{aligned}$$

To derive the limiting distribution, we need the similar result to Lemma 2.1. We use the following lemma, which is due to Nabeya (1989) and Tanaka (1996).

Lemma 2.3 *Consider the following statistic.*

$$S_N = \frac{1}{N}\sum_{j,k=1}^{N}\left\{K\left(\frac{j}{N},\frac{k}{N}\right) + \frac{\gamma}{N}\sum_{l=1}^{N} K\left(\frac{j}{N},\frac{l}{N}\right) K\left(\frac{l}{N},\frac{k}{N}\right)\right\} z_j' z_k,$$

where γ is a constant real value, $K(s,t)$ is a symmetric, continuous and positive definite function and $\{z_j\} \sim NID(0, I_q)$ with some fixed integer q.

Let $D(\lambda)$ be the Fredholm determinant of $K(s,t)$ and $\{\lambda_n\}$ a sequence of eigenvalues of $K(s,t)$. Then the limiting distribution of S_N is given by

$$S_N \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2}\right) Z_n' Z_n, \qquad (2-16)$$

where $Z_n \sim NID(0, I_q)$. The characteristic function of (2-16) is given by

$$\begin{aligned}
\lim_{N\to\infty} E\left[e^{i\theta S_N}\right] &= \left[\prod_{n=1}^{\infty}\left\{1 - 2i\theta\left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2}\right)\right\}^{-1/2}\right]^q \\
&= \left(D\left(i\theta + \sqrt{-\theta^2 + 2i\gamma\theta}\right) D\left(i\theta - \sqrt{-\theta^2 + 2i\gamma\theta}\right)\right)^{-q/2}.
\end{aligned}$$

Proof: See Theorem 5.13 of Tanaka (1996).

To use the above lemma, we need the similar result to (2-10). Note that

$$\begin{aligned}
&\frac{1}{N}\sum_{l=1}^{N}\frac{1}{N}(L_i'ML_i)(j,l)\frac{1}{N}(L_i'ML_i)(l,k) - \frac{1}{N}\sum_{l=1}^{N} F_i\left(\frac{j}{N},\frac{l}{N}\right) F_i\left(\frac{l}{N},\frac{k}{N}\right) \\
&= \frac{1}{N}\sum_{l=1}^{N}\left\{\left[\frac{1}{N}(L_i'ML_i)(j,l) - F_i\left(\frac{j}{N},\frac{l}{N}\right)\right]\frac{1}{N}(L_i'ML_i)(l,k)\right. \\
&\quad \left. + F_i\left(\frac{j}{N},\frac{l}{N}\right)\left[\frac{1}{N}(L_i'ML_i)(l,k) - F_i\left(\frac{l}{N},\frac{k}{N}\right)\right]\right\} \\
&= O(N^{-1}), \qquad (2\text{-}17)
\end{aligned}$$

for all j,k, since the equations in the square brackets are $O(N^{-1})$ by (2-10) and $(L_i'ML_i)(j,k)/N$ and $F_i(j/N,k/N)$ are bounded uniformly for all j,k,N. Then, by (2-10), (2-17) and Lemma 3 of Nabeya and Tanaka (1988), we have

$$E\left|\overline{S}_i - \tilde{S}_i\right| \to 0,$$

where

$$\tilde{S}_i = \frac{\kappa_i}{N}\sum_{j,k=1}^{N} z_j' \left\{ F_i\left(\frac{j}{N},\frac{k}{N}\right) + \frac{\kappa_i c^2}{N}\sum_{l=1}^{N} F_i\left(\frac{j}{N},\frac{l}{N}\right) F_i\left(\frac{l}{N},\frac{k}{N}\right)\right\} z_k, \qquad (2-18)$$

so that $\overline{S}_i - \tilde{S}_i$ converges to zero in probability. Then it is enough to consider the limiting distribution of $\tilde{S}_i$.

Using Lemma 2.2, we have the following theorem.

Theorem 2.4 *The LM test statistic S_i^A converges in distribution under H_1^i and its characteristic function, $\phi_i^A(\theta; H_1^i)$, is given by*

$$\begin{aligned}
\phi_1^A(\theta; H_1^1) &= \left[D_1\left(16(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_1\left(16(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2}. \\
\phi_2^A(\theta; H_1^2) &= \left[D_2\left(16(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(16(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2}. \\
\phi_3^A(\theta; H_1^3) &= \left[D_2\left(8(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(8(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1}. \\
\phi_4^A(\theta; H_1^4) &= \left[D_1\left(8(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_1\left(8(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(8(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(8(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2}. \\
\phi_5^A(\theta; H_1^5) &= \left[D_1\left(8(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_1\left(8(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1}. \\
\phi_6^A(\theta; H_1^6) &= \left[D_2\left(8(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(8(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1}. \\
\phi_7^A(\theta; H_1^7) &= \left[D_1\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_1\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-3/2}.
\end{aligned}$$

We can obtain the characteristic functions for the cases B and C in the same way as above and then we omit proofs of the following two theorems.

Theorem 2.5 *The LM test statistic S_i^B converges in distribution under H_1^i and its characteristic function, $\phi_i^B(\theta; H_1^i)$, is given by*

$$\phi_1^B(\theta; H_1^1) = \phi_2^B(\theta; H_1^2) = \left[D_2\left(16(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(16(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1/2}$$

$$\phi_3^B(\theta;H_1^3)=\phi_4^B(\theta;H_1^4) = \left[D_2\left(8(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_2\left(8(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1}.$$

$$\phi_5^B(\theta;H_1^5)=\phi_6^B(\theta;H_1^6) = \left[D_2\left(8(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_2\left(8(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1/2} \times\left[D_2\left(4(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_2\left(4(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1}.$$

$$\phi_7^B(\theta;H_1^7) = \left[D_2\left(4(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_2\left(4(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-2}.$$

Theorem 2.6 *The LM test statistic S_i^C converges in distribution under H_1^i and its characteristic function, $\phi_i^C(\theta;H_1^i)$, is given by*

$$\phi_1^C(\theta;H_1^1)=\phi_2^C(\theta;H_1^2) = \left[D_3\left(16(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_3\left(16(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1/2}$$

$$\phi_3^C(\theta;H_1^3)=\phi_4^C(\theta;H_1^4) = \left[D_3\left(8(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_3\left(8(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1}.$$

$$\phi_5^C(\theta;H_1^5)=\phi_6^C(\theta;H_1^6) = \left[D_3\left(8(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_3\left(8(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1/2} \times\left[D_3\left(4(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_3\left(4(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1}.$$

$$\phi_7^C(\theta;H_1^7) = \left[D_3\left(4(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_3\left(4(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-2}.$$

As in the previous section, we can obtain the limiting distribution functions, $G_i^A(x;H_1^i)$, $G_i^B(x;H_1^i)$, and $G_i^C(x;H_1^i)$, by inverting the characteristic functions. Then the limiting power functions are given by

$$g_i^j(c;H_1^i)=P\left(G_i^j(x;H_1^i)\geq x^*\right)=1-\frac{1}{\pi}\int_0^\infty Re\left[\frac{1-e^{-i\theta x^*}}{i\theta}\phi_i^j(\theta;H_1^i)\right]d\theta,$$

for $i=1,\cdots,7$ and $j=A,B,C$ as a function of c, where x^* denotes a percent point corresponding to the significance level. We calculate the limiting powers using the upper 5% points in Tables 1, 2 and 3. Note that the above tests are all consistent in the sense that $g_i^j(x;H_1^i)\to 1$ as $c\to\infty$ for $i=1,\cdots,7$ and $j=A,B,C$.

Figure 1a shows the limiting powers in the case A. Comparing with each limiting power of the test statistic, we can see that, for small values of c, S_2^A (corresponding to $g_2^A(c;H_1^2)$) is most powerful whereas S_1^A (corresponding to $g_1^A(c;H_1^1)$) is least powerful, that is, the LM test for a negative unit root is most powerful and the LM test for a non-seasonal unit root is least powerful in the case A. We also note that $g_2^A(c;H_1^2)$, $g_3^A(c;H_1^3)$, and $g_6^A(c;H_1^6)$, which are not affected by a linear trend, dominate the other power functions in the limit until the powers reach around 0.65.

Figures 1b and 1c show the limiting powers in the cases B and C, respectively. In both cases, the more the number of roots increases, the less powerful is the test for small values of c, whereas for large values of c, this relation is completely reversed and the test for the full unit roots is most powerful.

Figures 2a-2d show the limiting power function $g_i^j(c; H_1^i)$ for $i = 1,4,5,7$ and $j = A, B, C$, respectively. We can see that the more complicated the deterministic term is, the less powerful is the test. Note that $\phi_2^A(\theta; H_1^2) = \phi_1^B(\theta; H_1^1) = \phi_2^B(\theta; H_1^2)$, $\phi_3^A(\theta; H_1^3) = \phi_3^B(\theta; H_1^3) = \phi_4^B(\theta; H_1^4)$ and $\phi_6^A(\theta; H_1^6) = \phi_5^B(\theta; H_1^5) = \phi_6^B(\theta; H_1^6)$ and they are drawn in Figures 2a-2d.

4. Testing for Stationarity against Nonstationarity with Unknown Roots

In the previous section, we considered the null hypothesis of stationarity against the alternative hypothesis of nonstationarity with particular roots. We can test the hypothesis using the LM statistic S_i if we are interested in such particular roots.

However, in some cases, econometricians do not have interest in particular roots and wish to know only whether the observed variable is stationary or nonstationary. Then, in this section, we consider the test for the null hypothesis of stationarity against the alternative of nonstationarity with unknown roots, that is, we consider the following testing problem:

$$H_0 : \rho = 0 \quad v.s. \quad H_1 : \rho = \frac{c^2}{N^2}. \qquad (2-19)$$

We can interpret the alternative hypothesis H_1 as

$$H_1 = \bigcup_{i=1}^{7} H_1^i. \qquad (2-20)$$

Then H_1 is a composite alternative hypothesis, even if we fix the value of ρ, so that there exists no LBI test for the testing problem (2-19). That is, we can not derive an optimal test for (2-19).

Instead of the LBI test, let us consider the test which is LBI against a particular alternative H_1^i and consistent against the other hypotheses H_1^m for $m \neq i$. Then it seems that the possible test statistic is one of the LM tests derived in the previous section, since S_i is the LBI test against H_1^i and might be consistent against the other hypotheses H_1^m for $m \neq i$. If so, our interest may be which test statistic is desirable among S_i $(i = 1, \cdots, 7)$ in view of the power.

Let us consider the test statistic S_i^A for a fixed i. Since H_1 can be interpreted as (2-20), we investigate the power properties of S_i^A under each H_1^m for $m \neq i$. Note that $y \sim N(X\beta, \sigma_u^2(\kappa_m \rho L_m L_m' + I_T))$ under H_1^m and that $\tilde{\sigma}_u^2$ converges to σ_u^2 in probability as in the previous section. Then it is enough to consider the following statistic:

$$\begin{aligned}\overline{S}_{im} &= \frac{\kappa_i}{N^2} z' L_i' M(\kappa_m \rho L_m' L_m + I_T) M L_i z \\ &= \frac{\kappa_i}{N} \sum_{j,k=1}^{N} z_j' \left\{ (\frac{1}{N} L_i' M L_i)(j,k) + \frac{\kappa_m c^2}{N} \sum_{l=1}^{N} \frac{1}{N} (L_i' M L_m)(j,l) \frac{1}{N} (L_m' M L_i)(l,k) \right\} z_k.\end{aligned}$$

In the same way as the previous section, we can see that

$$\frac{1}{N}(L_i' M L_m)(j,k) = F_{im}\left(\frac{j}{N}, \frac{k}{N}\right) + O(N^{-1}), \qquad (2-21)$$

for all j, k, where

$$F_{im}\left(\frac{j}{N}, \frac{k}{N}\right) = K_2\left(\frac{j}{N}, \frac{k}{N}\right) V_{i1}' V_{m1} - \frac{1}{4}\left(K_2\left(\frac{j}{N}, \frac{k}{N}\right) - K_1\left(\frac{j}{N}, \frac{k}{N}\right)\right) V_{i1}' e_4 e_4' V_{m1}.$$

Then, from the same discussion as in the previous section, it is enough to consider

$$\begin{aligned}\tilde{S}_{im} &= \frac{\kappa_i}{N} \sum_{j,k=1}^{N} z_j' \left\{ F_i\left(\frac{j}{N}, \frac{k}{N}\right) + \frac{\kappa_m c^2}{N} \sum_{l=1}^{N} F_{im}\left(\frac{j}{N}, \frac{l}{N}\right) F_{mi}\left(\frac{l}{N}, \frac{k}{N}\right) \right\} z_k \\ &\stackrel{d}{=} \frac{\kappa_i}{N} \sum_{j,k=1}^{N} z_j' \left\{ P' F_i\left(\frac{j}{N}, \frac{k}{N}\right) P \right. \\ &\quad \left. + \frac{\kappa_m c^2}{N} \sum_{l=1}^{N} P' F_{im}\left(\frac{j}{N}, \frac{l}{N}\right) P P' F_{mi}\left(\frac{l}{N}, \frac{k}{N}\right) P \right\} z_k. \qquad (2\text{-}22)\end{aligned}$$

The following lemma is important to investigate the limiting power properties.

Lemma 2.4 *For each pair of* $(i,m) = (1,2), (1,3), (1,6), (2,3), (2,5),$ *and* $(3,4)$,

$$P' F_{im}\left(\frac{j}{N}, \frac{k}{N}\right) P = 0, \qquad (2-23)$$

for all j, k, N.

From this lemma, we can see that the second term in the braces of (2-22) vanishes for some pairs of (i, m), so that c has no influence on the limiting distribution of $\tilde{S}_{im}$ under the alternative. That is, the LM test statistic is inconsistent for these pairs of (i, m). For example, if we use the test statistic S_1^A and under the alternative of H_1^2 that $\{r_t\}$ has a negative unit root, the limiting power of S_1^A does not increase under H_1^2 and then S_1^A is an inconsistent test. Since $A_i(B)$ and $A_m(B)$ have no common roots for each pair of (i, m) in Lemma 2.4, we can say that the test is inconsistent if the alternative hypothesis we assume and the true data generating process (D.G.P.) have no common roots. This tendency has been observed in Canova and Hansen (1995), Hylleberg (1995) and Caner (1998) by Monte Carlo simulations. Their results show that if we consider the test against a negative unit root ($A_2(B) = (1+B)$) but the data generating process has annual unit roots ($A_3(B) = (1+B)^2$), the power of the test does not increase and vice versa. Our investigation of power functions supports their results theoretically.

Table 4 shows whether each pair of (i, m) leads to a consistent or inconsistent result. Since the same result as (i, m) holds for the pair of (m, i), the upper right of the table has no entry. From this table, we can see that if we suppose the alternative hypothesis (2-20), the LM test statistic S_i derived in the previous section, except for S_7, is inconsistent in the sense that the power will never increase under some specific alternatives. Then S_7^A is only a consistent test statistic against the alternative hypothesis H_1 in the limit. This also holds in the cases B and C, that is, S_7^B and S_7^C are only consistent tests among S_i^B and S_i^C for $i = 1, \cdots, 7$.

From the above result, our interest is concentrated on the power properties of S_7 under H_1. Then we investigate $\tilde{S}_{7m}$ for $m = 1, \cdots, 6$, since we have already studied the limiting properties for $m = 7$ in the previous section. To derive the limiting distribution and the characteristic function of $\tilde{S}_{7m}$ in the case A, we have to evaluate the equation in the braces of (2-22). From some algebra, we can prove the following lemma similar to Lemma 2.2.

Lemma 2.5 *$V_{71}'V_{m1}$ and $V_{71}'e_4e_4'V_{m1}$ $(m = 1, \cdots, 6)$ are block-diagonalized using the matrix P.*

$$P'V_{71}'V_{11}P = diag(4, 0, 0, 0). \qquad P'V_{71}'e_4e_4'V_{11}P = 4 \times diag(4, 0, 0, 0).$$

$$P'V_{71}'V_{21}P = diag(0,4,0,0). \qquad P'V_{71}'e_4e_4'V_{21}P = 0.$$

$$P'V_{71}'V_{31}P = diag(0,0,2,2). \qquad P'V_{71}'e_4e_4'V_{31}P = 0.$$

$$P'V_{71}'V_{41}P = diag(2,2,0,0). \qquad P'V_{71}'e_4e_4'V_{41}P = 4 \times diag(2,0,0,0).$$

$$P'V_{71}'V_{51}P = diag\left(2,0,\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right). \qquad P'V_{71}'e_4e_4'V_{51}P = 4 \times diag(2,0,0,0).$$

$$P'V_{71}'V_{61}P = diag\left(0,2,\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right). \qquad P'V_{71}'e_4e_4'V_{61}P = 0.$$

In the same way as the previous section, we can derive the limiting distribution of S_7^A under H_1^m for $m = 1, \cdots, 6$ and its characteristic function using the above lemma.

Theorem 2.7 *The LM test statistic S_7^A converges in distribution under H_1^m for $m = 1, \cdots, 6$ and its characteristic function, $\phi_7^A(\theta; H_1^m)$, is given by*

$$\begin{aligned}
\phi_7^A(\theta; H_1^1) &= \left[D_1\left(4(i\theta + \sqrt{-\theta^2 + 8ic^2\theta}\right) D_1\left(4(i\theta - \sqrt{-\theta^2 + 8ic^2\theta}\right)\right]^{-1/2} \times \left[D_2\left(8i\theta\right)\right]^{-3/2}. \\
\phi_7^A(\theta; H_1^2) &= \left[D_1\left(8i\theta\right)\right]^{-1/2} \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 8ic^2\theta}\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 8ic^2\theta}\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(8i\theta\right)\right]^{-1}. \\
\phi_7^A(\theta; H_1^3) &= \left[D_1\left(8i\theta\right)\right]^{-1/2} \times \left[D_2\left(8i\theta\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta}\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta}\right)\right]^{-1}. \\
\phi_7^A(\theta; H_1^4) &= \left[D_1\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta}\right) D_1\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta}\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta}\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta}\right)\right]^{-1/2} \times \left[D_2\left(8i\theta\right)\right]^{-1}. \\
\phi_7^A(\theta; H_1^5) &= \left[D_1\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta}\right) D_1\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta}\right)\right]^{-1/2} \times \left[D_2\left(8i\theta\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta}\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta}\right)\right]^{-1} \\
\phi_7^A(\theta; H_1^6) &= \left[D_1\left(8i\theta\right)\right]^{-1/2} \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta}\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta}\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta}\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta}\right)\right]^{-1}.
\end{aligned}$$

In the cases B and C, the following theorems can be obtained in the same way as in the case A.

Theorem 2.8 *The LM test statistic S_7^B converges in distribution under H_1^m for $m = 1, \cdots, 6$ and its characteristic function, $\phi_7^B(\theta; H_1^m)$, is given by*

$$\begin{aligned}
&\phi_7^B(\theta; H_1^1) = \phi_7^B(\theta; H_1^2) \\
&= \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 8ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 8ic^2\theta})\right)\right]^{-1/2} \times [D_2(8i\theta)]^{-3/2}. \\
&\phi_7^B(\theta; H_1^3) = \phi_7^B(\theta; H_1^4) \\
&= \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta})\right)\right]^{-1} \times [D_2(8i\theta)]^{-1}. \\
&\phi_7^B(\theta; H_1^5) = \phi_7^B(\theta; H_1^6) \\
&= \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta})\right)\right]^{-1/2} \\
&\quad \times \left[D_2\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_2\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1} \times [D_2(8i\theta)]^{-1/2}.
\end{aligned}$$

Theorem 2.9 *The LM test statistic S_7^C converges in distribution under H_1^m for $m = 1, \cdots, 6$ and its characteristic function, $\phi_7^C(\theta; H_1^m)$, is given by*

$$\begin{aligned}
&\phi_7^C(\theta; H_1^1) = \phi_7^C(\theta; H_1^2) \\
&= \left[D_3\left(4(i\theta + \sqrt{-\theta^2 + 8ic^2\theta})\right) D_3\left(4(i\theta - \sqrt{-\theta^2 + 8ic^2\theta})\right)\right]^{-1/2} \times [D_3(8i\theta)]^{-3/2}. \\
&\phi_7^C(\theta; H_1^3) = \phi_7^C(\theta; H_1^4) \\
&= \left[D_3\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta})\right) D_3\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta})\right)\right]^{-1} \times [D_3(8i\theta)]^{-1}. \\
&\phi_7^C(\theta; H_1^5) = \phi_7^C(\theta; H_1^6) \\
&= \left[D_3\left(4(i\theta + \sqrt{-\theta^2 + 4ic^2\theta})\right) D_3\left(4(i\theta - \sqrt{-\theta^2 + 4ic^2\theta})\right)\right]^{-1/2} \\
&\quad \times \left[D_3\left(4(i\theta + \sqrt{-\theta^2 + 2ic^2\theta})\right) D_3\left(4(i\theta - \sqrt{-\theta^2 + 2ic^2\theta})\right)\right]^{-1} \times [D_3(8i\theta)]^{-1/2}.
\end{aligned}$$

As in the previous section, we can obtain the limiting powers, $g_7^j(c; H_1^m)$ for $m = 1, \cdots, 7$ and $j = A, B, C$, by inverting the characteristic functions. Figures 3a-3c show the limiting powers $g_7^A(c; H_1^m)$, $g_7^B(c; H_1^m)$, and $g_7^C(c; H_1^m)$ for $m = 1, \cdots, 7$ in the cases A, B, and C, respectively. From Figure 3a, we can see that S_7^A under H_1^1 (corresponding to $g_7^A(c; H_1^1)$) is least powerful overall while S_7^A under H_1^6 (corresponding to $g_7^A(c; H_1^6)$) is most powerful. Figure 3b is similar to Figure 3c. They show that, for small values of c, the powers are almost same for each alternative and, as c increases, the test becomes more powerful as the number of roots included under the alternative increases. Then we can say that the test statistic S_7 is least powerful under H_1^1 in each case.

Next let us compare the power of S_7^A with that of S_m^A under the alternative of H_1^m in Figures 4a-4f. From Figure 4a, we can see that, under H_1^1, the test statistic S_7^A is rather less powerful than S_1^A while, from Figure 4f, S_7^A has almost the same power as S_6^A under H_1^6. These results are two extreme cases and, as shown in Figures 4b-4e for $m = 2, \cdots, 5$, S_7^A is less powerful than S_m^A under H_1^m but the divergence between them is moderate in comparison with the case of $m = 1$.

Figures 5a-5c and 6a-6c correspond to the cases B and C. In both cases, S_7 is less powerful than S_m under H_1^m for $m = 1, \cdots, 6$ as in the case A, though the divergence between them is not so much for all m.

5. Concluding Remarks

We have investigated the LM test for the null hypothesis of stationarity against the alternative hypothesis of nonstationarity using the quarterly seasonal model. If we are interested in particular roots, we can use the test statistic S_i while if we are interested in whether the observed variable is stationary or nonstationary, it is only S_7 that is consistent against the alternative H_1. Though these results were derived under the assumption of normality, we can relax this assumption, as is well known in the literature.

Since our interest is the limiting properties of seasonal and/or non-seasonal unit roots tests, we did not consider the serial correlation in the error term. If we want to apply our tests to a practical analysis, we have to modify them to have the same limiting distributions as derived in this chapter under the assumption of the serial correlation. This modification can be achieved in the same way as Canova and Hansen (1995) and Caner (1998).

Appendix 2.

Proof of Theorem 2.1: We prove only for $i = 5$. The other cases can be proved in an entirely analogous way and then a proof of them is omitted.

Since $\kappa_5 = 2$, we have, from the equation (2-11) and Lemma 2.2,

$$\begin{aligned}
\tilde{S}_5 &= \frac{2}{N}\sum_{j,k=1}^{N} z_j' F_5\left(\frac{j}{N},\frac{k}{N}\right) z_k \\
&\stackrel{d}{=} \frac{2}{N}\sum_{j,k=1}^{N} z_j' \left\{ K_2\left(\frac{j}{N},\frac{k}{N}\right) P'V_{51}'V_{51}P \right. \\
&\quad \left. -\frac{1}{4}\left(K_2\left(\frac{j}{N},\frac{k}{N}\right) - K_1\left(\frac{j}{N},\frac{k}{N}\right)\right) P'V_{51}'e_4e_4'V_{51}P \right\} z_k \\
&= \frac{2}{N}\sum_{j,k=1}^{N} z_j' \left\{ K_2\left(\frac{j}{N},\frac{k}{N}\right) diag(4,0,2,2) \right. \\
&\quad \left. -\left(K_2\left(\frac{j}{N},\frac{k}{N}\right) - K_1\left(\frac{j}{N},\frac{k}{N}\right)\right) diag(4,0,0,0) \right\} z_k \\
&= \frac{2}{N}\sum_{j,k=1}^{N} 4K_1\left(\frac{j}{N},\frac{k}{N}\right) z_{1j}z_{1k} + \frac{2}{N}\sum_{j,k=1}^{N} 2K_2\left(\frac{j}{N},\frac{k}{N}\right) z_{2j}'z_{2k},
\end{aligned}$$

where $\{z_{1j}\}$ and $\{z_{2j}\}$ are independent and $NID(0,1)$, $NID(0,I_2)$, respectively. Applying Lemma 2.1, we have

$$\begin{aligned}
\frac{2}{N}\sum_{j,k=1}^{N} 4K_1\left(\frac{j}{N},\frac{k}{N}\right) z_{1j}z_{1k} &\xrightarrow{d} \sum_{n=1}^{\infty}\frac{8}{\lambda_{1n}}Z_{1n}^2, \\
\frac{2}{N}\sum_{j,k=1}^{N} 2K_2\left(\frac{j}{N},\frac{k}{N}\right) z_{2j}'z_{2k} &\xrightarrow{d} \sum_{n=1}^{\infty}\frac{4}{\lambda_{2n}}Z_{2n}'Z_{2n},
\end{aligned}$$

where $\{Z_{1n}\}$ and $\{Z_{2n}\}$ are independent and $NID(0,1)$, $NID(0,I_2)$, respectively, and joint convergence of the above also applies since $\{z_{1j}\}$ and $\{z_{2j}\}$ are independent. Then, the limiting distribution of S_5 is given by

$$S_5 \xrightarrow{d} \sum_{n=1}^{\infty}\frac{8}{\lambda_{1n}}Z_{1n}^2 + \sum_{n=1}^{\infty}\frac{4}{\lambda_{2n}}Z_{2n}'Z_{2n}.$$

The characteristic function of the limiting distribution is given by

$$\begin{aligned}
\phi_5^A(\theta;H_0) &= \left[\prod_{n=1}^{\infty}\left(1-2i\theta\frac{8}{\lambda_{1n}}\right)^{-1/2}\right] \times \left[\prod_{n=1}^{\infty}\left(1-2i\theta\frac{4}{\lambda_{2n}}\right)^{-1/2}\right]^2 \\
&= \left[D_1(16i\theta)\right]^{-1/2} \times \left[D_2(8i\theta)\right]^{-1},
\end{aligned}$$

applying Lemma 2.1.□

Proof of Theorem 2.2: From the equation (2-15) and Lemma 2.2, we have

$$\begin{aligned}
\tilde{S}_5 &\stackrel{d}{=} \frac{2}{N}\sum_{j,k=1}^{N} z_j'\left\{K_2\left(\frac{j}{N},\frac{k}{N}\right)P'V_{51}'V_{51}P\right\}z_k \\
&= \frac{2}{N}\sum_{j,k=1}^{N} z_j'\left\{K_2\left(\frac{j}{N},\frac{k}{N}\right)diag(4,0,2,2)\right\}z_k \\
&= \frac{2}{N}\sum_{j,k=1}^{N} 4K_2\left(\frac{j}{N},\frac{k}{N}\right)z_{1j}z_{1k} + \frac{2}{N}\sum_{j,k=1}^{N} 2K_2\left(\frac{j}{N},\frac{k}{N}\right)z_{2j}'z_{2k},
\end{aligned}$$

where $\{z_{1j}\}$ and $\{z_{2j}\}$ are independent and $NID(0,1)$, $NID(0,I_2)$, respectively. Applying Lemma 2.1, we have

$$S_5 \xrightarrow{d} \sum_{n=1}^{\infty}\frac{8}{\lambda_{2n}}Z_{1n}^2 + \sum_{n=1}^{\infty}\frac{4}{\lambda_{2n}}Z_{2n}'Z_{2n},$$

where $\{Z_{1n}\}$ and $\{Z_{2n}\}$ are independent and $NID(0,1)$, $NID(0,I_2)$, respectively.

The characteristic function of the limiting distribution is given by

$$\begin{aligned}
\phi_5^B(\theta;H_0) &= \left[\prod_{n=1}^{\infty}\left(1-2i\theta\frac{8}{\lambda_{2n}}\right)^{-1/2}\right]\times\left[\prod_{n=1}^{\infty}\left(1-2i\theta\frac{4}{\lambda_{2n}}\right)^{-1/2}\right]^2 \\
&= [D_2(16i\theta)]^{-1/2}\times[D_2(8i\theta)]^{-1},
\end{aligned}$$

applying Lemma 2.1.

The other cases can be proved in an entirely analogous way.□

Proof of Theorem 2.4: Let us consider S_5. From the equation (2-18) and Lemma 2.2, we have

$$\begin{aligned}
\tilde{S}_5 &\stackrel{d}{=} \frac{2}{N}\sum_{j,k=1}^{N} z_j'P'\left\{F_5\left(\frac{j}{N},\frac{k}{N}\right)+\frac{2c^2}{N}\sum_{l=1}^{N}F_5\left(\frac{j}{N},\frac{l}{N}\right)F_5\left(\frac{l}{N},\frac{k}{N}\right)\right\}Pz_k \\
&= \frac{2}{N}\sum_{j,k=1}^{N}\left\{4K_1\left(\frac{j}{N},\frac{k}{N}\right)+\frac{32c^2}{N}\sum_{l=1}^{N}K_1\left(\frac{j}{N},\frac{l}{N}\right)K_1\left(\frac{l}{N},\frac{k}{N}\right)\right\}z_{1j}z_{1k} \\
&\quad +\frac{2}{N}\sum_{j,k=1}^{N}\left\{2K_2\left(\frac{j}{N},\frac{k}{N}\right)+\frac{8c^2}{N}\sum_{l=1}^{N}K_2\left(\frac{j}{N},\frac{l}{N}\right)K_2\left(\frac{l}{N},\frac{k}{N}\right)\right\}z_{2j}'z_{2k},
\end{aligned}$$

where $\{z_{1j}\}$ and $\{z_{2j}\}$ are independent and $NID(0,1)$, $NID(0,I_2)$, respectively. Applying Lemma 2.3, we have

$$S_5 \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{8}{\lambda_{1n}}+\frac{8^2c^2}{\lambda_{1n}^2}\right)Z_{1n}^2+\sum_{n=1}^{\infty}\left(\frac{4}{\lambda_{2n}}+\frac{4^2c^2}{\lambda_{2n}^2}\right)Z_{2n}'Z_{2n},$$

where $\{Z_{1n}\}$ and $\{Z_{2n}\}$ are independent and $NID(0,1)$, $NID(0,I_2)$, respectively.

The characteristic function of the limiting distribution is given by

$$\begin{aligned}\phi_5^A(\theta;H_1^5) &= \left[\prod_{n=1}^{\infty}\left\{1-2i\theta\left(\frac{8}{\lambda_{1n}}+\frac{8^2c^2}{\lambda_{1n}^2}\right)\right\}^{-1/2}\right]\left[\prod_{n=1}^{\infty}\left\{1-2i\theta\left(\frac{4}{\lambda_{2n}}+\frac{4^2c^2}{\lambda_{2n}^2}\right)\right\}^{-1/2}\right]^2 \\ &= \left[D_1\left(8(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_1\left(8(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1/2} \\ &\quad\times\left[D_2\left(4(i\theta+\sqrt{-\theta^2+2ic^2\theta})\right)D_2\left(4(i\theta-\sqrt{-\theta^2+2ic^2\theta})\right)\right]^{-1},\end{aligned}$$

applying Lemma 2.3.

The other results are obtained in an analogous way.□

Proof of Lemma 2.4: Some algebra reveals that, for each pair of (i,m), $P'V_{i1}'V_{m2}P=0$ and $P'V_{i1}'e_4e_4'V_{m2}P=0$, so that we obtain (2-23) from the definition of $F_{im}(j/N,k/N)$.□

Proof of Theorem 2.7: Let us consider when $m=5$. Note that, since $\kappa_5=2$ and $\kappa_7=4$,

$$\tilde{S}_{75}\stackrel{d}{=}\frac{4}{N}\sum_{j,k=1}^{N}z_j'\left\{P'F_7\left(\frac{j}{N},\frac{k}{N}\right)P+\frac{2c^2}{N}\sum_{l=1}^{N}P'F_{75}\left(\frac{j}{N},\frac{l}{N}\right)PP'F_{57}\left(\frac{l}{N},\frac{k}{N}\right)P\right\}z_k.$$

From Lemma 2.2, we can see that

$$\begin{aligned}P'F_7\left(\frac{j}{N},\frac{k}{N}\right)P &= K_2\left(\frac{j}{N},\frac{k}{N}\right)diag(1,1,1,1) \\ &\quad-\left(K_2\left(\frac{j}{N},\frac{k}{N}\right)-K_1\left(\frac{j}{N},\frac{k}{N}\right)\right)diag(1,0,0,0) \\ &= K_1\left(\frac{j}{N},\frac{k}{N}\right)diag(1,0,0,0)+K_2\left(\frac{j}{N},\frac{k}{N}\right)diag(0,1,1,1).\end{aligned}$$

In the same way, we have, from Lemma 2.5,

$$\begin{aligned}&P'F_{75}\left(\frac{j}{N},\frac{k}{N}\right)P \\ &= K_2\left(\frac{j}{N},\frac{k}{N}\right)diag\left(2,0,\begin{bmatrix}1&1\\-1&1\end{bmatrix}\right)-\left(K_2\left(\frac{j}{N},\frac{k}{N}\right)-K_1\left(\frac{j}{N},\frac{k}{N}\right)\right)diag(2,0,0,0) \\ &= K_1\left(\frac{j}{N},\frac{k}{N}\right)diag(2,0,0,0)+K_2\left(\frac{j}{N},\frac{k}{N}\right)diag\left(0,0,\begin{bmatrix}1&1\\-1&1\end{bmatrix}\right), \qquad (2\text{-}24)\end{aligned}$$

and then

$$\sum_{l=1}^{N} P'F_{75}\left(\frac{j}{N},\frac{l}{N}\right) PP'F_{57}\left(\frac{l}{N},\frac{k}{N}\right) P \tag{2-25}$$
$$= \sum_{l=1}^{N} K_1\left(\frac{j}{N},\frac{l}{N}\right) K_1\left(\frac{l}{N},\frac{k}{N}\right) diag(4,0,0,0) + K_2\left(\frac{j}{N},\frac{l}{N}\right) K_2\left(\frac{l}{N},\frac{k}{N}\right) diag(0,0,2,2).$$

From (2-24) and (2-25), $\tilde{S}_{75}$ can be expressed as

$$\begin{aligned}\tilde{S}_{75} \overset{d}{=} & \frac{4}{N}\sum_{j,k=1}^{N}\left\{K_1\left(\frac{j}{N},\frac{k}{N}\right)+\frac{8c^2}{N}\sum_{l=1}^{N}K_1\left(\frac{j}{N},\frac{l}{N}\right)K_1\left(\frac{l}{N},\frac{k}{N}\right)\right\}z_{1j}z_{1k}\\ &+\frac{4}{N}\sum_{j,k=1}^{N}K_2\left(\frac{j}{N},\frac{k}{N}\right)z'_{2j}z_{2k}\\ &+\frac{4}{N}\sum_{j,k=1}^{N}\left\{K_2\left(\frac{j}{N},\frac{k}{N}\right)+\frac{4c^2}{N}\sum_{l=1}^{N}K_2\left(\frac{j}{N},\frac{l}{N}\right)K_2\left(\frac{l}{N},\frac{k}{N}\right)\right\}z'_{3j}z_{3k},\end{aligned}$$

where $\{z_{1j}\}$, $\{z_{2j}\}$, and $\{z_{3j}\}$ are independent and $NID(0,1)$, $NID(0,1)$, and $NID(0,I_2)$, respectively. Then the limiting distribution of S_7 is given by

$$S_7 \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{4}{\lambda_{1n}}+\frac{32c^2}{\lambda_{1n}^2}\right)Z_{1n}^2+\sum_{n=1}^{\infty}\frac{4}{\lambda_{2n}}Z_{2n}^2+\sum_{n=1}^{\infty}\left(\frac{4}{\lambda_{2n}}+\frac{16c^2}{\lambda_{2n}^2}\right)Z'_{3n}Z_{3n},$$

where $\{Z_{1n}\}$, $\{Z_{2n}\}$ and $\{Z_{3n}\}$ are independent and $NID(0,1)$, $NID(0,1)$, and $NID(0,I_2)$, respectively.

The characteristic function is given by

$$\begin{aligned}\phi_7^A(\theta;H_1^5) = & \left[\prod_{n=1}^{\infty}\left\{1-2i\theta\left(\frac{4}{\lambda_{1n}}+\frac{32c^2}{\lambda_{1n}^2}\right)\right\}^{-1/2}\right]\times\left[\prod_{n=1}^{\infty}\left(1-2i\theta\frac{4}{\lambda_{2n}}\right)^{-1/2}\right]\\ &\times\left[\prod_{n=1}^{\infty}\left\{1-2i\theta\left(\frac{4}{\lambda_{2n}}+\frac{16c^2}{\lambda_{2n}^2}\right)\right\}^{-1}\right].\end{aligned}$$

and the result of the theorem is obtained.

We can prove the other results in an analogous way and then omit the proof.□

Table 1. Percent Points of the Limiting Null Distributions of S_i^A (Case A)

	0.01	0.05	0.1	0.9	0.95	0.99
$G_1^A(x;H_0)$	0.2763	0.3745	0.4462	1.9075	2.3662	3.4839
$G_2^A(x;H_0)$	0.3968	0.5850	0.7362	5.5569	7.3818	11.8953
$G_3^A(x;H_0)$	0.6306	0.8754	1.0578	4.8563	5.9802	8.5893
$G_4^A(x;H_0)$	0.4958	0.6612	0.7804	3.3763	4.2811	6.5296
$G_5^A(x;H_0)$	0.6304	0.8168	0.9459	3.0499	3.6179	4.9265
$G_6^A(x;H_0)$	0.7764	1.0392	1.2267	4.5421	5.4821	7.7221
$G_7^A(x;H_0)$	0.8001	1.0269	1.1829	3.6473	4.2843	5.7190

Table 2. Percent Points of the Limiting Null Distributions of S_i^B (Case B)

	0.01	0.05	0.1	0.9	0.95	0.99
$G_1^B(x;H_0)$	0.3968	0.5850	0.7362	5.5569	7.3818	11.8953
$G_3^B(x;H_0)$	0.6306	0.8754	1.0578	4.8563	5.9802	8.5893
$G_5^B(x;H_0)$	0.7764	1.0392	1.2267	4.5421	5.4821	7.7221
$G_7^B(x;H_0)$	0.9242	1.2026	1.3945	4.2524	4.9492	6.4905

Table 3. Percent Points of the Limiting Null Distributions of S_i^C (Case C)

	0.01	0.05	0.1	0.9	0.95	0.99
$G_1^C(x;H_0)$	0.5514	0.9034	1.2246	19.1331	26.4918	44.5993
$G_3^C(x;H_0)$	1.0153	1.5924	2.0825	16.4977	20.9924	31.4289
$G_5^C(x;H_0)$	1.3701	2.0713	2.6327	15.2650	19.0218	27.9910
$G_7^C(x;H_0)$	1.7504	2.5648	3.1854	14.1639	16.9358	23.0816

Table 4. Consistency and Inconsistency of S_i

		H_1^m						
		$H_1^1(1)$	$H_1^2(-1)$	$H_1^3(\pm i)$	$H_1^4(\pm 1)$	$H_1^5(1,\pm i)$	$H_1^6(-1,\pm i)$	$H_1^7(\pm 1,\pm i)$
	$H_1^1(1)$	+						
	$H_1^2(-1)$	−	+					
	$H_1^3(\pm i)$	−	−	+				
H_1^i	$H_1^4(\pm 1)$	+	+	−	+			
	$H_1^5(1,\pm i)$	+	−	+	+	+		
	$H_1^6(-1,\pm i)$	−	+	+	+	+	+	
	$H_1^7(\pm 1,\pm i)$	+	+	+	+	+	+	+

"+" and "−" denote that the pair of (i,m) leads to consistency and inconsistency, respectively.

The numbers in parences denote the corresponding unit roots.

Figure 1a. The Limiting Powers; $g_i^A(c; H_1^i)$ (Case A)

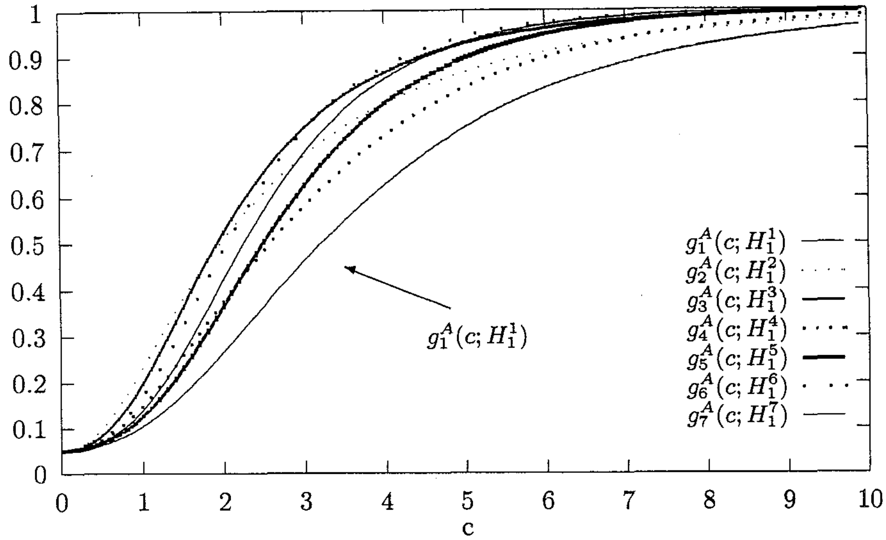


Figure 1b. The Limiting Powers; $g_i^B(c; H_1^i)$ (Case B)

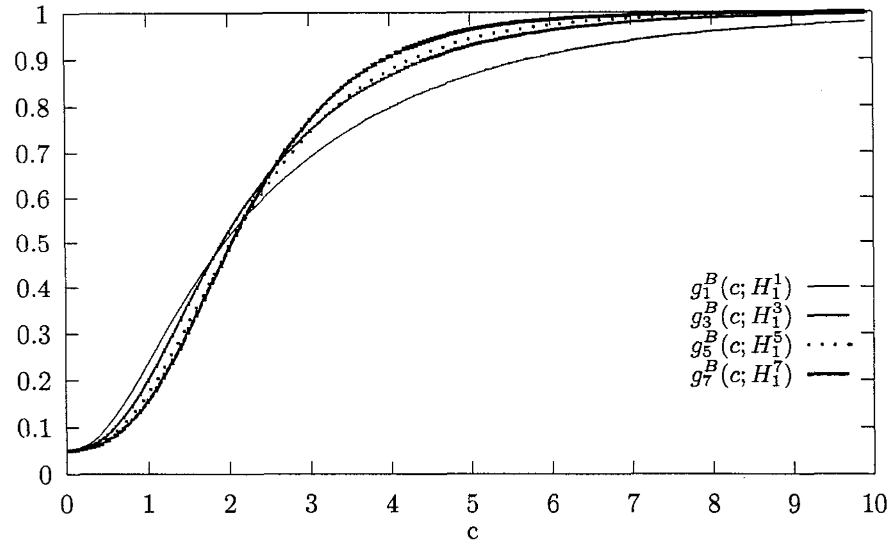


Figure 1c. The Limiting Powers; $g_i^C(c; H_1^i)$ (Case C)

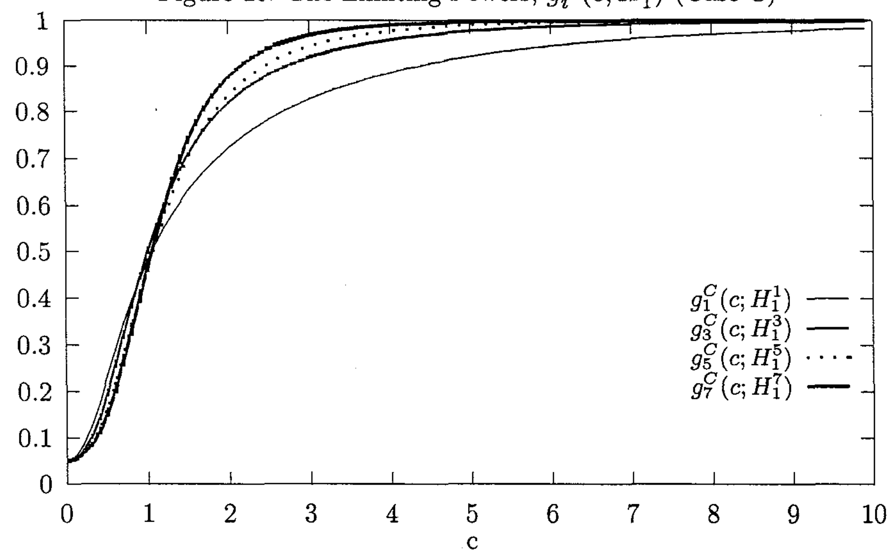


Figure 2a. The Limiting Powers; under H_1^1

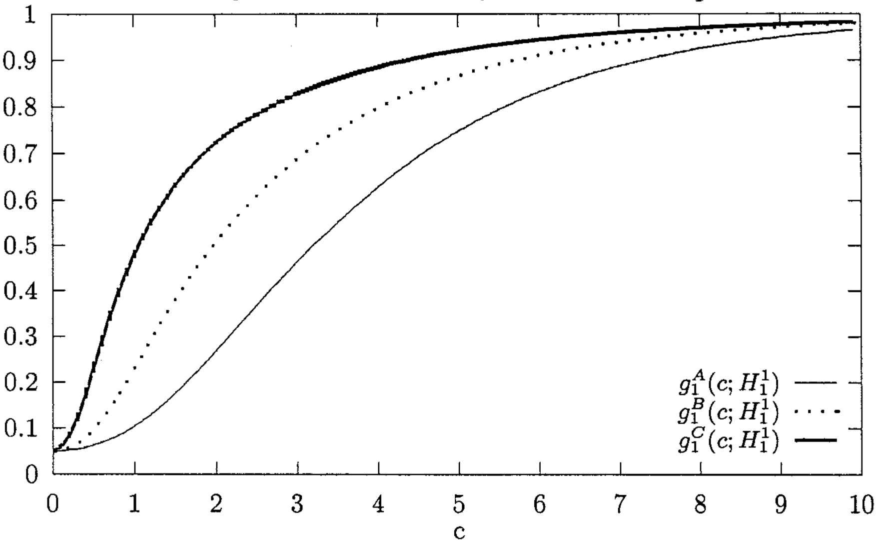


Figure 2b. The Limiting Powers; under H_1^4

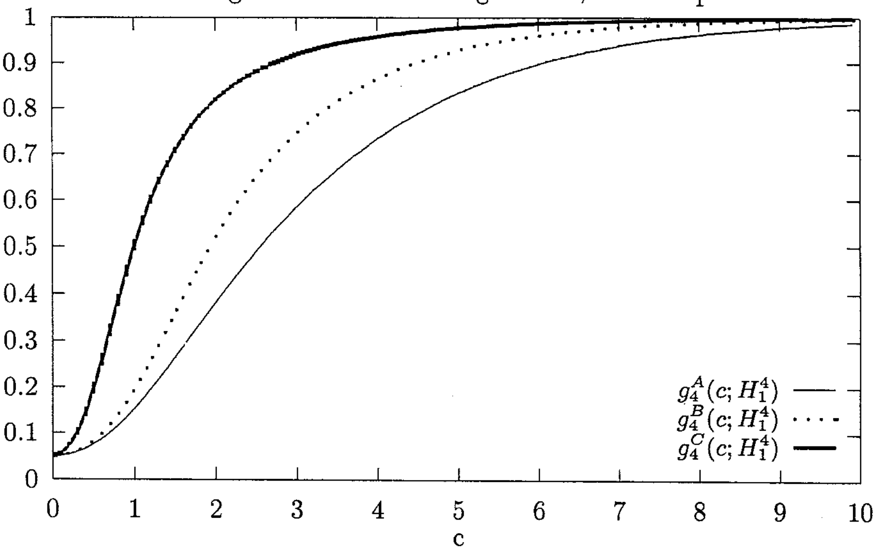


Figure 2c. The Limiting Powers; under H_1^5

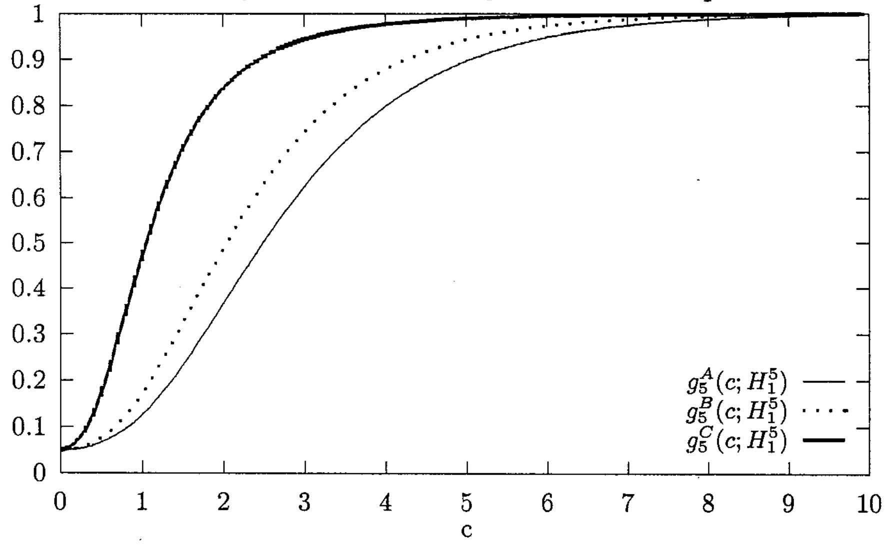


Figure 2d. The Limiting Powers; under H_1^7

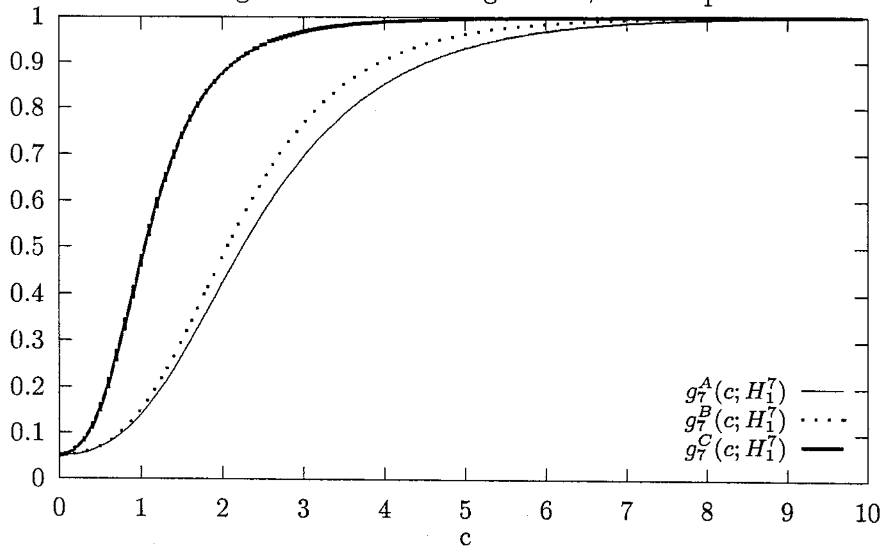


Figure 3a. The Limiting Powers; $g_7^A(c; H_1^m)$ (Case A)

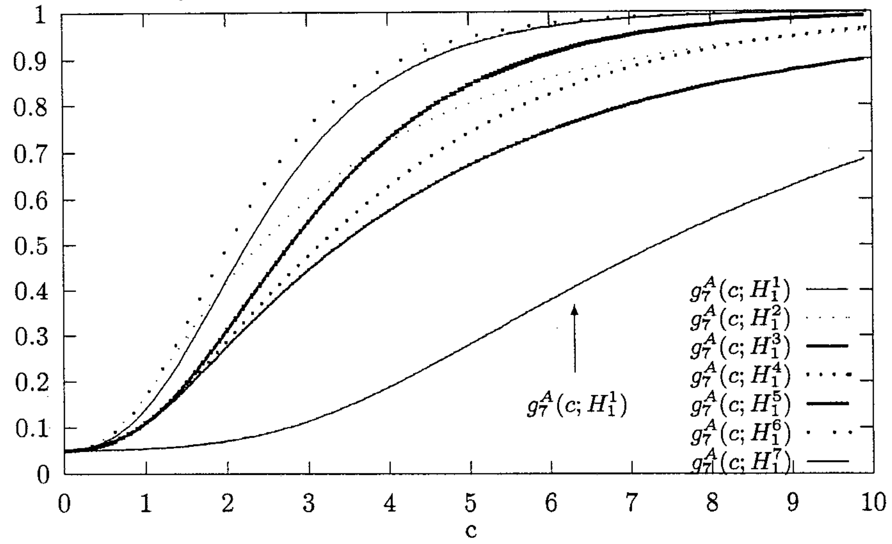


Figure 3b. The Limiting Powers; $g_7^B(c; H_1^m)$ (Case B)

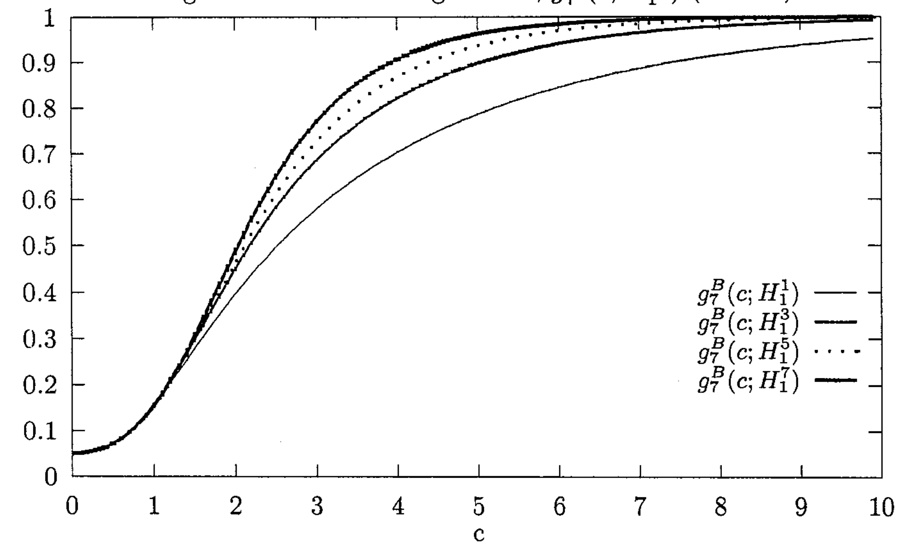


Figure 3c. The Limiting Powers; $g_7^C(c; H_1^m)$ (Case C)

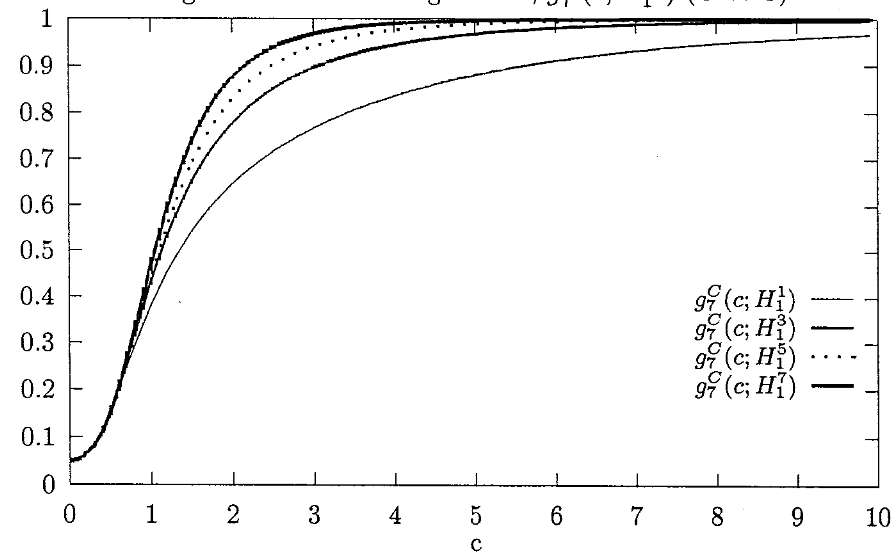


Figure 4a. The Limiting Powers; under H_1^1 (Case A)

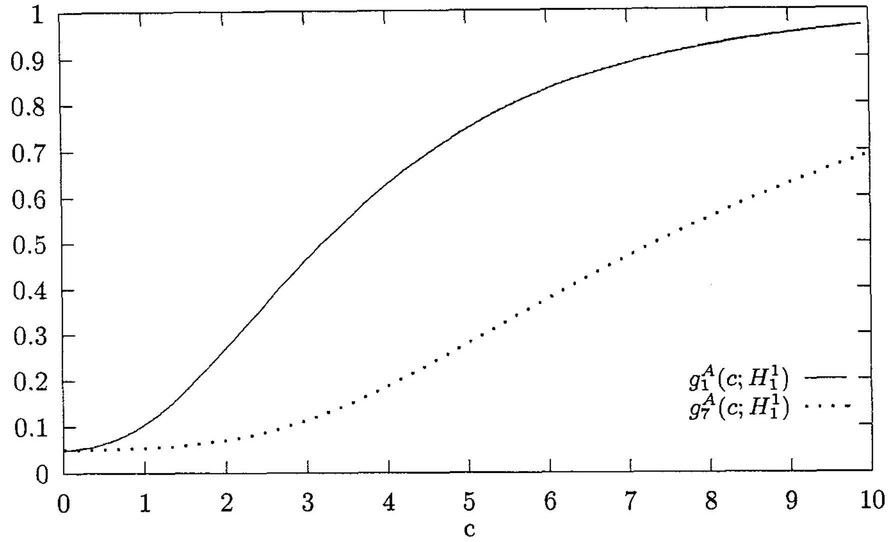


Figure 4b. The Limiting Powers; under H_1^2 (Case A)

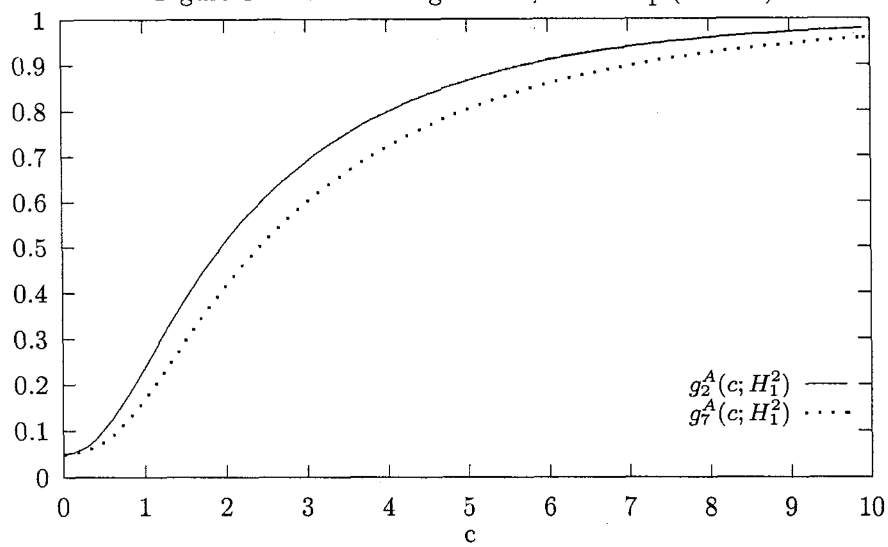


Figure 4c. The Limiting Powers; under H_1^3 (Case A)

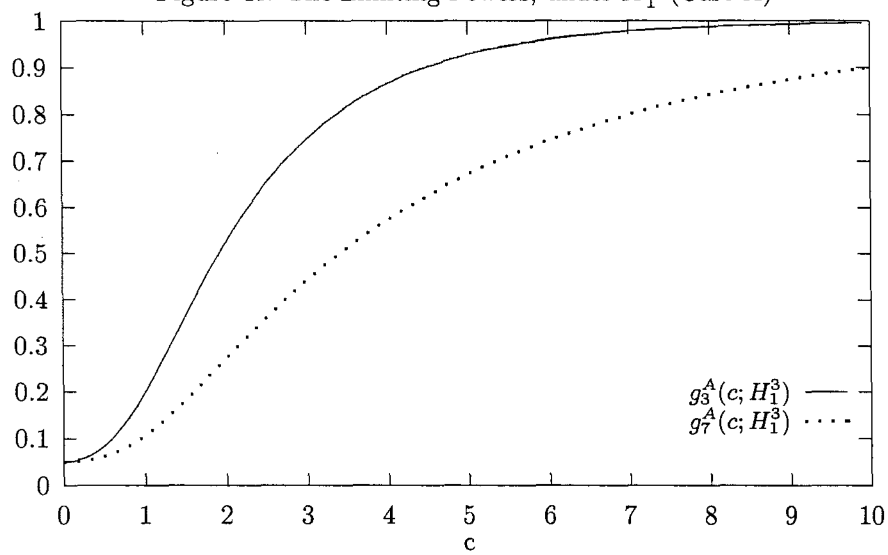


Figure 4d. The Limiting Powers; under H_1^4 (Case A)

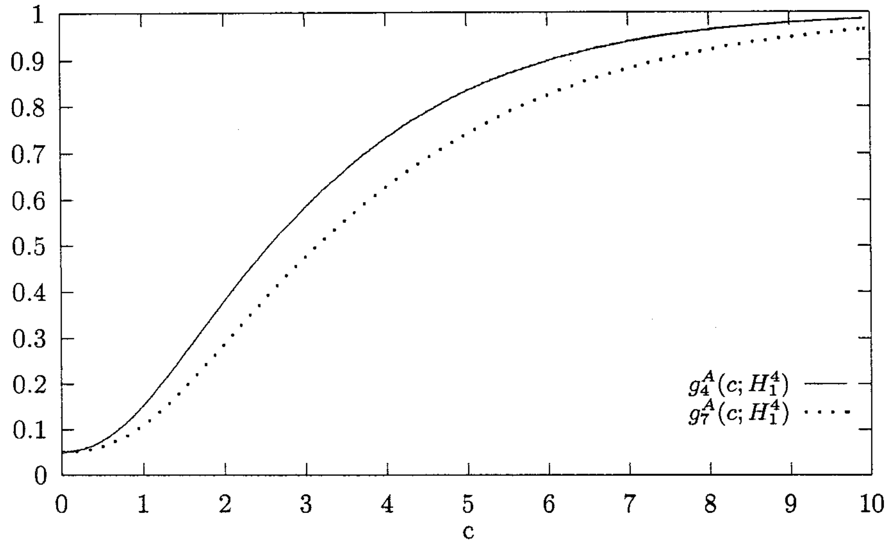


Figure 4e. The Limiting Powers; under H_1^5 (Case A)

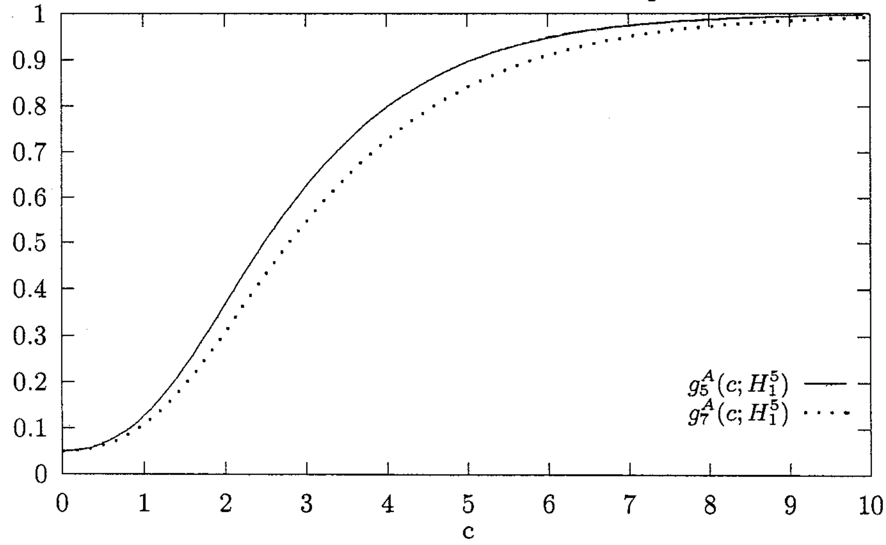


Figure 4f. The Limiting Powers; under H_1^6 (Case A)

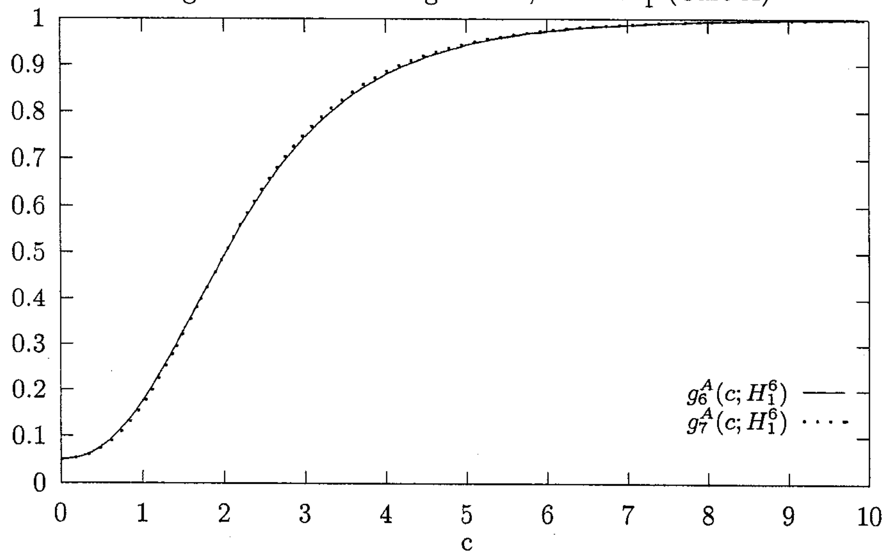


Figure 5a. The Limiting Powers; under H_1^1 (Case B)

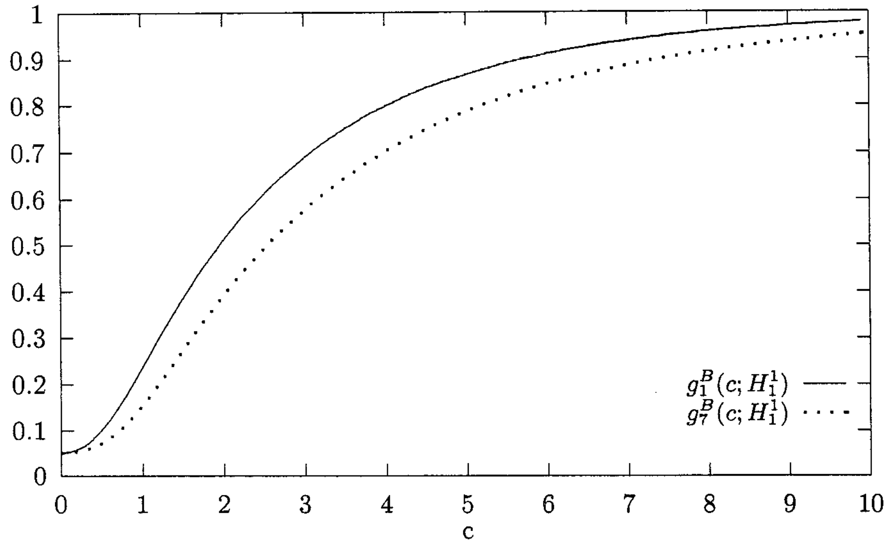


Figure 5b. The Limiting Powers; under H_1^3 (Case B)

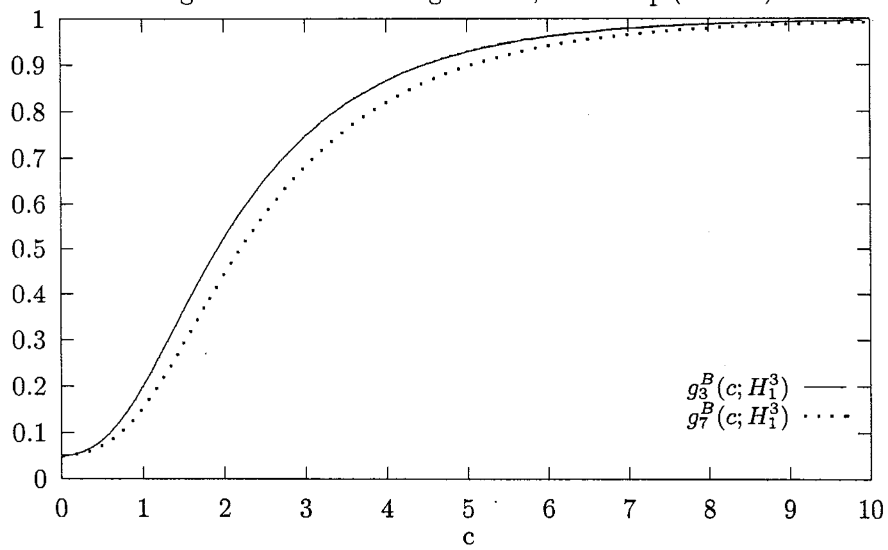


Figure 5c. The Limiting Powers; under H_1^5 (Case B)

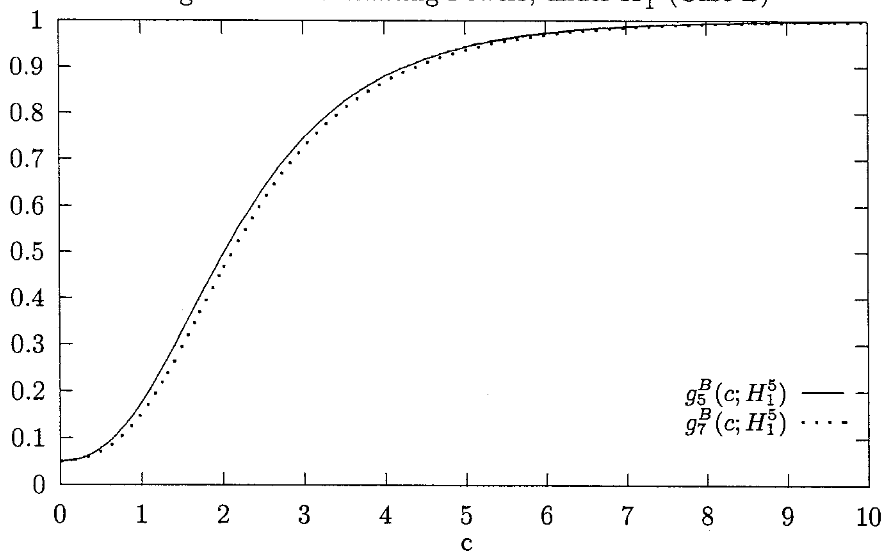


Figure 6a. The Limiting Powers; under H_1^1 (Case C)

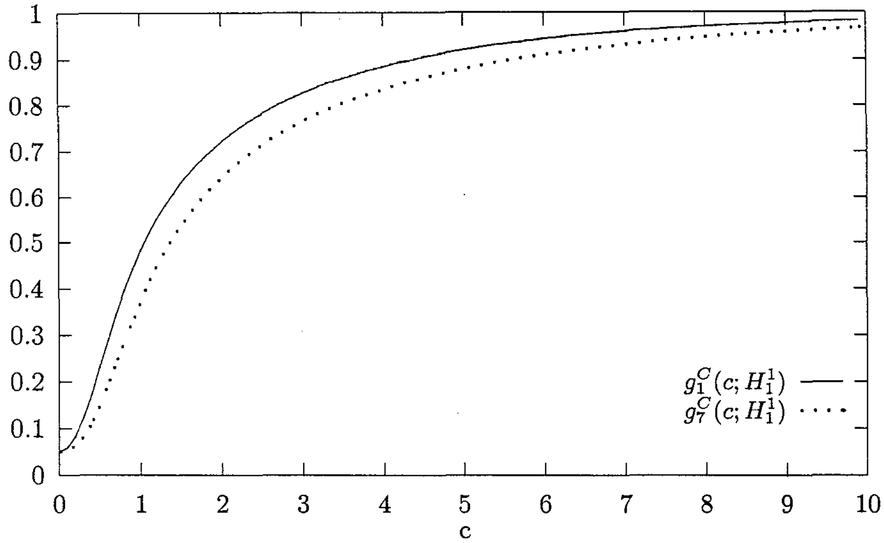


Figure 6b. The Limiting Powers; under H_1^3 (Case C)

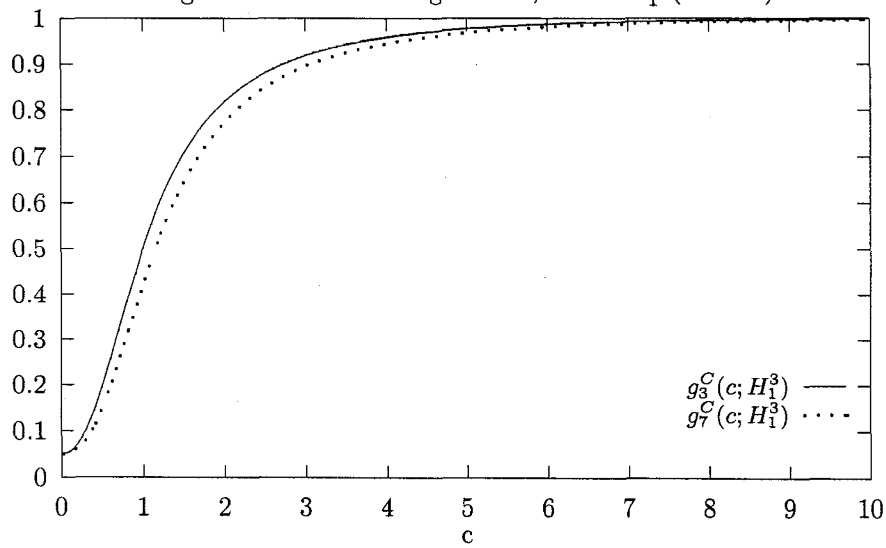
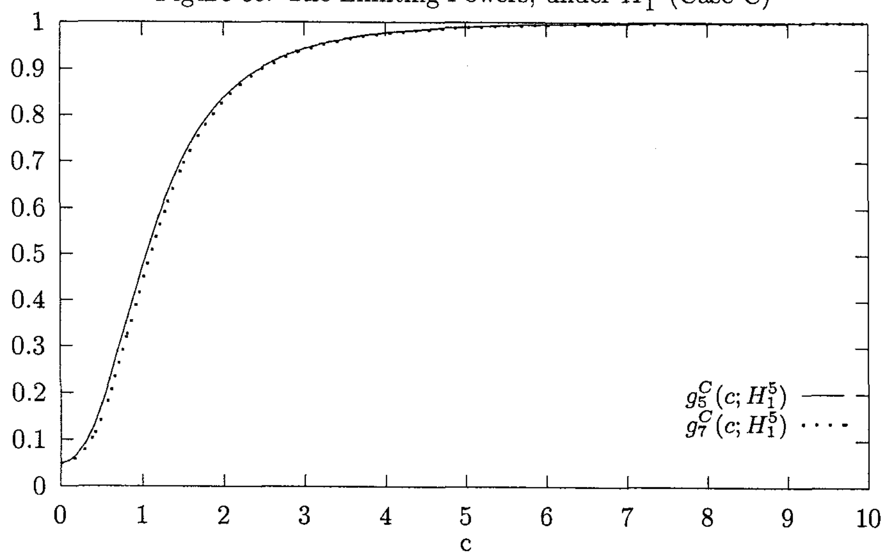


Figure 6c. The Limiting Powers; under H_1^5 (Case C)



Chapter 3.

Testing for Periodic Stationarity

In this chapter we investigate the test for the null hypothesis of periodic stationarity against the alternative hypothesis of periodic integration. We derive the limiting distribution of the test statistic and its characteristic function, which is used to tabulate the percent points of the limiting distribution by numerical integration. We find that some parameters, which we have to assume under the alternative, have an important role on the limiting power and we should carefully choose such parameters. The Monte Carlo simulation reveals that the test has the reasonable power but also is affected by the lag truncation parameter which is used for the nuisance parameter correction.

1. Introduction

In this chapter, we investigate the test for the null hypothesis of periodic stationarity against the alternative of periodic integration. The testing procedures for periodic integration are developed in such as Boswijk and Franses (1995, 1996), Boswijk, Franses and Haldrup (1997) and Franses and Paap (1995). The tests in these papers assume that the null of periodic integration against the alternative of no periodic integration, and, as in the case of the seasonal unit roots test, the test for the null of periodic stationarity seems to be useful and to complement the periodic integration tests investigated in the above papers. The derived test statistics in this chapter are free from nuisance parameters and critical points are tabulated.

The plan of this chapter is as follows. Section 2 briefly reviews the properties of periodic integration. Section 3 derives the test statistic using the error-components model. We investigate the limiting properties of the test both under the null and under a sequence of local alternatives. Section 4 considers the limiting behavior of the test statistic using the PAR model and also proposes the slightly modified test statistics. Section 5 investigates the finite sample properties of the tests through the Monte Carlo simulation, and the empirical applications are illustrated in Section 6. Section 7 concludes this chapter. All proofs are given in Appendix 3.

2. Periodic Integration

In this section, we briefly review the properties of periodic integration. Throughout the chapter, we will concentrate on the quarterly time series, but our results would be extended to the other seasonal models such as the monthly model. Let us consider the following quarterly periodic autoregressive model of order p (PAR(p)):

$$y_t = \sum_{s=1}^{4} \phi_{1s}^* D_{st} y_{t-1} + \cdots + \sum_{s=1}^{4} \phi_{ps}^* D_{st} y_{t-p} + u_t, \qquad (3-1)$$

where $\{u_t\} \sim i.i.d.(0, \sigma_u^2)$ and D_{st} is the seasonal dummy variable which takes 1 when the period t is in a season s and 0 otherwise. Note that the AR parameters vary with seasons, so that the structure of the model also varies with seasons. For example, the effect of the first quarter on the second one may be different from that of the third one on the fourth one.

For notational convenience, we represent the PAR(p) model (3-1) as, when t is in a season s,

$$y_t = \phi_{1s} y_{t-1} + \cdots + \phi_{ps} y_{t-p} + u_t, \tag{3-2}$$

where ϕ_{is}, $i = 1, \cdots, p$, are periodically varying parameters.

Stacking each variable to the annualized vector, such as $Y_j = [y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}]'$, we may represent (3-2) as

$$\Phi_0 Y_j = \Phi_1 Y_{j-1} + \cdots + \Phi_P Y_{j-P} + U_j, \tag{3-3}$$

where $P = [(p-1)/4] + 1$ with $[x]$ denoting the largest integer $\leq x$ and each Φ_i is defined appropriately. Following Franses (1994), we call $\{Y_j\}$ the vector of quarters (VQ) process of $\{y_t\}$. We can easily see that whether $\{Y_j\}$ is stationary or not depends on the roots of

$$\left| \Phi_0 - \Phi_1 z - \cdots - \Phi_P z^P \right| = 0. \tag{3-4}$$

We call $\{y_t\}$ is periodically stationary if all roots of (3-4) are outside the unit circle, and periodically integrated of order one, PI(1), if one root of (3-4) is 1 and the other roots are outside the unit circle. We also say that the periodically stationary process is periodically integrated of order zero, PI(0). See also Definition 1 of Boswijk and Franses (1996). Though we can think of a situation where (3-4) has more than one unit root (multiple unit roots), we will consider the PI(0) or PI(1) process in this chapter. Note that, in general, $\{y_t\}$ is not stationary even if it is periodically stationary.

The VQ process $\{Y_j\}$ has the error correction representation,

$$\triangle Y_j = \Gamma_1 \triangle Y_{j-1} + \cdots + \Gamma_{P-1} \triangle Y_{j-P+1} + \Pi Y_{j-1} + U_j^*, \tag{3-5}$$

where $\Pi = -\Phi_0^{-1} \sum_{i=1}^{P} \Phi_i$, $\Gamma_i = \Phi_0^{-1} \sum_{i=1}^{P} \Phi_i$, $U_j^* = \Phi_0^{-1} U_j$ and $\triangle Y_j = Y_j - Y_{j-1}$. Note that rank$(\Pi) = 3$ when $\{y_t\}$ is PI(1) so that we may interpret that $\{Y_j\}$ is cointegrated of order (1,1) with cointegrating rank 3. In this case, it is shown that, with some normalization, the 4×3 cointegrating matrix can be expressed as

$$\begin{bmatrix} -\delta_2 & 1 & 0 & 0 \\ 0 & -\delta_3 & 1 & 0 \\ 0 & 0 & -\delta_4 & 1 \end{bmatrix},$$

for some $\delta_s \neq 0$, $s = 2, 3, 4$. Defining $\delta_1 = 1/(\delta_2\delta_3\delta_4)$, Boswijk and Franses (1996) showed that $y_{4j+1} - \delta_1 y_{4j}$ becomes stationary. Then we may see that two successive seasons are cointegrated. In other words, $(y_t - \delta_s y_{t-1})$ has a stationary VQ representation with $\delta_1 = 1/(\delta_2\delta_3\delta_4)$, though $\{y_t\}$ is a nonstationary and nonstable process. According to Johansen (1991, 1992), $\{Y_j\}$ is expressed as $Y_j = \Pi^* \sum_{i=1}^{j} U_i^* + O_p(1)$ with $\text{rank}(\Pi^*) = 1$, and then we may see that $\{Y_j\}$ is driven by a single stochastic trend as in the case of a unit root process.

The PAR(p) model (3-2) can be extended to have the periodic moving average disturbance such as $u_t = \epsilon_t + \theta_{1s}\epsilon_{t-1} + \cdots + \theta_{qs}\epsilon_{t-q}$ where $\{\epsilon_t\} \sim i.i.d.(0, \sigma_\epsilon^2)$. This model may be seen as an extension of the ARMA model with seasonally varying parameters.

The properties of the PI(1) process are discussed in Boswijk and Franses (1995, 1996), Boswijk, Franses and Haldrup (1997), Franses (1994, 1996), Franses and Paap (1994), Ghysels, Hall and Lee (1996), while those of the PI(0) process are in Osborn (1991), Osborn and Smith (1989), Pagano (1978), Tiao and Grupe (1980), Troutman (1979), among others.

3. Testing for Periodic Stationarity

In this section, we derive the test statistic for the null of periodic stationarity and investigate its properties. Though our concern is the PAR model, we introduce the error-components model which is convenient to derive the test statistic, but slightly different from the pure PAR model. Such a model is considered in KPSS (1992), in which the test for stationarity is developed. We will see in the next section that the test derived in this section is also useful for the pure PAR model.

Let us consider the following error-components model.

$$y_t = x_t'\beta_s + w_t, \quad w_t = r_t + u_t, \quad \phi_{ps}(B)u_t = v_t, \quad r_t = \delta_s r_{t-1} + \epsilon_t, \qquad (3-6)$$

for $s = 1, \cdots, 4$ and $t = 1, \cdots, T$ with $N = T/4$ an integer, where $\{x_t\}$ is a seasonally varying deterministic component, $\phi_{ps} = 1 - \phi_{1s}B - \cdots - \phi_{ps}B^p$ with which $\{u_t\}$ is periodically stationary, $\delta_1, \delta_2, \delta_3, \delta_4$ satisfy the condition $\delta_1\delta_2\delta_3\delta_4 = 1$ so that $\{r_t\}$ is periodically integrated, and $\{v_t\}$ and $\{\epsilon_t\}$ are independent and $NID(0, \sigma_v^2)$ with $\sigma_v^2 > 0$, $NID(0, \sigma_\epsilon^2)$ with $\sigma_\epsilon^2 \geq 0$, respectively. We set $r_0 = 0$ without loss of generality, and for a while, we assume $\{\delta_s\}$ are known. The unknown case will be treated on page 62.

We denote the VQ representation of (3-6) as

$$Y_j = X_j\beta + W_j, \quad W_j = R_j + U_j, \quad \Phi(B)U_j = V_j, \quad \Theta_0 R_j = \Theta_1 R_{j-1} + E_j, \qquad (3-7)$$

for $j = 1, \cdots, N$, where, e.g., $Y_j = [y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j}]'$, $\Phi(B) = \Phi_0 - \Phi_1 B - \cdots - \Phi_P B^P$ with $P = [(p-1)/4] + 1$, $BU_j = U_{j-1}$ when B operates on the annualized vector U_j, and

$$\Theta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\delta_2 & 1 & 0 & 0 \\ 0 & -\delta_3 & 1 & 0 \\ 0 & 0 & -\delta_4 & 1 \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} 0 & 0 & 0 & \delta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (3-8)$$

Since $\{U_j\}$ is stationary, $\Phi(B)$ is invertible and we have the MA representation of $\{U_j\}$ as

$$U_j = \Phi(B)^{-1}V_j = \sum_{l=0}^{\infty} A_l V_{j-l}, \text{ say.} \qquad (3-9)$$

We define the deterministic term X_j as

Case 1 ; no deterministic,

Case 2 ; a seasonally varying constant; $X_j = I_4$,

Case 3 ; a seasonally varying constant and linear trend; $X_j = [I_4, jI_4]$,

where I_n denotes the $n \times n$ identity matrix.

Note that when $\sigma_\epsilon^2 = 0$, $w_t = u_t$ and the model (3-6) is expressed as $y_t = x_t'\beta_s + u_t$, or equivalently,

$$\phi_{ps}(B)y_t = x_t\beta_s^* + v_t.$$

Then, in this case, the model (3-6) is the pure PAR(p) model. On the other hand, when $\sigma_\epsilon^2 > 0$, we can easily show that a root of $|\Theta_0 - \Theta_1 z| = 0$ is 1, so that $\{r_t\}$ is periodically integrated of order one. Thus, $\{y_t\}$ has both PI(1) and PI(0) components in this case, though (3-6) can not be expressed as the pure PAR model.

Since $\{y_t\}$ is periodically stationary when $\sigma_\epsilon^2 = 0$ while periodically integrated when $\sigma_\epsilon^2 > 0$, we consider the following testing problem:

$$H_0 : \rho = 0 \text{ v.s. } H_1 : \rho = \frac{c^2}{N^2}, \qquad (3-10)$$

where $\rho = \sigma_\epsilon^2/\sigma_v^2$ and c is a constant, so that (3-10) signifies the testing problem, the null hypothesis of periodic stationarity against a sequence of local alternatives of periodic integration.

Stacking each variable from $j = 1$ to N, we have

$$y = X\beta + w, \quad w = r + u, \quad r = L\epsilon,$$

where, e.g., $y = [Y_1', \cdots, Y_N']'$, $X = [X_1', \cdots, X_N']'$, and

$$L = \begin{bmatrix} L_0 & & & \mathbf{0} \\ L_1 & L_0 & & \\ \vdots & \ddots & \ddots & \\ L_1 & \cdots & L_1 & L_0 \end{bmatrix},$$

with

$$L_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \delta_2 & 1 & 0 & 0 \\ \delta_2\delta_3 & \delta_3 & 1 & 0 \\ \delta_2\delta_3\delta_4 & \delta_3\delta_4 & \delta_4 & 1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 1 & \delta_1\delta_3\delta_4 & \delta_1\delta_4 & \delta_1 \\ \delta_2 & 1 & \delta_1\delta_2\delta_4 & \delta_1\delta_2 \\ \delta_2\delta_3 & \delta_3 & 1 & \delta_1\delta_2\delta_3 \\ \delta_2\delta_3\delta_4 & \delta_3\delta_4 & \delta_4 & 1 \end{bmatrix}. \qquad (3-11)$$

Note that L^{-1} has 1's in the diagonals, δ_s's immediately below the diagonals and 0's in the other elements. Then, each component of $L^{-1}r_t = \epsilon_t$ corresponds to $(1 - \delta_s B)r_t = \epsilon_t$.

To derive the LM test statistic for (3-10), we consider the special case when $\{u_t\} \sim NID(0, \sigma_v^2)$, as in KPSS (1992). This simplification is only for expository purpose, and we will investigate the derived test statistic assuming the model (3-6).

Since $y \sim N(X\beta, \sigma_v^2(\rho LL' + I_T))$ with the above simple model, it is easily derived that the LM test for (3-10) is given by

$$S_m = \frac{1}{\tilde{\sigma}_v^2} y'MLL'My = \frac{1}{\tilde{\sigma}_v^2} \sum_{j=1}^{N} G_j' G_j$$

as rejecting H_0 when S_m takes large values, for $m = 1, 2$ and 3 according to the definition of the deterministic component X, where $M = I_T$ for $m = 1$ and $M = I_T - X(X'X)^{-1}X'$ for $m = 2$ and 3, $\tilde{\sigma}_v^2 = y'My/T$, $G_j = L_0'\tilde{U}_j + L_1' \sum_{i=j+1}^{N} \tilde{U}_i$ for $j = 1, \cdots, N-1$ and $G_N = L_0'\tilde{U}_N$ with $\{\tilde{U}_j\}$ denoting the regression residuals of Y_j on a constant term, X_j. The second expression is convenient to calculate the test statistic in practice. Note that this LM test is equivalent to the locally best invariant test as discussed in such as King and Hillier (1985).

Now we investigates the limiting properties of the test statistic S_m with the model (3-6). As shown in the proof of Theorem 1, the test statistic S_m is of order N^2, and the limiting distribution of S_m/N^2 depends on the 'long-run variance' of $\{U_j\}$ as well as $\{\delta_s\}$ and is proportional to $\omega = \kappa'\Omega\kappa$, where $\kappa = \{1+1/\delta_2^2+1/(\delta_2^2\delta_3^2)+1/(\delta_2^2\delta_3^2\delta_4^2)\}^{1/2}[1, \delta_2, \delta_2\delta_3, \delta_2\delta_3\delta_4]'$ and Ω is the long-run variance of $\{U_j\}$, $\Omega = \sigma_v^2 AA'$ with $A = \sum_{l=0}^{\infty} A_l$. Then we consider the following test statistic:

$$S_{mI} = \frac{1}{\tilde{\omega}N^2} y'MLL'My, \qquad (3-12)$$

where $\tilde{\omega} = \kappa'\tilde{\Omega}\kappa$ and $\tilde{\Omega}$ is a consistent estimator of Ω of the form

$$\tilde{\Omega} = \tilde{\Gamma}(0) + \sum_{i=1}^{\ell_1} w(i, \ell_1)\left(\tilde{\Gamma}(i) + \tilde{\Gamma}(i)'\right),$$

with $\tilde{\Gamma}(i)$ a consistent estimator of the covariance matrix of $\{U_j\}$, $\tilde{\Gamma}(i) = \sum_{j=1}^{N-i} \tilde{U}_j\tilde{U}'_{j+i}/N$ and $w(i, \ell_1)$ the Bartlett kernel, $w(i, \ell_1) = 1 - i/(\ell_1 + 1)$ for $\ell_1 = o(N^{1/2})$.

The limiting distribution of (3-12) is given in the following theorem.

Theorem 3.1 *i) The LM test statistic S_{mI} converges in distribution under H_0,*

$$S_{mI} \xrightarrow{d} \sum_{n=1}^{\infty} \frac{1}{\lambda_{m,n}} Z_n^2, \qquad (3-13)$$

and its characteristic function, $\phi_m(\theta; H_0)$, is given by

$$\phi_m(\theta; H_0) = \lim_{N\to\infty} E\left[e^{i\theta S_{mI}}\right] = \left[D_m(2i\theta)\right]^{-1/2},$$

for m = 1, 2 and 3 according to the definition of the deterministic component X, where $\{Z_n\} \sim NID(0, 1)$ and both $\{\lambda_{m,n}\}$ and $D_m(\cdot)$ are defined in Appendix 3.

ii) The LM test statistic S_{mI} converges in distribution under H_1,

$$S_{mI} \xrightarrow{d} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{m,n}} + \frac{c^2\sigma_v^2\omega_1^2}{\omega\lambda_{m,n}^2}\right) Z_n^2, \qquad (3-14)$$

and its characteristic function, $\phi_m(\theta; H_1)$, is given by

$$\phi_m(\theta; H_1) = \left[D_m\left(i\theta + \sqrt{-\theta^2 + 2ic^2\sigma_v^2\omega_1^2\theta/\omega}\right) D_m\left(i\theta - \sqrt{-\theta^2 + 2ic^2\sigma_v^2\omega_1^2\theta/\omega}\right)\right]^{-1/2}, \qquad (3-15)$$

for $m = 1$, 2 and 3 according to the definition of the deterministic component X, where $\omega_1 = \kappa'\kappa$.

Though this theorem is proved using the assumption of normality, we can relax it as discussed in Nabeya and Tanaka (1988).

From the above theorem, the limiting distribution function, $F_m(x; H_0)$, can be obtained by inverting the characteristic function $\phi_m(\theta; H_0)$ using Lévy's inversion formula, that is,

$$F_m(x; H_0) = \frac{1}{\pi} \int_0^\infty Re\left[\frac{1 - e^{-i\theta x}}{i\theta}\phi_m(\theta; H_0)\right] d\theta,$$

for $m = 1$, 2 and 3. Table 1 shows percent points of the limiting distributions, which are calculated by numerical integration. From the table, the limiting distribution of S_{mI} shifts to the left as the deterministic terms become complicated.

As shown in Theorem 3.1 (ii), the limiting distribution under H_1 depends not only on c but also on $\{\delta_s\}$ and Ω through ω_1 and ω. To investigate the limiting properties of this test statistic under H_1, let us consider the simple case when $\{u_t\}$ is an independent sequence, $\{u_t\} = \{v_t\} \sim NID(0, \sigma_v^2)$. In this case, $A = I_4$ and then $\omega = \sigma_v^2\kappa'\kappa = \sigma_v^2\omega_1$, so that

$$S_{mI} \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{m,n}} + \frac{c^2\omega_1}{\lambda_{m,n}^2}\right) Z_n^2$$

under H_1 for $m = 1$, 2 and 3. Then, for $\omega_1^a > \omega_1^b$, we can easily see that

$$P\left(\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{m,n}} + \frac{c^2\omega_1^a}{\lambda_{m,n}^2}\right) Z_n^2 > x_\alpha\right) > P\left(\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{m,n}} + \frac{c^2\omega_1^b}{\lambda_{m,n}^2}\right) Z_n^2 > x_\alpha\right)$$

for a given critical value x_α, and then the test against the alternative of periodic integration with the larger ω_1 is more powerful than with the smaller one. Figures 1a-1c give the limiting power of S_{3I} as a function of c for some values of ω_1 in the cases 1, 2 and 3, respectively. These power functions are given by

$$1 - \frac{1}{\pi} \int_0^\infty Re\left[\frac{1 - e^{-i\theta x_\alpha}}{i\theta}\phi_m(\theta; H_1)\right] d\theta, \qquad (3-16)$$

as a function of c, where the characteristic function $\phi_m(\theta; H_1)$ is given by

$$\phi_m(\theta; H_1) = \left[D_m\left(i\theta + \sqrt{-\theta^2 + 2ic^2\omega_1\theta}\right) D_m\left(i\theta - \sqrt{-\theta^2 + 2ic^2\omega_1\theta}\right)\right]^{-1/2}.$$

We calculate (3-16) by numerical integration using the upper 5% point, $x_\alpha = x_{0.05}$. As discussed above, the limiting power function with the larger ω_1 dominates the power function with the smaller ω_1 for each case. Note that the above test is consistent in the sense that the limiting power reaches at 1 as c goes to infinity for each $m = 1$, 2 and 3.

In the above discussion,

$$\omega_1 = \kappa'\kappa = \left(1 + \delta_2^2 + \delta_2^2\delta_3^2 + \delta_2^2\delta_3^2\delta_4^2\right)\left(1 + \frac{1}{\delta_2^2} + \frac{1}{\delta_2^2\delta_3^2} + \frac{1}{\delta_2^2\delta_3^2\delta_4^2}\right)$$

plays an important role and it also affects the power of the test in the dependent case as shown in (3-14). From some algebra, we can see that $\omega_1 \geq 16$ and, especially, $\omega_1 = 16$ when $\{\delta_s\} \in \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ with

$$\begin{aligned}\mathcal{A}_1 &= \{\{1,1,1,1\},\{-1,-1,-1,-1\}\{-1,1,-1,1\},\{1,-1,1,-1\}\},\\ \mathcal{A}_2 &= \{\{-1,1,1,-1\},\{-1,-1,1,1\},\{1,1,-1,-1\},\{1,-1,-1,1\}\}.\end{aligned}$$

And the farther some of $|\delta_s|$'s deviate from one, the larger does ω_1 become. Then we might see ω_1 as a measure of deviation of $\{\delta_s\}$ from $\mathcal{A}$ and, especially, if all δ_s's are positive, we could regard ω_1 as a measure of *the degree of periodic integration*. For example, we say that the degree of periodic integration of the model with $\{\delta_1, \delta_2, \delta_3, \delta_4\} = \{1, 2, 1, 1/2\}$ ($\omega_1 = 25$) is stronger than that with $\{\delta_1, \delta_2, \delta_3, \delta_4\} = \{1, 5/4, 1, 4/5\}$ ($\omega_1 = 16.81$). Then, as discussed above, the more stronger the degree of periodic integration is, the more powerful is the test. In other words, when $\delta_s > 0$ for all s, the more $\{\delta_s\}$ fluctuate, the easier to detect periodic integration.

The above investigation of the limiting power is based on the assumption that $\{\delta_s\}$ of the true process are identical to $\{\delta_s\}$ we assumed under the alternative. Here, we investigate the effect of misspecifying the alternative hypothesis, that is, we assume $\{\delta_s\}$ under the alternative but the true process is the model (3-6) with $\{\delta_s^0\}$ instead of $\{\delta_s\}$. The following corollary gives the limiting power under such a situation.

Corollary 3.1 *Suppose that S_{mI} is constructed using $\{\delta_s\}$ and $\{y_t\}$ follows the model (3-6) with $\{\delta_s^0\}$ instead of $\{\delta_s\}$ and $c > 0$. Then,*

$$S_{mI} \xrightarrow{d} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{m,n}} + \frac{c^2\sigma_v^2\gamma}{\omega\lambda_{m,n}^2}\right) Z_n^2, \qquad (3-17)$$

$$\phi_m(\theta; H_1) = \left[D_m \left(i\theta + \sqrt{-\theta^2 + 2ic^2\sigma_v^2\gamma\theta/\omega} \right) D_m \left(i\theta - \sqrt{-\theta^2 + 2ic^2\sigma_v^2\gamma\theta/\omega} \right) \right]^{-1/2},$$

for $m = 1$, 2 *and* 3, *where* $\gamma = \kappa' L_1^0 L_1^{0\prime}\kappa$ *and* $L_1^0 = L_1$ *in (3-11) with* $\{\delta_s^0\}$ *instead of* $\{\delta_s\}$.

At first sight, this corollary is almost the same as Theorem 3.1 except for the difference between ω_1^2 and γ. However, γ has an important role on the limiting power since it is possible for γ to be zero, which means that the limiting power does not increase and remains at the significance level under the alternative. This would occur if κ is orthogonal to the space spanned by the columns in L_1^0. For example, if we assume the unit root ($\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$) under the alternative but the true process has a negative unit root ($\delta_1^0 = \delta_2^0 = \delta_3^0 = \delta_4^0 = -1$), then, we can see that $L_1^{0\prime}\kappa = 0$, so that the limiting power of the test does not increase. This non-increasing power was observed for the seasonal unit roots test in Caner (1998), Canova and Hansen (1995) and Hylleberg (1995) through the Monte Carlo simulation and analytically investigated in Chapter 2. In Corollary 3.1, we proved the similar result for the periodic integration test.

Note that if both $\{\delta_s\}$ and $\{\delta_s^0\}$ belong to $\mathcal{A}_1$ or $\mathcal{A}_2$ with $\{\delta_s\} \neq \{\delta_s^0\}$, we can show from direct calculation that $L_1^{0\prime}\kappa = 0$ which implies $\gamma = 0$. When $\{\delta_s\}$ belongs to $\mathcal{A}_1$ ($\mathcal{A}_2$) and $\{\delta_s^0\}$ to $\mathcal{A}_2$ ($\mathcal{A}_1$), $\gamma = 64$, which is smaller than 256 when $\{\delta_s\} = \{\delta_s^0\}$. Then, we deduce that γ takes a small value if signs of $\{\delta_s\}$ we assume are different from those of $\{\delta_s^0\}$ of the true process, so that the limiting power of S_{mI} would be very low. To illustrate how the power is affected by them, let us consider the simple case when $\{u_t\} = \{v_t\} \sim NID(0, \sigma_v^2)$ and $A = I_4$ as in the case of Figures 1a-1c and we construct S_{mI} with $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$. That is, we are going to test against the alternative of a unit root. In this case, the limiting distribution of S_{mI} in (3-17) becomes

$$S_{mI} \xrightarrow{d} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{m,n}} + \frac{c^2\gamma}{\omega_1 \lambda_{m,n}^2} \right) Z_n^2,$$

since $\omega = \sigma_v^2\omega_1$ in this case. Then the power depends on γ/ω_1 and c^2. Figures 2a-2c draw the limiting power as a function of c in the cases 1, 2 and 3 for $\{\delta_1^0, \delta_2^0, \delta_3^0, \delta_4^0\} = \{1, 1, 1, 1\}$, $\{1.5, 1.5, 2/3, 2/3\}$, $\{1.5, 1.5, -2/3, -2/3\}$ and $\{-1.5, -1.5, -2/3, -2/3\}$, and the corresponding values of γ/ω_1 are 16, 14.79, 4 and 0.24, respectively. As shown in figures, the misspecification of signs of $\{\delta_s\}$ affects severely on the limiting power. On the other hand, S_{mI} under the alternative of $\{\delta_1^0, \delta_2^0, \delta_3^0, \delta_4^0\} = \{1.5, 1.5, 2/3, 2/3\}$ is less powerful but still have a moderate power in comparison with the case when $\{\delta_1^0, \delta_2^0, \delta_3^0, \delta_4^0\} = \{1, 1, 1, 1\}$. Then, we should carefully assume signs of $\{\delta_s\}$ under the alternative in view of the limiting power.

4. Extension to a PAR Model

So far, we have considered the model (3-6) and investigated the LM test for periodic stationarity. However, in econometric analysis, it is often the case that the pure PAR model is considered instead of the model considered in the previous section. Then, in this section, we consider the limiting behavior of S_{mI} for the PAR model.

Let us consider the following model.

$$y_t = x_t'\beta_s + u_t, \quad \phi_{ps}(B)u_t = v_t, \tag{3-18}$$

where $\phi_{ps} = 1 - \phi_{1s}B - \cdots - \phi_{ps}B^p$, $\{v_t\} \sim i.i.d.(0, \sigma_v^2)$, and x_t denotes the same deterministic component as in the previous section. Since the above model can be expressed as $\phi_{ps}(B)(y_t - x_t'\beta_s) = v_t$, we may see (3-18) as the PAR(p) model with a periodic deterministic term. The VQ representation of (3-18) is expressed as

$$Y_j = X_j\beta + U_j, \quad \Phi(B)U_j = V_j, \tag{3-19}$$

where $\Phi(B)$ is the lag polynomial of order $P = [(p-1)/4] + 1$. Since we are interested in whether the model (3-18) is PI(0) or PI (1), not allowing it to be a explosive model, we assume that all the roots of $|\Phi(z)| = 0$ are outside the unit circle or equal to one and do not assume multiple unit roots.

Since we can not define the parameter ρ for the model (3-18), we consider the following testing problem:

$$H_0' : |\Phi(1)| \neq 0 \text{ v.s. } H_1' : |\Phi(1)| = 0.$$

Then, under the null hypothesis H_0', the model (3-18) is periodically stationary whereas, under the alternative hypothesis H_1', it is periodically integrated of order one.

Firstly we consider the limiting distribution of S_{mI} under H_0'. Since $\{u_t\}$ is periodically stationary under H_0', the model (3-18) is the same as the model (3-6), so that if we construct the test statistic S_{mI} in the same say as the previous section, we have the same limiting null distribution as Theorem 3.1 (i) and we can refer to Table 1 for the asymptotic critical point.

On the other hand, under H_1', $\{u_t\}$ is PI(1) and then, according to Boswijk and Franses (1996), the PAR(p) model of $\{u_t\}$ can be expressed as

$$u_t - \delta_s u_{t-1} = \sum_{i=1}^{p-1} \psi_{is}(u_{t-i} - \delta_{s-i} u_{t-i-1}) + v_t, \qquad (3-20)$$

where $\delta_1\delta_2\delta_3\delta_4 = 1$, $\delta_{s-4k} = \delta_s$ for a positive integer k and $(1 - \psi_{1s}B - \cdots - \psi_{p-1,s}B^{p-1})$ is a periodic autoregressive polynomial of order $p-1$ whose coefficients are defined from the following backward recursion,

$$\begin{aligned} \psi_{ps} &= 0, \quad \forall s, \\ \psi_{is} &= \frac{\psi_{i+1,s} - \delta_{i+1,s}}{\delta_{s-i}}, \quad i = p-1, \cdots, 1. \end{aligned}$$

Then the VQ representation of $\{u_t\}$ is given by

$$\Psi(B)\,(\Theta_0 - \Theta_1 B)\,U_j = V_j,$$

where Θ_0 and Θ_1 were defined in (3-8), and then

$$\Theta_0 U_j = \Theta_1 U_{j-1} + V_j^* \text{ with } V_j^* = \Psi(B)^{-1} V_j, \qquad (3-21)$$

because of invertibility of $\Psi(B)$. Noting that the VQ representation of $\{y_t\}$ is given by (3-19) with $\{U_j\}$ given by (3-21), we have, in the stacked form,

$$y = X\beta + u, \quad u = Lv^*.$$

Using this expression, the numerator of the test statistic S_{mI} is expressed as

$$y'MLL'My = v^{*\prime}L'MLL'MLv^*. \qquad (3-22)$$

In the same way as the proof of Theorem 3.1 (ii), we can show that (3-22) is of order N^4 while $\tilde{\Omega} = O_p(\ell_1 N)$ by Phillips (1991) and KPSS (1992). Then, as a whole, S_{mI} is $O_p(N/\ell_1)$ under H_1' and then the power of S_{mI} would increase to 1 as N goes to infinity since $\ell_1 = o(N^{1/2})$, that is, S_{mI} is consistent.

The above testing procedure includes the estimation of the long-run variance of $\{U_j\}$ using a nonparametric method, but sometimes this estimation procedure causes severe size distortions of the test under the null hypothesis in a small sample. Thus, instead of the nonparametric estimator of Ω, let us consider an autoregressive spectral density estimator. For simplicity, we assume $p \leq 4$. This assumption is not too restrictive since, according to Franses and Paap (1994) and Boswijk and Franses (1996), many economic time series are well approximated by the PAR(p) model with p at most 4, and our following procedure can be easily extended to the case when $p > 4$.

Under the assumption of $p \leq 4$, the VQ representation of $\{U_j\}$ is

$$\Phi_0 U_j = \Phi_1 U_{j-1} + V_j$$

and that of $\{y_t\}$ is given by

$$(\Phi_0 - \Phi_1 B)\, Y_j = (\Phi_0 - \Phi_1 B)\, X_j \beta + V_j.$$

Under the null hypothesis, $\{Y_j\}$ is a stationary process with a deterministic component and then by inverting $(\Phi_0 - \Phi_1 B) = \Phi_0(I_4 - \Phi_0^{-1}\Phi_1 B)$, we have

$$Y_j = X_j \beta + \sum_{i=0}^{\infty} \Psi^i \Phi_0^{-1} V_{j-i},$$

where $\Psi = \Phi_0^{-1}\Phi_1$. The long-run variance Ω is expressed as

$$\Omega = \sigma_v^2 CC', \quad \text{where} \quad C = \sum_{i=0}^{\infty} \Psi^i \Phi_0^{-1}.$$

Then the autoregressive spectral density estimator $\hat{\Omega}$ is given by $\hat{\Omega} = \hat{\sigma}_v^2 \hat{C}\hat{C}'$ with $\hat{C} = \sum_{i=0}^{\ell_2} \hat{\Psi}^i \hat{\Phi}_0^{-1}$, where $\hat{\sigma}_v^2$, $\hat{\Phi}_0$ and $\hat{\Psi}_1$ are consistent estimators of σ_v^2, Φ_0 and Ψ_1, respectively and $\ell_2 \to \infty$. The consistent estimator of each element of Φ_0 and Φ_1 will be obtained by regressions,

$$y_t = \hat{\beta}_{0s} + \hat{\beta}_{1s} j + \hat{\phi}_{1s} y_{t-1} + \cdots + \hat{\phi}_{ps} y_{t-p} + \hat{v}_t, \qquad (3-23)$$

where $j = [(t-1)/4] + 1$, and the consistent estimator of σ_v^2 is given by $\hat{\sigma}_v^2 = \sum_{t=1}^{T} \hat{v}_t^2 / T$. Note that we do not necessarily estimate all the elements of Φ_0 and Φ_1 but only the unknown elements of these matrices. For example, if we consider the PAR(1) model, we will obtain $\hat{\phi}_{1s}$ by regressions (3-23) and $\hat{\Phi}_0$ and $\hat{\Phi}_1$ should be constructed as

$$\hat{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\hat{\phi}_{12} & 1 & 0 & 0 \\ 0 & -\hat{\phi}_{13} & 1 & 0 \\ 0 & 0 & -\hat{\phi}_{14} & 1 \end{bmatrix}, \quad \hat{\Phi}_1 = \begin{bmatrix} 0 & 0 & 0 & \hat{\phi}_{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using these estimators, ω is consistently estimated by $\kappa'\hat{\Omega}\kappa$.

On the other hand, under the alternative, $|\Phi_0 - \Phi_1 z| = 0$ has a root on the unit circle and $(\Phi_0 - \Phi_1 B)$ is not invertible, which means that $\sum_{i=0}^{\ell_2} \Psi^i \to \infty$ as ℓ_2 goes to infinity. Since σ_v^2 and each element of Φ_0 and Φ_1 are consistently estimated by regressions (3-23) under the alternative, $\hat{C}$ and then $\hat{\Omega}$ will diverge, and the rate of divergence is important for the consistency of the test. Since $y'MLL'My/N^2$ in (3-12) is $O_p(N^2)$, $\hat{\Omega}$ should be of the smaller order than $O_p(N^2)$ in order for the test to be consistent. Note that if we use the truncation parameter ℓ_2 where ℓ_2 is of order N^d with $0 < d < 1$, $\hat{C} = \sum_{i=0}^{\ell_2} \hat{\Psi}^i \hat{\Phi}_0^{-1} = O_p(N^d)$ and $\hat{\Omega} = \hat{\sigma}_v^2 \hat{C}\hat{C}' = O_p(N^{2d})$. Then, the test statistic is $O_p(N^{2(1-d)})$ under the alternative, which means that the test is consistent. Since this $\hat{C}$ converges to C in probability under the null hypothesis even if we restrict $\ell_2 = O(N^d)$ and then $\hat{\Omega} = \hat{\sigma}_v^2 \hat{C}\hat{C}'$ is the consistent estimator of Ω under the null, we consider the test statistic S_{mII} defined by

$$S_{mII} = \frac{1}{\hat{\omega} N^2} y'MLL'My,$$

where $\hat{\omega} = \kappa'\hat{\Omega}\kappa$.

We also consider slight modification of S_{mII}. In the construction of $\hat{C}$, let us consider the weighted sum of $\hat{\Psi}$,

$$\check{C} = \sum_{i=0}^{\ell_3} w(i, \ell_3) \hat{\Psi}^i \hat{\Phi}_0^{-1},$$

where $w(i, \ell_3)$ is the kernel defined in the previous section and $\ell_3 = O(N^d)$ as ℓ_2. Then define the test statistic

$$S_{mIII} = \frac{1}{\check{\omega} N^2} y'MLL'My,$$

where $\check{\omega} = \hat{\sigma}_v^2 \kappa' \check{C}\check{C}'\kappa$. Though this modification has no effect asymptotically, it might enable us to avoid accumulation of the bias of the estimator $\hat{\Psi}^i$.

So far, we have investigated testing procedures assuming that $\{\delta_s\}$ are known, but in practical analyses, we often encounter the situation where we have interest in only whether the observed variable is periodically stationary or nonstationary and no interest in a particular set of $\{\delta_s\}$. Even in such a case, we have to set values of $\{\delta_s\}$ since we use them to construct the test statistics. As shown in the previous section, the power of the test crucially depends on $\{\delta_s\}$ assumed under the alternative and especially in some special cases, the limiting power might stay in the significance level set by the researcher and not increase. Then we have to carefully determine which set of values of $\{\delta_s\}$ should be assumed in construction of the test statistics developed above.

One candidate for $\{\delta_s\}$ used in the test statistics is the estimates of $\{\delta_s\}$ under the restriction of $\delta_1\delta_2\delta_3\delta_4 = 1$, since $\{\delta_s\}$ must satisfy this restriction. Since this restriction is adequate under the alternative, we can estimate $\{\delta_s\}$ consistently underH_1' by regressions of (3-23) with such a restriction, that is, by the nonlinear least squares (NLS) method. Then, we can easily see that the test statistics S_{mI}, S_{mII} and S_{mIII}, constructed by using the NLS estimates, $\{\hat{\delta}_s\}$, have the same limiting properties as those using the true $\{\delta_s\}$ under H_1'.

Turning to H_0', the NLS estimates of $\{\delta_s\}$ have no meaning since the restriction $\delta_1\delta_2\delta_3\delta_4 = 1$ is inadequate under H_0'. In practice, we would obtain the estimates of $\{\delta_s\}$ which have no meaning. However, for a given set of values of $\{\delta_s\}$, the limiting distributions of the test statistics do not depend on $\{\delta_s\}$ under H_0', so that we could expect that the finite sample distributions of the test statistics with $\{\hat{\delta}_s\}$ would be well approximated by the limiting distribution (3-13) if only $\{\hat{\delta}_s\}$ satisfy the restriction $\hat{\delta}_1\hat{\delta}_2\hat{\delta}_3\hat{\delta}_4 = 1$. Note that this strategy is similar to "cointegrating regression" discussed in Engle and Granger (1987), in which the cointegrating vector is identified only under the alternative.

In sum, for the PAR(p) model (3-18) with $\{\delta_s\}$ unknown, our testing procedure is that i) obtain $\{\hat{\delta}_s\}$ by NLS regressions of (3-23) with the restriction $\delta_1\delta_2\delta_3\delta_4 = 1$, ii) construct the test statistic using $\{\hat{\delta}_s\}$, iii) reject the null hypothesis when the test statistic is larger than a critical value (95%, say) in Table 1. Here we should keep in mind that the test statistic considered in this section is no longer the LM statistic, since the model (19) is different from (6). However, the test statistic is still consistent and then useful.

5. Finite Sample Properties

In this section, we investigate the finite sample properties of the test statistics derived in the previous section through the Monte Carlo simulation. We use the quarterly PAR(1) process as a simulation data generating process:

$$y_t = \delta_s y_{t-1} + \epsilon_t,$$

where $\{\epsilon_t\} \sim NID(0,1)$, $s = 1, \cdots, 4$ and the sample size T is 100 and 200. Note that $|\delta_1\delta_2\delta_3\delta_4| < 1$ under the null of periodic stationarity whereas $\delta_1\delta_2\delta_3\delta_4 = 1$ under the alternative of periodic integration. We set $\delta_s > 0$ for $s = 1, \cdots, 4$ since many economic time series are positively autocorrelated. From the discussion of Section 3, ω_1 may be seen as a measure of the degree of periodic integration and affect the limiting power under the alternative. Then, we conduct the simulation for several values of ω_1, and for a given ω_1, consider three sets of $\{\delta_s\}$, $\delta_1 = \delta_2 = \delta_3 = \delta_4$, $\delta_1 = 2\delta_2 = \delta_3 = 2\delta_4$ and $\delta_1 = \delta_2 = 2\delta_3 = 2\delta_4$. We select $\omega_1 = 16$, 18 and 20 under the alternative, which imply $\{\delta_1, \delta_2, \delta_3, \delta_4\} = \{1, 1, 1, 1\}$, $\{1.414, 0.707, 1.414, 0.707\}$ and $\{1.414, 1.414, 0.707, 0.707\}$, respectively. For the null hypothesis, we assume $\delta_1\delta_2\delta_3\delta_4 = 0.2$, 0.4, 0.6 and 0.8 and the relation among $\{\delta_s\}$ is proportional to the alternative. For example, we set $\{\delta_1, \delta_2, \delta_3, \delta_4\} = \{0.669, 0.669, 0.669, 0.669\}$, $\{0.946, 0.946, 0.473, 0.473\}$ and $\{0.946, 0.473, 0.946, 0, 473\}$ for $\delta_1\delta_2\delta_3\delta_4 = 0.2$. To construct the test statistics, we used not the true $\{\delta_s\}$ but the NLS estimates of them, $\{\hat{\delta}_s\}$. The upper 5% critical value is used and the level of significance is set equal to 0.05. The number of replication is 1,000 in all experiments.

Table 2 reports the frequencies of rejection of S_{mI}, S_{mII} and S_{mIII} for $T = 100$. We investigate each test statistic for several values of the lag truncation parameter. We used $\ell_1 = 2$, 4, 6 and 8 for S_{mI}, $\ell_2 = 1$, 2, 3 and 4 for S_{mII}, and $\ell_3 = 2$, 4, 6 and 8 for S_{mIII}, respectively. From the table, we can see that, under the null hypothesis of $\delta_1\delta_2\delta_3\delta_4 = 0.2$, 0.4, 0.6 and 0.8, the lag truncation parameter has much influence on the empirical size, especially the empirical size of S_{mI} is very sensitive to the truncation parameter. In most cases, the larger $\delta_1\delta_2\delta_3\delta_4$ is, the longer lag truncation parameter does each statistic need to have the empirical size close to 0.05.

The power of each test, corresponding to the row of $\delta_1\delta_2\delta_3\delta_4 = 1$, also varies according to the lag truncation parameter. Though we should carefully see the results since the empirical power is not size-adjusted, it seems that S_{mI} is less powerful than the other two test statistics which have the reasonable power. Comparing the cases 1, 2 and 3, the case 1 is most powerful and the case 3 is least powerful, like the other unit root tests.

Table 3 shows the result when $T = 200$. We used $\ell_1 = 4$, 6, 8 and 10 for S_{mI}, $\ell_2 = 1$, 2, 3 and 4 for S_{mII}, and $\ell_3 = 4$, 6, 8 and 10 for S_{mIII}, respectively. We can see that the relative performance of the tests is preserved without the fact that they require the longer lag truncation number to have the empirical size close to 0.05 compared with the case when $T = 100$.

6. Empirical Applications

In this section, we illustrate the empirical applications of the testing procedure derived in the previous section. Since our test assumes the null of periodic stationarity, it is useful to use our method in conjunction with the procedure developed in Boswijk and Franses (1996), in which the test for the null of periodic integration is proposed. We investigate two Japanese macroeconomic time series: Real national consumption expenditure (CP) and real disposable income of household (YDH) for 1955.1–1996.4 measured in logarithms. Figure 3 plots CP and YDH. Since both series have a seasonal pattern and increase with time, we include a seasonal dummy and a seasonal linear trend for the models of both series through all the testing procedures conducted below.

Firstly we test for the null of periodic integration against the alternative of no integration and next for the null of periodic stationarity. We take the same model selection procedure proposed in Franses and Paap (1994): i) decide the order p of the periodic autoregression, ii) test for the presence of periodicity, and iii) if the null of no periodicity is rejected, then test for periodic integration. By use of the Schwarz-Bayesian information criterion, we decide the order p of the PAR(p) model as two for both CP and YDH. Then, we assume the following PAR(2) model for both series:

$$y_t = \mu_{0s} + \mu_{1s} j + u_t, \quad u_t = \phi_{1s} u_{t-1} + \phi_{2s} u_{t-2} + v_t, \qquad (3-24)$$

where $j = [(t-1)/4] + 1$ and $s = 1, \cdots, 4$. The model of $\{u_t\}$ can also be expressed as

$$(1 - \delta_s B)(1 - \psi_s B) u_t = v_t, \qquad (3-25)$$

where $\delta_s + \psi_s = \phi_{1s}$ and $\delta_s \psi_s = -\phi_{2s}$ for $s = 1, \cdots, 4$. If $\{u_t\}$ is PI(0), both $(1 - \delta_s B)$ and $(1 - \phi_s B)$ constitute stationary VQ parameters whereas, if $\{u_t\}$ is PI(1), we assume that $(1 - \delta_s B)$ constitutes periodically integrated VQ parameters, that is, $\delta_1 \delta_2 \delta_3 \delta_4 = 1$.

Next step is to test for the null of no periodicity against the alternative of periodicity. The null hypothesis is expressed for the model (3-24) as

$$H_0^{NP} : \phi_{11} = \phi_{12} = \phi_{13} = \phi_{14} \text{ and } \phi_{21} = \phi_{22} = \phi_{23} = \phi_{24}.$$

The likelihood ratio (LR) test is investigated in Theorem 2 (i) of Boswijk and Franses (1996) and for both CP and YDH, we reject the null hypothesis H_0^{NP} with a 5% significance level, so that the PAR(2) model is appropriate for both series.

Now we test for the null of periodic integration. The LR test statistic is given by

$$LR = T \left(\ln \left(\frac{1}{T} \sum_{t=1}^{T} \tilde{v}_t^2 \right) - \ln \left(\frac{1}{T} \sum_{t=1}^{T} \hat{v}_t^2 \right) \right), \qquad (3-26)$$

where $T = 166$ is the number of observations, $\{\tilde{v}_t\}$ are the NLS residuals with the restriction $\delta_1 \delta_2 \delta_3 \delta_4 = 1$ and $\{\hat{v}_t\}$ are the (unrestricted) least squares residuals. Table 4 reports the values of the test statistic LR and the NLS estimates of $\{\delta_s\}$. According to Theorem 1 of Boswijk and Franses (1996), the null of PI(1) is rejected when LR is larger than the square of the critical value of Fuller's (1976) $\hat{\tau}_\tau$ statistic, and the null is not rejected both for CP and YDH with a 5% significance level.

Next, we test for the null of periodic stationarity against the alternative of periodic integration using S_{3I}, S_{3II} and S_{3III} developed in the previous section. Since we have to decide the values of $\{\delta_s\}$, we use the NLS estimates of $\{\delta_s\}$ in Table 4 to construct the test statistics. Note that, from our calculations, there seem to be many local minima in the objective function of the NLS, especially when we conduct the NLS using one of the $\mathcal{A}$ as starting values of $\{\delta_s\}$, we obtained the estimates of $\{\delta_s\}$ with same signs as the starting values. Since signs of $\{\delta_s\}$ have an important role on the power of the tests as discussed in Section 3, we should carefully estimate $\{\delta_s\}$ which minimizes the objective function globally.

Table 5 reports the values of the test statistics. Since the lag truncation parameters have an effect on the empirical size and power, we calculate the statistics for several values of them. Using the critical value of Table 1, all the test statistics reject the null of periodic stationarity for both CP and YDH with a 5% significance level. Then, with the results of Table 4, it seems that both CP and YDH are periodically integrated.

Though we rejected the null of no periodicity, it is possible that either of $\{\delta_s\}$ or $\{\psi_s\}$ in (3-25) are not periodic. Noting that integration in a usual sense is a special case of periodic integration, we test the null of a unit root, $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$, within the periodically integrated model, which is investigated in Theorem 2 (ii) of Boswijk and Franses (1996). The test statistic is the same as LR in (3-26) in which $\{\tilde{v}_t\}$ are the least squares residuals with the restriction $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$ and $\{\hat{v}_t\}$ are the NLS residuals with the restriction $\delta_1\delta_2\delta_3\delta_4 = 1$. Under the null hypothesis, the test statistic is asymptotically chi-squared distributed, and we reject the null for CP whereas do not reject for YDH, that is, YDH has a (non-periodic) unit root and the periodicity of YDH seems to be due to the stable parameters.

To conclude, both CP and YDH can be expressed as a PAR(2) process and CP is periodically integrated whereas YDH is integrated in a usual sense.

7. Concluding Remarks

We investigated the test for the null of periodic stationarity against the alternative of periodic integration. We derived the several test statistics which fundamentally are the same but different in the correction of nuisance parameters. We found that the limiting power of the test is much affected by signs of $\{\delta_s\}$, and in a finite sample, the lag truncation parameter affects both the size and the power through the Monte Carlo simulation.

Though we do not consider multiple unit roots which correspond to the case when $\text{rank}(\Pi) = 0$, 1 or 2 in the equation (3-5), our tests should have the considerable power against such a case. But we should carefully decide $\{\delta_s\}$ to be used for the test statistics so as not to lose the power by a bad selection of them as discussed in Section 3.

Appendix 3.

Proof of Theorem 3.1: (i) Firstly note that $W_j = U_j$ under H_0 and $\{U_j\}$ can be expressed as

$$U_j = AV_j + \tilde{V}_{j-1} - \tilde{V}_j,$$

where $\tilde{V}_j = \sum_{l=0}^{\infty} \tilde{A}_l V_{j-l}$ with $\tilde{A}_l = \sum_{k=l+1}^{\infty} A_k$, and in the stacked form,

$$u = \Omega_A v + \tilde{v}_{-1} - \tilde{v},$$

where Ω_A is a block diagonal matrix with each block A, $\Omega_A = diag\{A \cdots A\}$. Then, we can express the test statistic S_{mI} as

$$\begin{aligned}
S_{mI} &= \frac{1}{\tilde{\omega} N^2} w' M L L' M w \\
&= \frac{1}{\tilde{\omega} N^2} v' \Omega_A' M L L' M \Omega_A v + o_p(1) \\
&\stackrel{d}{=} \frac{\sigma_v^2}{\tilde{\omega} N^2} z' L' M \Omega_A \Omega_A' M L z + o_p(1) \\
&= \frac{\sigma_v^2}{\tilde{\omega} N} \sum_{j,k=1}^{N} z_j' \frac{1}{N} (L' M \Omega_A \Omega_A' M L)(j,k) z_k + o_p(1),
\end{aligned}$$

where $z = [z_1', z_2', \cdots, z_N']' \sim N(0, I_T)$, $\stackrel{d}{=}$ denotes equality in distribution, and the expression $(H)(j,k)$ denotes the (j,k) block of a $T \times T$ matrix H when H is decomposed into $N \times N$ blocks with each block a 4×4 matrix. The third equality in distribution is due to normality of v.

Let us consider the case 3. Note that

$$\begin{aligned}
\frac{1}{N}(L' M \Omega_A \Omega_A' M L)(j,k) &= \frac{1}{N}(L' \Omega_A \Omega_A' L)(j,k) - \frac{1}{N}(L' \Omega_A \Omega_A' X (X'X)^{-1} X' L)(j,k) \\
&\quad - \frac{1}{N}(L' X (X'X)^{-1} X' \Omega_A \Omega_A' L)(j,k) \\
&\quad + \frac{1}{N}(L' X (X'X)^{-1} X' \Omega_A \Omega_A' X (X'X)^{-1} X' L)(j,k), \qquad \text{(3-27)}
\end{aligned}$$

and from direct calculations, each term is expressed as

$$\frac{1}{N}(L' \Omega_A \Omega_A' L)(j,k) = \left(1 - \max\left(\frac{j}{N}, \frac{k}{N}\right)\right) L_1' A A' L_1 + O(N^{-1}),$$

$$\frac{1}{N}(L' \Omega_A \Omega_A' X (X'X)^{-1} X' L)(j,k)$$

$$
\begin{aligned}
= & \left\{4\left(1-\frac{j}{N}\right)\left(1-\frac{k}{N}\right)-3\left(1-\frac{j^2}{N^2}\right)\left(1-\frac{k}{N}\right)\right. \\
& \left.-3\left(1-\frac{j}{N}\right)\left(1-\frac{k^2}{N^2}\right)+3\left(1-\frac{j^2}{N^2}\right)\left(1-\frac{k^2}{N^2}\right)\right\} L_1'AA'L_1 \\
& +O(N^{-1}), \qquad (3\text{-}28)
\end{aligned}
$$

and the third and fourth terms of (3-27) have the same expression as (3-28) except for the $O(N^{-1})$ term. Then (3-27) can be expressed as

$$
\frac{1}{N}(L'M\Omega_A\Omega_A'ML)(j,k) = K_3\left(\frac{j}{N},\frac{k}{N}\right)L_1'AA'L_1 + O(N^{-1}),
$$

where

$$
\begin{aligned}
K_3(s,t) &= 1-\max(s,t)-4(1-s)(1-t)+3(1-s^2)(1-t) \\
&\quad +3(1-s)(1-t^2)-3(1-t^2)(1-s^2) \\
&= \min(s,t)-4st+3st(s+t)-3s^2t^2.
\end{aligned}
$$

From the above relation and Lemma 3 of Nabeya and Tanaka (1988), we have

$$
E\left|\frac{1}{N}\sum_{j,k=1}^{N} z_j'\frac{1}{N}(L'M\Omega_A\Omega_A'ML)(j,k)z_k - \frac{1}{N}\sum_{j,k=1}^{N} K_3\left(\frac{j}{N},\frac{k}{N}\right)z_j'L_1'AA'L_1z_k\right| \longrightarrow 0,
$$

so that the above difference in determinant converges to zero in probability by Markov's inequality. Then, since $\tilde{\Omega} \xrightarrow{p} \Omega$ and $\tilde{\omega} \xrightarrow{p} \omega$ under H_0 where $\xrightarrow{p}$ denotes convergence in probability, it is enough to consider the limiting distribution of

$$
S_{3I}^* = \frac{\sigma_v^2}{\omega N}\sum_{j,k=1}^{N} K_3\left(\frac{j}{N},\frac{k}{N}\right)z_j'L_1'AA'L_1z_k.
$$

To derive the limiting distribution of S_{3I}^*, we use Lemma 2.1, and to apply it to this case, we diagonalize the matrix $L_1'AA'L_1$ by the 4×4 orthogonal matrix P,

$$
\begin{aligned}
P &= [p_1,p_2,p_3,p_4] \\
&= \left[\frac{1}{\sqrt{q_1}}\begin{pmatrix}1\\ \delta_1\delta_3\delta_4\\ \delta_1\delta_4\\ \delta_1\end{pmatrix}, \frac{1}{\sqrt{q_2}}\begin{pmatrix}0\\ -1\\ 0\\ \delta_3\delta_4\end{pmatrix}, \frac{1}{\sqrt{q_3}}\begin{pmatrix}-1\\ 0\\ \delta_2\delta_3\\ 0\end{pmatrix}, \frac{1}{\sqrt{q_4}}\begin{pmatrix}-\delta_1\delta_2\delta_3(\delta_3^2\delta_4^2+1)\\ \delta_3\delta_4(\delta_2\delta_3+\delta_1\delta_4)\\ -\delta_1(\delta_3^2\delta_4^2+1)\\ \delta_2\delta_3+\delta_1\delta_4\end{pmatrix}\right],
\end{aligned}
$$

where each q_i is a normalizer for p_i to have a unit length. Especially, $q_1 = (1 + \delta_1^2 + \delta_1^2\delta_4^2 + \delta_1^2\delta_3^2\delta_4^2)$. Then, since $L_1 p_i = 0$ for $i = 2, 3, 4$, we have

$$P'L_1'AA'L_1P = diag(\kappa'AA'\kappa, 0, 0, 0), \qquad (3-29)$$

where $\kappa = L_1 p_1 = (1 + \delta_1^2 + \delta_1^2\delta_4^2 + \delta_1^2\delta_3^2\delta_4^2)^{1/2}[1, \delta_2, \delta_2\delta_3, \delta_2\delta_3\delta_4]'$. Note that $P'L_1'L_1P = diag(\kappa'\kappa, 0, 0, 0)$ and $\kappa'\kappa$ is a nonzero eigenvalue of $L_1'L_1$. By (3-29) and Lemma 2.1, we have

$$\begin{aligned} S_{3I}^* &= \frac{\sigma_v^2}{\omega N} \sum_{j,k=1}^{N} K_3\left(\frac{j}{N}, \frac{k}{N}\right) z_j' PP'L_1'AA'L_1PP'z_k \\ &\stackrel{d}{=} \frac{\sigma_v^2}{\omega N} \sum_{j,k=1}^{N} K_3\left(\frac{j}{N}, \frac{k}{N}\right) z_j' diag\{\kappa'AA'\kappa, 0, 0, 0\} z_k \\ &\xrightarrow{d} \frac{\sigma_v^2 \kappa AA'\kappa}{\omega} \sum_{n=1}^{\infty} \frac{1}{\lambda_{3,n}} Z_n^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{\lambda_{3,n}} Z_n^2, \end{aligned} \qquad (3\text{-}30)$$

since $\omega = \sigma_v^2 \kappa'AA'\kappa$, where $\{\lambda_{3,n}\}$ is a sequence of eigenvalues of $K_3(s,t)$. Since the Fredholm determinant of $K_3(s,t)$ is

$$D_3(\lambda) = \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right),$$

(see Nabeya and Tanaka, 1988), the characteristic function of (3-30) is $[D_3(2i\theta)]^{-1/2}$ by Lemma 2.1.

For the cases 1 and 2, we can show that

$$\frac{1}{N}(L'M\Omega_A\Omega_A'ML)(j,k) = K_m\left(\frac{j}{N}, \frac{k}{N}\right) L_1'AA'L_1 + O(N^{-1})$$

for $m = 1$ and 2, where

$$K_1(s,t) = 1 - \max(s,t), \quad K_2(s,t) = \min(s,t) - st,$$

and the corresponding Fredholm determinants are

$$D_1(\lambda) = \cos\sqrt{\lambda}, \quad D_2(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}.\square$$

(ii) Since, under H_1, $w = L\epsilon + \Omega_A v + \tilde{v}_{-1} - \tilde{v}$ and $(L\epsilon + \Omega_A v) \sim N(0, \sigma_v^2(\rho LL' + \Omega_A \Omega_A'))$, we have

$$\begin{aligned}
S_{mI} &= \frac{1}{\tilde{\omega} N^2} w' MLL'Mw \\
&\stackrel{d}{=} \frac{\sigma_v^2}{\tilde{\omega} N^2} z'(\rho LL' + \Omega_A \Omega_A')^{1/2} MLL'M(\rho LL' + \Omega_A \Omega_A')^{1/2} z + o_p(1) \\
&\stackrel{d}{=} \frac{\sigma_v^2}{\tilde{\omega} N^2} z' L'M(\rho LL' + \Omega_A \Omega_A') MLz + o_p(1) \\
&= \frac{\sigma_v^2}{\tilde{\omega} N} \sum_{j,k=1}^{N} z_j' \left(\frac{1}{N}(L'M\Omega_A\Omega_A'ML)(j,k) + \frac{\rho}{N}(L'MLL'ML)(j,k) \right) z_k + o_p(1) \\
&= \frac{\sigma_v^2}{\tilde{\omega} N} \sum_{j,k=1}^{N} z_j' \left(\frac{1}{N}(L'M\Omega_A\Omega_A'ML)(j,k) + \frac{c^2}{N} \sum_{l=1}^{N} \frac{1}{N}(L'ML)(j,l) \frac{1}{N}(L'ML)(l,k) \right) z_k + o_p(1)
\end{aligned}$$

In the same way as (i), we can derive that

$$\frac{1}{N}(L'ML)(j,k) = K_3\left(\frac{j}{N}, \frac{k}{N}\right) L_1' L_1 + O(N^{-1}), \tag{3-31}$$

for the case 3, and since $\tilde{\Omega} \xrightarrow{p} \Omega$ under H_1 and from the same discussion as (i), it is enough to investigate the limiting distribution of

$$S_{3I}^{**} = \frac{\sigma_v^2}{\omega N} \sum_{j,k=1}^{N} z_j' \left\{ K_3\left(\frac{j}{N}, \frac{k}{N}\right) L_1' AA' L_1 + \frac{c^2}{N} \left(\sum_{l=1}^{N} K_3\left(\frac{j}{N}, \frac{l}{N}\right) K_3\left(\frac{l}{N}, \frac{k}{N}\right) \right) (L_1' L_1)^2 \right\} z_k.$$

Since $P'L_1'AA'L_1P = diag(\kappa' AA' \kappa, 0, 0, 0)$ from (3-29) and $P'L_1'L_1P = diag(\kappa'\kappa, 0, 0, 0)$ by direct calculations, we have, using Lemma 2.3,

$$\begin{aligned}
S_{3I}^{**} &= \frac{\sigma_v^2}{\omega N} \sum_{j,k=1}^{N} z_j' P \left\{ K_3\left(\frac{j}{N}, \frac{k}{N}\right) P'L_1'AA'L_1P \right. \\
&\quad \left. + \frac{c^2}{N} \left(\sum_{l=1}^{N} K_3\left(\frac{j}{N}, \frac{l}{N}\right) K_3\left(\frac{l}{N}, \frac{k}{N}\right) \right) (P'L_1'L_1P)^2 \right\} P z_k \\
&\stackrel{d}{=} \frac{\sigma_v^2}{\omega N} \sum_{j,k=1}^{N} z_j' \left\{ K_3\left(\frac{j}{N}, \frac{k}{N}\right) diag(\kappa' AA' \kappa, 0, 0, 0) \right. \\
&\quad \left. + \frac{c^2}{N} \left(\sum_{l=1}^{N} K_3\left(\frac{j}{N}, \frac{l}{N}\right) K_3\left(\frac{l}{N}, \frac{k}{N}\right) \right) diag((\kappa'\kappa)^2, 0, 0, 0) \right\} P z_k \\
&\xrightarrow{d} \frac{\sigma_v^2}{\omega} \sum_{n=1}^{\infty} \left(\frac{\kappa' AA' \kappa}{\lambda_{3,n}} + \frac{c^2 (\kappa'\kappa)^2}{\lambda_{3,n}^2} \right) Z_n^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{3,n}} + \frac{c^2 \sigma_v^2 \omega_1^2}{\omega \lambda_{3,n}^2} \right) Z_n^2.
\end{aligned} \tag{3-32}$$

Then, by Lemma 2.3, the limiting characteristic function is given by (3-15).

For the cases 1 and 2, note that

$$\frac{1}{N}(L'ML)(j,k) = K_m\left(\frac{j}{N},\frac{k}{N}\right)L_1'L_1 + O(N^{-1}),$$

for $m = 1$ and 2, and the theorem is proved in the same way as above for these cases.□

Proof of Corollary 3.1: Since $w = L^0\epsilon + u$ where $L_0 = L$ with $\{\delta_s^0\}$ instead of $\{\delta_s\}$, we have, as in the proof of Theorem 3.1 (ii),

$$\begin{aligned} S_{mI} &\stackrel{d}{=} \frac{\sigma_v^2}{\tilde{\omega}N^2}z'L'M(\rho L^0L^{0\prime} + \Omega_A\Omega_A')MLz + o_p(1) \\ &= \frac{\sigma_v^2}{\tilde{\omega}N}\sum_{j,k=1}^{N} z_j'\left(\frac{1}{N}(L'M\Omega_A\Omega_A'ML)(j,k) + \frac{c^2}{N}\sum_{l=1}^{N}\frac{1}{N}(L'ML^0)(j,l)\frac{1}{N}(L^{0\prime}ML)(l,k)\right)z_k + o_p(1). \end{aligned}$$

In the same way as (3-31), we have

$$\frac{1}{N}(L'ML^0)(j,k) = K_m\left(\frac{j}{N},\frac{k}{N}\right)L_1'L_1^0 + O(N^{-1}),$$

for $m = 1$, 2 and 3. Then, since $L_1P = [\kappa, 0, 0, 0]$, we have

$$\begin{aligned} S_{mI} &\xrightarrow{d} \frac{\sigma_v^2}{\omega}\sum_{n=1}^{\infty}\left(\frac{\kappa'AA'\kappa}{\lambda_{m,n}} + \frac{c^2\kappa'L_1^0L_1^{0\prime}\kappa}{\lambda_{m,n}^2}\right)Z_n^2 \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{m,n}} + \frac{c^2\sigma_v^2\gamma}{\omega\lambda_{m,n}^2}\right)Z_n^2, \end{aligned}$$

in the same way as (3-32).□

Table 1. Percent Points of the Limiting Null Distribution of S_{mI}

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
S_{1I}	0.0345	0.0565	0.0765	0.2905	1.1958	1.6557	2.7875
S_{2I}	0.0248	0.0366	0.0460	0.1189	0.3473	0.4614	0.7435
S_{3I}	0.0173	0.0234	0.0279	0.0555	0.1192	0.1479	0.2177

Table 2. The Size and Power of S_{mI}, S_{mII} and S_{mIII} with 5% Asymptotic Critical Value: T=100

	S_{mI}				S_{mII}				S_{mIII}			
$\delta_1\delta_2\delta_3\delta_4$ (ratio)	ℓ_1=2	4	6	8	ℓ_2=1	2	3	4	ℓ_3=2	4	6	8
Case 1												
(1:1:1:1)	0.712	0.539	0.023	0.000	0.898	0.807	0.721	0.631	0.905	0.818	0.742	0.660
1 (2:1:2:1)	0.713	0.521	0.012	0.000	0.915	0.811	0.728	0.636	0.920	0.823	0.749	0.664
(2:2:1:1)	0.676	0.423	0.000	0.000	0.911	0.796	0.716	0.620	0.919	0.814	0.736	0.650
(1:1:1:1)	0.286	0.099	0.001	0.000	0.579	0.348	0.197	0.085	0.608	0.403	0.257	0.163
0.8 (2:1:2:1)	0.276	0.092	0.001	0.000	0.585	0.358	0.206	0.098	0.618	0.415	0.274	0.163
(2:2:1:1)	0.242	0.043	0.000	0.000	0.609	0.360	0.190	0.079	0.641	0.408	0.269	0.143
(1:1:1:1)	0.120	0.029	0.001	0.000	0.263	0.097	0.033	0.016	0.303	0.156	0.072	0.046
0.6 (2:1:2:1)	0.120	0.026	0.000	0.000	0.277	0.104	0.036	0.016	0.324	0.174	0.084	0.046
(2:2:1:1)	0.085	0.015	0.000	0.000	0.279	0.091	0.032	0.011	0.343	0.164	0.073	0.041
(1:1:1:1)	0.066	0.017	0.001	0.000	0.097	0.033	0.020	0.017	0.147	0.068	0.042	0.029
0.4 (2:1:2:1)	0.065	0.017	0.000	0.000	0.103	0.033	0.020	0.016	0.160	0.073	0.043	0.034
(2:2:1:1)	0.050	0.008	0.000	0.000	0.109	0.028	0.014	0.010	0.169	0.071	0.038	0.030
(1:1:1:1)	0.035	0.008	0.000	0.000	0.044	0.027	0.026	0.026	0.071	0.047	0.036	0.032
0.2 (2:1:2:1)	0.038	0.010	0.000	0.000	0.048	0.026	0.024	0.022	0.079	0.051	0.038	0.035
(2:2:1:1)	0.035	0.004	0.000	0.000	0.045	0.021	0.020	0.020	0.084	0.047	0.036	0.031
Case 2												
(1:1:1:1)	0.581	0.377	0.117	0.005	0.850	0.669	0.534	0.400	0.872	0.709	0.577	0.473
1 (2:1:2:1)	0.580	0.379	0.109	0.003	0.861	0.693	0.540	0.406	0.877	0.732	0.595	0.485
(2:2:1:1)	0.545	0.276	0.007	0.000	0.863	0.677	0.538	0.385	0.886	0.716	0.593	0.459
(1:1:1:1)	0.300	0.109	0.012	0.002	0.554	0.306	0.129	0.035	0.594	0.368	0.221	0.106
0.8 (2:1:2:1)	0.286	0.115	0.016	0.001	0.574	0.318	0.136	0.040	0.624	0.392	0.233	0.117
(2:2:1:1)	0.238	0.051	0.002	0.000	0.570	0.295	0.128	0.031	0.615	0.365	0.220	0.108
(1:1:1:1)	0.158	0.058	0.011	0.001	0.286	0.098	0.027	0.011	0.362	0.167	0.088	0.042
0.6 (2:1:2:1)	0.166	0.059	0.011	0.002	0.323	0.108	0.035	0.015	0.395	0.187	0.094	0.054
(2:2:1:1)	0.133	0.028	0.001	0.000	0.324	0.095	0.024	0.009	0.388	0.180	0.081	0.039
(1:1:1:1)	0.111	0.041	0.010	0.002	0.125	0.041	0.024	0.021	0.191	0.093	0.056	0.045
0.4 (2:1:2:1)	0.108	0.047	0.008	0.002	0.139	0.043	0.024	0.020	0.218	0.100	0.058	0.046
(2:2:1:1)	0.081	0.025	0.002	0.000	0.140	0.036	0.015	0.014	0.219	0.091	0.053	0.043
(1:1:1:1)	0.064	0.030	0.010	0.002	0.055	0.041	0.038	0.034	0.106	0.066	0.052	0.048
0.2 (2:1:2:1)	0.068	0.033	0.007	0.003	0.062	0.042	0.037	0.036	0.108	0.069	0.056	0.051
(2:2:1:1)	0.060	0.025	0.005	0.000	0.059	0.035	0.031	0.029	0.123	0.061	0.052	0.045
Case 3												
(1:1:1:1)	0.448	0.204	0.087	0.266	0.726	0.439	0.213	0.093	0.782	0.527	0.322	0.176
1 (2:1:2:1)	0.444	0.202	0.092	0.274	0.764	0.458	0.228	0.099	0.812	0.564	0.342	0.200
(2:2:1:1)	0.337	0.056	0.013	0.131	0.750	0.414	0.193	0.074	0.798	0.528	0.303	0.165
(1:1:1:1)	0.267	0.100	0.063	0.294	0.548	0.218	0.056	0.006	0.634	0.322	0.153	0.046
0.8 (2:1:2:1)	0.262	0.104	0.060	0.298	0.590	0.229	0.074	0.011	0.678	0.360	0.168	0.070
(2:2:1:1)	0.190	0.025	0.021	0.162	0.580	0.204	0.050	0.013	0.674	0.315	0.145	0.048
(1:1:1:1)	0.152	0.061	0.077	0.364	0.274	0.063	0.010	0.004	0.378	0.142	0.055	0.018
0.6 (2:1:2:1)	0.164	0.057	0.071	0.371	0.308	0.076	0.013	0.004	0.430	0.174	0.071	0.028
(2:2:1:1)	0.102	0.020	0.037	0.237	0.307	0.069	0.007	0.003	0.439	0.143	0.053	0.017
(1:1:1:1)	0.094	0.050	0.090	0.432	0.099	0.025	0.010	0.010	0.186	0.073	0.040	0.025
0.4 (2:1:2:1)	0.091	0.048	0.093	0.434	0.127	0.030	0.011	0.009	0.226	0.087	0.049	0.030
(2:2:1:1)	0.063	0.023	0.066	0.317	0.118	0.024	0.008	0.008	0.233	0.081	0.041	0.022
(1:1:1:1)	0.069	0.043	0.112	0.481	0.050	0.025	0.022	0.019	0.092	0.055	0.041	0.037
0.2 (2:1:2:1)	0.070	0.051	0.123	0.490	0.055	0.030	0.026	0.025	0.109	0.060	0.044	0.038
(2:2:1:1)	0.056	0.039	0.097	0.414	0.053	0.021	0.017	0.016	0.112	0.058	0.043	0.035

Table 3. The Size and Power of S_{mI}, S_{mII} and S_{mIII} with 5% Asymptotic Critical Value: T=200

	S_{mI}				S_{mII}				S_{mIII}			
$\delta_1\delta_2\delta_3\delta_4$ (ratio)	ℓ_1=2	4	6	8	ℓ_2=1	2	3	4	ℓ_3=2	4	6	8
Case 1												
(1:1:1:1)	0.756	0.690	0.597	0.463	0.965	0.925	0.877	0.828	0.931	0.883	0.839	0.797
1 (2:1:2:1)	0.750	0.685	0.600	0.474	0.978	0.918	0.878	0.833	0.927	0.883	0.843	0.802
(2:2:1:1)	0.728	0.643	0.520	0.327	0.976	0.933	0.872	0.815	0.944	0.884	0.835	0.788
(1:1:1:1)	0.205	0.130	0.072	0.028	0.630	0.413	0.278	0.188	0.456	0.332	0.243	0.187
0.8 (2:1:2:1)	0.209	0.121	0.073	0.028	0.643	0.426	0.289	0.182	0.478	0.345	0.257	0.190
(2:2:1:1)	0.167	0.097	0.038	0.008	0.645	0.404	0.258	0.166	0.452	0.311	0.224	0.162
(1:1:1:1)	0.096	0.055	0.032	0.008	0.302	0.137	0.081	0.048	0.192	0.133	0.096	0.082
0.6 (2:1:2:1)	0.094	0.054	0.025	0.005	0.325	0.145	0.080	0.050	0.212	0.135	0.094	0.081
(2:2:1:1)	0.073	0.037	0.015	0.003	0.315	0.132	0.069	0.039	0.191	0.126	0.098	0.071
(1:1:1:1)	0.062	0.033	0.016	0.005	0.130	0.057	0.043	0.037	0.111	0.078	0.064	0.056
0.4 (2:1:2:1)	0.057	0.031	0.014	0.003	0.136	0.061	0.044	0.033	0.110	0.079	0.065	0.061
(2:2:1:1)	0.044	0.019	0.010	0.002	0.125	0.057	0.034	0.029	0.101	0.074	0.065	0.054
(1:1:1:1)	0.040	0.023	0.012	0.003	0.064	0.043	0.040	0.036	0.066	0.057	0.053	0.051
0.2 (2:1:2:1)	0.040	0.021	0.012	0.002	0.070	0.038	0.036	0.036	0.075	0.064	0.057	0.052
(2:2:1:1)	0.034	0.016	0.007	0.001	0.062	0.035	0.030	0.030	0.066	0.060	0.056	0.052
Case 2												
(1:1:1:1)	0.625	0.525	0.446	0.340	0.960	0.892	0.799	0.716	0.909	0.832	0.750	0.679
1 (2:1:2:1)	0.621	0.535	0.445	0.340	0.969	0.903	0.796	0.708	0.923	0.833	0.753	0.672
(2:2:1:1)	0.596	0.479	0.362	0.210	0.965	0.883	0.794	0.700	0.900	0.828	0.750	0.656
(1:1:1:1)	0.215	0.131	0.068	0.034	0.701	0.417	0.254	0.160	0.484	0.341	0.234	0.166
0.8 (2:1:2:1)	0.218	0.139	0.071	0.037	0.734	0.446	0.267	0.159	0.511	0.345	0.242	0.175
(2:2:1:1)	0.166	0.090	0.041	0.014	0.734	0.424	0.247	0.142	0.513	0.323	0.215	0.150
(1:1:1:1)	0.099	0.067	0.044	0.027	0.329	0.138	0.072	0.045	0.206	0.129	0.096	0.075
0.6 (2:1:2:1)	0.109	0.068	0.042	0.020	0.353	0.147	0.080	0.046	0.222	0.136	0.106	0.082
(2:2:1:1)	0.074	0.050	0.026	0.011	0.355	0.131	0.063	0.043	0.216	0.126	0.086	0.068
(1:1:1:1)	0.064	0.052	0.038	0.024	0.139	0.054	0.040	0.034	0.109	0.081	0.058	0.051
0.4 (2:1:2:1)	0.067	0.050	0.036	0.019	0.150	0.062	0.042	0.035	0.117	0.086	0.065	0.057
(2:2:1:1)	0.057	0.043	0.023	0.010	0.139	0.058	0.044	0.038	0.110	0.076	0.063	0.057
(1:1:1:1)	0.051	0.044	0.034	0.019	0.061	0.041	0.037	0.036	0.072	0.057	0.054	0.048
0.2 (2:1:2:1)	0.055	0.041	0.032	0.018	0.069	0.043	0.040	0.039	0.075	0.065	0.058	0.051
(2:2:1:1)	0.050	0.038	0.023	0.011	0.067	0.042	0.040	0.038	0.075	0.059	0.054	0.051
Case 3												
(1:1:1:1)	0.512	0.373	0.247	0.147	0.953	0.822	0.653	0.487	0.868	0.719	0.587	0.473
1 (2:1:2:1)	0.505	0.368	0.251	0.148	0.969	0.826	0.667	0.497	0.880	0.741	0.604	0.477
(2:2:1:1)	0.435	0.267	0.113	0.022	0.959	0.821	0.630	0.461	0.875	0.712	0.567	0.441
(1:1:1:1)	0.207	0.123	0.060	0.045	0.792	0.453	0.223	0.108	0.563	0.357	0.216	0.129
0.8 (2:1:2:1)	0.212	0.116	0.064	0.047	0.825	0.479	0.239	0.119	0.606	0.375	0.224	0.147
(2:2:1:1)	0.155	0.063	0.026	0.008	0.824	0.466	0.218	0.093	0.572	0.355	0.202	0.121
(1:1:1:1)	0.103	0.056	0.049	0.043	0.409	0.134	0.046	0.017	0.232	0.134	0.080	0.051
0.6 (2:1:2:1)	0.092	0.059	0.047	0.043	0.445	0.143	0.048	0.025	0.256	0.142	0.083	0.056
(2:2:1:1)	0.063	0.031	0.024	0.012	0.445	0.132	0.041	0.016	0.248	0.125	0.071	0.042
(1:1:1:1)	0.057	0.036	0.041	0.046	0.138	0.040	0.016	0.011	0.108	0.071	0.043	0.032
0.4 (2:1:2:1)	0.053	0.040	0.040	0.045	0.158	0.041	0.018	0.014	0.127	0.075	0.053	0.037
(2:2:1:1)	0.039	0.025	0.027	0.022	0.166	0.042	0.013	0.010	0.118	0.079	0.048	0.035
(1:1:1:1)	0.035	0.033	0.039	0.044	0.053	0.025	0.018	0.016	0.057	0.043	0.036	0.034
0.2 (2:1:2:1)	0.034	0.032	0.042	0.049	0.057	0.025	0.018	0.018	0.066	0.049	0.039	0.037
(2:2:1:1)	0.036	0.025	0.027	0.034	0.060	0.020	0.016	0.015	0.077	0.053	0.033	0.031

Table 4. The Test for Periodic Integration

	δ_1	δ_2	δ_3	δ_4	LR
CP	0.964	1.033	1.025	0.980	0.206
YDH	0.997	0.989	1.041	0.973	0.573

Table 5. The Test for Periodic Stationarity

	S_{3I}			S_{3II}			S_{3III}		
	$\ell_1=4$	6	8	$\ell_2=2$	3	4	$\ell_3=8$	10	12
CP	0.246	0.194	0.167	10.284	5.637	3.577	3.664	2.559	1.900
YDH	0.244	0.190	0.164	4.978	2.711	1.729	1.822	1.293	0.978

Figure 1a. Limiting Powers in the Case 1

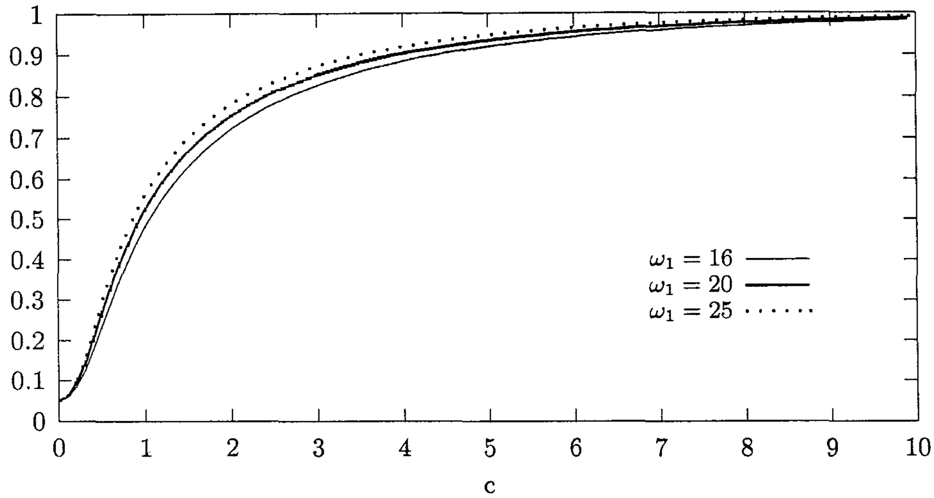


Figure 1b. Limiting Powers in the Case 2

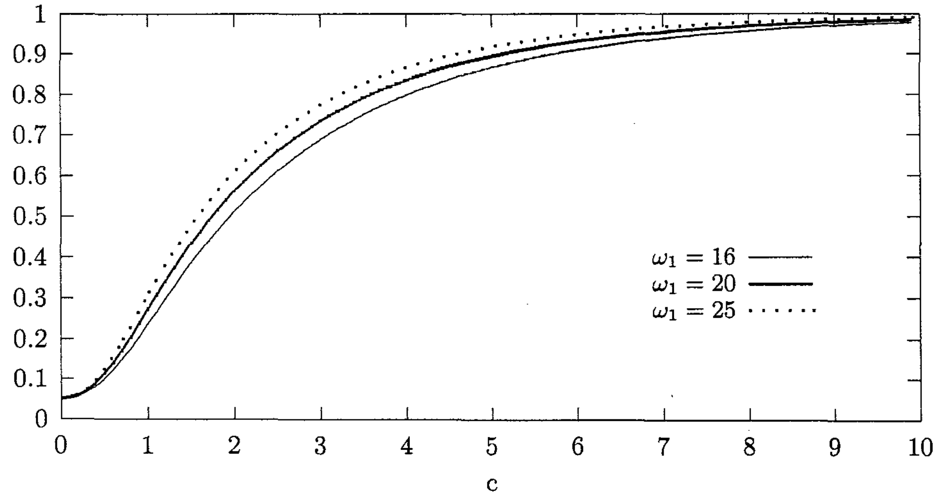


Figure 1c. Limiting Powers in the Case 3

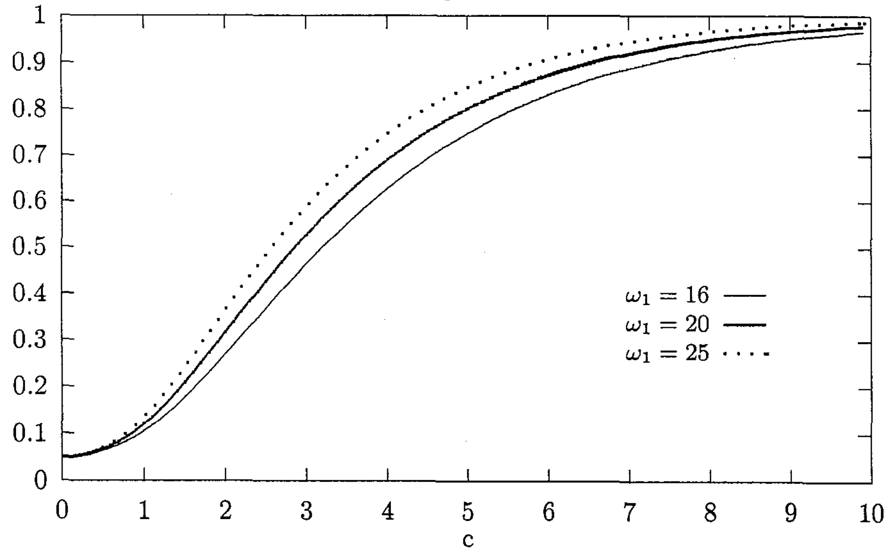


Figure 2a. Effects of Signs of $\{\delta_s\}$ in the Case 1

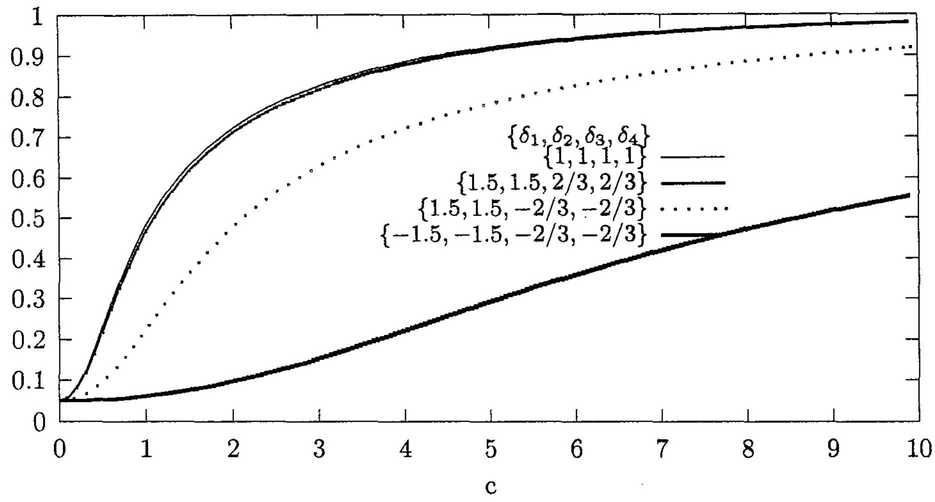


Figure 2b. Effects of Signs of $\{\delta_s\}$ in the Case 2

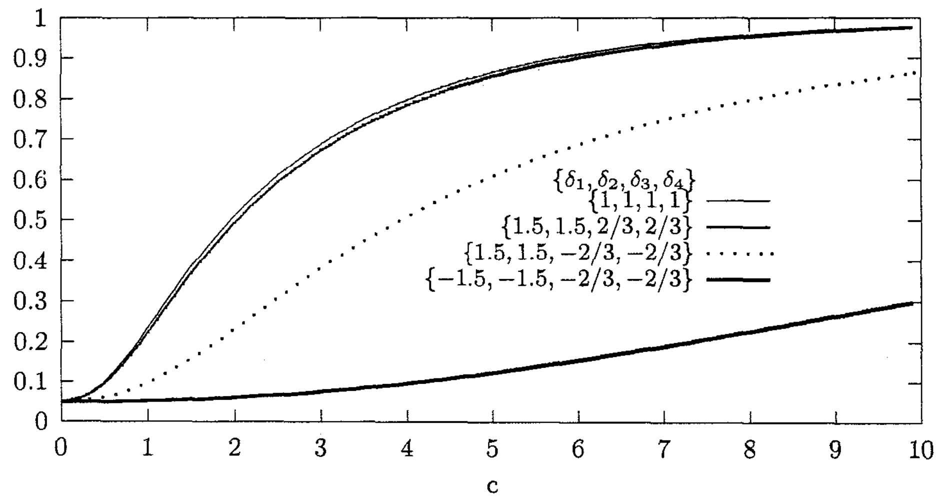


Figure 2c. Effects of Signs of $\{\delta_s\}$ in the Case 3

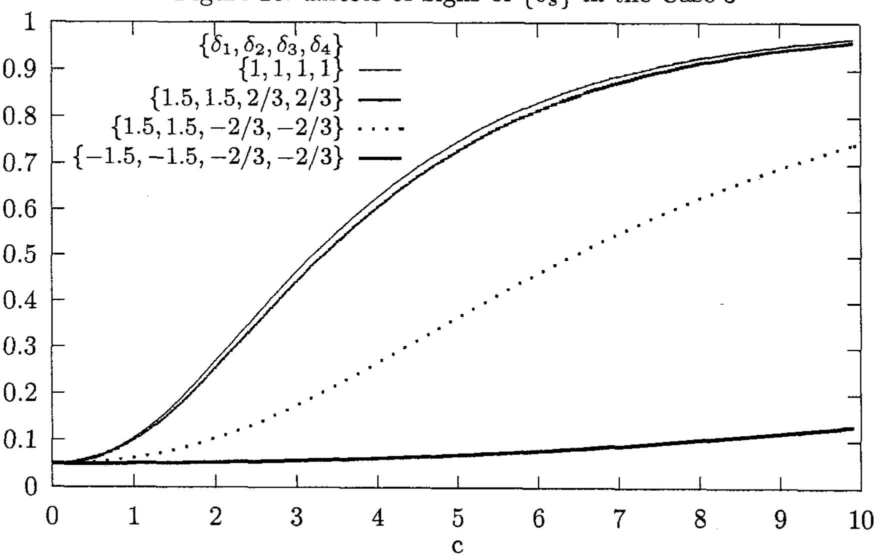
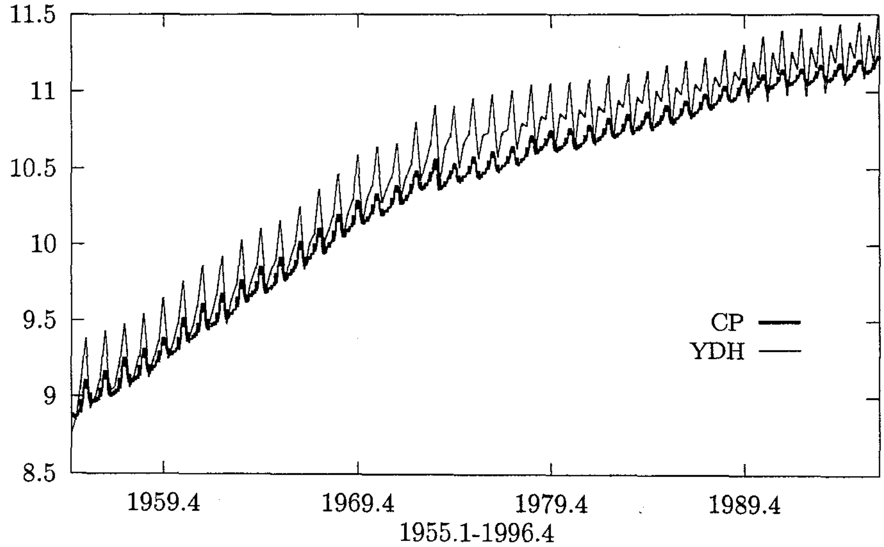


Figure 3. Log of Real National Consumption Expenditure and Log of Real Disposable Income of Household



Chapter 4.

Testing for Stationarity with a Break

In this chapter, we investigate the test for the null hypothesis of stationarity with a structural change against a unit root. We derive the limiting distribution of the LM test statistic and its characteristic function under a sequence of local alternatives. We also propose the test statistic which does not depend on the fraction of a break date to the sample size. Applying our tests to the Nelson-Plosser data, we find that for some macroeconomic time series, for which the tests proposed by Perron (1997) and Zivot and Andrews (1992) reject the null of a unit root, our tests accept the null of stationarity with a break.

1. Introduction

In this chapter, we propose testing procedures for the null hypothesis of stationarity with a structural change against a unit root. According to the empirical studies reviewed in Section 4 of Chapter 1, there are several cases that the null hypothesis of a unit root is rejected. In such cases, the researcher may guess that the time series obey a stationary process possibly with a break, but the rejection of a unit root does not necessarily imply stationarity of the data since the tests for a unit root may have the power against more general alternatives. Then, once the null of a unit root is rejected, the test for the null of stationarity with a break becomes of primary interest. As in the tests for the null of a unit root possibly with a break, we suppose that the fraction of the break date to the sample size is constant, and the limiting distribution of our test based on the LM principle also depends on its fraction. We also propose the test statistic which does not depend on the fraction under the null hypothesis in some cases as in Park and Sung (1994) (we call that test the PS test). The limiting properties of the tests proposed in this chapter are compared under a sequence of local alternatives, and, as suggested in theory that the LM test is locally best invariant (LBI) under the assumption of normality, the limiting power of the LM test dominates that of the PS test under the alternative close to the null, though this is not always the case when the local alternatives diverge from the null.

The plan of this chapter is as follows. Section 2 sets up the model and assumption. Two test statistics are proposed in Section 3 and the limiting properties of them are investigated. Finite sample properties are investigated in Section 4 and the tests proposed in this chapter are applied to the U.S. macroeconomic data in Section 5. Section 6 concludes the chapter.

2. The Model and the Testing Problem

Let us consider the following error-components model.

$$y_t = z_t'\beta + x_t, \quad x_t = \gamma_t + u_t, \quad \gamma_t = \gamma_{t-1} + \varepsilon_t, \quad u_t = v_t, \qquad (4-1)$$

where z_t denotes a deterministic component which includes a trend break, $\{v_t\} \sim NID(0, \sigma_v^2)$ with $\sigma_v^2 > 0$, $\{\varepsilon_t\} \sim NID(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon^2 \geq 0$, and $\{u_t\}$ and $\{\varepsilon_t\}$ are independent. We set $t = 1, \cdots, T$, and $\gamma_0 = 0$ without loss of generality since z_t includes a constant term as defined below. We suppose that a structural change occurred at time T_B $(1 < T_B < T)$, and that $\omega = T_B/T$ is fixed. For the deterministic component, z_t, we consider the following four cases.

Case 0 : a constant with a break; $z_t = [1, DU_t]'$,

Case 1 : a constant with a break and a linear trend; $z_t = [1, DU_t, t]'$,

Case 2 : a constant with no break and a linear trend with a break; $z_t = [1, t, DT_t]'$,

Case 3 : a constant and a linear trend both with a break; $z_t = [1, DU_t, t, DT_t]'$,

where $DU_t = 1(t > T_B)$ and $DT_t = 1(t > T_B) \times (t - T_B)$ with $1(\cdot)$ denoting an indicator function. The case 0 corresponds to the model without a linear trend such as an interest rate and the purchasing power parity as discussed in Perron (1990) and Perron and Vogelsang (1992a, b), whereas the cases 1 to 3 the model with a linear trend such as the gross domestic product and many macroeconomic variables. Perron (1989) called the case 1 the "crash model" while the case 2 the "changing growth model". The case 3 allows for a "sudden change in level followed by a different growth path".

Basically we will investigate the above "additive outlier model", that is, we suppose that a shock affects the observation only at one time, but we will later discuss the case that the effect of a structural change pervades the variables with lags, which may be called the 'innovational outlier model".

We also assume that the break point is known. As discussed in the literature, this assumption might be inadequate and the unknown break point emerges. However, when the observation obeys a stationary process, the testing procedures for a structural change are proposed in the literature and the consistent estimator of the break point has been developed and is available. Then, our analysis below can be established using such an estimator even with an unknown break point. See, for example, Andrews (1993), Andrews and Ploberger (1994) and Vogelsang (1997) for the tests of the structural change, and Bai (1994, 1998) and Nunes, Kuan and Newbold (1995) for the consistent estimation of the break point.

The model (4-1) can be expressed as the vectorized model by stacking each variable,

$$y = Z\beta + x, \quad x = \gamma + u, \quad \gamma = L\epsilon, \quad u = v,$$

where, e.g., $y = [y_1, \cdots, y_T]'$ and L is a lower triangular matrix with lower elements 1's,

$$L = \begin{bmatrix} 1 & & \mathbf{0} \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}.$$

To test for the null hypothesis of stationarity with a break, we consider the following testing problem:

$$H_0 : \rho = \frac{\sigma_\varepsilon^2}{\sigma_v^2} = 0 \quad \text{v.s.} \quad H_1 : \rho = \frac{c^2}{T^2}, \qquad (4-2)$$

where c is a constant. Then, under the null hypothesis, $\sigma_\varepsilon^2 = 0$ so that $\{y_t\}$ is (trend) stationary with a break. On the other hand, under H_1, x_t contains a unit root component γ so that $\{y_t\}$ becomes a unit root process with a break. By considering a sequence of the local alternatives, not a fixed alternative, we can derive the local limiting power functions and we will investigate the properties of the test statistics by drawing such functions.

3. Testing for Stationarity

3.1. The LM Test

For the testing problem (4-2), it is well known that the LM test statistic is proportional to $y'MLL'My$ where $M = I_T - Z(Z'Z)^{-1}Z'$. See, for example, Kwiatkowski, Phillips and Schmidt (1992) for its derivation. We will consider the limiting distribution of $y'MLL'My$ multiplied by T^{-2}. Note that, under the assumption of normality, the LM test is equivalent to the LBI test as discussed in King and Hillier (1985).

Here we allow for dependence of $\{u_t\}$ since the assumption of independence is too restrictive. We suppose that

$$u_t = \sum_{j=0}^{\infty} \alpha_j v_{t-j}, \quad \sum_{j=1}^{\infty} j|\alpha_j| < \infty.$$

We also assume that $\alpha = \sum_{j=0}^{\infty} \alpha_j \neq 0$. Note that a finite-order autoregressive moving average (ARMA) process satisfies the above condition.

Since the limiting distribution of the LM test statistic depends on a nuisance parameter as shown in the proof of Appendix 4, we consider the following statistic.

$$S_T = \frac{1}{\tilde{\sigma}^2 T^2} y'MLL'My = \frac{1}{\tilde{\sigma}^2 T^2} \sum_{j=1}^{T-1} \left(\sum_{t=1}^{j} \tilde{x}_t \right)^2, \qquad (4-3)$$

where

$$\tilde{\sigma}^2 = \tilde{\gamma}(0) + 2\sum_{i=1}^{\ell} w(i,\ell)\tilde{\gamma}(i), \qquad (4-4)$$

with $\tilde{\gamma}(i) = \sum_{t=1}^{T-i} \tilde{x}_t\tilde{x}_{t+i}/T$ and $w(i,\ell) = 1 - i/(\ell+1)$ the Bartlett window for $\ell = o(N^{1/2})$, and $\tilde{x}_t$ are regression residuals of y_t on z_t,

$$\tilde{x}_t = y_t - z_t'\left(\sum_{t=1}^{T} z_t z_t'\right)^{-1}\sum_{t=1}^{T} z_t y_t.$$

The second expression of (4-3) is convenient for the practical calculation of the test statistic, though we mainly use the first expression for the theoretical explanation.

The following theorem gives the limiting distribution of S_T and its characteristic function for each case. For notational convenience, we define the following functional of a standard Brownian motion as a generic form,

$$G(B;c^2) = \int_0^1 B(r)^2 dr - X(B)'\Lambda^{-1}X(B) + c^2\int_0^1 \left(\int_0^r B(s)ds - Z(r)'\Lambda^{-1}X(B)\right)^2 dr,$$

where $B(\cdot)$ is a standard Brownian motion and $X(B)$ denotes a functional of $B(\cdot)$. Since the null hypothesis is a special case of the alternative ($c = 0$), we give the result only under the alternative.

Theorem 4.1 *Consider the model (4-1). (i) For the cases 0 and 3, under a sequence of local alternatives, H_1,*

$$S_T \xrightarrow{d} \omega^2 G(B_1; c^2\omega^2/\alpha^2) + (1-\omega)^2 G(B_2, c^2(1-\omega^2)/\alpha^2), \qquad (4-5)$$

and its characteristic function is expressed as

$$\begin{aligned}\phi(\theta;c) \;=\; & \left[D\left(i\omega^2\theta + \sqrt{-\omega^4\theta^2 + 2ic^2\omega^4\theta/\alpha^2}\right) D\left(i\omega^2\theta - \sqrt{-\omega^4\theta^2 + 2ic^2\omega^4\theta/\alpha^2}\right)\right]^{-1/2} \\ & \left[D\left(i(1-\omega)^2\theta + \sqrt{-(1-\omega)^4\theta^2 + 2ic^2(1-\omega)^4\theta/\alpha^2}\right)\right. \\ & \left.D\left(i(1-\omega)^2\theta - \sqrt{-(1-\omega)^4\theta^2 + 2ic^2(1-\omega)^4\theta/\alpha^2}\right)\right]^{-1/2}, \qquad (4\text{-}6)\end{aligned}$$

where $B_1(\cdot)$ and $B_2(\cdot)$ are independent Brownian motions, $i = \sqrt{-1}$, and

(i-a) for the case 0,

$$X(B) = \int_0^1 B(r)dr, \quad Z(r) = r, \quad \Lambda = 1,$$

$$D(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$

(i-b) for the case 3,

$$X(B) = \begin{bmatrix} \int_0^1 B(r)dr \\ \int_0^1 rB(r)dr \end{bmatrix}, \quad Z(r) = \begin{bmatrix} r \\ \frac{r^2}{2} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix},$$

$$D(\lambda) = \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right).$$

(ii) For the cases 1 and 2, under a sequence of local alternatives, H_1,

$$S_T \xrightarrow{d} G(B; c^2/\alpha^2), \tag{4-7}$$

and its characteristic function is expressed as

$$\phi(\theta; c) = \left[D\left(i\theta + \sqrt{-\theta^2 + 2ic^2\theta/\alpha^2}\right) D\left(i\theta - \sqrt{-\theta^2 + 2ic^2\theta/\alpha^2}\right)\right]^{-1/2}, \tag{4-8}$$

where $B(\cdot)$ is a standard Brownian motion and

(ii-a) for the case 1,

$$X(B) = \begin{bmatrix} \int_0^1 B(r)dr \\ \int_\omega^1 B(r)dr \\ \int_0^1 rB(r)dr \end{bmatrix}, \quad Z(r) = \begin{bmatrix} r \\ dt_r \\ \frac{r^2}{2} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 1-\omega & \frac{1}{2} \\ 1-\omega & 1-\omega & \frac{1-\omega^2}{2} \\ \frac{1}{2} & \frac{1-\omega^2}{2} & \frac{1}{3} \end{bmatrix},$$

$$D(\lambda) = -12\frac{\sqrt{\lambda}\sin\sqrt{\lambda\omega^2}\sin\sqrt{\lambda(1-\omega)^2} + 2\left(\sin\sqrt{\lambda} - \sin\sqrt{\lambda\omega^2} - \sin\sqrt{\lambda(1-\omega)^2}\right)}{\lambda^{5/2}\omega(1-\omega)\{1-3\omega(1-\omega)\}},$$

(ii-b) for the case 2,

$$X(B) = \begin{bmatrix} \int_0^1 B(r)dr \\ \int_0^1 rB(r)dr \\ \int_\omega^1 (r-\omega)B(r)dr \end{bmatrix}, \quad Z(r) = \begin{bmatrix} r \\ \frac{r^2}{2} \\ 1(r>\omega)\frac{(r-\omega)^2}{2} \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 1 & \frac{1}{2} & \frac{(1-\omega)^2}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{(1-\omega)^2(\omega+2)}{6} \\ \frac{(1-\omega)^2}{2} & \frac{(1-\omega)^2(\omega+2)}{6} & \frac{(1-\omega)^3}{3} \end{bmatrix}, \quad D(\lambda) = \frac{D_1(\lambda) + D_2(\lambda) + D_3(\lambda)}{\lambda^{7/2}\omega^3(1-\omega)^3},$$

with

$$D_1(\lambda) = \lambda\omega(1-\omega)\sin\sqrt{\lambda},$$

$$D_2(\lambda) = 2\left\{\sin\sqrt{\lambda\omega^2} + \sin\sqrt{\lambda(1-\omega)^2} - \sin\sqrt{\lambda} - \lambda^{1/2}\left(\omega\cos\sqrt{\lambda\omega^2} + (1-\omega)\cos\sqrt{\lambda(1-\omega)^2}\right)\right\}$$

$$D_3(\lambda) = \lambda^{1/2}\left(\cos\sqrt{\lambda} + \cos\sqrt{\lambda\omega^2}\cos\sqrt{\lambda(1-\omega)^2}\right),$$

where $dt_r = 1(r > \omega) \times (r - \omega)$.

Remark 1: For the cases 0 and 3, the limiting distribution is expressed as the sum of two independent functionals, $G(B_1)$ and $G(B_2)$, so that its characteristic function is expressed as the product of two characteristic functions. This is because the test statistic S_T can be expressed as the sum of two functions, one is a function depending on the observation before the break and the other is after the break. See the proof of Appendix 4. Since S_T for the cases 1 and 2 can not be expressed in such a form, its characteristic function becomes a little complicated. Though the limiting distributions for the cases 0 and 3 can also be expressed as (4-7), the expression (4-5) may be more intuitive to understand why their characteristic functions have the form as (4-6).

Remark 2: Under the null hypothesis, $c = 0$ so that S_T converges in distribution to

$$\omega^2 \left(\int_0^1 B_1(r)^2 dr - X(B_1)'\Lambda^{-1}X(B_1) \right) + (1-\omega)^2 \left(\int_0^1 B_2(r)^2 dr - X(B_2)'\Lambda^{-1}X(B_2) \right), \tag{4-9}$$

for the cases 0 and 3, and

$$\int_0^1 B(r)^2 dr - X(B)'\Lambda^{-1}X(B), \tag{4-10}$$

for the cases 1 and 2. And their characteristic functions can be expressed more compactly as

$$\phi(\theta) = \left[D\left(2i\omega^2\theta\right) D\left(2i(1-\omega)^2\theta\right) \right]^{-1/2}, \tag{4-11}$$

for the cases 0 and 3, and

$$\phi(\theta) = \left[D\left(2i\theta\right) \right]^{-1/2}, \tag{4-12}$$

for the cases 1 and 2.

Remark 3: Though the proof of Theorem 4.1 depends on normality of disturbances, the null distribution can be derived only with i.i.d. assumption on $\{v_t\}$. For example, let us consider the case 0. From the second equality of (4-3), $S_T = T^{-1}\sum_{j=1}^{T-1}(\tilde{\sigma}^{-1}T^{-1/2}\sum_{t=1}^{j}\tilde{x}_t)^2$ and, as in the proof, we can see that $\tilde{\sigma}^{-1}T^{-1/2}\sum_{t=1}^{[Tr]}\tilde{x}_t \xrightarrow{d} B(r) - Z(r)'\Lambda^{-1}X^{**} \equiv V(r)$ with $X^{**} = [B(1), B(1) - B(\omega)]'$ under H_0. Then, using the continuous mapping theorem, S_T converges in distribution to $\int_0^1 V(r)^2 dr$, which is another expression of the limiting distribution different from (4-9). Here we used the invariance principle and then we have only to need the i.i.d. assumption, so that whether normality is assumed does not affect the limiting distribution. In this sense, the percentiles of the null distribution tabulated below can be used for the model with the more general assumption of disturbances.

From the above theorem, we can obtain the distribution function $F(x)$ in each case by inverting the characteristic function. Since the limiting distribution is nonnegative, we can calculate the percent points by numerical integration, using Lévy's inversion formula,

$$F(x) = \frac{1}{\pi} \int_0^\infty Re \left[\frac{1 - e^{-i\theta x}}{i\theta} \phi(\theta; c) \right] d\theta. \qquad (4-13)$$

Especially for the null distribution, we set $c = 0$, that is, we use the characteristic function (4-11) or (4-12).

Tables 1a-1d report the percent points for the cases 0 to 3. Since, as we can see from the characteristic function, the limiting distribution when $\omega = \omega^*$ is the same as when $\omega = 1 - \omega^*$, that is, it is symmetric around $\omega = 0.5$, we tabulate percentiles only for $\omega = 0.1$, 0.2, 0.3, 0.4 and 0.5. For $\omega > 0.5$, we can refer to the tables corresponding to $1 - \omega$.

From the tables, we can see that, for the cases 0, 2 and 3, the more centered the break point is, the further is the distribution function located to the left. But the properties of the distribution for the case 1 is different from the others. When ω increases from 0.1 to around 0.3, the distribution shifts to the left, whereas as ω goes up to around 0.5, it moves back to the right.

As in the case of the null distribution, the location of the break point, ω, also affects the limiting power properties. The limiting power function can also be calculated by the numerical integration and is given by $1 - F(x)$ as a function of c. Figures 1a to 1d draw the power functions for $\omega = 0.1$, 0.2, 0.3, 0.4 and 0.5. As in the case of the null distribution, the properties for the cases 0, 2 and 3 are similar when c is close to 0. For these cases, the power for the smaller ω dominates that for the larger ω (≤ 0.5) near the null hypothesis. On the other hand, for the case 1, the power function for $\omega = 0.1$ is located higher than that for $\omega = 0.3$, but the case of $\omega = 0.5$ is most powerful among five values of ω when c is close to 0. These properties seem to be only for the small values of c and as c increases, the above relation does not hold.

Next we compare the limiting power functions of four cases for a fixed ω. Figures 2a-2e draw them for $\omega = 0.1$, 0.2, 0.3, 0,4 and 0.5. As in the many other tests such as the Dickey-Fuller test, the more complicated the deterministic term becomes, the less powerful is the test statistic. We can see that the power function of the case 0 dominates the other three cases, and the test in the case 3 is least powerful. These differences among power functions tend to diminish as the value of ω decreases to 0.1, and especially when $\omega = 0.1$, the power functions of the cases 1, 2 and 3 are almost the same, though the power of the case 0 still dominates the others.

3.2. The Test Independent of the Break Point

As we can see from Theorem 4.1, the limiting distribution of the LM test statistic depends on the break point and then we tabulated the percent points of the null distribution for several ω's. In this section, we consider the test statistic whose limiting distribution does not depend on the value of ω. We can construct such a test not for all cases but only for the cases 0 and case 3, that is, for the cases when there is one time break in all deterministic components included in the model. Then we consider only the cases 0 and 3 in this section. Fundamentally our method is the same as Park and Sung (1994).

Firstly we make the weighted variable y_t^* following the idea of Park and Sung (1994).

$$y_t^* = \begin{cases} T/T_B y_t & : \ t = 1, \cdots, T_B, \\ T/(T-T_B) y_t & : \ t = T_B + 1, \cdots, T. \end{cases}$$

Then, using this variable, we construct the following statistic, which we call the PS statistic.

$$S_T^{ps} = \frac{1}{\tilde{\sigma}^2 T^2} y^{*\prime} M L L' M y^* = \frac{1}{\tilde{\sigma}^2 T^2} \sum_{j=1}^{T-1} \left(\sum_{t=1}^{j} \tilde{x}_t^* \right)^2,$$

where $\tilde{\sigma}^2$ is defined in (4-4) and $\tilde{x}_t^*$ are regression residuals of y_t^* on z_t,

$$\tilde{x}_t^* = y_t^* - z_t' \left(\sum_{t=1}^{T} z_t z_t' \right)^{-1} \sum_{t=1}^{T} z_t y_t^*.$$

The following theorem gives the limiting distribution of the PS statistic and its characteristic function.

Theorem 4.2 *Consider the model (4-1). For the cases 0 and 3, under a sequence of local alternatives,* H_1,

$$S_T^{ps} \xrightarrow{d} G(B_1; c^2\omega^2/\alpha^2) + G(B_2; c^2(1-\omega)^2/\alpha^2),$$

and its characteristic function is expressed as

$$\begin{aligned} \phi(\theta; c) &= \left[D\left(i\theta + \sqrt{-\theta^2 + 2ic^2\omega^2\theta}\right) D\left(i\theta - \sqrt{-\theta^2 + 2ic^2\omega^2\theta}\right)\right]^{-1/2} \qquad (4\text{-}14) \\ &\times \left[D\left(i\theta + \sqrt{-\theta^2 + 2ic^2(1-\omega)^2\theta}\right) D(i\theta - \sqrt{-\theta^2 + 2ic^2(1-\omega)^2\theta}\right)\right]^{-1/2}, \end{aligned}$$

where $G(B_1)$, $G(B_2)$ *and* $D(\lambda)$ *are defined as in Theorem 4.1 (i-a) and (i-b) for the cases 0 and 3, respectively.*

Remark 4: Though the above limiting distribution depends on the value of ω under H_1, we have, for $c = 0$,

$$S_T^{ps} \xrightarrow{d} \left(\int_0^1 B_1(r)^2 dr - X(B_1)'\Lambda^{-1}X(B_1)\right) + \left(\int_0^1 B_2(r)^2 dr - X(B_2)'\Lambda^{-1}X(B_2)\right),$$

and $\phi(\theta) = [D(2i\theta)]^{-1}$, so that the null distribution does not depend on the break point.

Remark 5: Note that when $\omega = 0.5$ for the cases 0 and 3, the characteristic function of the LM test (4-6) has the same structure as that of the PS test (4-14). Then, the PS test is equivalent to the LM test when the break point is located at center of the sample.

As in the case of the LM test, we can calculate the percentiles of the PS test under H_0 by numerical integration using the inversion formula (4-13). Table 2 reports each percent points of the PS test for the cases 0 and 3. As in the case of the LM test, the limiting distribution of the case 0 is located to the left compared with that of the case3, though both distributions of the PS tests are shifted to the right in comparison with those of the LM tests.

Though the null distribution does not depend on the break point, it depends on ω under a sequence of local alternatives as shown by Theorem 4.2, so that the power depends on the location of the break point. Figures 3a and 3b draw the limiting power functions of the PS tests for the cases 0 and 3. Again, as the characteristic function is symmetric around

$\omega = 0.5$, we consider only for the cases of $\omega = 0.1$, 0.2, 0.3, 0.4 and 0.5. The relation among power functions are very similar to the case of the LM test. That is, the power function corresponding to the smaller value of ω dominates that corresponding to the larger value of ω. However, the difference among the values of ω is not so much as the LM test for both cases 0 and 3.

Now we have two test statistics, S_T and S_T^{ps}, for the cases 0 and 3. Then, our interest is the difference of the powers of their limiting distributions and whether one dominates the other in view of the power. Figures 4a-4d depict the limiting power functions of the LM test and the PS test for the case 0 and Figures 5a-5d for the case 3. From Figures 4a and 4d, we can see that the power of the LM test dominates that of the PS test when $\omega = 0.1$, whereas when $\omega = 0.4$, such a relation holds for small values of c but that relation is reversed when c increases over 8, though the difference between their powers is slight. Since the LM test is LBI, the dominance of the LM test local to the null can be seen as a theoretical result. As discussed in Remark 5, their power functions are completely the same when $\omega = 0.5$ and then the PS test can be seen as the LBI test in such a case. For the case 3, the relation between the LM and the LBI tests are very similar to the case 0.

3.3. The Innovational Outlier Model

Until now we have investigated the additive outlier model, with which the structural change affects the observation only at one time. Here we discuss the innovational outlier model, that is, we consider the case when the shock is gradual.

Let us consider the following model.

$$y_t = z_{1t}'\beta_1 + \psi(B)(z_{2t}'\beta_2) + x_t, \qquad (4-15)$$

where $z_{1t} = 1$ or $[1, t]'$, $z_{2t} = DU_t$, DT_t, or $[DU_t, DT_t]'$ according to the cases 0 to 3, $\psi(B) = 1 + \psi_1 B + \cdots + \psi_m B^m$ is an m-th order lag polynomial, and x_t is defined as in the model (4-1). By introducing the lag polynomial $\psi(B)$, the shock of the structural change affects y_t gradually with lags.

To test the null of stationarity with a break, we put $z_t = [z_{1t}', z_{2t}^{*\prime}]'$ with $z_{2t}^* = [z_{2t}', z_{2t-1}', \cdots, z_{2t-m}']'$ and, as in the case of the additive outlier model, construct the test statistic as (4-3).

To consider the limiting distribution of the test statistic, we investigate $\tilde{x}_t$, regression residuals of y_t on z_t. Note that we can write

$$\psi(B)DU_t = \eta_0 DU_t + d(t, T_B)\eta^*,$$

$$\psi(B)DT_t = \gamma_0 DU_t + \gamma_1 DT_t + d(t, T_B)\gamma^*,$$

where η_0, $\eta^* = [\eta_1, \cdots, \eta_m]'$, γ_0, γ_1, and $\gamma^* = [\gamma_2, \cdots, \gamma_{m+1}]'$ are implicitly defined and $d(t, T_B) = [D(T_B)_t, \cdots, D(T_B)_{t-m}]$ with $D(T_B)_t = 1(t = T_B + 1)$. Some elements of η^* and γ^* might be zero. Then, $\tilde{x}_t$ is equivalent to regression residuals of y_t on z_t^*, where

$$\begin{aligned}
z_t^* &= [1, DU_t, d(t, T_B)]' \text{ for the case 0,} \\
z_t^* &= [1, t, DU_t, d(t, T_B)]' \text{ for the case 1,} \\
z_t^* &= [1, t, DU_t, DT_t, d(t, T_B)]' \text{ for the case 2,} \\
z_t^* &= [1, t, DU_t, DT_t, d(t, T_B)]' \text{ for the case 3.}
\end{aligned}$$

However, since $d(t, T_B)$ is asymptotically negligible, $\tilde{x}_t$ can be seen as regression residuals of y_t on $z_t^* = [1, DU_t]$, $[1, t, DU_t]$, $[1, t, DU_t, DT_t]$, and $[1, t, DU_t, DT_t]$ for the cases 0, 1, 2 and 3, respectively. Then, for the cases 0, 1 and 3, the limiting distributions of the test statistics with the innovational outlier model are the same as those with the additive outlier model, whereas, for the case 2, the limiting distribution is the same as in the case 3. Then, if we investigate the time series with the innovational outlier model, we can refer to Tables 1a, 1b, 1d, and 1d for the cases 0, 1, 2, and 3, respectively.

4. Finite Sample Properties

In this section, we investigate the finite sample behavior of the LM test statistic S_T and the PS test statistic S_T^{PS} for the sample size $T = 100$ and 200. Since the test statistics are invariant to β, we consider the following data generating process (D.G.P.) for all cases.

$$y_t = \gamma_t + u_t, \quad \gamma_t = \gamma_{t-1} + \varepsilon_t, \quad u_t = a u_{t-1} + v_t, \qquad (4-16)$$

where $\varepsilon_t \sim NID(0, \rho)$, $v_t \sim NID(0, 1)$, $\{\varepsilon_t\}$ and $\{v_t\}$ are independent, $\gamma_0 = 0$ and $u_0 = 0$. The size of the test depends on α and ω whereas the power is affected by ρ as well as those parameters. We set $a = 0$, ± 0.2, ± 0.5 and ± 0.8, $\omega = 0.1$ to 0.9 step by 0.1, and

$\rho = 0.01$, 1, and 100. In addition, both the size and the power depend on the lag truncation number ℓ in the equation (4-4), we consider the three values of ℓ as a function of T: $\ell 0 = 0$, $\ell 4 = [4(T/100)^{1/4}]$, and $\ell 12 = [12(T/100)^{1/4}]$, as in KPSS (1992). The number of replication is 1,000 in all experiments, performed by the GAUSS matrix programming language.

Table 3a reports the empirical sizes of the LM test and of the PS test. The rows corresponding to $\omega = 0.1$ to 0.9 are the size of the LM test, and those to "PS" are the size of the PS test. Since we used the upper 5% point as the critical value, the nominal size of the test is 0.05. From the table, we can see that the size of the LM test is much affected by the persistence of the stationary error, a, and the lag truncation number, ℓ. As a whole, there is tendency of the over-rejection when the value of a goes to 1 and of the under-rejection when a is a negative value. We can also say that when the absolute value of the AR parameter, $|a|$, is large, we need the longer lag truncation number to obtain the empirical size close to 0.05. The empirical size of the PS test also depends on the above parameters but not so much compared with the LM test, and seems stable especially when the sample size is 200.

Tables 3b-3d show the simulation results for the cases 1 to 3. They are similar to the case 0 and the relative performance of the tests is preserved. But for the case 3, the longer lag truncation number does not necessarily contribute to the correction of the size distortion and tends to cause the over-rejection of the LM test when $T = 100$ and of the PS test.

Tables 4a-4f report the power of each test (not size adjusted). For each ω, we consider, as the D.G.P., not only the error components model (4-16) but also the pure random walk model, $y_t = \gamma_t$, whose result corresponds to the rows labeled "R.W.".

Table 4a shows the empirical power of the case 0. The power increases when the sample size becomes large whereas it tends to decrease when we use the longer lag truncation number, except for the case when $\rho = 0.01$ and a is negative. We can also see that the larger value of the signal to noise ratio, ρ, entails the higher power. For a large value of ρ, the power of the test does not depend on a, especially when $\rho = 100$. This is because the large value of ρ means that the nonstationary behavior of the process, γ_t, dominates the stationary one, u_t. On the other hand, when $\rho = 0.01$, the empirical power much depends on the value of a. Note that when $a = 0$, the lag truncation number $\ell 0$ is chosen, and the sample size is 100, the powers, for example, for $\omega = 0.2$ and 0.8 are 0.477 and 0.491, respectively, which are larger than 0.413 for $\omega = 0.5$. Though we should carefully compare the powers since they are not size-adjusted, the above comparison is adequate since the empirical sizes for these cases are close to 0.05. This result is consistent with the previous section, that is, the limiting power is higher when the break point is not the middle but the ends of the sample for the case 0 when c is close to 0 (and when ρ is small). In addition, we can see that the above values, 0.477, 0.413, and 0.477 are not so far from the theoretical limiting powers, 0.513, 0.452, and 0.513 corresponding to the case when $c = 10$ ($\rho = c^2/T^2 = 10^2/100^2 = 0.01$). We also note that the powers for ω and $1 - \omega$ is very close, which is also indicated in the local limiting power analysis.

As is shown in Tables 4b-4f, we can see that the similar properties are established for the other cases.

5. Empirical Results

In this section, we apply the testing procedure developed in the previous section to the data series of Nelson and Plosser (1982). The Nelson Plosser data are used in various studies, and, especially, the existence of a unit root is one of the interesting issues and was analyzed in Perron (1997) and Zivot and Andrews (1992), assuming trend stationarity with a break under the alternative. Their results are very similar, that is, with the model corresponding to our case 1, the unit root hypothesis is rejected for 5 out of 11 macroeconomic time series, real GNP, nominal GNP, industrial production, employment and nominal wages, weakly rejected (at 10% level) for real per capita GNP, and, with the model corresponding to our case 3, the null of a unit root is rejected for common-stock prices. Here we should keep in mind that, though their tests are designed to have the power against the stationarity alternative, the rejection of a unit root hypothesis does not necessarily indicate stationarity of the series since the tests will have the power in detecting more general alternatives. Then, once the null of a unit root is rejected, our next interest may be whether the rejected time series are well specified as a stationary model with a break. We apply the tests proposed in the previous section to the above 7 macroeconomic time series as well as unemployment rate originally investigated in Nelson and Plosser (1982).

Firstly we should investigate whether the model with the structural change is adequate to these time series. Vogelsang (1997) proposed the conservative test for the null of no structural change for the I(0) or I(1) model and applied it to the Nelson Plosser data. For our 8 time series, their results indicate that the null of no break is rejected for nominal GNP and industrial product, weakly rejected for unemployment rate and common-stock prices, but is not rejected for the other 4 time series. However, since the Vogelsang's test is conservative, the null hypothesis may be accepted too much often. In fact, its conservative critical value is derived with the unit root model and, if the model is known to be the I(0) process, the critical value becomes more liberal (smaller than the conservative one). As mentioned above, the unit root model seems not adequate for our 8 time series and if we apply the critical value corresponding to the stationary model (Table 1 in Vogelsang, 1997), the null of no break is rejected for all of our time series, except for employment. Then, more or less, the no-break model seems not to be a good specification for those 7 time series, for which we proceed to test for the null of stationarity with a break.

Next we estimate the break point. We use the consistent estimator of the break point proposed in Nunes, Kuan and Newbold (1995). Note that it is enough for the estimator to be consistent under the assumption of stationarity (not under the alternative of a unit root) for our purpose, because we can see that the tests proposed in the previous section have the non-trivial power for the unit root model even when the break point is misspecified. The third and fourth columns in Table 5 report the estimated break point, T_B, and the fraction of the break, ω, respectively. The estimated break dates are around either 1929 or 1940.

Now using the above estimates of the break point, we apply the test for the stationarity with a break for 7 time series. The model of the case 1 is used for all the series except for common-stock prices, to which the model of the case 3 is applied. As was seen in the previous section, the test depends on the lag truncation number ℓ, we calculate the statistics for $\ell = \ell 4$ and $\ell 12$. We also calculate the PS test statistic for common-stock prices. From the table, we can not reject the null of stationarity for unemployment and common-stock price, whereas for real GNP, nominal GNP and nominal wages, there is a weak tendency against stationarity, but, since the tests tends to over-reject the null from the finite sample simulation of Section 4 when we use $\ell 4$ as the lag truncation number, the stationary model with a break may be adequate for these series. On the other hand, there is a strong tendency of rejection of stationarity for the real per capita GNP and industrial production. Then, for these series, both the null of a unit root and stationarity are rejected, and the further investigation may be required for them, possibly trying other models than the simple I(0) and (1) models.

6. Conclusion

In this chapter, we developed the testing procedure for the null hypothesis of stationarity with a break against nonstationarity. We proposed the LM test and also the PS test which does not depend on the fraction of the break point, ω, under the null hypothesis. The local limiting power is also investigated and the tests are shown to be consistent against the alternative of a unit root. The simulation experiment reveals that the finite sample properties depend on some parameters and, especially, we should be careful in selecting the lag truncation number. By applying our tests to the Nelson Plosser data, some of the time series for which the null of a unit root is rejected in Perron (1997) and Zivot and Andrews (1992) are well specified as the stationary process with a break, but the others are not.

Though the several testing procedures are proposed to test for the null of a unit root against stationarity with a break, our tests suppose the null of stationarity. Then, they do not compete but complement each other to investigate the persistence of the time series.

Appendix 4.

Proof of Theorem 4.1: Firstly we prove (4-5) and (4-7). Since we can easily see that $\tilde{\sigma}^2 \xrightarrow{p} \sigma^2 = \alpha^2\sigma_v^2$ under H_1, where $\xrightarrow{p}$ denotes convergence in probability, we can re-define $S_T = \sigma^{-2}T^{-2}y'MLL'My$ instead of (4-3) as far as the limiting distribution is concerned.

(i) Let us consider the case 0. Since $[1 - DU_t, DU_t]$ spans the same space as $[1, DU_t]$, we can replace $z_t = [1, DU_t]$ by $z_t = [1 - DU_t, DU_t]$ and then replace the orthogonal projection matrix M by $M_* = diag\{M_a, M_b\}$, where M_a and M_b are the $T_B \times T_B$ and $(T-T_B)\times(T-T_B)$ orthogonal projection matrices on a constant, $z_{at} = 1$ and $z_{bt} = 1$, respectively. Then, we have the relation $L'My = L'M_*y$. Hereafter, we use the subscripts a and b to denote that the vector or the matrix is associated with the data before a break and after a break, respectively.

The typical j-th element of $L'M_*y$ is $\sum_{t=j}^{T} \tilde{x}$ but from the property of the regression, we can see that $\sum_{t=T_B+1}^{T} \tilde{x}_t = 0$ so that $\sum_{t=j}^{T} \tilde{x}_t = \sum_{t=j}^{T_B} \tilde{x}_t$ for $j \le T_B$. Then, we have

$$L'M_*y = L'\tilde{x} = \begin{bmatrix} \sum_{t=1}^{T} \tilde{x}_t \\ \vdots \\ \sum_{t=T_B}^{T} \tilde{x}_t \\ \sum_{t=T_B+1}^{T} \tilde{x}_t \\ \vdots \\ \tilde{x}_T \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T_B} \tilde{x}_t \\ \vdots \\ \tilde{x}_{T_B} \\ \sum_{t=T_B+1}^{T} \tilde{x}_t \\ \vdots \\ \tilde{x}_T \end{bmatrix} = \begin{bmatrix} L_a' & 0 \\ 0 & L_b' \end{bmatrix} \begin{bmatrix} \tilde{x}_a \\ \tilde{x}_b \end{bmatrix} = L_*'M_*y, \qquad (4-17)$$

where L_a and L_b are the $T_B \times T_B$ and $(T-T_B)\times(T-T_B)$ matrices with the same structure as L, $L_* = diag\{L_a, L_b\}$, $\tilde{x}_a = [\tilde{x}_1, \cdots, \tilde{x}_{T_B}]'$ and $\tilde{x}_b = [\tilde{x}_{T_B+1}, \cdots, \tilde{x}_T]'$.

Next we decompose the stationary component u_t as

$$u_t = \alpha v_t + \tilde{v}_{t-1} - \tilde{v}_t,$$

where $\tilde{v}_t = \sum_{j=0}^{\infty} \tilde{\alpha}_j v_{t-j}$ with $\tilde{\alpha}_j = \sum_{k=j+1}^{\infty} \alpha_k$. Then we can write the stochastic component of y_t as $x_t = \gamma_t + \alpha v_t + \tilde{v}_{t-1} - \tilde{v}_t$, of which the last two terms are asymptotically negligible.

Noting that under H_1, $\gamma_t + \alpha v_t \sim N(0, \sigma_v^2(\alpha^2 I_T + \rho LL'))$, we have, using the relation (4-17),

$$S_T = \frac{1}{\sigma^2 T^2} x'M_*L_*L_*'M_*x$$

$$
\begin{aligned}
&\stackrel{d}{=} \frac{\sigma_v^2}{\sigma^2 T^2}\nu'(\alpha^2 I_T + \rho LL')^{1/2} M_* L_* L_*' M_* (\alpha^2 I_T + \rho LL')^{1/2}\nu + o_p(1) \\
&\stackrel{d}{=} \frac{\sigma_v^2}{\sigma^2 T^2}\nu' L_*' M_* (\alpha^2 I_T + \rho LL') M_* L_* \nu + o_p(1) \\
&= \frac{1}{T^2}\nu' L_*' M_* L_* \nu + \frac{c^2}{\alpha^2 T^4}\nu' L_*' M_* LL' M_* L_* \nu + o_p(1), \qquad (4\text{-}18)
\end{aligned}
$$

where $\nu = [\nu_a', \nu_b']' \sim N(0, I_T)$ and $\stackrel{d}{=}$ denotes equality in distribution. The third relation holds because of normality of ν. From the definition, we have

$$
L_*' M_* L_* = \begin{bmatrix} L_a' M_a L_a & 0 \\ 0 & L_b' M_b L_b \end{bmatrix}. \qquad (4-19)
$$

In addition, in the same discussion as the equation (4-17), since $M_* L_* \nu$ is the regression residual of $L_*\nu$ on the space spanned by $[1 - DU_t, DU_t]$, the following equivalence holds.

$$
L' M_* L_* \nu = L' \begin{bmatrix} M_a L_a \nu_a \\ M_b L_b \nu_b \end{bmatrix} = \begin{bmatrix} L_a' M_a L_a \nu_a \\ L_b' M_b L_b \nu_b \end{bmatrix}. \qquad (4-20)
$$

Using (4-18), (4-19) and (4-20), the LM test statistic is expressed as

$$
\begin{aligned}
S_T &\stackrel{d}{=} \frac{1}{T^2}\nu_a' \left\{ L_a' M_a L_a + \frac{c^2}{\alpha^2 T^2}(L_a' M_a L_a)^2 \right\} \nu_a \\
&\quad + \frac{1}{T^2}\nu_b' \left\{ L_b' M_b L_b + \frac{c^2}{\alpha^2 T^2}(L_b' M_b L_b)^2 \right\} \nu_b + o_p(1) \\
&= S_{aT} + S_{bT} + o_p(1), \quad \text{say}. \qquad (4\text{-}21)
\end{aligned}
$$

Since ν_a and ν_b are independent, we can investigate the limiting distributions of S_a and S_b separately. We first consider the limiting distribution of S_{aT}. Denoting the t-th element of $L_a \nu_a$ as η_{at}, we have

$$
\frac{1}{\sqrt{T_B}}\eta_{a[T_B r]} = \frac{1}{\sqrt{T_B}}\sum_{j=1}^{[T_B r]} \nu_j \stackrel{d}{\longrightarrow} B_1(r),
$$

where $B_1(\cdot)$ is a standard Brownian motion and $[p]$ denotes the largest integer $\leq p$. We can also see that

$$
\frac{1}{T_B}\sum_{t=1}^{T_B} \Upsilon z_{at}\eta_{at} \stackrel{d}{\longrightarrow} \int_0^1 B_1(r)dr \equiv X(B_1), \quad \sum_{t=1}^{T_B} \Upsilon z_{at} z_{at}' \Upsilon = 1 \equiv \Lambda,
$$

where $\Upsilon = T_B^{-1/2}$. Then,

$$
\begin{aligned}
\frac{1}{T_B^2}\nu_a' L_a' M_a L_a \nu_a &= \frac{1}{T_B^2}\sum_{t=1}^{T_B}\eta_{at}^2 - \left(\frac{1}{T_B}\sum_{t=1}^{T_B}\eta_{at} z_{at}' \Upsilon\right)\left(\sum_{t=1}^{T_B}\Upsilon z_{at} z_{at}' \Upsilon\right)^{-1}\left(\frac{1}{T_B}\sum_{t=1}^{T_B}\Upsilon z_{at}\eta_{at}\right) \\
&\stackrel{d}{\longrightarrow} \int_0^1 B_1(r)^2 dr - X(B_1)'\Lambda^{-1}X(B_1). \qquad (4\text{-}22)
\end{aligned}
$$

Next, denoting regression residuals of η_{at} on z_{at} as $\tilde{\eta}_{at}$, we have

$$\begin{aligned}\frac{1}{T_B^{3/2}}\sum_{t=1}^{[T_Br]}\tilde{\eta}_{at} &= \frac{1}{T_B^{3/2}}\left\{\sum_{t=1}^{[T_Br]}\eta_{at}-\left(\sum_{t=1}^{[T_Br]}z_{at}'\Upsilon\right)\left(\sum_{t=1}^{T_B}\Upsilon z_{at}z_{at}'\Upsilon\right)^{-1}\left(\sum_{t=1}^{T_B}\Upsilon z_{at}\eta_{at}\right)\right\}\\ &\xrightarrow{d} \int_0^r B_1(s)ds - Z(r)'\Lambda^{-1}X(B_1).\end{aligned}$$

Since the typical t-th element of $L_a'M_aL_a\nu_a = L_a'\tilde{\eta}_a$ is $\sum_{j=t}^{T_B}\tilde{\eta}_{aj} = -\sum_{j=1}^{t-1}\tilde{\eta}_{aj}$ because $\sum_{j=1}^{T_B}\tilde{\eta}_j = 0$ in the same reason as (4-17), we have

$$\begin{aligned}\frac{c^2}{\alpha^2T_B^4}\nu_a'(L_a'M_aL_a)^2\nu_a &= \frac{c^2}{\alpha^2T_B^4}\sum_{t=1}^{T_B-1}\left(\sum_{j=1}^{t}\tilde{\eta}_{aj}\right)^2\\ &\xrightarrow{d} \frac{c^2}{\alpha^2}\int_0^1\left(\int_0^r B_1(s)ds - Z(r)'\Lambda^{-1}X(B_1)\right)^2dr. \qquad (4\text{-}23)\end{aligned}$$

From (4-22) and (4-23), we obtain

$$\begin{aligned}S_{aT} &= \frac{T_B^2}{T^2}\frac{1}{T_B^2}\nu_a'L_a'M_aL_a\nu_a + \frac{T_B^4}{T^4}\frac{c^2}{\alpha^2T_B^4}\nu_a'(L_a'M_aL_a)^2\nu_a\\ &\xrightarrow{d} \omega^2\left\{\int_0^1 B_1(r)^2dr - X(B_1)'\Lambda^{-1}X(B_1) + \frac{c^2\omega^2}{\alpha^2}\int_0^1\left(\int_0^r B_1(s)ds - Z(r)'\Lambda^{-1}X(B_1)\right)^2dr\right\}\\ &= \omega^2G(B_1; c^2\omega^2/\alpha^2).\end{aligned}$$

Completely in the same way as S_{aT}, we obtain $S_{bT} \xrightarrow{d} (1-\omega)^2G(B_2; c^2(1-\omega)^2/\alpha^2)$ with $\Upsilon = (T-T_B)^{-1/2}$, where $B_2(\cdot)$ is a standard Brownian motion independent of $B_1(\cdot)$, and then (4-5) is established.

For the case 3, we can replace $z_t = [1, DU_t, t, DT_t]$ by $z_t = [1-DU_t, DU_t, 1(t \leq T_B)\times t, DT_t]$ and then replace the orthogonal projection matrix M by $M_* = diag\{M_a, M_b\}$, where M_a and M_b are the orthogonal projection matrices on a constant and a linear trend, $z_{at} = [1, t]$ and $z_{bt} = [1, t]$, respectively. Then, in an analogous way as the case 0, putting $\Upsilon = diag\{T_B^{-1/2}, T_B^{-3/2}\}$ or $diag\{(T-T_B)^{-1/2}, (T-T_B)^{-3/2}\}$, the relation (4-5) can be established.

(ii) For the cases 1 and 2, we can not decompose M as $diag\{M_a, M_b\}$. But in the same way as (4-18), we have,

$$S_T \overset{d}{=} \frac{\sigma_v^2}{\sigma^2T^2}\nu'(\alpha^2I_T+\rho LL')^{1/2}MLL'M(\alpha^2I_T+\rho LL')^{1/2}\nu + o_p(1)$$

$$\begin{aligned}
&\stackrel{d}{=} \frac{\sigma_v^2}{\sigma^2 T^2}\nu' L' M(\alpha^2 I_T + \rho LL')ML\nu + o_p(1) \\
&= \frac{1}{T^2}\nu' L' ML\nu + \frac{c^2}{\alpha^2 T^4}\nu'(L'ML)^2\nu + o_p(1). \qquad (4\text{-}24)
\end{aligned}$$

We also have, as (4-22) and (4-23),

$$\frac{1}{T^2}\nu' L' ML\nu \xrightarrow{d} \int_0^1 B(r)^2 dr - X(B)'\Lambda^{-1}X(B),$$

and

$$\frac{c^2}{\alpha^2 T^4}\nu'(L'ML)^2\nu = \frac{c^2}{\alpha^2 T^4}\sum_{t=1}^{T-1}\left(\sum_{j=1}^{t}\tilde{\eta}_j\right)^2 \xrightarrow{d} \frac{c^2}{\alpha^2}\int_0^1\left(\int_0^r B(s)ds - Z(r)'\Lambda^{-1}X(B)\right)^2 dr,$$

using $T^{-1/2}\sum_{t=1}^{[Tr]}\nu_t \xrightarrow{d} B(r)$, where $B(\cdot)$ is a standard Brownian motion and $\tilde{\eta}_t$ is constructed as $\tilde{\eta}_{at}$ with the full sample. Then, (4-7) is established.

Next we derive the characteristic function of the limiting distribution. Note that, in general, as shown in Theorem 5.13 of Tanaka (1996), if S_T^* is defined by

$$S_T^* = \frac{1}{T}\nu' B_T \nu + \frac{\gamma}{T^2}\nu' B_T^2 \nu, \qquad (4-25)$$

where $\nu = [\nu_1, \cdots, \nu_T]'$, $\{\nu_t\} \sim i.i.d.(0,1)$ and B_T satisfies

$$\lim_{T\to\infty}\max_{j,k}\left|B_T(j,k) - K\left(\frac{j}{T},\frac{k}{T}\right)\right| = 0, \qquad (4-26)$$

with $K(s,t)(\not\equiv 0)$ a symmetric, continuous and nearly definite function, Lemma (2.3) can be applied to S_T^*. Then, we have only to check (4-25) and (4-26) so as to apply Lemma 2.3 to S_T.

(i) For the cases 0 and 3, $\omega^{-2}S_{aT}$ has the same expression as (4-25) with $B_T = T_B^{-1}L_a' M_a L_a$ and $\gamma = c^2\omega^2/\alpha^2$. Moreover, from some algebra, we can see that the (j,k)-th element of $T_B^{-1}L_a' M_a L_a$ is expressed as $K(j/T_B, k/T_B) + O(T_B^{-1})$ with

$$K(s,t) = \min(s,t) - st, \quad \text{and} \quad K(s,t) = \min(s,t) - 4st + 3st(s+t) - 3s^2t^2,$$

for the cases 1 and 3, respectively, so that both $K(s,t)$'s satisfy the condition (4-26). Then, by Lemma 2.3, the characteristic function of the limiting distribution of S_{aT} is given by

$$\begin{aligned}
\lim_{T\to\infty}\left[e^{i\theta S_{aT}}\right] &= \lim_{T\to\infty}\left[e^{i(\omega^2\theta)(\omega^{-2}S_{aT})}\right] \\
&= \left[D\left(i\omega^2\theta + \sqrt{-\omega^4\theta^2 + 2ic^2\omega^4\theta/\alpha^2}\right) D\left(i\omega^2\theta - \sqrt{-\omega^4\theta^2 + 2ic^2\omega^4\theta/\alpha^2}\right)\right]^{-1/2},
\end{aligned}$$

where the Fredholm determinants of $K_0(s,t)$ and $K_3(s,t)$ are given in the Theorem 4.1, as shown by Theorem 6 of Nabeya and Tanaka (1988) and by the equations (5.34) and (9.94) of Tanaka (1996, p.139 and p.369).

The characteristic function corresponding to S_{bT} is obtained similarly, and since S_{aT} and S_{bT} are independent, we have the expression (4-6).

(ii) For the case 1, S_T in (4-24) has the same expression as (4-25) with $B_T = T^{-1}L'ML$ and $\gamma = c^2/\alpha^2$, and we can find the kernel $K(s,t)$ satisfying the condition (4-26). Then the next step is to find out the Fredholm determinant of $K(s,t)$. Here note that the characteristic function of the limiting distribution of S_T^* in (4-25) with $\gamma = 0$ is given by $[D(2i\theta)]^{-1/2}$ as in Lemma 2.1. Then, if we derive the characteristic function of the null distribution of S_T corresponding to the case when $c = 0$, we can obtain the Fredholm determinant $D(\lambda)$.

To derive the characteristic function under the null, we follow the method used by Perron (1991), and use the expression of the limiting distribution (4-7) with $c = 0$. Denote by μ_B and μ_Y the measures induced by the processes $B(\cdot)$ and $Y(\cdot)$ which is generated by the following stochastic differential equation:

$$dY(t) = -bY(t)dt + dB(t), \quad Y(0) = B(0) = 0.$$

Then the measures μ_B and μ_Y are equivalent and the Radon-Nikodym derivative $d\mu_B/d\mu_Y$ evaluated at y is given by

$$d\mu_B/d\mu_Y(y) = exp\left[b\int_0^1 y(t)dy(t) + b^2/2\int_0^1 y(t)^2dt\right].$$

See, for example, Liptser and Shiryayev (1977) and Theorem 4.1 of Tanaka (1996). Then, we obtain

$$\begin{aligned}
\phi(\theta;0) &= E\left[exp\left\{\theta\int_0^1 B(r)^2dr - \theta X(B)'\Lambda^{-1}X(B)\right\}\right] \\
&= E\left[exp\left\{\theta\int_0^1 Y(r)^2dr - \theta X(Y)'\Lambda^{-1}X(Y) + \frac{b}{2}(Y(1)^2-1) + \frac{b^2}{2}\int_0^1 Y(t)^2dt\right\}\right] \\
&= e^{-b/2}E\left[exp\left\{\frac{b}{2}Y(1)^2 - \theta X(Y)'\Lambda^{-1}X(Y)\right\}\right] \\
&= e^{-b/2}E\left[exp\left\{F'AF\right\}\right] \\
&= \left(e^b|I-2\Sigma A|\right)^{-1/2},
\end{aligned}$$

where we put $b^2 = -2\theta$, $F = [Y(1), X(Y)']'$, $A = diag\{b/2, -\theta\Lambda^{-1}\}$, and Σ is the variance-covariance matrix of F. The last equality follows from normality of F. Making use of the computerized algebra MAPLE V, we obtain the characteristic function $\phi(\theta; 0) = D(2i\theta)^{-1/2}$ where $D(\lambda)$ is given in Theorem 4.1.

The characteristic function for the case 2 can be obtained similarly and we omit the proof.□.

Proof of Theorem 4.2: As in the equation (4-21) of the LM test statistic, we have

$$\begin{aligned} S_T^{ps} &\stackrel{d}{=} \frac{1}{T^2}\nu_a^{ps\prime}\left\{L_a'M_aL_a + \frac{c^2}{\alpha^2T^2}(L_a'M_aL_a)^2\right\}\nu_a^{ps} \\ &+\frac{1}{T^2}\nu_b^{ps\prime}\left\{L_b'M_bL_b + \frac{c^2}{\alpha^2T^2}(L_b'M_bL_b)^2\right\}\nu_b^{ps} + o_p(1) \\ &= S_{aT}^{ps} + S_{bT}^{ps} + o_p(1), \quad \text{say,} \end{aligned}$$

where $\nu_a^{ps} = T/T_B\nu_a$ and $\nu_b^{ps} = T/(T-T_B)\nu_b$. Then,

$$\begin{aligned} S_{aT}^{ps} &= \frac{1}{T_B^2}\nu_a' L_a M_a L_a \nu_a + \frac{c^2T_B^2}{\alpha^2T^2}\frac{1}{T_B^4}\nu_a'(L_a'M_aL_a)^2\nu_a \\ &\xrightarrow{d} \int_0^1 B_1(r)^2dr - X(B_1)'\Lambda X(B_1) + \frac{c^2\omega^2}{\alpha^2}\int_0^1\left(\int_0^r B_1(s)ds - Z(r)'\Lambda^{-1}X(B_1)\right)^2 dr \\ &= G(B_1; c^2\omega^2/\alpha^2). \end{aligned}$$

Similarly,

$$\begin{aligned} S_{bT}^{ps} &\xrightarrow{d} \int_0^1 B_2(r)^2dr - X(B_2)'\Lambda X(B_2) + \frac{c^2(1-\omega)^2}{\alpha^2}\int_0^1\left(\int_0^r B_2(s)ds - Z(r)'\Lambda^{-1}X(B_2)\right)^2 dr \\ &= G(B_2; c^2(1-\omega)^2/\alpha^2), \end{aligned}$$

and then the limiting distribution of S_T^{ps} can be derived.

The characteristic function of the limiting distribution is obtained in an analogous way as the LM test and we omit the proof.□

Table 1a. Percent Points of the Null Distribution of the LM test: Case 0

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
$\omega = 0.1$	0.02160	0.03123	0.03892	0.09797	0.28299	0.37538	0.60388
$\omega = 0.2$	0.02049	0.02895	0.03548	0.08302	0.22915	0.30212	0.48265
$\omega = 0.3$	0.02001	0.02796	0.03396	0.07440	0.18678	0.24247	0.38052
$\omega = 0.4$	0.01978	0.02749	0.03326	0.07050	0.16007	0.20106	0.30162
$\omega = 0.5$	0.01971	0.02736	0.03305	0.06939	0.15176	0.18688	0.26842

Table 1b. Percent Points of the Null Distribution of the LM test: Case 1

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
$\omega = 0.1$	0.01544	0.02057	0.02426	0.04680	0.09840	0.12162	0.17821
$\omega = 0.2$	0.01517	0.02005	0.02350	0.04343	0.08537	0.10376	0.14839
$\omega = 0.3$	0.01525	0.02019	0.02370	0.04412	0.08579	0.10304	0.14291
$\omega = 0.4$	0.01541	0.02050	0.02415	0.04623	0.09736	0.12080	0.17842
$\omega = 0.5$	0.01549	0.02066	0.02439	0.04741	0.10551	0.13378	0.20405

Table 1c. Percent Points of the Null Distribution of the LM test: Case 2

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
$\omega = 0.1$	0.01536	0.02064	0.02448	0.04816	0.10263	0.12716	0.18696
$\omega = 0.2$	0.01441	0.01907	0.02242	0.04267	0.08879	0.10956	0.16020
$\omega = 0.3$	0.01394	0.01825	0.02129	0.03907	0.07815	0.09563	0.13829
$\omega = 0.4$	0.01371	0.01784	0.02073	0.03712	0.07138	0.08643	0.12299
$\omega = 0.5$	0.01364	0.01772	0.02056	0.03651	0.06909	0.08318	0.11727

Table 1d. Percent Points of the Null Distribution of the LM test: Case 3

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
$\omega = 0.1$	0.01463	0.01962	0.02325	0.04566	0.09724	0.12046	0.17704
$\omega = 0.2$	0.01331	0.01744	0.02039	0.03826	0.07903	0.09737	0.14208
$\omega = 0.3$	0.01267	0.01634	0.01889	0.03343	0.06485	0.07889	0.11308
$\omega = 0.4$	0.01237	0.01582	0.01817	0.03095	0.05570	0.06615	0.09122
$\omega = 0.5$	0.01228	0.01566	0.01796	0.03022	0.05267	0.06163	0.08216

Table 2. Percent Points of the Null Distribution of the PS test

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
Case 0	0.07883	0.10942	0.13222	0.27757	0.60704	0.74752	1.07366
Case 3	0.04912	0.06265	0.07184	0.12087	0.21067	0.24654	0.32862

Table 3a. The Size of the Case 0

		$T = 100$			$T = 200$		
	a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.1$	0.8	0.794	0.256	0.101	0.845	0.283	0.122
	0.5	0.356	0.089	0.053	0.346	0.097	0.062
	0.2	0.108	0.047	0.044	0.122	0.062	0.056
	0	0.039	0.040	0.044	0.044	0.050	0.048
	-0.2	0.013	0.031	0.042	0.009	0.036	0.045
	-0.5	0.000	0.022	0.029	0.002	0.020	0.039
	-0.8	0.000	0.002	0.014	0.000	0.002	0.018
$\omega = 0.2$	0.8	0.814	0.253	0.107	0.864	0.287	0.110
	0.5	0.356	0.098	0.068	0.359	0.091	0.072
	0.2	0.117	0.057	0.055	0.122	0.049	0.055
	0	0.049	0.038	0.046	0.040	0.040	0.050
	-0.2	0.014	0.032	0.036	0.012	0.031	0.040
	-0.5	0.000	0.021	0.027	0.001	0.017	0.030
	-0.8	0.000	0.001	0.022	0.000	0.006	0.016
$\omega = 0.3$	0.8	0.862	0.263	0.114	0.921	0.308	0.113
	0.5	0.399	0.089	0.057	0.412	0.106	0.067
	0.2	0.124	0.056	0.052	0.129	0.057	0.052
	0	0.048	0.044	0.045	0.047	0.047	0.050
	-0.2	0.013	0.038	0.037	0.011	0.033	0.047
	-0.5	0.000	0.021	0.035	0.000	0.017	0.039
	-0.8	0.000	0.000	0.019	0.000	0.003	0.023
$\omega = 0.4$	0.8	0.929	0.277	0.105	0.953	0.347	0.098
	0.5	0.478	0.101	0.062	0.502	0.090	0.053
	0.2	0.153	0.053	0.055	0.129	0.050	0.044
	0	0.054	0.041	0.050	0.038	0.038	0.041
	-0.2	0.014	0.032	0.047	0.009	0.031	0.038
	-0.5	0.000	0.025	0.044	0.000	0.011	0.025
	-0.8	0.000	0.002	0.033	0.000	0.002	0.014
$\omega = 0.5$	0.8	0.934	0.323	0.064	0.958	0.365	0.075
	0.5	0.514	0.113	0.044	0.508	0.096	0.038
	0.2	0.179	0.054	0.041	0.138	0.045	0.031
	0	0.056	0.038	0.035	0.037	0.034	0.031
	-0.2	0.011	0.029	0.032	0.005	0.022	0.030
	-0.5	0.000	0.018	0.027	0.000	0.015	0.025
	-0.8	0.000	0.003	0.023	0.000	0.001	0.018

Table 3a. The Size of the Case 0 (continued)

		$T=100$			$T=200$		
	a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
	0.8	0.922	0.305	0.085	0.953	0.326	0.095
	0.5	0.498	0.103	0.060	0.473	0.105	0.052
	0.2	0.164	0.055	0.051	0.139	0.059	0.045
$\omega=0.6$	0	0.047	0.040	0.049	0.045	0.042	0.040
	-0.2	0.016	0.034	0.044	0.008	0.032	0.036
	-0.5	0.000	0.018	0.034	0.000	0.015	0.030
	-0.8	0.000	0.002	0.018	0.000	0.001	0.018
	0.8	0.876	0.280	0.110	0.916	0.274	0.105
	0.5	0.419	0.106	0.065	0.396	0.101	0.061
	0.2	0.144	0.058	0.059	0.123	0.057	0.050
$\omega=0.7$	0	0.054	0.046	0.049	0.045	0.042	0.046
	-0.2	0.013	0.036	0.043	0.010	0.037	0.045
	-0.5	0.001	0.022	0.035	0.000	0.027	0.039
	-0.8	0.000	0.002	0.019	0.000	0.000	0.025
	0.8	0.833	0.291	0.122	0.866	0.266	0.101
	0.5	0.392	0.101	0.056	0.359	0.095	0.054
	0.2	0.130	0.058	0.039	0.115	0.049	0.043
$\omega=0.8$	0	0.043	0.039	0.034	0.040	0.039	0.038
	-0.2	0.013	0.028	0.031	0.011	0.029	0.037
	-0.5	0.000	0.015	0.023	0.000	0.019	0.029
	-0.8	0.000	0.001	0.014	0.000	0.001	0.010
	0.8	0.806	0.276	0.088	0.846	0.289	0.098
	0.5	0.353	0.083	0.047	0.376	0.092	0.048
	0.2	0.111	0.047	0.033	0.122	0.052	0.039
$\omega=0.9$	0	0.035	0.032	0.031	0.045	0.040	0.035
	-0.2	0.008	0.022	0.024	0.010	0.030	0.032
	-0.5	0.000	0.014	0.019	0.001	0.017	0.028
	-0.8	0.000	0.001	0.008	0.000	0.003	0.017
	0.8	0.763	0.113	0.010	0.866	0.203	0.055
	0.5	0.377	0.058	0.051	0.449	0.092	0.067
	0.2	0.138	0.057	0.084	0.165	0.064	0.077
PS	0	0.056	0.057	0.099	0.054	0.058	0.084
	-0.2	0.013	0.054	0.114	0.012	0.048	0.083
	-0.5	0.000	0.046	0.143	0.000	0.029	0.083
	-0.8	0.000	0.021	0.187	0.000	0.006	0.078

Table 3b. The Size of the Case 1

	a	$T=100$ $\ell 0$	$\ell 4$	$\ell 12$	$T=200$ $\ell 0$	$\ell 4$	$\ell 12$
	0.8	0.964	0.377	0.140	0.978	0.395	0.120
	0.5	0.541	0.139	0.085	0.565	0.103	0.056
	0.2	0.190	0.078	0.074	0.146	0.060	0.047
$\omega=0.1$	0	0.058	0.061	0.064	0.047	0.044	0.044
	-0.2	0.011	0.042	0.061	0.011	0.034	0.040
	-0.5	0.000	0.021	0.052	0.000	0.019	0.033
	-0.8	0.000	0.003	0.031	0.000	0.003	0.024
	0.8	0.968	0.358	0.126	0.989	0.413	0.113
	0.5	0.611	0.120	0.079	0.617	0.111	0.066
	0.2	0.197	0.069	0.077	0.161	0.054	0.051
$\omega=0.2$	0	0.067	0.053	0.073	0.039	0.041	0.043
	-0.2	0.008	0.040	0.069	0.008	0.027	0.038
	-0.5	0.000	0.028	0.071	0.000	0.017	0.031
	-0.8	0.000	0.001	0.053	0.000	0.001	0.018
	0.8	0.978	0.364	0.102	0.993	0.445	0.100
	0.5	0.634	0.115	0.081	0.636	0.103	0.053
	0.2	0.205	0.063	0.080	0.153	0.050	0.048
$\omega=0.3$	0	0.061	0.048	0.080	0.039	0.038	0.043
	-0.2	0.011	0.039	0.084	0.008	0.032	0.041
	-0.5	0.000	0.023	0.077	0.000	0.015	0.031
	-0.8	0.000	0.004	0.067	0.000	0.003	0.017
	0.8	0.951	0.322	0.126	0.982	0.378	0.123
	0.5	0.552	0.107	0.091	0.568	0.103	0.067
	0.2	0.160	0.058	0.076	0.154	0.056	0.049
$\omega=0.4$	0	0.056	0.046	0.074	0.036	0.037	0.047
	-0.2	0.014	0.038	0.063	0.008	0.031	0.041
	-0.5	0.002	0.019	0.062	0.000	0.015	0.034
	-0.8	0.000	0.006	0.034	0.000	0.001	0.020
	0.8	0.940	0.341	0.175	0.959	0.339	0.120
	0.5	0.501	0.136	0.104	0.499	0.097	0.070
	0.2	0.165	0.078	0.086	0.130	0.058	0.053
$\omega=0.5$	0	0.057	0.058	0.078	0.045	0.044	0.047
	-0.2	0.013	0.040	0.067	0.008	0.033	0.044
	-0.5	0.002	0.021	0.055	0.000	0.017	0.031
	-0.8	0.000	0.007	0.026	0.000	0.003	0.015

Table 3b. The Size of the Case 1 (continued)

	a	$T = 100$ $\ell 0$	$\ell 4$	$\ell 12$	$T = 200$ $\ell 0$	$\ell 4$	$\ell 12$
	0.8	0.952	0.359	0.178	0.979	0.353	0.115
	0.5	0.543	0.127	0.092	0.529	0.088	0.053
	0.2	0.168	0.076	0.081	0.129	0.053	0.048
$\omega = 0.6$	0	0.059	0.057	0.077	0.041	0.041	0.045
	-0.2	0.015	0.044	0.075	0.008	0.033	0.045
	-0.5	0.000	0.033	0.061	0.000	0.016	0.034
	-0.8	0.000	0.006	0.043	0.000	0.000	0.019
	0.8	0.974	0.367	0.111	0.991	0.431	0.109
	0.5	0.603	0.108	0.083	0.651	0.098	0.045
	0.2	0.181	0.049	0.077	0.172	0.049	0.033
$\omega = 0.7$	0	0.052	0.038	0.074	0.038	0.034	0.032
	-0.2	0.010	0.027	0.070	0.005	0.028	0.031
	-0.5	0.000	0.018	0.070	0.000	0.013	0.028
	-0.8	0.000	0.001	0.067	0.000	0.001	0.019
	0.8	0.970	0.341	0.121	0.992	0.436	0.117
	0.5	0.599	0.109	0.092	0.642	0.121	0.067
	0.2	0.166	0.055	0.088	0.172	0.060	0.057
$\omega = 0.8$	0	0.045	0.042	0.086	0.053	0.048	0.051
	-0.2	0.013	0.031	0.084	0.010	0.036	0.050
	-0.5	0.000	0.017	0.076	0.000	0.018	0.043
	-0.8	0.000	0.001	0.058	0.000	0.001	0.023
	0.8	0.952	0.345	0.107	0.982	0.427	0.127
	0.5	0.546	0.110	0.073	0.587	0.116	0.073
	0.2	0.161	0.065	0.063	0.163	0.054	0.064
$\omega = 0.9$	0	0.058	0.046	0.059	0.042	0.043	0.057
	-0.2	0.016	0.036	0.058	0.008	0.033	0.049
	-0.5	0.000	0.018	0.045	0.000	0.018	0.038
	-0.8	0.000	0.004	0.025	0.000	0.000	0.017

Table 3c. The Size of the Case 2

	a	$T=100$ $\ell0$	$\ell4$	$\ell12$	$T=200$ $\ell0$	$\ell4$	$\ell12$
	0.8	0.961	0.394	0.131	0.981	0.391	0.111
	0.5	0.557	0.136	0.077	0.560	0.109	0.061
	0.2	0.183	0.075	0.062	0.144	0.061	0.047
$\omega=0.1$	0	0.065	0.058	0.059	0.049	0.045	0.044
	-0.2	0.009	0.043	0.051	0.011	0.034	0.039
	-0.5	0.000	0.022	0.039	0.000	0.018	0.031
	-0.8	0.000	0.001	0.016	0.000	0.003	0.015
	0.8	0.968	0.378	0.163	0.986	0.407	0.124
	0.5	0.558	0.134	0.091	0.569	0.104	0.065
	0.2	0.182	0.075	0.084	0.146	0.059	0.057
$\omega=0.2$	0	0.068	0.056	0.076	0.051	0.047	0.051
	-0.2	0.012	0.043	0.066	0.012	0.036	0.042
	-0.5	0.000	0.021	0.049	0.000	0.017	0.039
	-0.8	0.000	0.001	0.019	0.000	0.001	0.020
	0.8	0.964	0.368	0.157	0.988	0.420	0.123
	0.5	0.598	0.127	0.092	0.600	0.103	0.062
	0.2	0.178	0.067	0.074	0.157	0.052	0.049
$\omega=0.3$	0	0.057	0.047	0.064	0.045	0.038	0.045
	-0.2	0.009	0.032	0.057	0.009	0.029	0.038
	-0.5	0.000	0.021	0.053	0.000	0.017	0.029
	-0.8	0.000	0.002	0.032	0.000	0.003	0.016
	0.8	0.974	0.376	0.137	0.994	0.441	0.102
	0.5	0.629	0.105	0.080	0.651	0.103	0.057
	0.2	0.180	0.053	0.070	0.169	0.050	0.049
$\omega=0.4$	0	0.052	0.040	0.068	0.046	0.034	0.043
	-0.2	0.009	0.034	0.061	0.005	0.025	0.042
	-0.5	0.000	0.017	0.055	0.000	0.015	0.029
	-0.8	0.000	0.002	0.041	0.000	0.001	0.017
	0.8	0.976	0.359	0.120	0.994	0.463	0.114
	0.5	0.633	0.093	0.074	0.672	0.102	0.059
	0.2	0.175	0.053	0.066	0.175	0.048	0.047
$\omega=0.5$	0	0.052	0.041	0.056	0.049	0.038	0.041
	-0.2	0.006	0.029	0.056	0.003	0.027	0.040
	-0.5	0.000	0.012	0.052	0.000	0.013	0.032
	-0.8	0.000	0.001	0.041	0.000	0.001	0.011

Table 3c. The Size of the Case 2 (continued)

		$T = 100$			$T = 200$		
	a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.6$	0.8	0.974	0.338	0.108	0.992	0.449	0.125
	0.5	0.614	0.095	0.077	0.654	0.118	0.070
	0.2	0.172	0.053	0.069	0.179	0.051	0.058
	0	0.048	0.043	0.063	0.043	0.039	0.053
	-0.2	0.011	0.033	0.058	0.006	0.022	0.045
	-0.5	0.000	0.012	0.051	0.000	0.010	0.030
	-0.8	0.000	0.002	0.036	0.000	0.002	0.010
$\omega = 0.7$	0.8	0.965	0.332	0.115	0.990	0.432	0.129
	0.5	0.571	0.094	0.077	0.632	0.116	0.062
	0.2	0.140	0.053	0.063	0.178	0.062	0.053
	0	0.045	0.036	0.061	0.053	0.042	0.050
	-0.2	0.012	0.030	0.056	0.008	0.029	0.048
	-0.5	0.000	0.014	0.051	0.000	0.014	0.043
	-0.8	0.000	0.002	0.025	0.000	0.001	0.016
$\omega = 0.8$	0.8	0.957	0.333	0.113	0.982	0.433	0.133
	0.5	0.557	0.113	0.072	0.579	0.118	0.072
	0.2	0.155	0.055	0.067	0.169	0.060	0.052
	0	0.049	0.042	0.061	0.054	0.047	0.046
	-0.2	0.015	0.034	0.057	0.011	0.030	0.042
	-0.5	0.000	0.016	0.046	0.000	0.016	0.035
	-0.8	0.000	0.002	0.016	0.000	0.001	0.013
$\omega = 0.9$	0.8	0.955	0.347	0.108	0.982	0.425	0.130
	0.5	0.547	0.114	0.064	0.576	0.113	0.062
	0.2	0.159	0.065	0.053	0.152	0.050	0.048
	0	0.057	0.045	0.047	0.042	0.039	0.041
	-0.2	0.014	0.036	0.043	0.010	0.033	0.036
	-0.5	0.001	0.018	0.029	0.000	0.020	0.025
	-0.8	0.000	0.002	0.013	0.000	0.001	0.012

Table 3d. The Size of the Case 3

		$T = 100$			$T = 200$		
	a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.1$	0.8	0.965	0.393	0.150	0.977	0.415	0.126
	0.5	0.546	0.152	0.090	0.566	0.109	0.061
	0.2	0.197	0.085	0.087	0.150	0.061	0.053
	0	0.062	0.070	0.072	0.047	0.047	0.049
	-0.2	0.014	0.048	0.066	0.011	0.036	0.044
	-0.5	0.000	0.023	0.054	0.000	0.021	0.035
	-0.8	0.000	0.002	0.032	0.000	0.003	0.022
$\omega = 0.2$	0.8	0.957	0.393	0.196	0.980	0.421	0.155
	0.5	0.562	0.145	0.128	0.563	0.115	0.082
	0.2	0.181	0.088	0.107	0.153	0.056	0.066
	0	0.064	0.062	0.101	0.042	0.039	0.060
	-0.2	0.008	0.048	0.095	0.009	0.030	0.051
	-0.5	0.000	0.029	0.083	0.000	0.018	0.045
	-0.8	0.000	0.004	0.057	0.000	0.002	0.024
$\omega = 0.3$	0.8	0.969	0.390	0.235	0.993	0.422	0.150
	0.5	0.606	0.140	0.171	0.623	0.120	0.075
	0.2	0.185	0.084	0.160	0.176	0.053	0.065
	0	0.063	0.070	0.142	0.040	0.038	0.065
	-0.2	0.016	0.059	0.136	0.009	0.030	0.058
	-0.5	0.000	0.035	0.143	0.000	0.016	0.053
	-0.8	0.000	0.005	0.132	0.000	0.001	0.027
$\omega = 0.4$	0.8	0.984	0.396	0.248	0.999	0.491	0.140
	0.5	0.692	0.133	0.228	0.731	0.136	0.091
	0.2	0.201	0.075	0.245	0.202	0.063	0.072
	0	0.059	0.052	0.258	0.045	0.043	0.070
	-0.2	0.012	0.041	0.256	0.010	0.032	0.066
	-0.5	0.000	0.025	0.270	0.000	0.020	0.057
	-0.8	0.000	0.007	0.292	0.000	0.002	0.038
$\omega = 0.5$	0.8	0.991	0.402	0.225	0.999	0.548	0.143
	0.5	0.735	0.125	0.278	0.773	0.150	0.087
	0.2	0.221	0.056	0.321	0.234	0.067	0.072
	0	0.056	0.045	0.341	0.054	0.041	0.069
	-0.2	0.007	0.033	0.366	0.008	0.030	0.065
	-0.5	0.000	0.011	0.393	0.000	0.016	0.056
	-0.8	0.000	0.001	0.450	0.000	0.000	0.036

Table 3d. The Size of the Case 3 (continued)

		$T=100$			$T=200$		
	a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega=0.6$	0.8	0.982	0.391	0.244	0.998	0.503	0.147
	0.5	0.704	0.137	0.221	0.745	0.130	0.079
	0.2	0.204	0.072	0.245	0.208	0.065	0.071
	0	0.047	0.054	0.251	0.043	0.045	0.066
	-0.2	0.009	0.038	0.256	0.008	0.028	0.061
	-0.5	0.000	0.019	0.267	0.000	0.013	0.050
	-0.8	0.000	0.001	0.268	0.000	0.000	0.023
$\omega=0.7$	0.8	0.958	0.367	0.206	0.995	0.437	0.143
	0.5	0.600	0.139	0.165	0.651	0.117	0.089
	0.2	0.169	0.075	0.160	0.164	0.055	0.080
	0	0.050	0.052	0.162	0.045	0.039	0.071
	-0.2	0.014	0.036	0.156	0.009	0.032	0.064
	-0.5	0.000	0.020	0.137	0.000	0.015	0.045
	-0.8	0.000	0.002	0.120	0.000	0.001	0.029
$\omega=0.8$	0.8	0.956	0.391	0.182	0.988	0.437	0.160
	0.5	0.558	0.129	0.128	0.602	0.135	0.089
	0.2	0.165	0.071	0.117	0.163	0.075	0.079
	0	0.049	0.055	0.116	0.055	0.055	0.074
	-0.2	0.014	0.042	0.108	0.010	0.042	0.066
	-0.5	0.000	0.023	0.084	0.000	0.019	0.052
	-0.8	0.000	0.001	0.056	0.000	0.002	0.024
$\omega=0.9$	0.8	0.955	0.366	0.124	0.981	0.453	0.147
	0.5	0.550	0.126	0.081	0.583	0.127	0.081
	0.2	0.168	0.073	0.068	0.167	0.061	0.067
	0	0.059	0.051	0.060	0.045	0.045	0.062
	-0.2	0.018	0.040	0.053	0.008	0.035	0.051
	-0.5	0.000	0.020	0.043	0.000	0.018	0.039
	-0.8	0.000	0.005	0.021	0.000	0.000	0.014
PS	0.8	0.909	0.123	0.010	0.959	0.212	0.024
	0.5	0.452	0.054	0.056	0.601	0.079	0.074
	0.2	0.175	0.077	0.144	0.180	0.064	0.118
	0	0.060	0.094	0.216	0.054	0.058	0.149
	-0.2	0.018	0.118	0.334	0.006	0.053	0.176
	-0.5	0.001	0.183	0.518	0.000	0.049	0.257
	-0.8	0.000	0.194	0.668	0.000	0.023	0.400

Table 4a. The Power of the LM Test: the Case 0

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.996	0.838	0.607	0.999	0.947	0.704
		0.8	0.849	0.334	0.157	0.893	0.504	0.287
		0.5	0.643	0.331	0.234	0.827	0.588	0.476
		0.2	0.567	0.421	0.327	0.808	0.699	0.582
	$\rho = 0.01$	0.0	0.532	0.474	0.382	0.795	0.749	0.621
		-0.2	0.511	0.512	0.416	0.783	0.772	0.642
		-0.5	0.428	0.550	0.440	0.732	0.804	0.660
		-0.8	0.247	0.504	0.443	0.587	0.770	0.662
		0.8	0.982	0.777	0.542	0.996	0.922	0.704
		0.5	0.987	0.799	0.584	0.998	0.930	0.706
		0.2	0.985	0.816	0.587	0.999	0.934	0.707
$\omega = 0.1$	$\rho = 1$	0.0	0.985	0.824	0.592	0.999	0.936	0.705
		-0.2	0.984	0.826	0.593	0.999	0.937	0.705
		-0.5	0.979	0.826	0.597	0.999	0.937	0.705
		-0.8	0.962	0.825	0.603	0.996	0.937	0.706
		0.8	0.996	0.835	0.606	1.000	0.943	0.704
		0.5	0.996	0.836	0.606	0.999	0.945	0.704
		0.2	0.996	0.837	0.607	0.999	0.946	0.703
	$\rho = 100$	0.0	0.996	0.837	0.608	0.999	0.946	0.703
		-0.2	0.996	0.836	0.607	0.999	0.946	0.703
		-0.5	0.996	0.836	0.607	0.999	0.946	0.703
		-0.8	0.996	0.837	0.607	0.999	0.946	0.704
		R.W.	0.996	0.802	0.589	1.000	0.926	0.712
		0.8	0.852	0.347	0.171	0.919	0.470	0.263
		0.5	0.589	0.294	0.218	0.817	0.566	0.458
		0.2	0.525	0.369	0.295	0.797	0.665	0.572
	$\rho = 0.01$	0.0	0.477	0.431	0.342	0.780	0.709	0.605
		-0.2	0.445	0.475	0.378	0.758	0.749	0.632
		-0.5	0.367	0.505	0.412	0.690	0.778	0.662
		-0.8	0.196	0.454	0.426	0.525	0.743	0.657
		0.8	0.974	0.753	0.540	1.000	0.915	0.703
		0.5	0.979	0.780	0.572	1.000	0.917	0.709
		0.2	0.981	0.789	0.588	1.000	0.923	0.711
$\omega = 0.2$	$\rho = 1$	0.0	0.980	0.797	0.592	1.000	0.923	0.710
		-0.2	0.981	0.803	0.592	1.000	0.923	0.711
		-0.5	0.976	0.800	0.589	0.999	0.925	0.710
		-0.8	0.953	0.794	0.587	0.997	0.922	0.711
		0.8	0.994	0.807	0.591	1.000	0.927	0.710
		0.5	0.995	0.804	0.588	1.000	0.926	0.713
		0.2	0.996	0.803	0.588	1.000	0.926	0.712
	$\rho = 100$	0.0	0.996	0.803	0.587	1.000	0.926	0.712
		-0.2	0.996	0.803	0.587	1.000	0.926	0.712
		-0.5	0.996	0.803	0.588	1.000	0.926	0.712
		-0.8	0.996	0.803	0.588	1.000	0.926	0.711

Table 4a. The Power of the LM Test: the Case 0 (continued)

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega=0.3$		R.W.	0.996	0.767	0.530	1.000	0.945	0.688
	$\rho=0.01$	0.8	0.903	0.326	0.154	0.949	0.450	0.221
		0.5	0.620	0.261	0.179	0.815	0.512	0.388
		0.2	0.479	0.316	0.249	0.774	0.629	0.497
		0.0	0.434	0.365	0.291	0.762	0.678	0.558
		-0.2	0.391	0.418	0.325	0.735	0.715	0.587
		-0.5	0.301	0.465	0.359	0.648	0.761	0.618
		-0.8	0.123	0.402	0.366	0.469	0.717	0.621
	$\rho=1$	0.8	0.988	0.722	0.484	0.999	0.914	0.675
		0.5	0.986	0.742	0.519	1.000	0.937	0.682
		0.2	0.986	0.748	0.528	1.000	0.939	0.689
		0.0	0.987	0.751	0.533	1.000	0.939	0.689
		-0.2	0.985	0.755	0.537	1.000	0.939	0.689
		-0.5	0.982	0.759	0.534	1.000	0.941	0.689
		-0.8	0.962	0.758	0.535	0.997	0.939	0.692
	$\rho=100$	0.8	0.996	0.765	0.534	1.000	0.944	0.687
		0.5	0.996	0.765	0.531	1.000	0.947	0.688
		0.2	0.996	0.766	0.530	1.000	0.946	0.688
		0.0	0.996	0.768	0.530	1.000	0.946	0.688
		-0.2	0.996	0.768	0.530	1.000	0.946	0.689
		-0.5	0.996	0.768	0.530	1.000	0.946	0.689
		-0.8	0.996	0.768	0.530	1.000	0.946	0.689
$\omega=0.4$		R.W.	0.998	0.821	0.483	1.000	0.961	0.712
	$\rho=0.01$	0.8	0.929	0.313	0.120	0.963	0.462	0.191
		0.5	0.655	0.234	0.126	0.827	0.472	0.334
		0.2	0.480	0.263	0.187	0.795	0.600	0.450
		0.0	0.404	0.305	0.233	0.764	0.669	0.505
		-0.2	0.342	0.351	0.270	0.728	0.723	0.552
		-0.5	0.225	0.402	0.301	0.637	0.770	0.588
		-0.8	0.080	0.339	0.305	0.405	0.725	0.597
	$\rho=1$	0.8	0.996	0.742	0.424	1.000	0.940	0.671
		0.5	0.997	0.776	0.465	1.000	0.957	0.703
		0.2	0.996	0.793	0.468	1.000	0.958	0.704
		0.0	0.995	0.799	0.472	1.000	0.959	0.706
		-0.2	0.995	0.803	0.475	1.000	0.960	0.710
		-0.5	0.991	0.806	0.475	1.000	0.959	0.712
		-0.8	0.977	0.802	0.477	1.000	0.961	0.711
	$\rho=100$	0.8	0.999	0.813	0.478	1.000	0.964	0.712
		0.5	0.999	0.815	0.484	1.000	0.963	0.712
		0.2	0.998	0.817	0.484	1.000	0.962	0.713
		0.0	0.998	0.818	0.484	1.000	0.962	0.713
		-0.2	0.998	0.818	0.484	1.000	0.962	0.714
		-0.5	0.998	0.820	0.484	1.000	0.962	0.714
		-0.8	0.998	0.820	0.484	1.000	0.962	0.714

Table 4a. The Power of the LM Test: the Case 0 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.5$		R.W.	0.999	0.862	0.508	1.000	0.979	0.763
	$\rho = 0.01$	0.8	0.950	0.343	0.069	0.973	0.485	0.164
		0.5	0.671	0.205	0.093	0.840	0.513	0.314
		0.2	0.484	0.265	0.130	0.788	0.649	0.474
		0.0	0.413	0.322	0.164	0.773	0.713	0.547
		-0.2	0.339	0.375	0.199	0.745	0.747	0.600
		-0.5	0.203	0.412	0.242	0.659	0.795	0.630
		-0.8	0.041	0.311	0.241	0.396	0.757	0.638
	$\rho = 1$	0.8	0.996	0.794	0.427	1.000	0.954	0.726
		0.5	0.995	0.830	0.483	1.000	0.961	0.763
		0.2	0.995	0.843	0.498	1.000	0.966	0.768
		0.0	0.995	0.850	0.499	1.000	0.966	0.769
		-0.2	0.996	0.850	0.497	1.000	0.968	0.770
		-0.5	0.993	0.855	0.497	1.000	0.971	0.768
		-0.8	0.979	0.852	0.496	1.000	0.971	0.768
	$\rho = 100$	0.8	1.000	0.858	0.504	1.000	0.974	0.769
		0.5	0.999	0.860	0.503	1.000	0.976	0.765
		0.2	0.999	0.862	0.504	1.000	0.976	0.764
		0.0	0.999	0.863	0.507	1.000	0.976	0.764
		-0.2	0.999	0.863	0.507	1.000	0.976	0.764
		-0.5	0.999	0.863	0.507	1.000	0.976	0.764
		-0.8	0.999	0.862	0.507	1.000	0.976	0.763
$\omega = 0.6$		R.W.	1.000	0.847	0.493	1.000	0.963	0.695
	$\rho = 0.01$	0.8	0.921	0.320	0.089	0.957	0.486	0.174
		0.5	0.649	0.229	0.111	0.838	0.495	0.321
		0.2	0.473	0.285	0.186	0.779	0.626	0.462
		0.0	0.409	0.325	0.227	0.763	0.692	0.532
		-0.2	0.350	0.370	0.263	0.734	0.736	0.553
		-0.5	0.239	0.405	0.302	0.650	0.771	0.591
		-0.8	0.072	0.340	0.294	0.407	0.730	0.601
	$\rho = 1$	0.8	0.996	0.766	0.426	1.000	0.940	0.673
		0.5	0.995	0.804	0.466	1.000	0.953	0.687
		0.2	0.996	0.821	0.483	1.000	0.958	0.691
		0.0	0.995	0.827	0.488	1.000	0.959	0.691
		-0.2	0.994	0.828	0.489	1.000	0.959	0.694
		-0.5	0.992	0.828	0.491	0.999	0.958	0.693
		-0.8	0.978	0.827	0.489	0.997	0.958	0.693
	$\rho = 100$	0.8	1.000	0.845	0.492	1.000	0.962	0.697
		0.5	1.000	0.847	0.491	1.000	0.962	0.693
		0.2	1.000	0.849	0.489	1.000	0.963	0.694
		0.0	1.000	0.849	0.489	1.000	0.963	0.694
		-0.2	1.000	0.849	0.489	1.000	0.963	0.694
		-0.5	1.000	0.848	0.490	1.000	0.963	0.694
		-0.8	1.000	0.848	0.490	1.000	0.963	0.694

Table 4a. The Power of the LM Test: the Case 0 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.997	0.791	0.558	1.000	0.949	0.681
		0.8	0.896	0.313	0.119	0.938	0.444	0.224
		0.5	0.610	0.259	0.160	0.800	0.511	0.380
		0.2	0.483	0.325	0.237	0.766	0.621	0.502
	$\rho = 0.01$	0.0	0.435	0.366	0.292	0.745	0.672	0.540
		-0.2	0.396	0.413	0.338	0.720	0.710	0.567
		-0.5	0.295	0.453	0.377	0.642	0.738	0.593
		-0.8	0.130	0.398	0.373	0.450	0.709	0.609
		0.8	0.988	0.729	0.499	1.000	0.920	0.658
		0.5	0.993	0.749	0.534	0.999	0.937	0.671
		0.2	0.993	0.768	0.543	0.999	0.946	0.677
$\omega = 0.7$	$\rho = 1$	0.0	0.994	0.770	0.546	0.999	0.947	0.680
		-0.2	0.992	0.773	0.550	0.999	0.947	0.683
		-0.5	0.989	0.777	0.550	0.999	0.946	0.680
		-0.8	0.976	0.774	0.550	0.997	0.947	0.683
		0.8	0.996	0.788	0.550	1.000	0.949	0.682
		0.5	0.997	0.793	0.555	1.000	0.950	0.682
		0.2	0.997	0.791	0.555	1.000	0.949	0.681
	$\rho = 100$	0.0	0.997	0.790	0.556	1.000	0.949	0.681
		-0.2	0.997	0.790	0.556	1.000	0.949	0.681
		-0.5	0.997	0.790	0.556	1.000	0.949	0.681
		-0.8	0.997	0.790	0.556	1.000	0.949	0.681
		R.W.	0.992	0.796	0.595	1.000	0.933	0.715
		0.8	0.854	0.333	0.145	0.907	0.454	0.255
		0.5	0.602	0.287	0.189	0.812	0.561	0.441
		0.2	0.525	0.379	0.292	0.803	0.669	0.547
	$\rho = 0.01$	0.0	0.491	0.432	0.346	0.788	0.704	0.585
		-0.2	0.460	0.466	0.395	0.756	0.749	0.617
		-0.5	0.365	0.496	0.427	0.702	0.781	0.645
		-0.8	0.188	0.458	0.428	0.507	0.751	0.654
		0.8	0.973	0.762	0.558	0.999	0.914	0.699
		0.5	0.981	0.770	0.588	0.999	0.925	0.716
		0.2	0.983	0.777	0.594	0.999	0.930	0.710
$\omega = 0.8$	$\rho = 1$	0.0	0.983	0.782	0.602	0.999	0.930	0.714
		-0.2	0.983	0.780	0.603	0.999	0.930	0.714
		-0.5	0.977	0.783	0.600	0.998	0.931	0.714
		-0.8	0.961	0.786	0.598	0.996	0.931	0.715
		0.8	0.989	0.796	0.595	1.000	0.934	0.714
		0.5	0.991	0.798	0.593	1.000	0.933	0.713
		0.2	0.991	0.796	0.594	1.000	0.932	0.713
	$\rho = 100$	0.0	0.991	0.796	0.594	1.000	0.932	0.713
		-0.2	0.991	0.796	0.594	1.000	0.932	0.714
		-0.5	0.991	0.799	0.593	1.000	0.932	0.715
		-0.8	0.991	0.799	0.593	1.000	0.932	0.715

Table 4a. The Power of the LM Test: the Case 0 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.993	0.828	0.603	0.999	0.937	0.710
		0.8	0.847	0.361	0.144	0.893	0.496	0.273
		0.5	0.621	0.349	0.225	0.838	0.597	0.477
		0.2	0.551	0.434	0.331	0.816	0.682	0.571
	$\rho = 0.01$	0.0	0.533	0.487	0.387	0.803	0.731	0.622
		-0.2	0.505	0.521	0.430	0.781	0.775	0.637
		-0.5	0.429	0.551	0.461	0.738	0.801	0.658
		-0.8	0.242	0.511	0.461	0.558	0.776	0.659
		0.8	0.980	0.778	0.559	0.996	0.913	0.697
		0.5	0.984	0.804	0.586	0.997	0.934	0.713
		0.2	0.988	0.813	0.597	0.996	0.936	0.712
$\omega = 0.9$	$\rho = 1$	0.0	0.988	0.813	0.601	0.997	0.935	0.714
		-0.2	0.988	0.814	0.602	0.997	0.937	0.715
		-0.5	0.988	0.814	0.602	0.997	0.937	0.715
		-0.8	0.965	0.814	0.602	0.995	0.937	0.715
		0.8	0.993	0.821	0.604	0.999	0.938	0.715
		0.5	0.992	0.824	0.605	0.999	0.936	0.712
		0.2	0.992	0.825	0.602	0.999	0.935	0.712
	$\rho = 100$	0.0	0.992	0.825	0.602	0.999	0.935	0.712
		-0.2	0.992	0.825	0.602	0.999	0.935	0.712
		-0.5	0.992	0.827	0.602	0.999	0.936	0.712
		-0.8	0.992	0.828	0.602	0.999	0.936	0.712

Table 4b. The Power of the PS Test: the Case 0

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.1$		R.W.	0.989	0.712	0.249	0.998	0.886	0.570
	$\rho = 0.01$	0.8	0.803	0.195	0.018	0.915	0.405	0.135
		0.5	0.625	0.232	0.073	0.832	0.519	0.346
		0.2	0.534	0.332	0.130	0.783	0.618	0.456
		0.0	0.485	0.391	0.157	0.760	0.661	0.489
		-0.2	0.432	0.436	0.178	0.731	0.692	0.512
		-0.5	0.326	0.462	0.204	0.660	0.724	0.529
		-0.8	0.149	0.407	0.229	0.481	0.689	0.534
	$\rho = 1$	0.8	0.977	0.648	0.194	0.998	0.859	0.561
		0.5	0.973	0.688	0.233	0.997	0.875	0.570
		0.2	0.975	0.699	0.243	0.997	0.879	0.568
		0.0	0.972	0.706	0.242	0.997	0.878	0.567
		-0.2	0.970	0.706	0.242	0.997	0.882	0.567
		-0.5	0.965	0.705	0.244	0.997	0.883	0.568
		-0.8	0.943	0.705	0.246	0.995	0.882	0.568
	$\rho = 100$	0.8	0.985	0.712	0.246	0.998	0.888	0.568
		0.5	0.987	0.711	0.248	0.998	0.887	0.568
		0.2	0.987	0.710	0.249	0.998	0.886	0.570
		0.0	0.988	0.710	0.249	0.998	0.887	0.570
		-0.2	0.989	0.710	0.249	0.998	0.887	0.570
		-0.5	0.989	0.710	0.249	0.998	0.887	0.570
		-0.8	0.989	0.710	0.249	0.998	0.887	0.570
$\omega = 0.2$		R.W.	0.995	0.767	0.331	1.000	0.934	0.646
	$\rho = 0.01$	0.8	0.885	0.275	0.055	0.966	0.452	0.157
		0.5	0.670	0.253	0.096	0.867	0.524	0.344
		0.2	0.535	0.305	0.168	0.817	0.651	0.444
		0.0	0.455	0.360	0.190	0.787	0.693	0.505
		-0.2	0.388	0.404	0.213	0.753	0.737	0.533
		-0.5	0.286	0.436	0.239	0.657	0.770	0.562
		-0.8	0.112	0.373	0.241	0.442	0.724	0.569
	$\rho = 1$	0.8	0.982	0.710	0.297	1.000	0.916	0.619
		0.5	0.987	0.731	0.334	1.000	0.928	0.641
		0.2	0.988	0.740	0.338	1.000	0.934	0.652
		0.0	0.988	0.748	0.339	1.000	0.937	0.650
		-0.2	0.988	0.749	0.338	1.000	0.935	0.648
		-0.5	0.986	0.752	0.338	0.999	0.936	0.646
		-0.8	0.964	0.751	0.334	0.997	0.935	0.646
	$\rho = 100$	0.8	0.994	0.764	0.329	1.000	0.935	0.644
		0.5	0.994	0.767	0.330	1.000	0.935	0.646
		0.2	0.994	0.766	0.331	1.000	0.933	0.647
		0.0	0.994	0.766	0.331	1.000	0.933	0.646
		-0.2	0.994	0.766	0.331	1.000	0.933	0.646
		-0.5	0.994	0.765	0.331	1.000	0.934	0.645
		-0.8	0.994	0.766	0.331	1.000	0.934	0.645

Table 4b. The Power of the PS Test: the Case 0 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.997	0.812	0.428	1.000	0.955	0.712
		0.8	0.943	0.324	0.052	0.963	0.480	0.174
		0.5	0.688	0.231	0.091	0.851	0.511	0.320
		0.2	0.518	0.282	0.142	0.809	0.641	0.468
	$\rho = 0.01$	0.0	0.448	0.339	0.181	0.772	0.701	0.527
		-0.2	0.361	0.383	0.212	0.741	0.738	0.571
		-0.5	0.229	0.422	0.246	0.659	0.769	0.610
		-0.8	0.080	0.332	0.239	0.402	0.734	0.618
		0.8	0.992	0.761	0.363	1.000	0.924	0.678
		0.5	0.985	0.786	0.398	1.000	0.945	0.700
		0.2	0.988	0.799	0.408	1.000	0.953	0.701
$\omega = 0.3$	$\rho = 1$	0.0	0.989	0.802	0.413	1.000	0.953	0.706
		-0.2	0.988	0.803	0.415	1.000	0.954	0.706
		-0.5	0.985	0.803	0.416	0.999	0.953	0.706
		-0.8	0.964	0.801	0.416	0.998	0.952	0.705
		0.8	0.997	0.810	0.426	1.000	0.955	0.708
		0.5	0.997	0.815	0.428	1.000	0.955	0.708
		0.2	0.997	0.814	0.426	1.000	0.955	0.709
	$\rho = 100$	0.0	0.997	0.813	0.427	1.000	0.956	0.711
		-0.2	0.997	0.812	0.427	1.000	0.956	0.711
		-0.5	0.997	0.812	0.427	1.000	0.955	0.711
		-0.8	0.997	0.812	0.426	1.000	0.955	0.711
		R.W.	0.999	0.832	0.482	1.000	0.968	0.766
		0.8	0.945	0.348	0.067	0.962	0.478	0.174
		0.5	0.665	0.213	0.074	0.833	0.492	0.324
		0.2	0.492	0.271	0.127	0.786	0.612	0.460
	$\rho = 0.01$	0.0	0.413	0.315	0.167	0.764	0.689	0.516
		-0.2	0.328	0.365	0.202	0.732	0.742	0.570
		-0.5	0.207	0.406	0.241	0.644	0.784	0.611
		-0.8	0.046	0.309	0.242	0.388	0.733	0.620
		0.8	0.995	0.769	0.421	1.000	0.941	0.710
		0.5	0.998	0.798	0.449	1.000	0.961	0.753
		0.2	0.997	0.811	0.468	1.000	0.964	0.758
$\omega = 0.4$	$\rho = 1$	0.0	0.996	0.818	0.474	1.000	0.963	0.758
		-0.2	0.995	0.824	0.474	1.000	0.963	0.759
		-0.5	0.991	0.825	0.479	1.000	0.967	0.760
		-0.8	0.979	0.823	0.477	1.000	0.967	0.763
		0.8	1.000	0.829	0.481	1.000	0.968	0.762
		0.5	1.000	0.828	0.481	1.000	0.968	0.764
		0.2	0.999	0.829	0.481	1.000	0.967	0.765
	$\rho = 100$	0.0	0.999	0.830	0.482	1.000	0.968	0.765
		-0.2	0.999	0.831	0.482	1.000	0.968	0.765
		-0.5	0.999	0.831	0.482	1.000	0.968	0.765
		-0.8	0.999	0.830	0.483	1.000	0.968	0.766

Table 4b. The Power of the PS Test: the Case 0 (continued)

			$T = 100$			$T = 200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.999	0.862	0.508	1.000	0.979	0.763
		0.8	0.950	0.343	0.069	0.973	0.485	0.164
		0.5	0.671	0.205	0.093	0.840	0.513	0.314
		0.2	0.484	0.265	0.130	0.788	0.649	0.474
	$\rho = 0.01$	0.0	0.413	0.322	0.164	0.773	0.713	0.547
		-0.2	0.339	0.375	0.199	0.745	0.747	0.600
		-0.5	0.203	0.412	0.242	0.659	0.795	0.630
		-0.8	0.041	0.311	0.241	0.396	0.757	0.638
		0.8	0.996	0.794	0.427	1.000	0.954	0.726
		0.5	0.995	0.830	0.483	1.000	0.961	0.763
		0.2	0.995	0.843	0.498	1.000	0.966	0.768
$\omega = 0.5$	$\rho = 1$	0.0	0.995	0.850	0.499	1.000	0.966	0.769
		-0.2	0.996	0.850	0.497	1.000	0.968	0.770
		-0.5	0.993	0.855	0.497	1.000	0.971	0.768
		-0.8	0.979	0.852	0.496	1.000	0.971	0.768
		0.8	1.000	0.858	0.504	1.000	0.974	0.769
		0.5	0.999	0.860	0.503	1.000	0.976	0.765
		0.2	0.999	0.862	0.504	1.000	0.976	0.764
	$\rho = 100$	0.0	0.999	0.863	0.507	1.000	0.976	0.764
		-0.2	0.999	0.863	0.507	1.000	0.976	0.764
		-0.5	0.999	0.863	0.507	1.000	0.976	0.764
		-0.8	0.999	0.862	0.507	1.000	0.976	0.763
		R.W.	1.000	0.860	0.478	1.000	0.961	0.752
		0.8	0.930	0.361	0.065	0.967	0.504	0.167
		0.5	0.677	0.219	0.080	0.856	0.517	0.313
		0.2	0.509	0.276	0.133	0.809	0.654	0.478
	$\rho = 0.01$	0.0	0.417	0.328	0.176	0.788	0.717	0.558
		-0.2	0.352	0.373	0.213	0.748	0.753	0.608
		-0.5	0.210	0.409	0.248	0.669	0.781	0.653
		-0.8	0.050	0.326	0.258	0.407	0.750	0.660
		0.8	0.998	0.800	0.413	1.000	0.939	0.708
		0.5	0.993	0.828	0.460	1.000	0.954	0.737
		0.2	0.992	0.839	0.468	1.000	0.958	0.741
$\omega = 0.6$	$\rho = 1$	0.0	0.993	0.844	0.468	1.000	0.958	0.743
		-0.2	0.993	0.848	0.471	1.000	0.959	0.744
		-0.5	0.993	0.851	0.472	1.000	0.960	0.746
		-0.8	0.982	0.849	0.473	0.999	0.961	0.749
		0.8	1.000	0.861	0.481	1.000	0.964	0.750
		0.5	1.000	0.858	0.478	1.000	0.962	0.753
		0.2	1.000	0.858	0.479	1.000	0.962	0.753
	$\rho = 100$	0.0	1.000	0.858	0.479	1.000	0.962	0.752
		-0.2	1.000	0.859	0.479	1.000	0.962	0.752
		-0.5	1.000	0.859	0.479	1.000	0.962	0.752
		-0.8	1.000	0.859	0.480	1.000	0.962	0.752

Table 4b. The Power of the PS Test: the Case 0 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.998	0.814	0.420	1.000	0.958	0.684
		0.8	0.927	0.330	0.085	0.959	0.492	0.168
		0.5	0.666	0.244	0.090	0.856	0.512	0.327
		0.2	0.500	0.291	0.141	0.809	0.636	0.456
	$\rho = 0.01$	0.0	0.429	0.340	0.164	0.775	0.693	0.509
		-0.2	0.362	0.372	0.196	0.736	0.727	0.542
		-0.5	0.239	0.407	0.230	0.651	0.767	0.578
		-0.8	0.066	0.338	0.221	0.398	0.726	0.597
		0.8	0.998	0.747	0.373	1.000	0.938	0.664
		0.5	0.994	0.771	0.410	1.000	0.943	0.678
		0.2	0.995	0.791	0.419	0.999	0.952	0.681
$\omega = 0.7$	$\rho = 1$	0.0	0.994	0.792	0.420	0.999	0.953	0.683
		-0.2	0.994	0.797	0.424	0.999	0.955	0.688
		-0.5	0.993	0.800	0.423	0.999	0.958	0.688
		-0.8	0.978	0.798	0.417	0.998	0.955	0.688
		0.8	0.998	0.815	0.419	1.000	0.958	0.686
		0.5	0.998	0.816	0.419	1.000	0.958	0.687
		0.2	0.998	0.816	0.420	1.000	0.958	0.686
	$\rho = 100$	0.0	0.998	0.816	0.419	1.000	0.958	0.686
		-0.2	0.998	0.815	0.420	1.000	0.958	0.686
		-0.5	0.998	0.815	0.420	1.000	0.958	0.686
		-0.8	0.998	0.814	0.420	1.000	0.958	0.685
		R.W.	0.995	0.761	0.329	1.000	0.933	0.616
		0.8	0.895	0.288	0.069	0.952	0.455	0.176
		0.5	0.677	0.236	0.100	0.858	0.521	0.324
		0.2	0.519	0.303	0.153	0.802	0.617	0.441
	$\rho = 0.01$	0.0	0.457	0.351	0.176	0.770	0.669	0.477
		-0.2	0.381	0.396	0.189	0.736	0.718	0.506
		-0.5	0.269	0.434	0.217	0.653	0.749	0.536
		-0.8	0.109	0.374	0.226	0.420	0.705	0.543
		0.8	0.981	0.713	0.291	1.000	0.908	0.590
		0.5	0.982	0.731	0.317	1.000	0.922	0.609
		0.2	0.984	0.753	0.317	0.998	0.923	0.612
$\omega = 0.8$	$\rho = 1$	0.0	0.986	0.755	0.319	0.998	0.925	0.612
		-0.2	0.985	0.755	0.320	0.998	0.928	0.612
		-0.5	0.979	0.753	0.322	0.997	0.928	0.612
		-0.8	0.963	0.750	0.324	0.996	0.927	0.614
		0.8	0.996	0.759	0.328	0.999	0.932	0.613
		0.5	0.996	0.763	0.327	1.000	0.932	0.616
		0.2	0.996	0.760	0.325	1.000	0.932	0.616
	$\rho = 100$	0.0	0.996	0.760	0.325	1.000	0.932	0.616
		-0.2	0.996	0.759	0.326	1.000	0.932	0.616
		-0.5	0.996	0.759	0.327	1.000	0.932	0.616
		-0.8	0.995	0.759	0.327	1.000	0.932	0.616

Table 4b. The Power of the PS Test: the Case 0 (continued)

			$T = 100$			$T = 200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.991	0.715	0.238	0.999	0.876	0.567
		0.8	0.836	0.200	0.026	0.914	0.403	0.142
		0.5	0.635	0.228	0.065	0.859	0.517	0.330
		0.2	0.539	0.337	0.113	0.798	0.612	0.434
	$\rho = 0.01$	0.0	0.489	0.389	0.145	0.766	0.661	0.470
		-0.2	0.441	0.431	0.174	0.739	0.696	0.487
		-0.5	0.340	0.482	0.210	0.650	0.719	0.514
		-0.8	0.141	0.471	0.278	0.454	0.681	0.532
		0.8	0.973	0.654	0.196	0.998	0.860	0.549
		0.5	0.979	0.689	0.230	0.997	0.874	0.558
		0.2	0.981	0.707	0.237	0.998	0.872	0.562
$\omega = 0.9$	$\rho = 1$	0.0	0.981	0.710	0.241	0.998	0.870	0.563
		-0.2	0.977	0.710	0.243	0.996	0.872	0.562
		-0.5	0.970	0.708	0.241	0.996	0.872	0.561
		-0.8	0.949	0.706	0.244	0.995	0.870	0.562
		0.8	0.989	0.711	0.238	0.998	0.873	0.560
		0.5	0.990	0.712	0.238	0.998	0.876	0.563
		0.2	0.990	0.713	0.240	0.999	0.876	0.563
	$\rho = 100$	0.0	0.990	0.714	0.241	0.999	0.876	0.564
		-0.2	0.990	0.715	0.240	0.999	0.876	0.564
		-0.5	0.990	0.715	0.238	0.999	0.876	0.565
		-0.8	0.990	0.715	0.240	0.999	0.876	0.565

Table 4c. The Power of the LM Test: the Case 1

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.998	0.834	0.441	1.000	0.957	0.654
		0.8	0.958	0.393	0.130	0.992	0.445	0.176
		0.5	0.660	0.197	0.109	0.809	0.382	0.249
		0.2	0.418	0.205	0.144	0.704	0.511	0.375
	$\rho=0.01$	0.0	0.307	0.231	0.169	0.649	0.573	0.432
		-0.2	0.218	0.260	0.197	0.596	0.631	0.467
		-0.5	0.098	0.286	0.217	0.481	0.682	0.507
		-0.8	0.014	0.198	0.211	0.234	0.617	0.510
		0.8	0.993	0.742	0.360	1.000	0.914	0.615
		0.5	0.995	0.768	0.412	1.000	0.937	0.639
		0.2	0.993	0.787	0.421	1.000	0.943	0.649
$\omega=0.1$	$\rho=1$	0.0	0.992	0.803	0.427	1.000	0.946	0.650
		-0.2	0.990	0.807	0.427	1.000	0.946	0.649
		-0.5	0.986	0.813	0.428	0.999	0.948	0.651
		-0.8	0.967	0.807	0.430	0.999	0.948	0.650
		0.8	0.998	0.829	0.432	1.000	0.954	0.652
		0.5	0.998	0.830	0.441	1.000	0.958	0.651
		0.2	0.998	0.833	0.442	1.000	0.958	0.652
	$\rho=100$	0.0	0.998	0.836	0.442	1.000	0.958	0.652
		-0.2	0.998	0.837	0.443	1.000	0.958	0.652
		-0.5	0.998	0.837	0.443	1.000	0.957	0.652
		-0.8	0.998	0.837	0.443	1.000	0.957	0.652
		R.W.	0.998	0.816	0.419	1.000	0.976	0.630
		0.8	0.964	0.366	0.137	0.994	0.484	0.154
		0.5	0.680	0.161	0.130	0.842	0.354	0.204
		0.2	0.369	0.173	0.150	0.694	0.463	0.320
	$\rho=0.01$	0.0	0.247	0.199	0.171	0.643	0.548	0.389
		-0.2	0.160	0.217	0.195	0.577	0.613	0.442
		-0.5	0.066	0.242	0.216	0.435	0.655	0.491
		-0.8	0.010	0.151	0.211	0.149	0.581	0.489
		0.8	0.994	0.722	0.355	1.000	0.937	0.581
		0.5	0.996	0.744	0.390	1.000	0.960	0.615
		0.2	0.995	0.780	0.405	1.000	0.971	0.618
$\omega=0.2$	$\rho=1$	0.0	0.996	0.791	0.408	1.000	0.972	0.621
		-0.2	0.996	0.799	0.414	1.000	0.974	0.625
		-0.5	0.994	0.800	0.418	1.000	0.972	0.623
		-0.8	0.965	0.793	0.415	0.998	0.970	0.624
		0.8	0.998	0.819	0.421	1.000	0.975	0.628
		0.5	0.998	0.818	0.416	1.000	0.976	0.631
		0.2	0.998	0.817	0.418	1.000	0.976	0.631
	$\rho=100$	0.0	0.998	0.816	0.418	1.000	0.976	0.631
		-0.2	0.998	0.816	0.418	1.000	0.976	0.629
		-0.5	0.998	0.816	0.418	1.000	0.976	0.629
		-0.8	0.998	0.816	0.418	1.000	0.976	0.629

Table 4c. The Power of the LM Test: the Case 1 (continued)

			T = 100			T = 200		
		a	$\ell0$	$\ell4$	$\ell12$	$\ell0$	$\ell4$	$\ell12$
$\omega = 0.3$		R.W.	1.000	0.851	0.450	1.000	0.974	0.705
	$\rho = 0.01$	0.8	0.973	0.396	0.102	0.990	0.534	0.136
		0.5	0.706	0.176	0.098	0.851	0.385	0.203
		0.2	0.380	0.176	0.123	0.715	0.487	0.326
		0.0	0.261	0.196	0.154	0.640	0.569	0.407
		-0.2	0.166	0.216	0.182	0.585	0.625	0.450
		-0.5	0.059	0.239	0.212	0.445	0.674	0.508
		-0.8	0.005	0.160	0.187	0.133	0.608	0.526
	$\rho = 1$	0.8	0.999	0.756	0.365	1.000	0.934	0.643
		0.5	0.997	0.799	0.425	1.000	0.960	0.691
		0.2	0.998	0.821	0.431	1.000	0.968	0.706
		0.0	0.998	0.830	0.433	1.000	0.968	0.708
		-0.2	0.998	0.834	0.440	1.000	0.969	0.706
		-0.5	0.997	0.839	0.439	1.000	0.972	0.706
		-0.8	0.974	0.834	0.446	0.999	0.971	0.706
	$\rho = 100$	0.8	1.000	0.848	0.445	1.000	0.975	0.708
		0.5	1.000	0.847	0.453	1.000	0.973	0.709
		0.2	1.000	0.846	0.451	1.000	0.973	0.706
		0.0	1.000	0.847	0.451	1.000	0.973	0.705
		-0.2	1.000	0.848	0.451	1.000	0.973	0.705
		-0.5	1.000	0.849	0.451	1.000	0.973	0.705
		-0.8	1.000	0.850	0.451	1.000	0.973	0.705
$\omega = 0.4$		R.W.	0.999	0.796	0.476	1.000	0.946	0.640
	$\rho = 0.01$	0.8	0.961	0.347	0.154	0.979	0.453	0.172
		0.5	0.631	0.197	0.144	0.795	0.373	0.239
		0.2	0.392	0.217	0.183	0.688	0.481	0.344
		0.0	0.302	0.237	0.210	0.635	0.553	0.402
		-0.2	0.221	0.269	0.229	0.577	0.603	0.446
		-0.5	0.111	0.302	0.257	0.447	0.642	0.487
		-0.8	0.021	0.226	0.248	0.216	0.576	0.497
	$\rho = 1$	0.8	0.992	0.699	0.418	1.000	0.903	0.588
		0.5	0.994	0.744	0.461	0.999	0.933	0.614
		0.2	0.992	0.763	0.471	1.000	0.942	0.625
		0.0	0.992	0.774	0.471	1.000	0.943	0.626
		-0.2	0.991	0.777	0.469	1.000	0.945	0.629
		-0.5	0.983	0.782	0.471	1.000	0.945	0.632
		-0.8	0.957	0.782	0.468	1.000	0.945	0.632
	$\rho = 100$	0.8	1.000	0.796	0.471	1.000	0.947	0.634
		0.5	0.999	0.795	0.472	1.000	0.946	0.639
		0.2	0.999	0.796	0.473	1.000	0.946	0.640
		0.0	0.999	0.796	0.474	1.000	0.946	0.640
		-0.2	0.999	0.797	0.474	1.000	0.946	0.639
		-0.5	0.999	0.797	0.474	1.000	0.946	0.639
		-0.8	0.999	0.797	0.475	1.000	0.946	0.640

Table 4c. The Power of the LM Test: the Case 1 (continued)

			T = 100			T = 200		
		a	$\ell0$	$\ell4$	$\ell12$	$\ell0$	$\ell4$	$\ell12$
		R.W.	0.998	0.758	0.507	1.000	0.944	0.641
		0.8	0.938	0.344	0.194	0.969	0.425	0.196
		0.5	0.610	0.201	0.159	0.786	0.394	0.278
		0.2	0.407	0.225	0.203	0.666	0.495	0.404
	$\rho = 0.01$	0.0	0.318	0.254	0.242	0.614	0.553	0.445
		-0.2	0.243	0.291	0.274	0.574	0.590	0.484
		-0.5	0.137	0.323	0.300	0.480	0.633	0.512
		-0.8	0.033	0.258	0.286	0.270	0.579	0.518
		0.8	0.990	0.685	0.461	1.000	0.893	0.586
		0.5	0.990	0.710	0.483	1.000	0.920	0.618
		0.2	0.992	0.734	0.495	1.000	0.929	0.629
$\omega = 0.5$	$\rho = 1$	0.0	0.990	0.747	0.501	1.000	0.931	0.636
		-0.2	0.989	0.748	0.503	1.000	0.933	0.641
		-0.5	0.984	0.752	0.505	1.000	0.934	0.641
		-0.8	0.955	0.749	0.504	0.999	0.936	0.640
		0.8	0.999	0.761	0.508	1.000	0.942	0.641
		0.5	0.999	0.764	0.512	1.000	0.944	0.641
		0.2	0.998	0.762	0.508	1.000	0.945	0.640
	$\rho = 100$	0.0	0.998	0.760	0.508	1.000	0.945	0.640
		-0.2	0.998	0.758	0.507	1.000	0.945	0.640
		-0.5	0.998	0.758	0.507	1.000	0.945	0.641
		-0.8	0.998	0.758	0.507	1.000	0.945	0.641
		R.W.	0.998	0.803	0.470	1.000	0.946	0.651
		0.8	0.953	0.357	0.179	0.985	0.451	0.181
		0.5	0.635	0.188	0.135	0.816	0.398	0.257
		0.2	0.390	0.206	0.182	0.690	0.511	0.380
	$\rho = 0.01$	0.0	0.286	0.225	0.209	0.654	0.575	0.436
		-0.2	0.200	0.263	0.230	0.602	0.631	0.468
		-0.5	0.107	0.291	0.256	0.471	0.674	0.493
		-0.8	0.017	0.212	0.248	0.223	0.596	0.499
		0.8	0.997	0.708	0.417	1.000	0.911	0.601
		0.5	0.990	0.752	0.447	1.000	0.925	0.626
		0.2	0.992	0.782	0.458	1.000	0.934	0.639
$\omega = 0.6$	$\rho = 1$	0.0	0.991	0.789	0.460	1.000	0.940	0.643
		-0.2	0.991	0.795	0.468	1.000	0.942	0.643
		-0.5	0.987	0.796	0.470	1.000	0.941	0.648
		-0.8	0.950	0.786	0.468	0.998	0.941	0.648
		0.8	0.998	0.809	0.473	1.000	0.946	0.649
		0.5	0.998	0.809	0.472	1.000	0.944	0.651
		0.2	0.998	0.806	0.471	1.000	0.945	0.650
	$\rho = 100$	0.0	0.998	0.804	0.470	1.000	0.945	0.650
		-0.2	0.998	0.804	0.468	1.000	0.945	0.650
		-0.5	0.998	0.803	0.469	1.000	0.946	0.650
		-0.8	0.998	0.803	0.469	1.000	0.946	0.650

Table 4c. The Power of the LM Test: the Case 1 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.999	0.851	0.455	1.000	0.974	0.689
		0.8	0.972	0.412	0.124	0.990	0.490	0.131
		0.5	0.717	0.179	0.111	0.864	0.370	0.195
		0.2	0.405	0.174	0.146	0.719	0.480	0.322
	$\rho = 0.01$	0.0	0.262	0.199	0.180	0.651	0.560	0.400
		-0.2	0.170	0.224	0.203	0.591	0.629	0.452
		-0.5	0.059	0.252	0.218	0.427	0.692	0.503
		-0.8	0.003	0.167	0.220	0.144	0.601	0.518
		0.8	0.998	0.759	0.381	1.000	0.942	0.632
		0.5	0.994	0.806	0.425	1.000	0.964	0.671
		0.2	0.994	0.828	0.436	1.000	0.965	0.681
$\omega = 0.7$	$\rho = 1$	0.0	0.994	0.835	0.441	1.000	0.965	0.681
		-0.2	0.991	0.839	0.445	1.000	0.966	0.683
		-0.5	0.988	0.841	0.446	1.000	0.967	0.684
		-0.8	0.968	0.839	0.444	0.999	0.966	0.681
		0.8	0.999	0.854	0.454	1.000	0.971	0.682
		0.5	0.999	0.853	0.456	1.000	0.972	0.684
		0.2	0.999	0.852	0.456	1.000	0.973	0.685
	$\rho = 100$	0.0	0.999	0.852	0.458	1.000	0.974	0.686
		-0.2	0.999	0.853	0.458	1.000	0.974	0.688
		-0.5	0.999	0.852	0.457	1.000	0.974	0.688
		-0.8	0.999	0.852	0.457	1.000	0.974	0.688
		R.W.	1.000	0.814	0.436	1.000	0.969	0.637
		0.8	0.976	0.360	0.133	0.988	0.485	0.149
		0.5	0.650	0.183	0.126	0.829	0.366	0.204
		0.2	0.376	0.190	0.162	0.680	0.468	0.338
	$\rho = 0.01$	0.0	0.271	0.209	0.186	0.610	0.530	0.373
		-0.2	0.177	0.231	0.204	0.549	0.586	0.414
		-0.5	0.071	0.251	0.220	0.403	0.635	0.463
		-0.8	0.007	0.162	0.213	0.161	0.556	0.474
		0.8	0.995	0.711	0.372	1.000	0.925	0.572
		0.5	0.990	0.740	0.406	1.000	0.953	0.618
		0.2	0.991	0.767	0.420	1.000	0.957	0.626
$\omega = 0.8$	$\rho = 1$	0.0	0.991	0.778	0.424	1.000	0.959	0.627
		-0.2	0.993	0.789	0.426	1.000	0.960	0.631
		-0.5	0.990	0.793	0.431	1.000	0.964	0.632
		-0.8	0.961	0.789	0.434	0.998	0.961	0.633
		0.8	0.999	0.803	0.431	1.000	0.969	0.634
		0.5	1.000	0.811	0.437	1.000	0.968	0.638
		0.2	1.000	0.811	0.439	1.000	0.968	0.638
	$\rho = 100$	0.0	1.000	0.811	0.437	1.000	0.968	0.638
		-0.2	1.000	0.811	0.437	1.000	0.968	0.638
		-0.5	1.000	0.812	0.437	1.000	0.968	0.638
		-0.8	1.000	0.812	0.438	1.000	0.968	0.638

Table 4c. The Power of the LM Test: the Case 1 (continued)

		a	$T=100$ $\ell 0$	$\ell 4$	$\ell 12$	$T=200$ $\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.999	0.815	0.454	1.000	0.953	0.648
		0.8	0.953	0.370	0.132	0.985	0.474	0.167
		0.5	0.645	0.200	0.113	0.806	0.384	0.239
		0.2	0.410	0.211	0.142	0.685	0.489	0.344
	$\rho=0.01$	0.0	0.305	0.239	0.177	0.638	0.554	0.399
		-0.2	0.226	0.277	0.198	0.593	0.617	0.443
		-0.5	0.100	0.302	0.238	0.467	0.655	0.491
		-0.8	0.020	0.209	0.233	0.227	0.606	0.504
		0.8	0.993	0.734	0.386	1.000	0.911	0.589
		0.5	0.995	0.769	0.429	1.000	0.935	0.623
		0.2	0.994	0.793	0.439	1.000	0.946	0.635
$\omega=0.9$	$\rho=1$	0.0	0.993	0.798	0.445	1.000	0.947	0.637
		-0.2	0.991	0.807	0.449	1.000	0.949	0.638
		-0.5	0.989	0.811	0.447	1.000	0.947	0.638
		-0.8	0.966	0.804	0.448	0.998	0.946	0.639
		0.8	0.999	0.818	0.452	1.000	0.955	0.643
		0.5	0.999	0.817	0.454	1.000	0.954	0.645
		0.2	0.999	0.818	0.454	1.000	0.954	0.646
	$\rho=100$	0.0	0.999	0.817	0.454	1.000	0.954	0.646
		-0.2	0.999	0.818	0.455	1.000	0.954	0.647
		-0.5	0.999	0.818	0.455	1.000	0.954	0.647
		-0.8	0.999	0.816	0.455	1.000	0.954	0.649

Table 4d. The Power of the LM Test: the Case 2

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.999	0.829	0.424	1.000	0.958	0.647
		0.8	0.960	0.404	0.117	0.992	0.470	0.172
		0.5	0.675	0.199	0.103	0.827	0.391	0.250
		0.2	0.426	0.212	0.138	0.706	0.526	0.373
	$\rho = 0.01$	0.0	0.327	0.240	0.165	0.661	0.590	0.419
		-0.2	0.229	0.271	0.195	0.615	0.647	0.464
		-0.5	0.113	0.306	0.217	0.496	0.689	0.502
		-0.8	0.013	0.211	0.198	0.255	0.627	0.512
		0.8	0.993	0.746	0.352	1.000	0.923	0.595
		0.5	0.996	0.778	0.399	1.000	0.945	0.641
		0.2	0.996	0.802	0.409	1.000	0.952	0.651
$\omega = 0.1$	$\rho = 1$	0.0	0.995	0.800	0.412	0.999	0.955	0.651
		-0.2	0.994	0.809	0.413	0.999	0.953	0.651
		-0.5	0.992	0.816	0.421	0.999	0.954	0.652
		-0.8	0.969	0.813	0.421	0.998	0.952	0.652
		0.8	0.999	0.827	0.425	1.000	0.957	0.648
		0.5	0.999	0.830	0.425	1.000	0.958	0.649
		0.2	0.999	0.832	0.425	1.000	0.958	0.649
	$\rho = 100$	0.0	0.999	0.830	0.425	1.000	0.959	0.648
		-0.2	0.999	0.830	0.426	1.000	0.958	0.648
		-0.5	0.999	0.829	0.425	1.000	0.958	0.647
		-0.8	0.999	0.828	0.425	1.000	0.958	0.648
		R.W.	0.997	0.815	0.435	1.000	0.956	0.654
		0.8	0.955	0.392	0.151	0.988	0.453	0.171
		0.5	0.643	0.190	0.123	0.808	0.363	0.229
		0.2	0.394	0.199	0.134	0.679	0.487	0.359
	$\rho = 0.01$	0.0	0.279	0.221	0.159	0.634	0.559	0.424
		-0.2	0.193	0.247	0.192	0.579	0.623	0.458
		-0.5	0.086	0.267	0.221	0.459	0.657	0.496
		-0.8	0.011	0.185	0.209	0.199	0.599	0.509
		0.8	0.992	0.727	0.372	1.000	0.915	0.604
		0.5	0.992	0.749	0.415	1.000	0.933	0.637
		0.2	0.993	0.776	0.425	1.000	0.948	0.646
$\omega = 0.2$	$\rho = 1$	0.0	0.993	0.787	0.431	1.000	0.953	0.651
		-0.2	0.990	0.787	0.432	1.000	0.954	0.651
		-0.5	0.980	0.793	0.434	1.000	0.954	0.649
		-0.8	0.954	0.790	0.432	0.998	0.954	0.650
		0.8	0.998	0.805	0.438	1.000	0.957	0.653
		0.5	0.998	0.813	0.435	1.000	0.957	0.651
		0.2	0.997	0.813	0.436	1.000	0.958	0.652
	$\rho = 100$	0.0	0.997	0.812	0.435	1.000	0.957	0.652
		-0.2	0.997	0.813	0.435	1.000	0.956	0.652
		-0.5	0.997	0.813	0.435	1.000	0.956	0.652
		-0.8	0.997	0.814	0.435	1.000	0.956	0.653

Table 4d. The Power of the LM Test: the Case 2 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.3$		R.W.	0.998	0.782	0.414	1.000	0.952	0.608
	$\rho = 0.01$	0.8	0.965	0.373	0.156	0.992	0.464	0.162
		0.5	0.652	0.172	0.114	0.821	0.342	0.205
		0.2	0.360	0.172	0.133	0.670	0.448	0.319
		0.0	0.238	0.193	0.156	0.613	0.526	0.388
		-0.2	0.166	0.212	0.182	0.554	0.589	0.432
		-0.5	0.064	0.232	0.210	0.417	0.633	0.483
		-0.8	0.009	0.153	0.203	0.144	0.560	0.494
	$\rho = 1$	0.8	0.993	0.700	0.348	1.000	0.907	0.576
		0.5	0.990	0.727	0.394	1.000	0.934	0.598
		0.2	0.989	0.761	0.406	1.000	0.941	0.605
		0.0	0.989	0.762	0.409	1.000	0.946	0.604
		-0.2	0.988	0.768	0.413	1.000	0.949	0.606
		-0.5	0.979	0.774	0.414	1.000	0.949	0.610
		-0.8	0.951	0.767	0.410	0.997	0.947	0.609
	$\rho = 100$	0.8	0.999	0.784	0.417	1.000	0.953	0.610
		0.5	0.998	0.785	0.416	1.000	0.953	0.608
		0.2	0.998	0.783	0.413	1.000	0.953	0.609
		0.0	0.998	0.782	0.414	1.000	0.953	0.609
		-0.2	0.998	0.782	0.414	1.000	0.952	0.609
		-0.5	0.998	0.782	0.414	1.000	0.952	0.607
		-0.8	0.998	0.782	0.414	1.000	0.952	0.608
$\omega = 0.4$		R.W.	1.000	0.777	0.395	1.000	0.960	0.585
	$\rho = 0.01$	0.8	0.971	0.382	0.141	0.995	0.491	0.134
		0.5	0.711	0.171	0.105	0.833	0.330	0.184
		0.2	0.347	0.151	0.121	0.659	0.423	0.290
		0.0	0.220	0.166	0.157	0.580	0.500	0.349
		-0.2	0.136	0.184	0.179	0.511	0.565	0.391
		-0.5	0.049	0.200	0.203	0.350	0.622	0.440
		-0.8	0.007	0.125	0.204	0.105	0.528	0.453
	$\rho = 1$	0.8	0.996	0.675	0.329	1.000	0.914	0.545
		0.5	0.994	0.713	0.369	1.000	0.940	0.580
		0.2	0.992	0.741	0.380	1.000	0.950	0.587
		0.0	0.991	0.753	0.380	1.000	0.956	0.585
		-0.2	0.990	0.759	0.384	1.000	0.957	0.587
		-0.5	0.984	0.762	0.385	1.000	0.960	0.588
		-0.8	0.946	0.755	0.384	0.999	0.959	0.587
	$\rho = 100$	0.8	0.999	0.778	0.383	1.000	0.963	0.588
		0.5	0.999	0.777	0.387	1.000	0.961	0.586
		0.2	1.000	0.778	0.391	1.000	0.961	0.585
		0.0	1.000	0.778	0.392	1.000	0.961	0.585
		-0.2	1.000	0.777	0.393	1.000	0.961	0.586
		-0.5	1.000	0.777	0.393	1.000	0.961	0.586
		-0.8	1.000	0.777	0.396	1.000	0.960	0.586

Table 4d. The Power of the LM Test: the Case 2 (continued)

			T = 100			T = 200		
		a	$\ell0$	$\ell4$	$\ell12$	$\ell0$	$\ell4$	$\ell12$
		R.W.	0.999	0.776	0.368	1.000	0.965	0.562
		0.8	0.985	0.383	0.133	0.995	0.507	0.126
		0.5	0.730	0.158	0.105	0.847	0.324	0.170
		0.2	0.351	0.150	0.131	0.640	0.391	0.256
	$\rho = 0.01$	0.0	0.209	0.159	0.150	0.564	0.475	0.296
		-0.2	0.125	0.174	0.174	0.486	0.539	0.343
		-0.5	0.046	0.193	0.195	0.327	0.579	0.392
		-0.8	0.002	0.121	0.192	0.097	0.482	0.393
		0.8	0.992	0.678	0.315	1.000	0.919	0.502
		0.5	0.992	0.719	0.343	1.000	0.947	0.536
		0.2	0.992	0.742	0.361	1.000	0.957	0.545
$\omega = 0.5$	$\rho = 1$	0.0	0.990	0.754	0.357	1.000	0.959	0.551
		-0.2	0.988	0.764	0.355	1.000	0.962	0.556
		-0.5	0.983	0.765	0.360	1.000	0.964	0.559
		-0.8	0.957	0.759	0.358	1.000	0.961	0.557
		0.8	0.999	0.779	0.362	1.000	0.963	0.557
		0.5	0.999	0.779	0.362	1.000	0.965	0.560
		0.2	0.999	0.777	0.364	1.000	0.965	0.562
	$\rho = 100$	0.0	0.999	0.776	0.364	1.000	0.964	0.561
		-0.2	0.999	0.777	0.363	1.000	0.964	0.561
		-0.5	0.999	0.777	0.363	1.000	0.965	0.561
		-0.8	0.999	0.776	0.365	1.000	0.964	0.561
		R.W.	0.999	0.771	0.355	1.000	0.955	0.568
		0.8	0.978	0.352	0.137	0.993	0.487	0.124
		0.5	0.696	0.160	0.109	0.830	0.321	0.169
		0.2	0.365	0.156	0.137	0.638	0.394	0.255
	$\rho = 0.01$	0.0	0.225	0.168	0.156	0.544	0.464	0.306
		-0.2	0.137	0.197	0.172	0.472	0.529	0.353
		-0.5	0.054	0.211	0.207	0.330	0.576	0.387
		-0.8	0.003	0.132	0.187	0.110	0.481	0.399
		0.8	0.998	0.672	0.331	1.000	0.917	0.501
		0.5	0.993	0.706	0.342	1.000	0.934	0.543
		0.2	0.995	0.731	0.359	1.000	0.945	0.555
$\omega = 0.6$	$\rho = 1$	0.0	0.996	0.741	0.359	1.000	0.949	0.557
		-0.2	0.993	0.747	0.360	1.000	0.950	0.557
		-0.5	0.984	0.757	0.358	1.000	0.951	0.564
		-0.8	0.944	0.747	0.356	1.000	0.948	0.562
		0.8	0.999	0.768	0.359	1.000	0.952	0.565
		0.5	0.999	0.770	0.358	1.000	0.953	0.567
		0.2	0.999	0.769	0.357	1.000	0.954	0.566
	$\rho = 100$	0.0	0.999	0.770	0.358	1.000	0.954	0.569
		-0.2	0.999	0.771	0.358	1.000	0.955	0.569
		-0.5	0.999	0.768	0.356	1.000	0.955	0.568
		-0.8	0.999	0.769	0.355	1.000	0.955	0.568

Table 4d. The Power of the LM Test: the Case 2 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	1.000	0.782	0.406	1.000	0.949	0.587
		0.8	0.965	0.353	0.144	0.985	0.454	0.148
		0.5	0.662	0.179	0.124	0.809	0.341	0.195
		0.2	0.362	0.179	0.147	0.631	0.426	0.289
	$\rho = 0.01$	0.0	0.258	0.205	0.171	0.556	0.482	0.345
		-0.2	0.166	0.235	0.190	0.496	0.534	0.382
		-0.5	0.071	0.241	0.209	0.379	0.595	0.420
		-0.8	0.005	0.151	0.204	0.145	0.513	0.432
		0.8	0.995	0.699	0.349	1.000	0.907	0.536
		0.5	0.990	0.729	0.385	1.000	0.927	0.569
		0.2	0.989	0.751	0.384	1.000	0.938	0.580
$\omega = 0.7$	$\rho = 1$	0.0	0.989	0.759	0.393	1.000	0.942	0.581
		-0.2	0.988	0.765	0.401	1.000	0.942	0.581
		-0.5	0.976	0.769	0.402	0.999	0.945	0.583
		-0.8	0.942	0.762	0.402	0.997	0.944	0.582
		0.8	1.000	0.780	0.404	1.000	0.950	0.587
		0.5	1.000	0.779	0.404	1.000	0.951	0.586
		0.2	1.000	0.780	0.404	1.000	0.949	0.588
	$\rho = 100$	0.0	1.000	0.781	0.405	1.000	0.949	0.587
		-0.2	1.000	0.782	0.407	1.000	0.950	0.587
		-0.5	1.000	0.782	0.407	1.000	0.950	0.588
		-0.8	1.000	0.782	0.407	1.000	0.950	0.588
		R.W.	1.000	0.806	0.428	1.000	0.942	0.636
		0.8	0.960	0.363	0.134	0.983	0.460	0.163
		0.5	0.634	0.191	0.119	0.797	0.371	0.220
		0.2	0.390	0.204	0.149	0.642	0.461	0.335
	$\rho = 0.01$	0.0	0.280	0.230	0.172	0.589	0.511	0.378
		-0.2	0.197	0.242	0.192	0.538	0.571	0.407
		-0.5	0.085	0.265	0.221	0.420	0.629	0.448
		-0.8	0.013	0.190	0.211	0.194	0.562	0.467
		0.8	0.993	0.723	0.366	1.000	0.907	0.559
		0.5	0.990	0.747	0.405	1.000	0.931	0.608
		0.2	0.990	0.769	0.417	1.000	0.935	0.626
$\omega = 0.8$	$\rho = 1$	0.0	0.992	0.778	0.419	1.000	0.935	0.629
		-0.2	0.990	0.781	0.424	1.000	0.936	0.633
		-0.5	0.984	0.785	0.424	1.000	0.938	0.633
		-0.8	0.960	0.783	0.422	0.997	0.937	0.635
		0.8	0.999	0.803	0.425	1.000	0.940	0.638
		0.5	0.999	0.808	0.427	1.000	0.942	0.639
		0.2	1.000	0.809	0.427	1.000	0.941	0.639
	$\rho = 100$	0.0	1.000	0.807	0.426	1.000	0.941	0.638
		-0.2	1.000	0.807	0.425	1.000	0.941	0.638
		-0.5	1.000	0.808	0.425	1.000	0.942	0.638
		-0.8	0.999	0.808	0.425	1.000	0.942	0.638

Table 4d. The Power of the LM Test: the Case 2 (continued)

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.998	0.816	0.425	1.000	0.957	0.639
		0.8	0.957	0.379	0.128	0.981	0.470	0.168
		0.5	0.640	0.202	0.099	0.811	0.393	0.244
		0.2	0.428	0.207	0.130	0.697	0.497	0.351
	$\rho = 0.01$	0.0	0.318	0.245	0.163	0.655	0.577	0.411
		-0.2	0.229	0.277	0.202	0.607	0.632	0.454
		-0.5	0.108	0.300	0.230	0.486	0.680	0.494
		-0.8	0.023	0.220	0.220	0.234	0.621	0.508
		0.8	0.991	0.724	0.368	1.000	0.917	0.574
		0.5	0.998	0.766	0.412	0.999	0.946	0.611
		0.2	0.995	0.793	0.424	0.999	0.953	0.624
$\omega = 0.9$	$\rho = 1$	0.0	0.995	0.801	0.424	1.000	0.954	0.627
		-0.2	0.993	0.803	0.428	1.000	0.956	0.627
		-0.5	0.991	0.808	0.429	0.999	0.958	0.631
		-0.8	0.972	0.799	0.428	0.999	0.957	0.634
		0.8	0.998	0.818	0.427	1.000	0.961	0.637
		0.5	0.998	0.820	0.428	1.000	0.960	0.638
		0.2	0.998	0.817	0.427	1.000	0.959	0.639
	$\rho = 100$	0.0	0.998	0.817	0.427	1.000	0.959	0.640
		-0.2	0.998	0.817	0.426	1.000	0.959	0.639
		-0.5	0.998	0.817	0.426	1.000	0.959	0.639
		-0.8	0.998	0.817	0.426	1.000	0.958	0.640

Table 4e. The Power of the LM Test: the Case 3

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.1$		R.W.	0.998	0.843	0.452	1.000	0.961	0.662
	$\rho = 0.01$	0.8	0.959	0.418	0.147	0.991	0.460	0.191
		0.5	0.669	0.214	0.125	0.806	0.395	0.266
		0.2	0.424	0.222	0.151	0.705	0.525	0.393
		0.0	0.314	0.245	0.177	0.655	0.587	0.447
		-0.2	0.223	0.274	0.206	0.598	0.639	0.476
		-0.5	0.101	0.298	0.238	0.480	0.689	0.520
		-0.8	0.014	0.203	0.216	0.236	0.624	0.521
	$\rho = 1$	0.8	0.994	0.760	0.365	1.000	0.918	0.627
		0.5	0.994	0.786	0.418	1.000	0.942	0.653
		0.2	0.993	0.810	0.434	1.000	0.949	0.659
		0.0	0.992	0.821	0.437	1.000	0.954	0.658
		-0.2	0.990	0.824	0.438	1.000	0.954	0.661
		-0.5	0.987	0.828	0.442	0.999	0.954	0.664
		-0.8	0.967	0.827	0.445	0.999	0.954	0.664
	$\rho = 100$	0.8	0.998	0.835	0.443	1.000	0.959	0.664
		0.5	0.998	0.841	0.450	1.000	0.961	0.665
		0.2	0.998	0.842	0.449	1.000	0.961	0.664
		0.0	0.998	0.843	0.450	1.000	0.961	0.662
		-0.2	0.998	0.843	0.450	1.000	0.961	0.661
		-0.5	0.998	0.842	0.451	1.000	0.961	0.661
		-0.8	0.998	0.842	0.452	1.000	0.961	0.662
$\omega = 0.2$		R.W.	0.997	0.820	0.485	1.000	0.951	0.688
	$\rho = 0.01$	0.8	0.951	0.414	0.208	0.990	0.481	0.181
		0.5	0.633	0.194	0.167	0.799	0.355	0.244
		0.2	0.356	0.201	0.197	0.657	0.466	0.365
		0.0	0.243	0.209	0.219	0.603	0.543	0.441
		-0.2	0.155	0.241	0.250	0.549	0.605	0.480
		-0.5	0.072	0.253	0.272	0.427	0.642	0.523
		-0.8	0.010	0.166	0.256	0.152	0.575	0.531
	$\rho = 1$	0.8	0.990	0.739	0.420	1.000	0.909	0.618
		0.5	0.989	0.757	0.467	0.999	0.940	0.652
		0.2	0.985	0.785	0.477	0.999	0.948	0.665
		0.0	0.986	0.794	0.481	0.999	0.950	0.670
		-0.2	0.982	0.807	0.480	0.999	0.951	0.670
		-0.5	0.979	0.815	0.483	0.999	0.952	0.674
		-0.8	0.949	0.805	0.478	0.997	0.948	0.674
	$\rho = 100$	0.8	0.997	0.819	0.485	1.000	0.954	0.678
		0.5	0.997	0.820	0.487	1.000	0.951	0.680
		0.2	0.997	0.820	0.487	1.000	0.951	0.685
		0.0	0.997	0.820	0.485	1.000	0.952	0.685
		-0.2	0.997	0.821	0.485	1.000	0.952	0.685
		-0.5	0.997	0.821	0.484	1.000	0.952	0.688
		-0.8	0.997	0.822	0.484	1.000	0.952	0.687

Table 4e. The Power of the LM Test: the Case 3 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.3$		R.W.	0.998	0.770	0.494	1.000	0.951	0.623
	$\rho = 0.01$	0.8	0.977	0.413	0.247	0.995	0.461	0.175
		0.5	0.681	0.179	0.210	0.809	0.327	0.217
		0.2	0.334	0.167	0.221	0.622	0.399	0.324
		0.0	0.209	0.175	0.241	0.528	0.469	0.387
		-0.2	0.117	0.195	0.263	0.459	0.526	0.423
		-0.5	0.042	0.207	0.290	0.321	0.571	0.461
		-0.8	0.006	0.129	0.293	0.087	0.487	0.483
	$\rho = 1$	0.8	0.997	0.693	0.431	1.000	0.897	0.573
		0.5	0.995	0.718	0.472	1.000	0.928	0.613
		0.2	0.990	0.735	0.488	1.000	0.943	0.613
		0.0	0.989	0.750	0.492	1.000	0.945	0.617
		-0.2	0.984	0.757	0.494	1.000	0.945	0.620
		-0.5	0.975	0.769	0.494	1.000	0.947	0.620
		-0.8	0.922	0.760	0.492	0.997	0.946	0.620
	$\rho = 100$	0.8	0.998	0.780	0.491	1.000	0.951	0.622
		0.5	0.997	0.777	0.495	1.000	0.951	0.621
		0.2	0.997	0.774	0.497	1.000	0.951	0.622
		0.0	0.997	0.773	0.495	1.000	0.951	0.623
		-0.2	0.997	0.771	0.494	1.000	0.950	0.623
		-0.5	0.997	0.770	0.493	1.000	0.950	0.622
		-0.8	0.997	0.770	0.495	1.000	0.950	0.623
$\omega = 0.4$		R.W.	0.999	0.777	0.463	1.000	0.961	0.606
	$\rho = 0.01$	0.8	0.988	0.414	0.268	0.998	0.508	0.155
		0.5	0.747	0.172	0.263	0.843	0.307	0.182
		0.2	0.351	0.133	0.294	0.627	0.357	0.257
		0.0	0.180	0.153	0.314	0.498	0.411	0.301
		-0.2	0.090	0.165	0.329	0.401	0.472	0.340
		-0.5	0.025	0.181	0.361	0.222	0.532	0.379
		-0.8	0.000	0.101	0.373	0.039	0.410	0.398
	$\rho = 1$	0.8	0.998	0.662	0.407	1.000	0.921	0.522
		0.5	0.993	0.686	0.437	1.000	0.938	0.570
		0.2	0.988	0.718	0.449	1.000	0.944	0.586
		0.0	0.987	0.737	0.452	1.000	0.948	0.589
		-0.2	0.985	0.745	0.455	1.000	0.949	0.590
		-0.5	0.979	0.750	0.455	1.000	0.951	0.597
		-0.8	0.914	0.735	0.456	1.000	0.948	0.602
	$\rho = 100$	0.8	0.999	0.781	0.464	1.000	0.961	0.597
		0.5	0.999	0.777	0.462	1.000	0.961	0.603
		0.2	0.999	0.778	0.460	1.000	0.961	0.605
		0.0	0.999	0.777	0.460	1.000	0.960	0.607
		-0.2	0.999	0.776	0.459	1.000	0.959	0.607
		-0.5	0.999	0.776	0.459	1.000	0.959	0.607
		-0.8	0.999	0.777	0.459	1.000	0.959	0.607

Table 4e. The Power of the LM Test: the Case 3 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.998	0.790	0.454	1.000	0.980	0.629
		0.8	0.994	0.411	0.236	0.998	0.564	0.150
		0.5	0.758	0.152	0.300	0.874	0.284	0.163
		0.2	0.351	0.120	0.345	0.635	0.327	0.222
	$\rho = 0.01$	0.0	0.171	0.123	0.374	0.488	0.389	0.281
		-0.2	0.069	0.139	0.390	0.358	0.453	0.328
		-0.5	0.006	0.146	0.425	0.173	0.517	0.379
		-0.8	0.000	0.062	0.473	0.026	0.368	0.394
		0.8	1.000	0.688	0.397	1.000	0.948	0.538
		0.5	0.993	0.709	0.435	1.000	0.966	0.591
		0.2	0.989	0.739	0.449	1.000	0.969	0.609
$\omega = 0.5$	$\rho = 1$	0.0	0.990	0.755	0.454	1.000	0.972	0.612
		-0.2	0.988	0.771	0.460	1.000	0.978	0.617
		-0.5	0.977	0.774	0.460	1.000	0.977	0.617
		-0.8	0.926	0.761	0.461	0.999	0.974	0.618
		0.8	0.999	0.792	0.454	1.000	0.980	0.628
		0.5	0.999	0.791	0.453	1.000	0.981	0.630
		0.2	0.999	0.791	0.455	1.000	0.981	0.632
	$\rho = 100$	0.0	0.999	0.790	0.455	1.000	0.981	0.632
		-0.2	0.999	0.789	0.455	1.000	0.981	0.631
		-0.5	0.999	0.789	0.455	1.000	0.981	0.628
		-0.8	0.998	0.790	0.457	1.000	0.981	0.628
		R.W.	1.000	0.761	0.470	1.000	0.957	0.576
		0.8	0.983	0.396	0.246	0.998	0.515	0.166
		0.5	0.754	0.165	0.264	0.855	0.296	0.187
		0.2	0.363	0.152	0.299	0.621	0.344	0.262
	$\rho = 0.01$	0.0	0.201	0.168	0.318	0.492	0.399	0.309
		-0.2	0.098	0.182	0.333	0.381	0.463	0.356
		-0.5	0.018	0.184	0.369	0.229	0.516	0.401
		-0.8	0.000	0.098	0.374	0.042	0.417	0.407
		0.8	0.998	0.679	0.434	1.000	0.933	0.519
		0.5	0.990	0.707	0.458	1.000	0.944	0.547
		0.2	0.990	0.724	0.462	1.000	0.952	0.558
$\omega = 0.6$	$\rho = 1$	0.0	0.990	0.731	0.465	1.000	0.955	0.560
		-0.2	0.987	0.733	0.468	1.000	0.957	0.562
		-0.5	0.981	0.744	0.466	1.000	0.958	0.564
		-0.8	0.923	0.728	0.468	1.000	0.952	0.564
		0.8	1.000	0.769	0.470	1.000	0.958	0.570
		0.5	1.000	0.764	0.472	1.000	0.959	0.575
		0.2	1.000	0.764	0.468	1.000	0.958	0.574
	$\rho = 100$	0.0	1.000	0.764	0.470	1.000	0.957	0.574
		-0.2	1.000	0.764	0.470	1.000	0.957	0.573
		-0.5	1.000	0.762	0.469	1.000	0.957	0.574
		-0.8	1.000	0.762	0.471	1.000	0.957	0.574

Table 4e. The Power of the LM Test: the Case 3 (continued)

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.997	0.780	0.504	1.000	0.954	0.628
		0.8	0.965	0.395	0.231	0.996	0.470	0.177
		0.5	0.666	0.178	0.207	0.813	0.312	0.216
		0.2	0.350	0.177	0.246	0.618	0.395	0.306
	$\rho=0.01$	0.0	0.228	0.191	0.258	0.533	0.461	0.351
		-0.2	0.131	0.207	0.287	0.444	0.510	0.400
		-0.5	0.039	0.233	0.318	0.304	0.554	0.446
		-0.8	0.003	0.142	0.295	0.105	0.459	0.456
		0.8	0.991	0.697	0.459	1.000	0.893	0.561
		0.5	0.985	0.722	0.482	1.000	0.919	0.605
		0.2	0.981	0.750	0.497	1.000	0.935	0.612
$\omega=0.7$	$\rho=1$	0.0	0.982	0.759	0.498	1.000	0.939	0.621
		-0.2	0.979	0.767	0.497	1.000	0.943	0.621
		-0.5	0.968	0.768	0.500	1.000	0.943	0.622
		-0.8	0.909	0.767	0.494	0.998	0.943	0.623
		0.8	0.998	0.778	0.506	1.000	0.949	0.627
		0.5	0.997	0.778	0.507	1.000	0.954	0.630
		0.2	0.997	0.777	0.507	1.000	0.953	0.628
	$\rho=100$	0.0	0.997	0.778	0.505	1.000	0.953	0.628
		-0.2	0.997	0.779	0.504	1.000	0.953	0.628
		-0.5	0.997	0.779	0.504	1.000	0.953	0.628
		-0.8	0.997	0.779	0.504	1.000	0.953	0.628
		R.W.	1.000	0.819	0.487	1.000	0.950	0.675
		0.8	0.962	0.396	0.212	0.987	0.478	0.188
		0.5	0.624	0.209	0.193	0.785	0.379	0.250
		0.2	0.369	0.213	0.215	0.639	0.466	0.372
	$\rho=0.01$	0.0	0.258	0.227	0.242	0.576	0.527	0.417
		-0.2	0.179	0.260	0.256	0.535	0.585	0.453
		-0.5	0.077	0.278	0.286	0.394	0.624	0.489
		-0.8	0.010	0.184	0.264	0.171	0.554	0.496
		0.8	0.989	0.734	0.426	1.000	0.900	0.622
		0.5	0.983	0.762	0.465	1.000	0.930	0.653
		0.2	0.984	0.785	0.481	1.000	0.938	0.660
$\omega=0.8$	$\rho=1$	0.0	0.986	0.793	0.488	1.000	0.944	0.666
		-0.2	0.986	0.801	0.488	1.000	0.946	0.669
		-0.5	0.978	0.806	0.490	0.999	0.944	0.670
		-0.8	0.938	0.795	0.490	0.996	0.944	0.670
		0.8	0.998	0.822	0.489	1.000	0.948	0.675
		0.5	1.000	0.822	0.493	1.000	0.950	0.674
		0.2	1.000	0.822	0.493	1.000	0.950	0.673
	$\rho=100$	0.0	1.000	0.821	0.492	1.000	0.949	0.673
		-0.2	1.000	0.822	0.491	1.000	0.949	0.673
		-0.5	1.000	0.820	0.489	1.000	0.949	0.673
		-0.8	1.000	0.820	0.489	1.000	0.949	0.674

Table 4e. The Power of the LM Test: the Case 3 (continued)

			$T = 100$			$T = 200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	1.000	0.825	0.465	1.000	0.958	0.656
		0.8	0.959	0.397	0.148	0.983	0.490	0.186
		0.5	0.654	0.216	0.122	0.808	0.399	0.253
		0.2	0.415	0.225	0.158	0.686	0.502	0.367
	$\rho = 0.01$	0.0	0.311	0.260	0.192	0.643	0.565	0.416
		-0.2	0.226	0.294	0.224	0.594	0.620	0.463
		-0.5	0.100	0.316	0.249	0.469	0.665	0.504
		-0.8	0.020	0.216	0.245	0.230	0.612	0.517
		0.8	0.993	0.750	0.395	1.000	0.915	0.605
		0.5	0.995	0.781	0.437	1.000	0.944	0.635
		0.2	0.994	0.805	0.445	1.000	0.952	0.650
$\omega = 0.9$	$\rho = 1$	0.0	0.993	0.812	0.452	1.000	0.954	0.652
		-0.2	0.992	0.821	0.454	1.000	0.956	0.655
		-0.5	0.989	0.823	0.453	1.000	0.956	0.655
		-0.8	0.970	0.812	0.455	0.998	0.953	0.656
		0.8	0.999	0.828	0.462	1.000	0.961	0.656
		0.5	0.999	0.827	0.461	1.000	0.957	0.657
		0.2	0.999	0.826	0.466	1.000	0.957	0.657
	$\rho = 100$	0.0	0.999	0.826	0.465	1.000	0.957	0.657
		-0.2	0.999	0.827	0.464	1.000	0.957	0.657
		-0.5	0.999	0.826	0.464	1.000	0.957	0.657
		-0.8	0.999	0.826	0.463	1.000	0.958	0.657

Table 4f. The Power of the PS Test: the Case 3

		a	T = 100 ℓ0	T = 100 ℓ4	T = 100 ℓ12	T = 200 ℓ0	T = 200 ℓ4	T = 200 ℓ12
		R.W.	0.994	0.514	0.000	1.000	0.849	0.306
		0.8	0.913	0.119	0.006	0.962	0.264	0.036
		0.5	0.555	0.086	0.047	0.810	0.265	0.114
		0.2	0.331	0.125	0.106	0.663	0.398	0.183
	$\rho = 0.01$	0.0	0.219	0.162	0.152	0.587	0.465	0.218
		-0.2	0.139	0.221	0.193	0.495	0.536	0.249
		-0.5	0.048	0.300	0.280	0.331	0.589	0.281
		-0.8	0.005	0.274	0.430	0.125	0.510	0.325
		0.8	0.980	0.428	0.002	1.000	0.802	0.267
		0.5	0.979	0.470	0.002	1.000	0.831	0.296
		0.2	0.980	0.497	0.001	1.000	0.846	0.304
$\omega = 0.1$	$\rho = 1$	0.0	0.979	0.499	0.000	1.000	0.849	0.303
		-0.2	0.975	0.505	0.000	1.000	0.849	0.305
		-0.5	0.964	0.510	0.000	0.999	0.851	0.305
		-0.8	0.916	0.507	0.000	0.995	0.848	0.304
		0.8	0.995	0.521	0.000	1.000	0.850	0.303
		0.5	0.995	0.517	0.000	1.000	0.848	0.304
		0.2	0.995	0.516	0.000	1.000	0.849	0.304
	$\rho = 100$	0.0	0.994	0.515	0.000	1.000	0.848	0.304
		-0.2	0.994	0.515	0.000	1.000	0.848	0.304
		-0.5	0.993	0.515	0.000	1.000	0.849	0.304
		-0.8	0.993	0.515	0.000	1.000	0.849	0.305
		R.W.	1.000	0.608	0.040	1.000	0.907	0.389
		0.8	0.963	0.224	0.067	0.994	0.397	0.090
		0.5	0.686	0.137	0.183	0.859	0.279	0.150
		0.2	0.362	0.149	0.232	0.668	0.366	0.219
	$\rho = 0.01$	0.0	0.207	0.166	0.256	0.556	0.448	0.247
		-0.2	0.105	0.186	0.263	0.464	0.506	0.275
		-0.5	0.028	0.207	0.295	0.286	0.555	0.296
		-0.8	0.003	0.129	0.385	0.076	0.449	0.319
		0.8	0.992	0.514	0.061	1.000	0.863	0.349
		0.5	0.992	0.544	0.051	1.000	0.886	0.375
		0.2	0.990	0.578	0.041	1.000	0.900	0.383
$\omega = 0.2$	$\rho = 1$	0.0	0.987	0.585	0.045	1.000	0.901	0.383
		-0.2	0.984	0.590	0.043	1.000	0.900	0.385
		-0.5	0.975	0.595	0.042	1.000	0.901	0.384
		-0.8	0.916	0.589	0.043	0.997	0.899	0.384
		0.8	1.000	0.607	0.045	1.000	0.905	0.383
		0.5	1.000	0.604	0.041	1.000	0.905	0.387
		0.2	1.000	0.605	0.041	1.000	0.905	0.389
	$\rho = 100$	0.0	1.000	0.605	0.040	1.000	0.905	0.389
		-0.2	1.000	0.606	0.041	1.000	0.906	0.389
		-0.5	1.000	0.605	0.040	1.000	0.907	0.389
		-0.8	1.000	0.608	0.040	1.000	0.907	0.389

Table 4f. The Power of the PS Test: the Case 3 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
$\omega = 0.3$		R.W.	0.999	0.700	0.205	1.000	0.943	0.483
	$\rho = 0.01$	0.8	0.982	0.359	0.188	0.996	0.486	0.124
		0.5	0.737	0.154	0.276	0.869	0.286	0.155
		0.2	0.373	0.134	0.317	0.651	0.356	0.236
		0.0	0.192	0.139	0.328	0.526	0.421	0.279
		-0.2	0.079	0.153	0.326	0.411	0.485	0.306
		-0.5	0.022	0.153	0.357	0.228	0.526	0.348
		-0.8	0.002	0.074	0.420	0.042	0.417	0.360
	$\rho = 1$	0.8	0.998	0.606	0.183	1.000	0.902	0.442
		0.5	0.995	0.632	0.205	1.000	0.924	0.471
		0.2	0.994	0.675	0.203	1.000	0.932	0.478
		0.0	0.992	0.684	0.198	1.000	0.933	0.480
		-0.2	0.990	0.686	0.200	1.000	0.935	0.480
		-0.5	0.982	0.687	0.203	1.000	0.935	0.483
		-0.8	0.922	0.675	0.208	0.999	0.934	0.486
	$\rho = 100$	0.8	0.999	0.698	0.203	1.000	0.941	0.483
		0.5	0.999	0.702	0.204	1.000	0.941	0.484
		0.2	0.999	0.701	0.202	1.000	0.943	0.484
		0.0	0.999	0.700	0.203	1.000	0.942	0.486
		-0.2	0.999	0.701	0.204	1.000	0.942	0.486
		-0.5	0.999	0.701	0.206	1.000	0.942	0.485
		-0.8	0.999	0.701	0.208	1.000	0.942	0.484
$\omega = 0.4$		R.W.	0.999	0.762	0.375	1.000	0.956	0.595
	$\rho = 0.01$	0.8	0.992	0.419	0.233	0.999	0.523	0.145
		0.5	0.772	0.146	0.297	0.870	0.302	0.163
		0.2	0.362	0.127	0.341	0.634	0.358	0.235
		0.0	0.173	0.139	0.359	0.502	0.418	0.283
		-0.2	0.082	0.159	0.376	0.397	0.477	0.331
		-0.5	0.011	0.167	0.404	0.187	0.525	0.365
		-0.8	0.000	0.069	0.446	0.025	0.399	0.371
	$\rho = 1$	0.8	0.998	0.661	0.335	1.000	0.927	0.530
		0.5	0.994	0.666	0.355	1.000	0.943	0.576
		0.2	0.995	0.710	0.362	1.000	0.955	0.590
		0.0	0.994	0.721	0.366	1.000	0.956	0.591
		-0.2	0.991	0.730	0.371	1.000	0.957	0.597
		-0.5	0.978	0.738	0.377	1.000	0.957	0.599
		-0.8	0.929	0.718	0.380	1.000	0.956	0.597
	$\rho = 100$	0.8	0.999	0.757	0.371	1.000	0.961	0.599
		0.5	0.999	0.759	0.368	1.000	0.959	0.598
		0.2	0.999	0.763	0.367	1.000	0.958	0.598
		0.0	0.999	0.764	0.369	1.000	0.957	0.597
		-0.2	0.999	0.765	0.371	1.000	0.958	0.597
		-0.5	0.999	0.765	0.373	1.000	0.956	0.597
		-0.8	0.999	0.763	0.374	1.000	0.956	0.597

Table 4f. The Power of the PS Test: the Case 3 (continued)

			T = 100			T = 200		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.998	0.790	0.454	1.000	0.980	0.629
		0.8	0.994	0.411	0.236	0.998	0.564	0.150
		0.5	0.758	0.152	0.300	0.874	0.284	0.163
		0.2	0.351	0.120	0.345	0.635	0.327	0.222
	$\rho = 0.01$	0.0	0.171	0.123	0.374	0.488	0.389	0.281
		-0.2	0.069	0.139	0.390	0.358	0.453	0.328
		-0.5	0.006	0.146	0.425	0.173	0.517	0.379
		-0.8	0.000	0.062	0.473	0.026	0.368	0.394
		0.8	1.000	0.688	0.397	1.000	0.948	0.538
		0.5	0.993	0.709	0.435	1.000	0.966	0.591
		0.2	0.989	0.739	0.449	1.000	0.969	0.609
$\omega = 0.5$	$\rho = 1$	0.0	0.990	0.755	0.454	1.000	0.972	0.612
		-0.2	0.988	0.771	0.460	1.000	0.978	0.617
		-0.5	0.977	0.774	0.460	1.000	0.977	0.617
		-0.8	0.926	0.761	0.461	0.999	0.974	0.618
		0.8	0.999	0.792	0.454	1.000	0.980	0.628
		0.5	0.999	0.791	0.453	1.000	0.981	0.630
		0.2	0.999	0.791	0.455	1.000	0.981	0.632
	$\rho = 100$	0.0	0.999	0.790	0.455	1.000	0.981	0.632
		-0.2	0.999	0.789	0.455	1.000	0.981	0.631
		-0.5	0.999	0.789	0.455	1.000	0.981	0.628
		-0.8	0.998	0.790	0.457	1.000	0.981	0.628
		R.W.	1.000	0.769	0.376	1.000	0.967	0.575
		0.8	0.988	0.435	0.268	0.998	0.553	0.159
		0.5	0.785	0.184	0.327	0.885	0.283	0.172
		0.2	0.384	0.141	0.371	0.643	0.335	0.231
	$\rho = 0.01$	0.0	0.194	0.150	0.389	0.494	0.407	0.273
		-0.2	0.078	0.165	0.402	0.377	0.475	0.307
		-0.5	0.006	0.165	0.431	0.190	0.513	0.351
		-0.8	0.000	0.069	0.486	0.025	0.386	0.358
		0.8	0.998	0.687	0.358	1.000	0.941	0.501
		0.5	0.992	0.704	0.380	1.000	0.948	0.543
		0.2	0.990	0.737	0.385	1.000	0.959	0.556
$\omega = 0.6$	$\rho = 1$	0.0	0.989	0.745	0.381	1.000	0.961	0.564
		-0.2	0.987	0.753	0.376	1.000	0.962	0.563
		-0.5	0.981	0.753	0.376	1.000	0.963	0.567
		-0.8	0.930	0.741	0.379	1.000	0.962	0.567
		0.8	1.000	0.774	0.368	1.000	0.967	0.569
		0.5	1.000	0.770	0.371	1.000	0.966	0.574
		0.2	1.000	0.770	0.370	1.000	0.965	0.575
	$\rho = 100$	0.0	1.000	0.769	0.369	1.000	0.965	0.575
		-0.2	1.000	0.769	0.371	1.000	0.965	0.575
		-0.5	1.000	0.769	0.372	1.000	0.965	0.575
		-0.8	1.000	0.770	0.374	1.000	0.966	0.574

Table 4f. The Power of the PS Test: the Case 3 (continued)

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.999	0.708	0.206	1.000	0.954	0.468
		0.8	0.977	0.340	0.194	0.996	0.504	0.158
		0.5	0.745	0.158	0.297	0.865	0.311	0.168
		0.2	0.379	0.146	0.323	0.646	0.361	0.234
	$\rho=0.01$	0.0	0.188	0.157	0.342	0.524	0.414	0.265
		-0.2	0.089	0.175	0.359	0.401	0.472	0.294
		-0.5	0.021	0.183	0.391	0.229	0.514	0.326
		-0.8	0.001	0.093	0.462	0.048	0.400	0.343
		0.8	0.992	0.620	0.209	1.000	0.906	0.417
		0.5	0.991	0.634	0.210	1.000	0.930	0.457
		0.2	0.990	0.669	0.210	1.000	0.942	0.461
$\omega=0.7$	$\rho=1$	0.0	0.991	0.677	0.208	1.000	0.942	0.465
		-0.2	0.987	0.681	0.205	1.000	0.946	0.466
		-0.5	0.974	0.686	0.204	1.000	0.948	0.468
		-0.8	0.923	0.671	0.204	0.996	0.945	0.469
		0.8	0.999	0.711	0.202	1.000	0.953	0.465
		0.5	0.999	0.707	0.201	1.000	0.953	0.470
		0.2	0.999	0.706	0.200	1.000	0.954	0.470
	$\rho=100$	0.0	0.999	0.705	0.200	1.000	0.955	0.471
		-0.2	0.999	0.706	0.200	1.000	0.955	0.472
		-0.5	0.999	0.706	0.200	1.000	0.955	0.471
		-0.8	0.999	0.705	0.203	1.000	0.955	0.470
		R.W.	0.998	0.606	0.050	1.000	0.899	0.372
		0.8	0.958	0.246	0.088	0.994	0.425	0.109
		0.5	0.683	0.155	0.185	0.835	0.313	0.156
		0.2	0.378	0.152	0.231	0.645	0.387	0.212
	$\rho=0.01$	0.0	0.211	0.172	0.251	0.552	0.440	0.249
		-0.2	0.120	0.202	0.269	0.447	0.491	0.272
		-0.5	0.033	0.224	0.331	0.285	0.536	0.298
		-0.8	0.002	0.160	0.473	0.070	0.442	0.320
		0.8	0.992	0.519	0.064	1.000	0.856	0.318
		0.5	0.989	0.553	0.068	1.000	0.874	0.350
		0.2	0.984	0.576	0.058	1.000	0.890	0.358
$\omega=0.8$	$\rho=1$	0.0	0.980	0.581	0.056	1.000	0.896	0.363
		-0.2	0.977	0.587	0.054	1.000	0.899	0.364
		-0.5	0.969	0.595	0.054	1.000	0.898	0.367
		-0.8	0.924	0.584	0.055	0.997	0.897	0.370
		0.8	0.998	0.600	0.047	1.000	0.905	0.371
		0.5	0.998	0.604	0.051	1.000	0.903	0.372
		0.2	0.998	0.603	0.051	1.000	0.902	0.372
	$\rho=100$	0.0	0.998	0.605	0.052	1.000	0.901	0.371
		-0.2	0.998	0.606	0.051	1.000	0.900	0.372
		-0.5	0.998	0.606	0.050	1.000	0.900	0.372
		-0.8	0.998	0.606	0.051	1.000	0.901	0.372

Table 4f. The Power of the PS Test: the Case 3 (continued)

			$T=100$			$T=200$		
		a	$\ell 0$	$\ell 4$	$\ell 12$	$\ell 0$	$\ell 4$	$\ell 12$
		R.W.	0.995	0.530	0.000	1.000	0.854	0.307
		0.8	0.886	0.117	0.002	0.965	0.286	0.045
		0.5	0.576	0.081	0.044	0.797	0.291	0.122
		0.2	0.352	0.138	0.119	0.642	0.391	0.187
	$\rho=0.01$	0.0	0.243	0.181	0.156	0.570	0.452	0.213
		-0.2	0.151	0.235	0.205	0.488	0.497	0.238
		-0.5	0.059	0.340	0.313	0.336	0.564	0.269
		-0.8	0.004	0.420	0.530	0.113	0.516	0.341
		0.8	0.979	0.441	0.003	1.000	0.784	0.251
		0.5	0.986	0.485	0.003	0.999	0.818	0.292
		0.2	0.985	0.508	0.003	0.999	0.836	0.297
$\omega=0.9$	$\rho=1$	0.0	0.985	0.515	0.002	0.999	0.838	0.298
		-0.2	0.982	0.517	0.002	0.999	0.844	0.302
		-0.5	0.977	0.530	0.000	0.998	0.849	0.303
		-0.8	0.920	0.528	0.000	0.994	0.849	0.304
		0.8	0.995	0.525	0.001	1.000	0.856	0.303
		0.5	0.995	0.524	0.000	1.000	0.855	0.302
		0.2	0.995	0.528	0.000	1.000	0.855	0.303
	$\rho=100$	0.0	0.995	0.528	0.000	1.000	0.854	0.305
		-0.2	0.995	0.528	0.000	1.000	0.854	0.305
		-0.5	0.995	0.529	0.000	1.000	0.854	0.305
		-0.8	0.995	0.529	0.000	1.000	0.854	0.305

Table 5. Test for Stationarity

Series	T	T_B	ω	$\ell 4$	$\ell 12$
Real GNP	62	1929	0.339	0.11842**	0.09202*
Nominal GNP	62	1929	0.339	0.10164*	0.07659
Real per capita GNP	62	1940	0.516	0.18685**	0.15879**
Industrial Production	111	1941	0.739	0.22738***	0.13382**
Unemployment	81	1929	0.494	0.07498	0.06684
Nominal Wages	71	1930	0.437	0.12002*	0.09013
Common-stock Prices	100	1939	0.690	0.03514	0.05237
Common-stock Prices (the PS test)				0.11146	0.16611

Figure 1a. The Limiting Powers (Case 0)

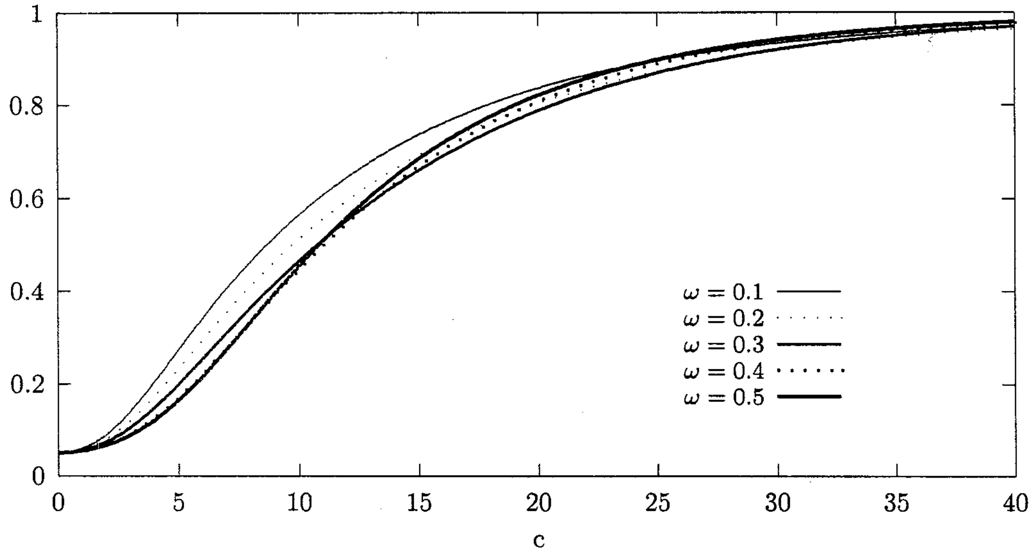


Figure 1b. The Limiting Powers (Case 1)

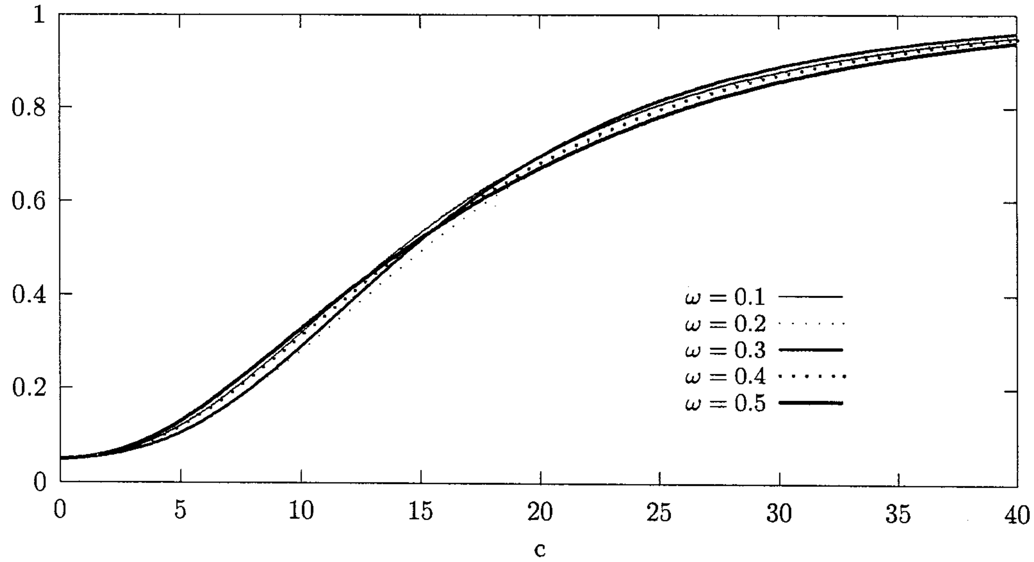


Figure 1c. The Limiting Powers (Case 2)

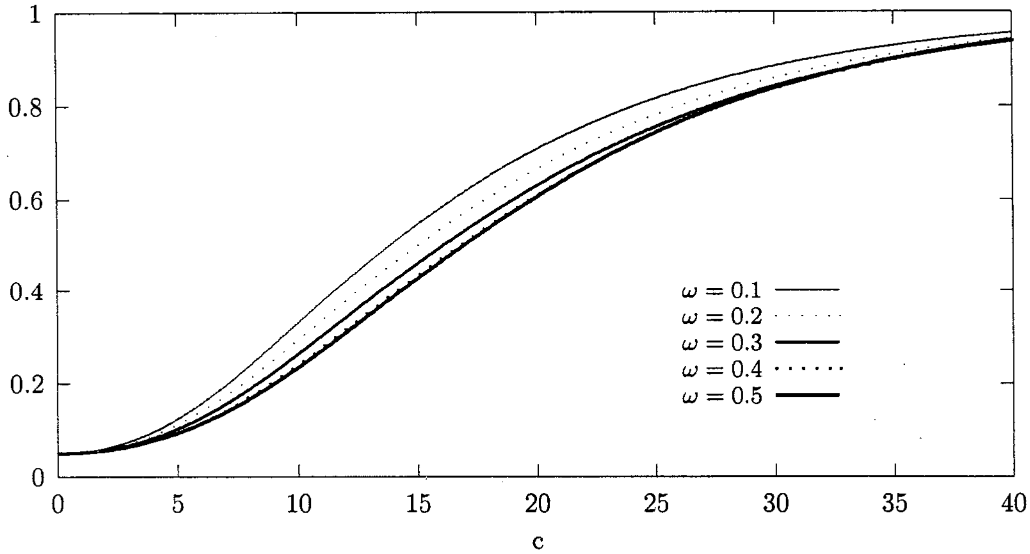


Figure 1d. The Limiting Powers (Case 3)

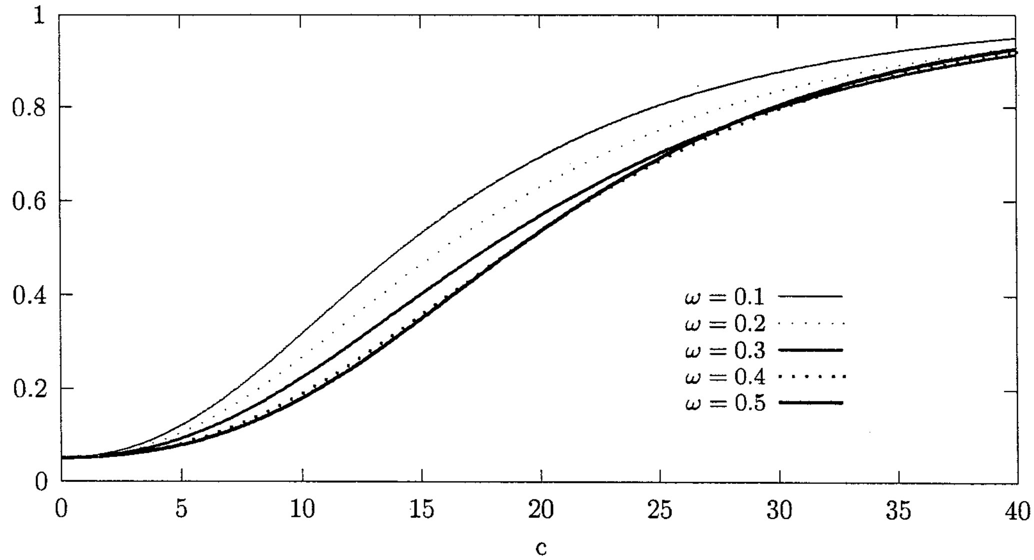


Figure 2a. The Limiting Powers ($\omega = 0.1$)

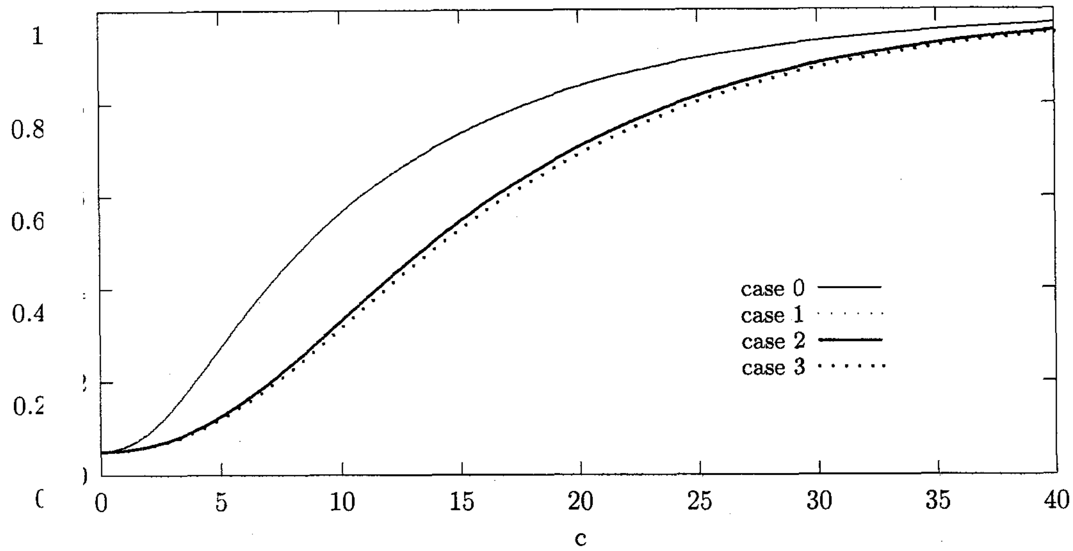


Figure 2b. The Limiting Powers ($\omega = 0.2$)

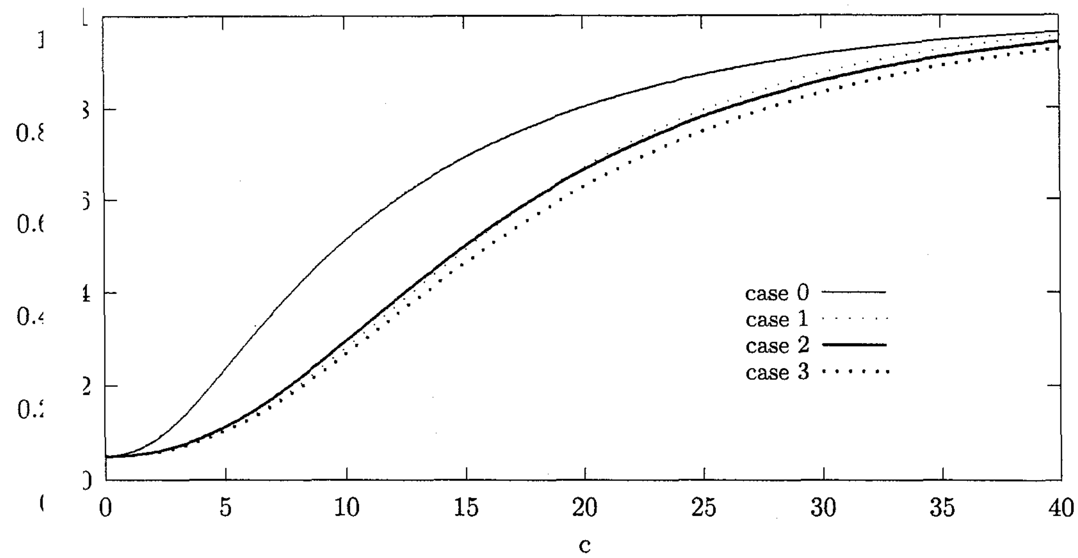


Figure 2c. The Limiting Powers ($\omega = 0.3$)

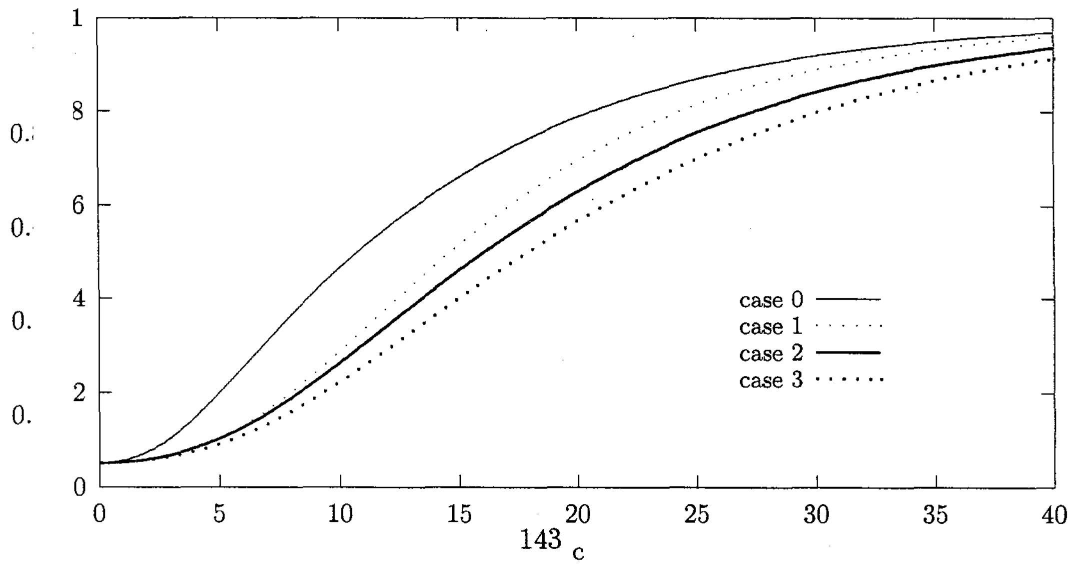


Figure 2d. The Limiting Powers ($\omega = 0.4$)

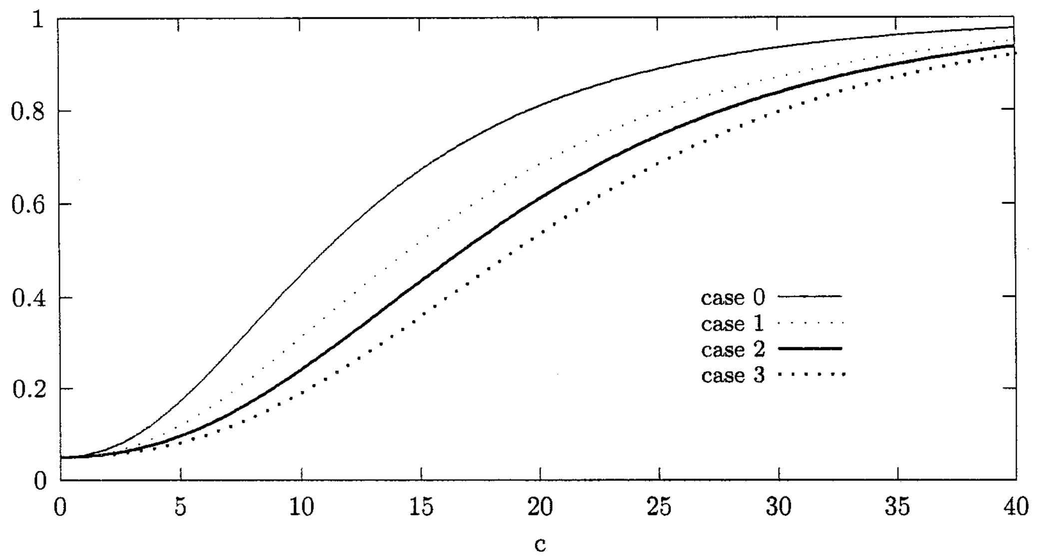


Figure 2e. The Limiting Powers ($\omega = 0.5$)

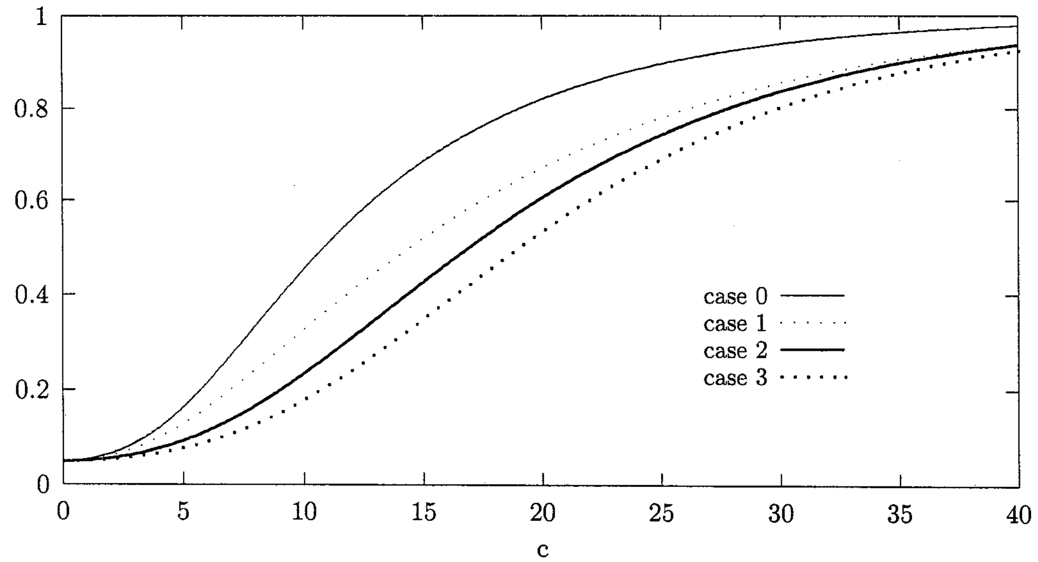


Figure 3a. The Limiting Powers (Case 0:PS)

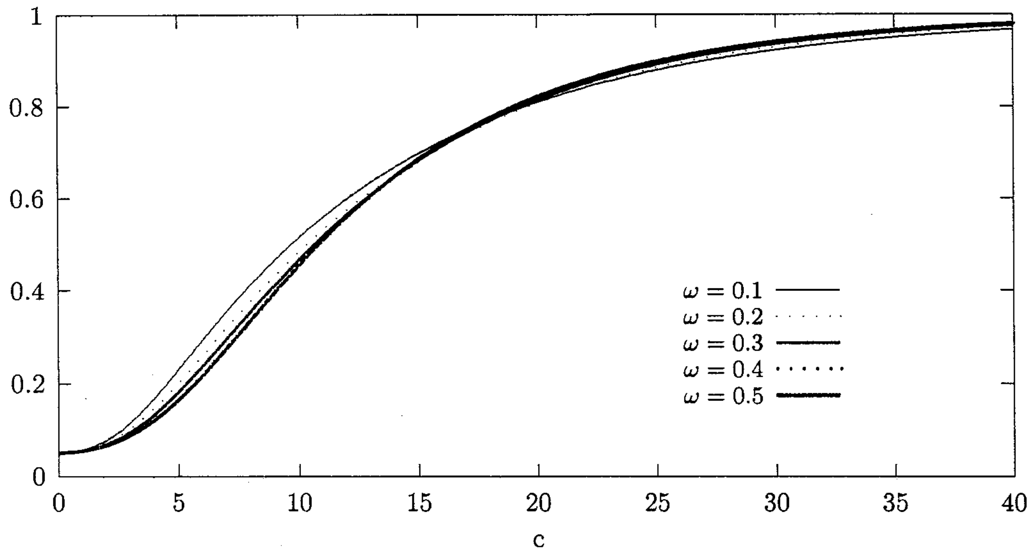


Figure 3b. The Limiting Powers (Case 3:PS)

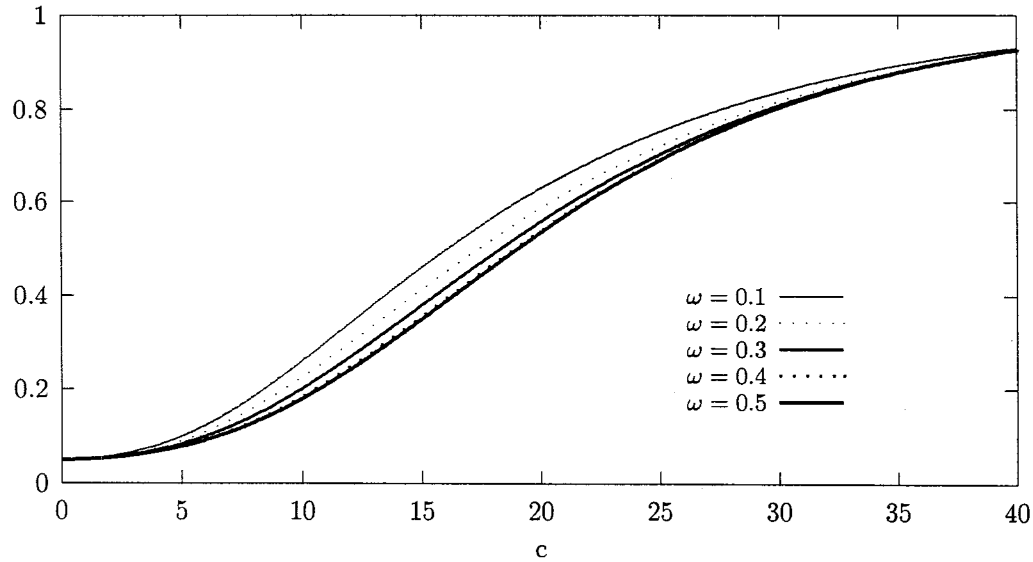


Figure 4a. The Limiting Powers (Case 0: $\omega = 0.1$)

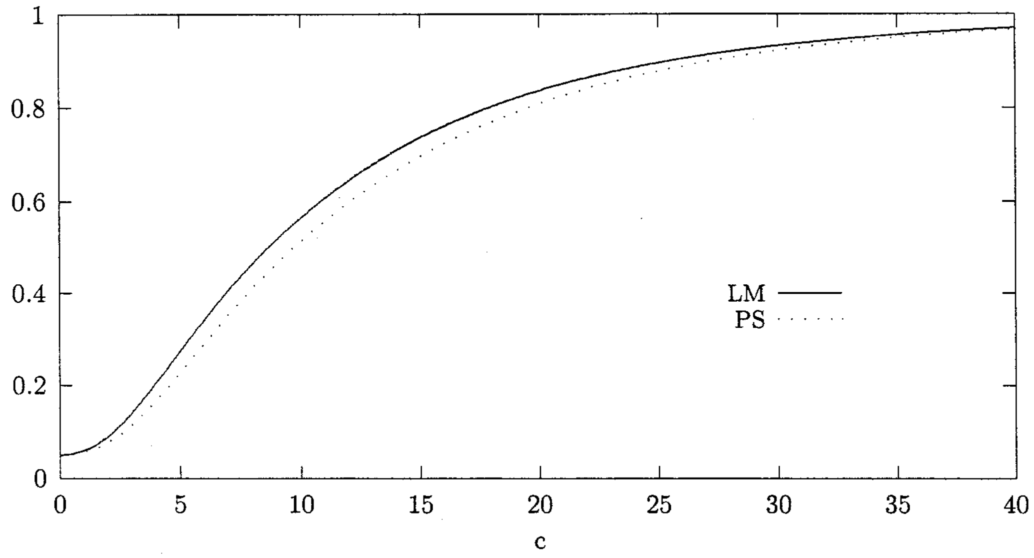


Figure 4b. The Limiting Powers (Case 0: $\omega = 0.2$)

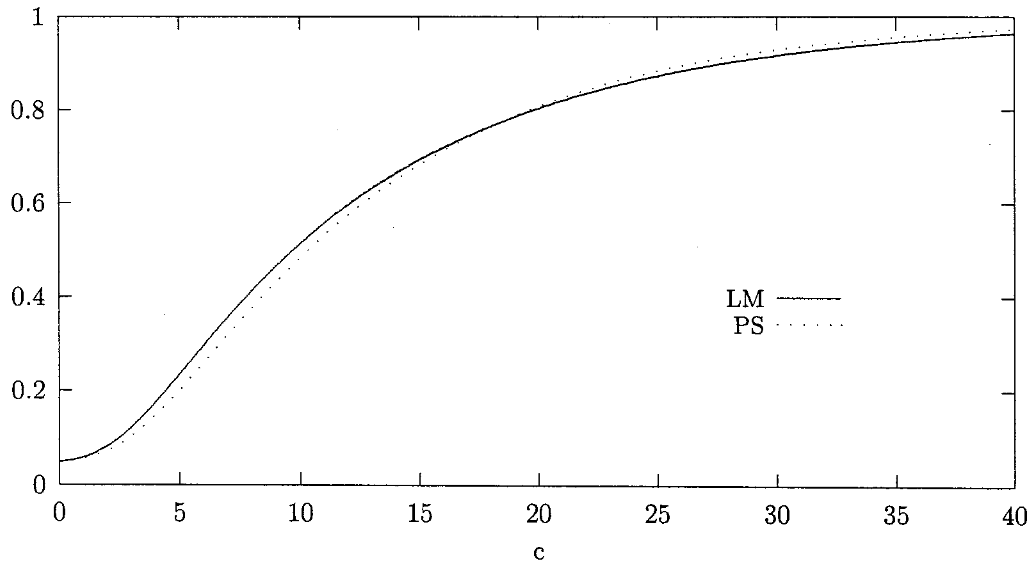


Figure 4c. The Limiting Powers (Case 0: $\omega = 0.3$)

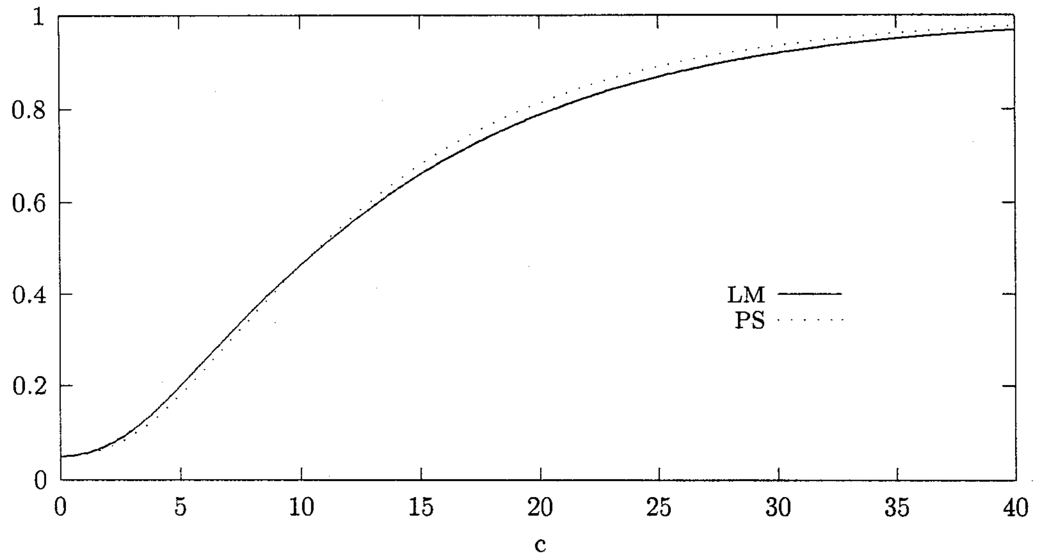


Figure 4d. The Limiting Powers (Case 0: $\omega = 0.4$)

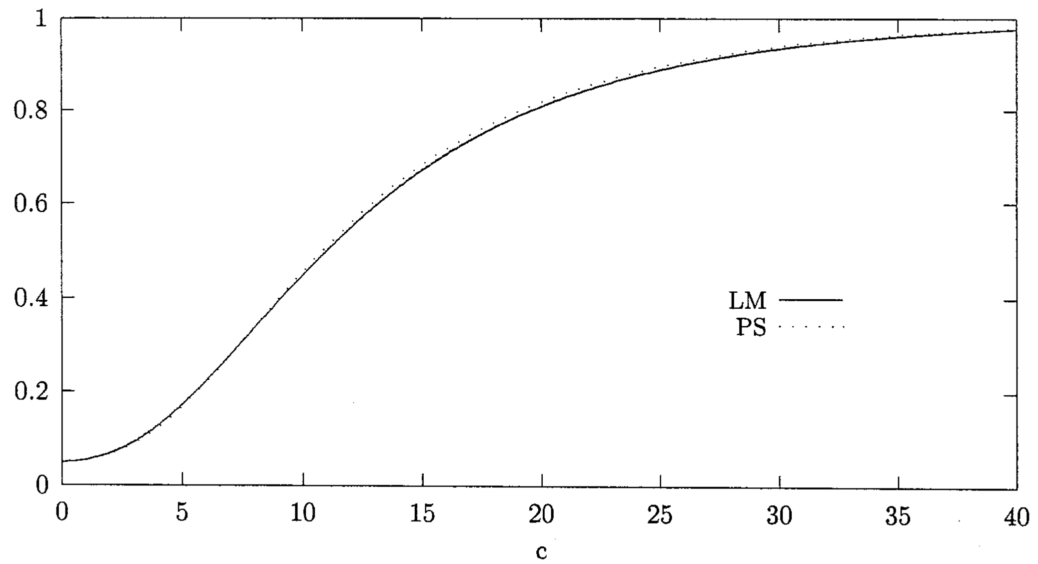


Figure 5a. The Limiting Powers (Case 3: $\omega = 0.1$)

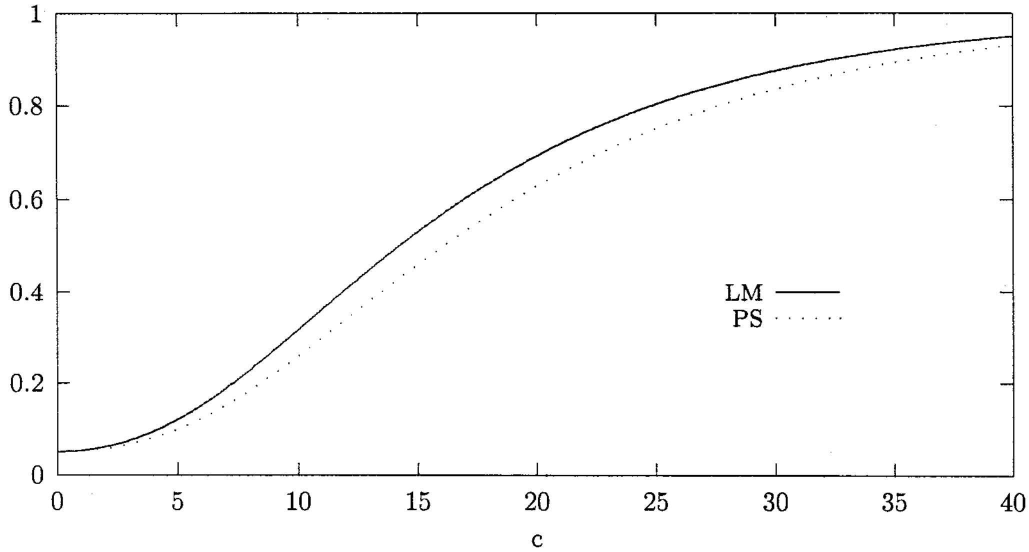


Figure 5b. The Limiting Powers (Case 3: $\omega = 0.2$)

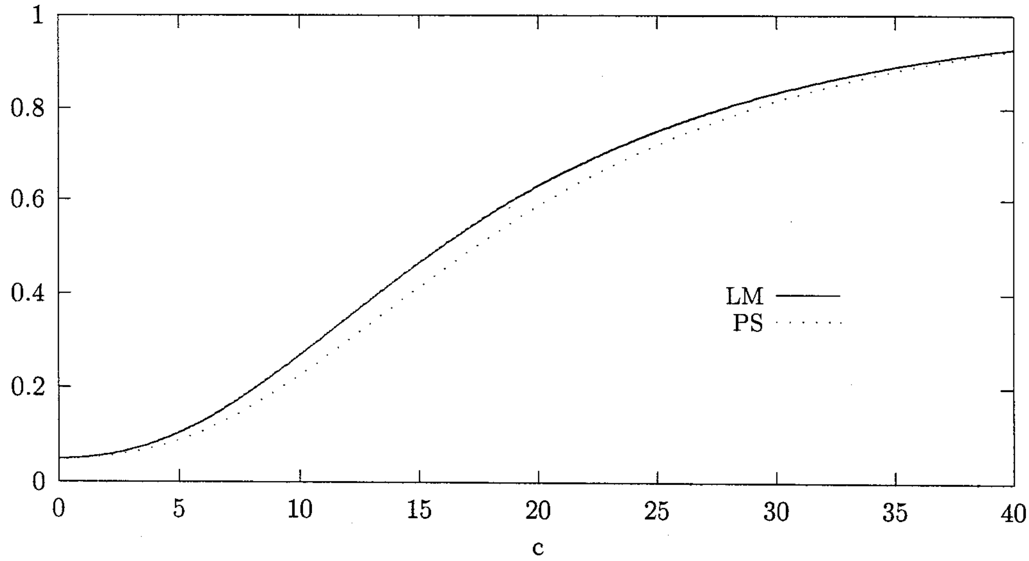


Figure 5c. The Limiting Powers (Case 3: $\omega = 0.3$)

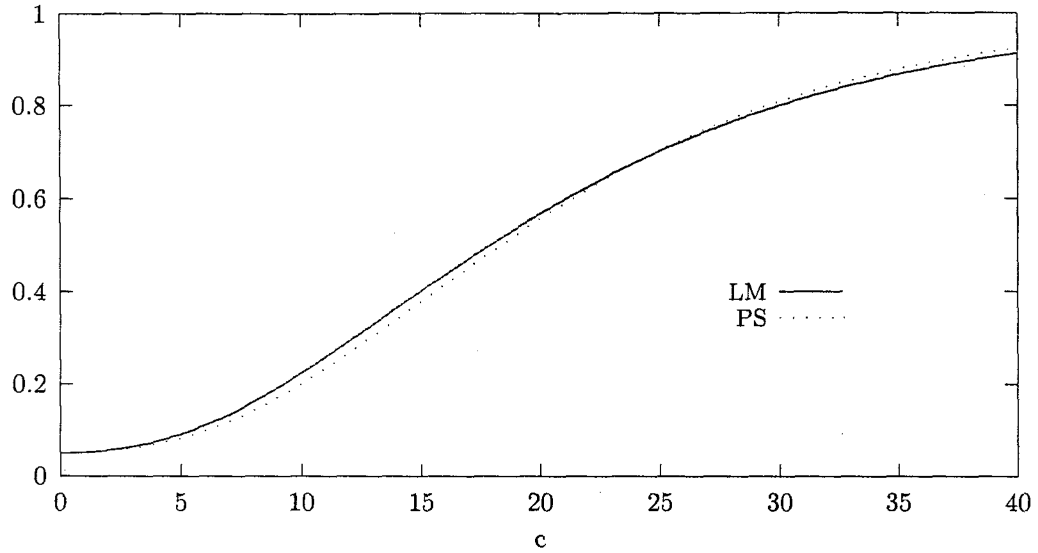
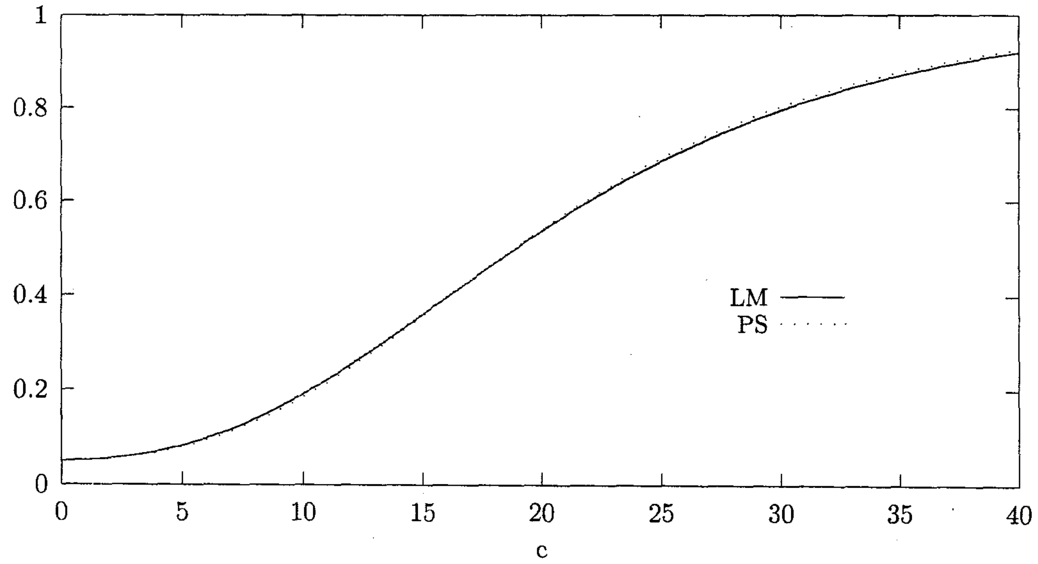


Figure 5d. The Limiting Powers (Case 3: $\omega = 0.4$)



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