Infinite-horizon choice functions*

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This version: January 23, 2008

*An earlier version of the paper was presented at Seoul National University, Victoria University of Wellington, the University of Auckland, the 2005 polarization and conflict conference in Milan, the 2006 workshop on intergenerational resource allocation in Montréal and the University of New South Wales. Thanks are due to two anonymous referees of this journal for their incisive comments which greatly helped us to improve this paper. Financial support through grants from the Social Sciences and Humanities Research Council of Canada, the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche of Québec, the Research Council of Norway, and a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan is gratefully acknowledged. Asheim thanks Université de Montréal and Cornell University for the hospitality of these institutions while working on this project.

Abstract

We analyze infinite-horizon choice functions within the setting of a simple technology. Efficiency and time consistency are characterized by stationary consumption and inheritance functions, as well as a transversality condition. In addition, we consider the equity axioms Suppes-Sen, Pigou-Dalton, and resource monotonicity. We show that Suppes-Sen and Pigou-Dalton imply that the consumption and inheritance functions are monotone with respect to time—thus justifying sustainability—while resource monotonicity implies that the consumption and inheritance functions are monotone with respect to the resource. Examples illustrate the characterization results.

Journal of Economic Literature Classification Numbers: D63, D71.

Keywords: Intergenerational resource allocation, infinite-horizon choice.

1 Introduction

The literature on ranking infinite consumption (or utility) streams has produced a number of negative results in the form of the incompatibility of seemingly mild axioms. For example, following Koopmans (1960), Diamond (1965) establishes that anonymity is incompatible with the strong Pareto principle. Finite anonymity weakens anonymity by restricting the application of the standard anonymity requirement to situations where utility streams differ in at most a finite number of components. Diamond (1965) goes on to show that strong Pareto, finite anonymity and a continuity requirement are incompatible if the social relation is required to be transitive and complete. Instead of requiring the Sidgwickean principle of finite anonymity, Hara, Shinotsuka, Suzumura and Xu (2007) focus on two principles of distributional egalitarianism along the line of the Pigou-Dalton transfer principle and the Lorenz domination principle, and show that there exists no social evaluation relation satisfying one of these egalitarian principles and a weakened continuity condition even in the absence of the Pareto principle and completeness. Basu and Mitra (2003) show that strong Pareto, finite anonymity and representability by a real-valued function are incompatible. Epstein (1986) establishes the incompatibility of a set of standard axioms and a substitution property requiring the possibility to improve upon any given constant stream by means of a stream with lower initial consumption.

The main purpose of this paper is to explore an alternative approach that may provide a promising way to address issues involving intergenerational allocation problems with an infinite horizon. Instead of searching for a *ranking* of infinite streams, we examine a choice-theoretic model where a *choice function* is used to select a consumption stream from each set of feasible streams. Because our focus is on the choice-theoretic aspect of the model, we deliberately consider a simple setting where there is a single resource and a stationary technology. This implies that the feasibility of a consumption stream is determined by the initial amount of the resource available, and the choice function assigns a consumption stream (the chosen consumption stream, given the feasibility constraint) to each possible initial amount.

We begin with an analysis of two fundamental properties whose versions formulated for orderings have been used extensively in the literature, namely, *efficiency* and *time consistency.* We provide characterizations of all infinite-horizon choice functions satisfying either of the two axioms and, moreover, identify all choice functions with both properties. We then consider *equity* properties that are choice-theoretic versions of the Suppes-Sen principle, the Pigou-Dalton transfer principle and resource monotonicity (see

Asheim, Mitra and Tungodden, 2007; Bossert, Sprumont and Suzumura, 2007; Hara, Shinotsuka, Suzumura and Xu, 2007, for equity properties imposed on rankings of infinite streams). Again, classes of infinite-horizon choice functions possessing one of these properties are characterized, and further axiomatizations are obtained by adding efficiency or time consistency.

The results we obtain are promising. Unlike in the case of orderings of infinite utility streams, impossibilities can be avoided and rich classes of infinite-horizon choice functions satisfying several desirable properties do exist. In particular, our choice-theoretic version of the Suppes-Sen principle imposes full anonymity rather than merely finite anonymity and our choice functions may be continuous in the initial endowment. Moreover, it turns out that the notion of *sustainability,* which has played a major role in the literature on intergenerational resource allocation, is closely linked to the Suppes-Sen and Pigou-Dalton principles. Our conclusion from these results is that the choice-theoretic approach to intergenerational resource allocation provides an interesting and viable alternative to the models based on establishing orderings of infinite utility streams, and we propose to explore this approach further.[1]

It is true that optimal growth theory also studies choice functions under the name of *optimal policy functions* in the infinite-horizon setting, but it derives choice functions from specified quasi-orderings such as the *catching up principle* over the set of infinite utility streams by looking at maximal elements from the set of feasible alternatives. In contrast, our approach considers a choice-theoretic setting which does not postulate any specific optimality criterion. Instead, we propose several axioms to be satisfied by choice functions and characterize their implications, thereby encompassing different choice functions which

[1]One of the referees of this journal points out that some of the impossibilities in the literature on ranking infinite utility streams arise from requiring a complete ordering, or even a numerical representability of such an ordering, or from postulating certain continuity conditions on the ranking. It is asserted that these impossibility theorems are not negative results, as they provide a justification for focusing on social welfare quasi-orderings as a basic object of study with intertemporal social choice. This is a valuable and largely agreeable observation, but two remarks seem to be in order here. In the first place, to find an escape route from impossibility theorems by weakening the requirements on social welfare orderings is also relevant in the context of Arrovian impossibility theorems. However, in order to secure meaningful possibility theorems in this context, we may have to go beyond quasi-orderings and weaken transitivity as well as completeness of social evaluation relations. For a recent vindication of this fact, see Bossert and Suzumura (2008). In the second place, a recent study by Hara, Shinotsuka, Suzumura and Xu (2007) shows that impossibility theorems persist in the context of evaluating infinite utility streams if only we retain weak vestiges of continuity, even when we dispose of completeness and coherence requirements on social evaluation relations altogether.

may or may not be generated by a variety of optimality criteria.

Section 2 contains some basic definitions and a first well-known observation characterizing sets of feasible consumption streams. In Section 3, we examine the fundamental axioms of efficiency and time consistency. We characterize all efficient infinite-horizon choice functions, all time-consistent infinite-horizon choice functions, and the class of infinite-horizon choice functions satisfying both requirements. Section 4 deals with the equity axioms à la Suppes-Sen, Pigou-Dalton and resource monotonicity. We characterize all infinite-horizon choice functions satisfying: (i) Suppes-Sen; (ii) efficiency and Pigou-Dalton; (iii) time consistency and Suppes-Sen; (iv) efficiency, time consistency and Pigou-Dalton; (v) efficiency, time consistency and resource monotonicity. As a by-product of our analysis, we show that the conjunction of efficiency and Pigou-Dalton is equivalent to Suppes-Sen. Section 5 provides some examples and Section 6 concludes.

2 Preliminaries

Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of all non-negative real numbers and the set of all positive real numbers, respectively. Analogously, $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ denote the set of all non-negative integers and the set of all positive integers, respectively.

Define the set $\mathcal{Y} = \mathbb{R}_+^{\mathbb{Z}_+}$ to be the set of all sequences $y = (y_0, y_1, \ldots, y_t, \ldots)$. We interpret y as a consumption stream, where y_t is the amount of a single resource consumed in period $t \in \mathbb{Z}_+$. Time is measured relative to the present: period t is the t^{th} period after today. We use the following notation for inequalities in $\mathcal{Y}$. For all $y, z \in \mathcal{Y}$, $y \geq z$ if and only if $y_t \geq z_t$ for all $t \in \mathbb{Z}_+$, and $y > z$ if and only if $y \geq z$ and $y \neq z$.

The initial amount of the resource is $x \in \mathbb{R}_+$. We assume a stationary technology. For $x \in \mathbb{R}_+$ and $y \in \mathcal{Y}$, the sequence of resource stocks

$$X(x, y) = (X_0(x, y), X_1(x, y), \ldots, X_t(x, y), \ldots) \in \mathbb{R}_+^{\mathbb{Z}_+}$$

generated by x and y is *x-feasible* if $y_0 \leq X_0(x, y) = x$ and

$$y_t \leq X_t(x, y) = f(X_{t-1}(x, y) - y_{t-1})$$

for all $t \in \mathbb{Z}_{++}$, where the *gross output function* f is assumed to satisfy

$$f(0) = 0\,, \quad f \text{ is increasing, differentiable and concave on } \mathbb{R}_+\,, \tag{F1}$$

$$r := \inf\nolimits_{k \geq 0} f'(k) - 1 \ > \ 0\,. \tag{F2}$$

Clearly, the case in which $f(k) = k(1+r)$ for all $k \in \mathbb{R}_+$ (with $r > 0$) is a special case of a gross output function satisfying (F1) and (F2).[2]

For $x \in \mathbb{R}_+$, the set of x-feasible consumption streams is

$$\mathcal{S}(x) = \{y \in \mathcal{Y} \mid X(x, y) \text{ is } x\text{-feasible } \}.$$

Associated with any x-feasible consumption stream y, there is a sequence of shadow prices

$$P(x, y) = (P_0(x, y), P_1(x, y), \ldots, P_t(x, y), \ldots) \in \mathbb{R}_{++}^{\mathbb{Z}_+}$$

defined by $P_0(x, y) = 1$ and

$$P_t(x, y) = \frac{P_{t-1}(x, y)}{f'(X_{t-1}(x, y) - y_{t-1})}$$

for all $t \in \mathbb{Z}_{++}$. In the case in which $f(k) = k(1+r)$, we have that $P_t(x, y) = 1/(1+r)^t$.

As shown by Mitra (1979, Lemma 2) and stated in the following lemma, any x-feasible consumption stream has finite value under (F1) and (F2).

Lemma 1 *For all $x \in \mathbb{R}_+$ and for all $y \in \mathcal{S}(x)$,*

$$\sum_{t=0}^{\infty} P_t(x, y) y_t < \infty\,.$$

If $f(k) = k(1+r)$, then $\mathcal{S}(x) = \{y \in \mathcal{Y} \mid \sum_{t=0}^{\infty} P_t(x, y) y_t \leq x\}$; see, for instance, Epstein (1986) who made this observation in his analysis of the linear model in an intertemporal social choice setting.

3 Efficient and time-consistent choice

An *infinite-horizon choice function* is a mapping $C \colon \mathbb{R}_+ \to \mathcal{Y}$ such that $C(x) \in \mathcal{S}(x)$ for all $x \in \mathbb{R}_+$. This function assigns a consumption stream to any given initial amount of a single resource available in the economy. Note that consumption streams are undated: whether the choice takes place today or tomorrow makes no difference if the same initial endowment is present. This time-independence feature of a choice function ensures that the choice of a starting period is irrelevant. For all $t \in \mathbb{Z}_+$, we write $C_t(x)$ for the t^{th} component of the sequence $C(x)$.

[2]Our original analysis was based on linear gross output functions. We owe thanks to a referee of this journal for the suggestion to adopt the more general class of gross output functions characterized by (F1) and (F2).

The first fundamental property of an infinite-horizon choice function is the familiar *efficiency* axiom. It requires that no x-feasible consumption stream Pareto dominates the chosen consumption stream with initial stock x.

Efficiency. For all $x \in \mathbb{R}_+$ and for all $y \in \mathcal{Y}$,

$$y > C(x) \quad \Rightarrow \quad y \notin \mathcal{S}(x).$$

Given Lemma 1, a corollary to the theorem of Cass and Yaari (1971, pp. 337–338) implies that the class of efficient choice functions can be characterized as follows.

Lemma 2 *An infinite-horizon choice function C satisfies efficiency if and only if*

$$\sum_{t=0}^{\infty} P_t(x, C(x))C_t(x) \geq \sum_{t=0}^{\infty} P_t(x, C(x))y_t \quad \text{for all } x \in \mathbb{R}_+ \text{ and for all } y \in \mathcal{S}(x). \tag{C1}$$

Time consistency prevents deviations from chosen consumption streams as time progresses. Thus, for any $x \in \mathbb{R}_+$ and for any $t, \tau \in \mathbb{Z}_+$, the consumption $C_{t+\tau}(x)$ in period $t+\tau$ for the initial endowment x should be the same as the consumption $C_\tau(X_t(x, C(x)))$ in period τ for the initial endowment $X_t(x, C(x))$.

Time consistency. For all $x \in \mathbb{R}_+$ and for all $t, \tau \in \mathbb{Z}_+$,

$$C_{t+\tau}(x) = C_\tau(X_t(x, C(x))).$$

We now characterize all infinite-horizon choice functions satisfying time consistency. In order to express this class of choice functions, we use a function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ that indicates, for each initial level of the resource, the amount of the resource that is available in the next period after the present consumption has taken place. Hence, we may refer to g as the *inheritance function*.

For any function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$, let the function $g^0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $g^0(x) = x$ for all $x \in \mathbb{R}_+$ and, for all $t \in \mathbb{Z}_{++}$, define the function $g^t \colon \mathbb{R}_+ \to \mathbb{R}_+$ by letting $g^t(x) = g(g^{t-1}(x))$ for all $x \in \mathbb{R}_+$. As will become clear once our characterization of time consistency is stated, the functions g^t have a natural interpretation: they identify the amount of the resource available in period t as a function of the initial endowment x only. Because all these functions are determined once a function g is chosen, it is sufficient to specify, for any initial endowment, the amount of the resource remaining at the beginning of period one.

The following lemma characterizes all time-consistent choice functions.

Lemma 3 *An infinite-horizon choice function C satisfies time consistency if and only if there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$g(x) \le f(x) \quad \text{for all } x \in \mathbb{R}_+ \tag{G1}$$

and

$$g^{t+1}(x) = f(g^t(x) - C_t(x)) \quad \text{for all } t \in \mathbb{Z}_+ \text{ and for all } x \in \mathbb{R}_+ \,. \tag{CG}$$

Proof. 'If.' Let C be an infinite-horizon choice function and suppose there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (G1) and (CG) are satisfied. Let $x \in \mathbb{R}_+$ and $t \in \mathbb{Z}_+$. By (G1), it follows that

$$g^{t+1}(x) = g(g^t(x)) \le f(g^t(x))$$

and, together with (F1) and (CG), that

$$0 \le C_t(x) \le g^t(x) \,.$$

Using (CG) and the definition of X, this observation entails that $X(x, C(x))$ is an x-feasible sequence of resource stocks satisfying

$$X_t(x, C(x)) = g^t(x) \quad \text{for all } t \in \mathbb{Z}_+ \,. \tag{1}$$

Hence, $C(x) \in \mathcal{S}(x)$, and C is a well-defined infinite-horizon choice function.

To establish time consistency, let $x \in \mathbb{R}_+$ and $t, \tau \in \mathbb{Z}_+$. By (CG),

$$f(g^\tau(g^t(x)) - C_\tau(g^t(x))) = g^{\tau+1}(g^t(x)) = g^{t+\tau+1}(x) = f(g^{t+\tau}(x) - C_{t+\tau}(x)) \,,$$

which, together with (F1) and (1), proves that C is time consistent.

'Only if.' Suppose C is an infinite-horizon choice function that satisfies time consistency. Define the function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ by letting

$$g(x) = f(x - C_0(x)) \tag{2}$$

for all $x \in \mathbb{R}_+$. By feasibility, $C_0(x) \in [0, x]$, and the definition of g immediately implies $g(x) \in [0, f(x)]$ for all $x \in \mathbb{R}_+$, establishing that g indeed maps into $\mathbb{R}_+$ and that (G1) is satisfied.

It remains to be shown that (CG) is satisfied. We proceed by induction. Suppose

$$g^{t+1}(x) = f(g^t(x) - C_t(x)) \tag{3}$$

for some $t \in \mathbb{Z}_+$. Since, by definition, $g(x) = f(x - C_0(x)) = X_1(x, C(x))$, and C satisfies time consistency, it follows that $C_t(g(x)) = C_t(X_1(x, C(x))) = C_{t+1}(x)$. Thus, using (3), we obtain

$$g^{t+2}(x) = g^{t+1}(g(x)) = f(g^t(g(x)) - C_t(g(x))) = f(g^{t+1}(x) - C_{t+1}(x)),$$

which combined with (2) completes the proof. ■

If (G1) is satisfied, then g determines a bequest function $f^{-1} \circ g \colon \mathbb{R}_+ \to \mathbb{R}_+$. For each initial level of the resource, $f^{-1}(g(x)) \in [0, x]$ is the bequest that is left behind, and $x - f^{-1}(g(x))$ is the present consumption.

We now characterize all infinite-horizon choice functions satisfying both efficiency and time consistency.

Theorem 1 *An infinite-horizon choice function C satisfies efficiency and time consistency if and only if there exists a function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that* (G1), (CG) *and*

$$\lim_{t\to\infty} P_t(x, C(x)) g^t(x) = 0 \quad \textit{for all } x \in \mathbb{R}_+ \tag{G2}$$

are satisfied.

Proof. 'If.' Let C be an infinite-horizon choice function and suppose there exists a function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (G1), (CG) and (G2) are satisfied. Then, by Lemma 3, C is a well-defined infinite-horizon choice function that satisfies time consistency. Let $y \in \mathcal{S}(x)$. By (F1) and the definition of P, it holds for all $t \in \mathbb{Z}_{++}$ that

$$\begin{aligned} g^t(x) - X_t(x, y) &= f(g^{t-1}(x) - C_{t-1}(x)) - f(X_{t-1}(x, y) - y_{t-1}) \\ &\geq f'(g^{t-1}(x) - C_{t-1}(x)) \big((g^{t-1}(x) - C_{t-1}(x)) - (X_{t-1}(x, y) - y_{t-1}) \big) \\ &= \frac{P_{t-1}(x, C(x))}{P_t(x, C(x))} \big((g^{t-1}(x) - C_{t-1}(x)) - (X_{t-1}(x, y) - y_{t-1}) \big) , \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &P_{t-1}(x, C(x))(C_{t-1}(x) - y_{t-1}) \\ &\geq P_t(x, C(x))(X_t(x, y) - g^t(x)) - P_{t-1}(x, C(x))(X_{t-1}(x, y) - g^{t-1}(x)) . \end{aligned}$$

Hence, since $g^0(x) = x = X_0(x, y)$,

$$\sum_{t=0}^{\tau-1} P_t(x, C(x))(C_t(x) - y_t) \geq P_\tau(x, C(x))(X_\tau(x, y) - g^\tau(x)) .$$

By invoking Lemma 2, this, combined with (G2), implies that C satisfies efficiency, since $P(x, C(x)) \in \mathbb{R}^{\mathbb{Z}_+}_{++}$ and $X(x, y) \in \mathbb{R}^{\mathbb{Z}_+}_{+}$.

'Only if.' (This proof builds on Mitra, 1979, Theorem 2.) Suppose C is an infinite-horizon choice function that satisfies efficiency and time consistency. Then, by Lemma 3, there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (G1) and (CG) are satisfied. Suppose g does not satisfy (G2); i.e., there exists a subsequence of periods, and $m > 0$, such that $P_t(x, C(x))g^t(x) \geq m$ for this subsequence. By Lemma 1, we can choose τ' such that $\sum_{t=\tau'}^{\infty} P_t(x, C(x))C_t(x) \leq \frac{1}{2}m$. Choose $\tau \geq \tau'$ such that $P_\tau(x, C(x))g^\tau(x) \geq m$, and define $y \in \mathcal{Y}$ as follows: $y_t = C_t(x)$ for $0 \leq t < \tau$, $y_\tau = g^\tau(x)$, and $y_t = 0$ for $t > \tau$. Clearly, $y \in \mathcal{S}(x)$ with

$$\sum_{t=0}^{\infty} P_t(x, C(x))y_t \geq \sum_{t=0}^{\tau-1} P_t(x, C(x))y_t + m \geq \sum_{t=0}^{\infty} P_t(x, C(x))C_t(x) + \tfrac{1}{2}m\,.$$

By Lemma 2, this contradicts that C satisfies efficiency. Hence, g satisfies (G2). ■

Condition (G2) is of course a capital value transversality condition, which has been used to characterize efficient capital accumulation at least since Malinvaud (1953).

The properties (G1) and (G2) of a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ are independent, as is straightforward to verify. That (CG) must be satisfied is a consequence of the time-consistency requirement, and (G1) ensures that this is done without violating the resource constraints. Property (G2) is required for the efficiency axiom.

4 Imposing equity axioms

We now examine the consequences of imposing certain equity axioms, in addition to efficiency and time consistency.

The first of the equity axioms that we consider—*Suppes-Sen*—requires that no x-feasible consumption stream has a permutation which Pareto dominates the chosen consumption stream with initial stock x. The term 'permutation' signifies a bijective mapping π of $\mathbb{Z}_+$ onto itself. The Suppes-Sen axiom is a straightforward adaptation of the Suppes-Sen principle for orderings (cf. Suppes, 1966; Sen, 1970) to the present infinite-horizon choice-theoretic setting.

Suppes-Sen. For all $x \in \mathbb{R}_+$ and for all $y, y' \in \mathcal{Y}$, if y' is a permutation of y, then

$$y' > C(x) \quad \Rightarrow \quad y \notin \mathcal{S}(x).$$

Clearly, the Suppes-Sen axiom implies efficiency. Note that we do not restrict the scope of the axiom to finite permutations (that is, permutations π with the property that there is a $t \in \mathbb{Z}_+$ such that $\pi(\tau) = \tau$ for all $\tau \geq t$). In contrast to the Suppes-Sen axiom formulated for orderings of infinite utility streams, allowing for infinite permutations does not lead to an impossibility in the choice-theoretic setting, given our technological environment. This is established by combining our next result, which characterizes all choice functions satisfying the Suppes-Sen principle, with the fact that, for any initial resource stock, there exists a non-empty set of efficient and non-decreasing streams.

Lemma 4 *An infinite-horizon choice function C satisfies Suppes-Sen if and only if* (C1) *and*

$$C_t(x) \leq C_{t+1}(x) \quad \text{for all } x \in \mathbb{R}_+ \text{ and for all } t \in \mathbb{Z}_+ \tag{C2}$$

are satisfied.

Proof. 'If.' Assume (C1) and (C2) are satisfied. Let y' be a permutation π of y, and suppose $y' > C(x)$. Denote by y'' the permutation π^{-1} of $C(x)$. Then

$$\sum_{t=0}^{\infty} P_t(x, C(x)) y_t'' \geq \sum_{t=0}^{\infty} P_t(x, C(x)) C_t(x) \,,$$

since the sequence $\langle P_t(x, C(x)) \rangle_{t \in \mathbb{Z}_+}$ is decreasing by virtue of (F1) and (F2), and the sequence $\langle C_t(x) \rangle_{t \in \mathbb{Z}_+}$ is non-decreasing. By construction, $y > y''$; hence,

$$\sum_{t=0}^{\infty} P_t(x, C(x)) y_t > \sum_{t=0}^{\infty} P_t(x, C(x)) C_t(x) \,.$$

By (C1), $y \notin \mathcal{S}(x)$. Thus, C satisfies Suppes-Sen.

'Only if.' Suppose that C satisfies Suppes-Sen. Let $x \in \mathbb{R}_+$. By way of contradiction, suppose first that there exists $y' \in \mathcal{S}(x)$ such that

$$\sum_{t=0}^{\infty} P_t(x, C(x)) C_t(x) < \sum_{t=0}^{\infty} P_t(x, C(x)) y_t' \,.$$

Then by Lemma 2, there exists $y \in \mathcal{S}(x)$ such that $y > C(x)$. Thus, there is an x-feasible consumption stream which Pareto-dominates the chosen consumption stream with initial stock x, entailing that C does not satisfy Suppes-Sen. Hence, (C1) must be satisfied.

By way of contradiction, suppose that (C2) is not satisfied, i.e., there exists $\tau \in \mathbb{Z}_+$ such that $C_\tau(x) > C_{\tau+1}(x)$. Construct $y \in \mathcal{Y}$ as follows:

$$y_t = \begin{cases} C_t(x) & \text{if } t \notin \{\tau, \tau+1\}, \\ C_{\tau+1}(x) & \text{if } t = \tau, \\ C_{\tau+1}(x) + f(X_\tau(x, C(x)) - C_{\tau+1}(x)) - f(X_\tau(x, C(x)) - C_\tau(x)) & \text{if } t = \tau + 1. \end{cases}$$

Then $X_{\tau+1}(x,y) - y_{\tau+1} = f(X_\tau(x,y) - y_\tau) - y_{\tau+1} = f(X_\tau(x,C(x)) - C_{\tau+1}(x)) - y_{\tau+1} = f(X_\tau(x,C(x)) - C_\tau(x)) - C_{\tau+1}(x) = X_{\tau+1}(x,C(x)) - C_{\tau+1}(x)$, where the second equality follows from the definition of y_τ and the third equality follows from the definition of $y_{\tau+1}$. This entails that $X_t(x,y) = X_t(x,C(x))$ for all $t \neq \tau+1$, and implies that $y \in \mathcal{S}(x)$. Construct $y' \in \mathcal{Y}$ from y by permuting y_τ and $y_{\tau+1}$. It follows that $y'_\tau = y_{\tau+1} > C_\tau(x)$ since, by (F2), $f(k+\varepsilon) > f(k) + \varepsilon$ whenever $\varepsilon > 0$, while $y'_t = C_t(x)$ holds otherwise; hence, we have that $y' > C(x)$. Thus, there is an x-feasible consumption stream with a permutation which Pareto-dominates the chosen consumption stream with initial stock x, entailing that C does not satisfy Suppes-Sen. Hence, (C2) must be satisfied. ■

Note that the necessity of (C2) requires only that $f(k+\varepsilon) > f(k)+\varepsilon$ whenever $\varepsilon > 0$. Hence, since the Suppes-Sen axiom implies efficiency, the statement "An infinite-horizon choice function C satisfies Suppes-Sen only if efficiency and (C2) are satisfied" remains true even if (F2) is weakened to

$$f'(k) > 1 \quad \text{for all } k \in \mathbb{R}_+ \,. \tag{F2$'$}$$

The converse statement, "An infinite-horizon choice function C satisfies Suppes-Sen if efficiency and (C2) are satisfied", also remains true even if (F2) is weakened to (F2$'$), as shown by the following argument. Assume that efficiency and (C2) are satisfied. Let y' be a permutation π of y, and suppose $y' > C(x)$. Denote by y'' the permutation π^{-1} of $C(x)$. Then

$$\liminf_{T\to\infty} \sum_{t=0}^{T} P_t(x,C(x))\,(y''_t - C_t(x)) \geq 0\,,$$

since the sequence $\langle P_t(x,C(x))\rangle_{t\in\mathbb{Z}_+}$ is decreasing by virtue of (F1) and (F2), and the sequence $\langle C_t(x)\rangle_{t\in\mathbb{Z}_+}$ is non-decreasing. By construction, $y > y''$; hence,

$$\liminf_{T\to\infty} \sum_{t=0}^{T} P_t(x,C(x))\,(y_t - C_t(x)) > 0\,.$$

By the theorem of Cass and Yaari (1971, p. 337), $y \notin \mathcal{S}(x)$. Thus, C satisfies Suppes-Sen.

As is apparent from the proof of Lemma 4, the Suppes-Sen principle as stated in the lemma can be replaced with its finite counterpart, restricting its conclusion to finite permutations. In our setting, the two properties are equivalent and we chose to use the general version in order to illustrate that, unlike the model based on orderings of infinite streams, our approach does not lead to an impossibility when infinite permutations are permitted.

The observation that the Suppes-Sen axiom can allow for infinite permutations without leading to an impossibility in the choice-theoretic setting is robust with respect to modifications in our technological assumptions. To see this, consider the technological assumptions of *immediate productivity* and *eventual productivity*, as defined by Asheim, Buchholz and Tungodden (2001, p. 259). The assumption of immediate productivity states that if $y \in \mathcal{Y}$ with $y_\tau > y_{\tau+1}$ for some $\tau \in \mathbb{Z}_+$ is feasible, then $y' \in \mathcal{Y}$ constructed by

$$y'_t = \begin{cases} y_t & \text{if} \quad t \notin \{\tau, \tau+1\}, \\ y_{\tau+1} & \text{if} \quad t = \tau, \\ y_\tau & \text{if} \quad t = \tau + 1 \end{cases}$$

is feasible and *inefficient*. The assumption of eventual productivity states that, for any initial resource stock(s) and time, there exists an efficient and equally-distributed stream. The class of technologies that satisfy the assumptions of immediate productivity and eventual productivity includes the simple stationary technologies that we consider throughout this paper. However, the former class of technologies is wider than the latter class, as illustrated by Asheim, Buchholz and Tungodden (2001, Examples 1–3).

In a technology satisfying eventual productivity, the choice function assigning to any initial resource stock(s) and time the efficient and equally-distributed stream is an efficient, time consistent choice function satisfying even the infinite permutation Suppes-Sen axiom. Hence, provided that the assumption of eventual productivity is satisfied, the Suppes-Sen axiom can allow for infinite permutations without leading to an impossibility in the choice-theoretic setting. If we add immediate productivity, we obtain a generalization of Lemma 4: *An infinite-horizon choice function satisfies Suppes-Sen if and only if, for any initial resource stock(s) and time, the chosen stream is efficient and non-decreasing.* Also the latter result allows for the version of Suppes-Sen axiom that includes infinite permutations, although it continues to hold if the axiom is replaced by its finite permutation counterpart.

The second of the equity axioms—*Pigou-Dalton*—requires that no x-feasible consumption stream can be generated from the chosen consumption stream with initial stock x through a transfer of consumption from a better-off to a worse-off generation. The axiom is a straightforward adaptation of the Pigou-Dalton transfer principle (cf. Pigou, 1912; Dalton, 1920) for social welfare orderings to the present choice-theoretic setting.

Pigou-Dalton. For all $x \in \mathbb{R}_+$ and for all $y, y' \in \mathcal{Y}$, if there exist $\varepsilon \in \mathbb{R}_{++}$ and $\tau, \tau' \in \mathbb{Z}_+$ such that $y_\tau = y'_\tau - \varepsilon \geq y'_{\tau'} + \varepsilon = y_{\tau'}$ and $y_t = y'_t$ for all $t \in \mathbb{Z}_+ \backslash \{\tau, \tau'\}$, then

$$y' = C(x) \quad \Rightarrow \quad y \notin \mathcal{S}(x).$$

Unlike the Suppes-Sen principle, Pigou-Dalton does not imply efficiency. However, it rules out all violations of efficiency that do not involve equally-distributed streams. As will become clear in the proof of the following theorem, efficiency could therefore be replaced with a weaker axiom that applies to equal distributions only. We chose to keep the standard efficiency axiom for clarity and ease of exposition.

We now characterize all infinite-horizon choice functions satisfying efficiency and the Pigou-Dalton principle. Interestingly, this is the same class as the one identified in the previous lemma.

Lemma 5 *An infinite-horizon choice function C satisfies efficiency and Pigou-Dalton if and only if* (C1) *and* (C2) *are satisfied.*

Proof. 'If.' Assume (C1) and (C2) are satisfied. By Lemma 2, C satisfies efficiency. Since the sequence $\langle P_t(x, C(x))\rangle_{t\in\mathbb{Z}_+}$ is decreasing and the sequence $\langle C_t(x)\rangle_{t\in\mathbb{Z}_+}$ is non-decreasing, if $y_\tau = C_\tau(x) - \varepsilon \geq C_{\tau'}(x) + \varepsilon = y_{\tau'}$ for some $\varepsilon \in \mathbb{R}_{++}$ and $y_t = C_t(x)$ for all $t \in \mathbb{Z}_+ \setminus \{\tau, \tau'\}$, then

$$\sum_{t=0}^{\infty} P_t(x, C(x)) y_t > \sum_{t=0}^{\infty} P_t(x, C(x)) C_t(x)\,.$$

By (C1), $y \notin \mathcal{S}(x)$. Thus, C satisfies Pigou-Dalton.

'Only if.' Suppose that C satisfies efficiency and Pigou-Dalton. Let $x \in \mathbb{R}_+$. By way of contradiction, suppose first that there exists $y' \in \mathcal{S}(x)$ such that

$$\sum_{t=0}^{\infty} P_t(x, C(x)) C_t(x) < \sum_{t=0}^{\infty} P_t(x, C(x)) y'_t\,.$$

This, by Lemma 2, contradicts the efficiency of C. Hence, (C1) must be satisfied.

By way of contradiction, suppose that (C2) is not satisfied, i.e., there exists $\tau \in \mathbb{Z}_+$ such that $C_\tau(x) > C_{\tau+1}(x)$. Construct $y \in \mathcal{Y}$ as follows:

$$y_t = \begin{cases} C_t(x) & \text{if} \quad t \notin \{\tau, \tau+1\}, \\ C_\tau(x) - \varepsilon & \text{if} \quad t = \tau, \\ C_{\tau+1}(x) + \varepsilon & \text{if} \quad t = \tau + 1, \end{cases}$$

where $0 < \varepsilon \leq (C_\tau(x) - C_{\tau+1}(x))/2$, so that $y_\tau = C_\tau(x) - \varepsilon \geq C_{\tau+1}(x) + \varepsilon = y_{\tau+1}$. Then

$$\begin{aligned} X_{\tau+1}(x, y) - y_{\tau+1} &= f(X_\tau(x, y) - y_\tau) - y_{\tau+1} \\ &= f(X_\tau(x, C(x)) - C_\tau(x) + \varepsilon) - C_{\tau+1}(x) - \varepsilon \\ &> f(X_\tau(x, C(x)) - C_\tau(x)) - C_{\tau+1}(x) = X_{\tau+1}(x, C(x)) - C_{\tau+1}(x) \end{aligned}$$

since, by (F2), $f(k+\varepsilon) > f(k)+\varepsilon$ whenever $\varepsilon > 0$. This entails that $X_t(x,y) = X_t(x,C(x))$ for all $t < \tau+1$ and, by (F1), $X_t(x,y) > X_t(x,C(x))$ for all $t \geq \tau+1$, and implies that $y \in \mathcal{S}(x)$. Thus, an x-feasible consumption stream can be generated from the chosen consumption stream with initial stock x through a transfer of consumption from a better-off to a worse-off generation, entailing that C does not satisfy Pigou-Dalton. Hence, (C2) must be satisfied. ■

A similar remark as the one made subsequent to Lemma 4 applies to Lemma 5. Hence, the statement "An efficient infinite horizon choice function C satisfies Pigou-Dalton if and only if (C2) is satisfied" remains true even if (F2) is weakened to (F2′).

The following corollary is an immediate consequence of the previous two lemmas.

Corollary 1 *An infinite-horizon choice function C satisfies Suppes-Sen if and only if C satisfies efficiency and Pigou-Dalton.*

The following theorem identifies all choice functions satisfying time consistency in addition to Suppes-Sen (or, equivalently, in addition to efficiency and Pigou-Dalton).

Theorem 2 *An infinite-horizon choice function C satisfies time consistency and Suppes-Sen (or efficiency, time consistency and Pigou-Dalton) if and only if there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that* (G1), (CG), (G2),

$$x \;\leq\; g(x) \quad \textit{for all}\; x \in \mathbb{R}_+ \tag{G3}$$

and

$$x - f^{-1}(g(x)) \;\leq\; g(x) - f^{-1}(g(g(x))) \quad \textit{for all}\; x \in \mathbb{R}_+ \tag{G4}$$

are satisfied.

Proof. 'If.' Suppose there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (G1), (CG), (G2), (G3) and (G4) are satisfied. By Theorem 1, C satisfies efficiency and time consistency. Thus, by Lemma 2, (C1) is satisfied. By (CG) and (G4),

$$\begin{aligned} C_t(x) \;&=\; g^t(x) - f^{-1}(g^{t+1}(x)) \\ &\leq\; g(g^t(x)) - f^{-1}(g(g^{t+1}(x))) \;=\; g^{t+1}(x) - f^{-1}(g^{t+2}(x)) \;=\; C_{t+1}(x) \end{aligned}$$

for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$. Hence, by Lemma 4, C satisfies Suppes-Sen.

'Only if.' Assume that C satisfies time consistency and Suppes-Sen. By Lemma 4, (C1) and (C2) are satisfied and, by Lemma 2, C satisfies efficiency. By Theorem 1, there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (G1), (CG) and (G2).

To show (G3), suppose there exists $x \in \mathbb{R}_+$ such that $x > g(x)$. By (C2),

$$\sum_{t=0}^{\infty} P_t(x, C(x)) C_t(x) \;\le\; \sum_{t=0}^{\infty} P_t(x, C(x)) C_{t+1}(x)\,. \tag{4}$$

Construct $y \in \mathcal{S}(x)$ by y_0 satisfying $g(g(x)) = f(x - y_0)$ and $y_t = C_{t+1}(x)$ for all $t \in \mathbb{Z}_{++}$. Since $x > g(x)$ and, by (CG), $f(x - y_0) = g(g(x)) = f(g(x) - C_1(x))$, it follows from (F1) that $y_0 > C_1(x)$. Hence,

$$\sum_{t=0}^{\infty} P_t(x, C(x)) C_{t+1}(x) \;<\; \sum_{t=0}^{\infty} P_t(x, C(x)) y_t\,. \tag{5}$$

However, (4) and (5) combined with $y \in \mathcal{S}(x)$ contradict (C1).

To show (G4), suppose there exists $x \in \mathbb{R}_+$ such that

$$x - f^{-1}(g(x)) > g(x) - f^{-1}(g(g(x))).$$

By (CG),

$$C_0(x) \;=\; x - f^{-1}(g(x)) \;>\; g(x) - f^{-1}(g(g(x))) \;=\; g(x) - f^{-1}(g^2(x)) \;=\; C_1(x)\,,$$

contradicting (C2). ■

Condition (G3) ensures sustainable development in the sense that the current consumption can potentially be shared by all future generations. In the context of a stationary technology with only one resource (or capital good), this requires that the resource stock is maintained from the current period to the next, which is just what condition (G3) entails. Condition (G4) complements (G3) by requiring that the potential for sharing present consumption with future generations actually materializes. Hence, Theorem 2 means that both the Suppes-Sen axiom and the Pigou-Dalton axiom can be used to justify sustainability in the present choice-theoretic setting.

Theorem 2 thereby echoes similar results when infinite-horizon social choice is analyzed through social welfare relations.

- In particular, Asheim, Buchholz and Tungodden (2001) show how the Suppes-Sen principle for social welfare relations can be used to rule out unsustainable consumption streams as maximal elements under technological conditions satisfied by the simple model considered here. Given such technological assumptions, this result also implies that social welfare relations like those considered in Asheim and Tungodden (2004), Basu and Mitra (2007), and Bossert, Sprumont and Suzumura (2007), which all satisfy the Suppes-Sen principle, yield sustainable consumption streams as maximal elements as long as maximal elements exist.

- Asheim (1991) shows in a similar way how the Pigou-Dalton principle for social welfare relations can be used to rule out unsustainable consumption streams.

Another equity axiom that appears to be natural in this context is *resource monotonicity*. It requires that no one should be worse off as a consequence of an increase in the initial level of the resource. See Thomson (2006) for a discussion of resource monotonicity in a variety of economic models and further references. Formulated for infinite-horizon choice functions, the axiom is defined as follows.

Resource monotonicity. For all $x, x' \in \mathbb{R}_+$,

$$x > x' \;\Rightarrow\; C(x) \geq C(x').$$

Adding resource monotonicity to efficiency and time consistency leads to the choice functions characterized in the following theorem.

Theorem 3 *An infinite-horizon choice function C satisfies efficiency, time consistency and resource monotonicity if and only if there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that* (G1), (CG), (G2),

$$g \quad \text{is non-decreasing in } x \tag{G5}$$

and

$$x \mapsto x - f^{-1}(g(x)) \quad \text{is non-decreasing in } x \tag{G6}$$

are satisfied.

Proof. 'If.' Assume that there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (G1), (CG), (G2), (G5) and (G6) are satisfied. By Theorem 1, C satisfies efficiency and time consistency. Let $x > x'$. By (G5), we have that

$$g^t(x) \geq g^t(x')$$

for all $t \in \mathbb{Z}_+$. Consequently, by (CG) and (G6),

$$\begin{aligned} C_t(x) &= g^t(x) - f^{-1}(g^{t+1}(x)) = g^t(x) - f^{-1}(g(g^t(x))) \\ &\geq g^t(x') - f^{-1}(g(g^t(x'))) = g^t(x') - f^{-1}(g^{t+1}(x')) = C_t(x') \end{aligned}$$

for all $t \in \mathbb{Z}_+$. Hence, C satisfies resource monotonicity.

'Only if.' Assume that C satisfies efficiency, time consistency and resource monotonicity. By Theorem 1, there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that (G1), (CG) and (G2) are satisfied.

To show (G5), suppose there exist $x, x' \in \mathbb{R}_+$ such that $x > x'$, but $g(x) < g(x')$. By $x > x'$ and resource monotonicity,

$$\sum_{t=1}^{\infty} P_t(x', C(x'))C_t(x) \;\geq\; \sum_{t=1}^{\infty} P_t(x', C(x'))C_t(x')\,. \tag{6}$$

Construct $y \in \mathcal{S}(x')$ by y_0 satisfying $g(x) = f(x' - y_0)$ and $y_t = C_t(x)$ for all $t \in \mathbb{Z}_{++}$. Since, by (CG), $f(x' - y_0) = g(x) < g(x') = f(x' - C_0(x'))$, it follows from (F1) that $y_0 > C_0(x')$. Hence,

$$\sum_{t=0}^{\infty} P_t(x', C(x'))y_t \;>\; \sum_{t=0}^{\infty} P_t(x', C(x'))C_t(x')\,. \tag{7}$$

However, (6) and (7) combined with $y \in \mathcal{S}(x')$ contradict (C1).

To show (G6), suppose there exist $x, x' \in \mathbb{R}_+$ such that $x > x'$, but

$$x - f^{-1}(g(x)) < x' - f^{-1}(g(x'))\,.$$

By (CG),

$$C_0(x) \;=\; x - f^{-1}(g(x)) \;<\; x' - f^{-1}(g(x')) \;=\; C_0(x')\,,$$

contradicting resource monotonicity. ■

Note that the proof of (G5) relies on efficiency, whereas (G6) is established without using this axiom.

It follows from Theorems 2 and 3 that the classes of choice functions characterized in Theorem 1 can be narrowed down considerably by adding equity axioms. However, Suppes-Sen or Pigou-Dalton, on the one hand, and resource monotonicity, on the other hand, do so in different ways.

- By Theorem 2, Suppes-Sen or efficiency and Pigou-Dalton in combination with time consistency imply that, for given $x \in \mathbb{R}_+$, $g^t(x)$ and $g^t(x) - f^{-1}(g^{t+1}(x))$ are monotone with respect to t, while

- by Theorem 3, resource monotonicity in combination with efficiency and time consistency implies that $g^t(x)$ and $g^t(x) - f^{-1}(g^{t+1}(x))$ are monotone with respect to x for given $t \in \mathbb{Z}_+$.

5 Examples

Consider first the *steady-state* example, where consumption is equalized across generations.

Example 1. The infinite-horizon choice function C^1 of this example corresponds to the case in which the function g is the identity mapping, defined by $g(x) = x$ for all $x \in \mathbb{R}_+$. This implies $g^t(x) = x$ for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$. (G1) and (G2) are satisfied because

$$g(x) = x \leq f(x)$$

and

$$\lim_{t\to\infty} P_t(x, C^1(x))g^t(x) = \lim_{t\to\infty} \frac{g^t(x)}{\prod_{\tau=1}^{t}[f'(f^{-1}(g^\tau(x)))]} = \lim_{t\to\infty} \frac{x}{[f'(f^{-1}(x))]^t} = 0$$

for all $x \in \mathbb{R}_+$, where use is made of (F1) and (F2). According to (CG), $C_t^1(x)$ satisfies

$$x = g^{t+1}(x) = f(g^t(x) - C_t^1(x)) = f(x - C_t^1(x)) \tag{8}$$

for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$, that is, every generation consumes the same amount.

In addition to satisfying efficiency and time consistency, the infinite-horizon choice function C^1 is characterized by a g-function for which the conditions of (G3) and (G4) hold with equality. By Theorem 2 this entails that C^1 satisfies both Suppes-Sen and Pigou-Dalton. Furthermore, both $g(x)$ and $x - f^{-1}(g(x))$ are non-decreasing in x. Hence, by Theorem 3, the choice function satisfies resource monotonicity, as can easily be verified directly from (8), by invoking (F2).

The remaining examples are formulated for a linear technology, i.e., $f(k) = k(1 + r)$, with $r > 0$. The renewal rate r is assumed to be positive to satisfy (F2) and ensure that all choice functions are well-defined. It is possible to generalize these examples by considering certain non-linear technologies but we restrict attention to the linear case for the sake of simplicity of exposition. With $f(k) = k(1 + r)$, we have that

(i) $P_t(x, y) = 1/(1 + r)^t$ for all $t \in \mathbb{Z}_+$, for all $x \in \mathbb{R}_+$ and for all $y \in \mathcal{S}(x)$, and

(ii) $f^{-1}(x) = x/(1 + r)$ for all $x \in \mathbb{R}_+$ so that (CG) can be written as

$$C_t(x) = g^t(x) - \frac{g^{t+1}(x)}{1 + r} \quad \text{for all } t \in \mathbb{Z}_+ \text{ and for all } x \in \mathbb{R}_+ \,. \tag{CG$'$}$$

In the case where the technology is linear, the choice function C^1 of Example 1 can be generalized by letting g be a linear function such that both $g(x)$ and $x - f^{-1}(g(x)) = x - (g(x)/(1+r))$ are non-decreasing in x, so that resource monotonicity is satisfied. This is investigated in the following example.

Example 2. The infinite-horizon choice function $C^{2,a}$ of this example is obtained by letting $g(x) = ax$ for all $x \in \mathbb{R}_+$, where $a \in [0, 1+r]$ is a parameter. Obviously, the steady-state case is obtained for $a = 1$. It follows that $g^t(x) = a^t x$ for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$. Clearly, (G1) is satisfied because

$$g(x) = ax \leq x(1+r)$$

for all $x \in \mathbb{R}_+$. (G2) is satisfied if and only if $a < 1+r$, because

$$\lim_{t\to\infty} P_t(x, C^{2,a}(x))g^t(x) = \lim_{t\to\infty} \frac{a^t x}{(1+r)^t} = \lim_{t\to\infty} \left(\frac{a}{1+r}\right)^t x = 0.$$

Hence, the case where $a = 1+r$ illustrates how (G2) can be violated by excessive accumulation of the resource.

Substituting into (CG′), it follows that

$$C_t^{2,a}(x) = g^t(x) - \frac{g^{t+1}(x)}{1+r} = a^t x - \frac{a^{t+1}x}{1+r} = \frac{a^t(1+r-a)x}{1+r} \tag{9}$$

for all $x \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$.

In addition to satisfying efficiency and time consistency for $a < 1+r$, the infinite-horizon choice function $C^{2,a}$ is characterized by a g-function for which the conditions of (G3) and (G4) hold if and only if $a \geq 1$. By Theorem 2 this entails that $C^{2,a}$ satisfies efficiency, time consistency, Suppes-Sen and Pigou-Dalton if and only if $a \in [1, 1+r)$. If $a \in (1, 1+r)$, then consumption is increasing in t, and the consumption of generations t such that

$$t > \frac{\ln(r) - \ln(1+r-a)}{\ln(a)}$$

is higher than that of the steady-state, at the expense of earlier generations. Moreover, the consumption of generation t approaches infinity as t approaches infinity.

Both $g(x)$ and $x - (g(x)/(1+r))$ are non-decreasing in x for any $a \in [0, 1+r]$. Hence, by Theorem 3, the choice function satisfies efficiency, time consistency and resource monotonicity if and only if $a \in [0, 1+r)$, as can easily be verified directly from (9). Therefore, $C^{2,a}$ satisfies resource monotonicity, but not Suppes-Sen and Pigou-Dalton, if and only if $a \in [0,1)$. If $a \in (0,1)$, then consumption is decreasing in t, and the consumption of generations t such that

$$t < \frac{\ln(r) - \ln(1 + r - a)}{\ln(a)}$$

is higher than that of the steady-state, at the expense of later generations. Moreover, the consumption of generation t approaches zero as t approaches infinity.

Example 2 shows, in the case where $a < 1$, that $g^t(x)$ and $g^t(x) - (g^{t+1}(x)/(1 + r))$ can be non-decreasing with respect to x, without $g^t(x)$ and $g^t(x) - (g^{t+1}(x)/(1+r))$ being non-decreasing with respect to t. In particular, a choice function can satisfy resource monotonicity without satisfying Suppes-Sen and Pigou-Dalton. In the following pair of examples, we show that a choice function can satisfy Suppes-Sen and Pigou-Dalton without satisfying resource monotonicity.

Example 3. The infinite-horizon choice function C^3 of this example is obtained by setting $r = 1$, so that $1 + r = 2$, and by letting g be given by:

$$g(x) \; = \; \begin{cases} \frac{3}{2}x & \text{if} \quad 0 \leq x \leq 1, \\ \frac{4}{3}x & \text{if} \quad x > 1\,. \end{cases}$$

Clearly, (G1) is satisfied. Also, $x \leq g(x)$ for all $x \in \mathbb{R}_+$ so that (G3) is satisfied, and $x - g(x)/2$ is an increasing function of x so that (G6) is satisfied. By combining these observations we obtain that $x - g(x)/2 \leq g(x) - g^2(x)/2$ for all $x \in \mathbb{R}_+$ so that (G4) is satisfied. Furthermore, if $x \in \mathbb{R}_{++}$, then C^3 behaves as $C^{2,a}$ with $a \in (0, 1 + r)$ when t goes to infinity, implying that (G2) is satisfied. If $x = 0$, then (G2) is trivially satisfied. Hence, it follows from Theorem 2 that the infinite-horizon choice function C^3 satisfies efficiency, time consistency, Suppes-Sen and Pigou-Dalton. However,

$$g(1) \; = \; \tfrac{3}{2} \; > \; \tfrac{17}{12} \; = \; g\big(\tfrac{17}{16}\big).$$

Hence, (G5) does not hold, and it follows from Theorem 3 that C^3 does not satisfy resource monotonicity.

Example 4. The infinite-horizon choice function C^4 of this example is obtained by setting $r = 1$, so that $1 + r = 2$, and by letting g be given by:

$$g(x) \; = \; \begin{cases} \frac{4}{3}x & \text{if} \quad 0 \leq x \leq 1, \\ \frac{3}{2}x & \text{if} \quad x > 1. \end{cases}$$

Clearly, (G1) is satisfied. Also, $x \leq g(x)$ for all $x \in \mathbb{R}_+$ so that (G3) is satisfied, and $g(x)$ is an increasing function of x so that (G5) is satisfied. Furthermore, if $x \in \mathbb{R}_{++}$, then C^4 behaves as $C^{2,a}$ with $a \in (0, 1+r)$ when t goes to infinity, implying that (G2) is satisfied. If $x = 0$, then (G2) is trivially satisfied. To verify that (G4) is satisfied, note that

$$\begin{aligned}
x - \tfrac{g(x)}{2} &= \big(1 - \tfrac{2}{3}\big)x = \tfrac{1}{3}x \leq \tfrac{4}{9}x = \big(\tfrac{4}{3} - \tfrac{8}{9}\big)x = g(x) - \tfrac{g^2(x)}{2} && \text{if } 0 \leq x \leq \tfrac{3}{4}, \\
x - \tfrac{g(x)}{2} &= \big(1 - \tfrac{2}{3}\big)x = \tfrac{1}{3}x = \tfrac{1}{3}x = \big(\tfrac{4}{3} - 1\big)x = g(x) - \tfrac{g^2(x)}{2} && \text{if } \tfrac{4}{3} < x \leq 1, \\
x - \tfrac{g(x)}{2} &= \big(1 - \tfrac{3}{4}\big)x = \tfrac{1}{4}x \leq \tfrac{3}{8}x = \big(\tfrac{3}{2} - \tfrac{9}{8}\big)x = g(x) - \tfrac{g^2(x)}{2} && \text{if } x > 1.
\end{aligned}$$

Hence, it follows from Theorem 2 that the infinite-horizon choice function C^4 satisfies efficiency, time consistency, Suppes-Sen and Pigou-Dalton. However,

$$1 - \tfrac{g(1)}{2} = 1 - \tfrac{2}{3} = \tfrac{1}{3} > \tfrac{5}{18} = \tfrac{10}{9} - \tfrac{5}{6} = \tfrac{10}{9} - \tfrac{g(10/9)}{2}.$$

Hence, (G6) does not hold, and it follows from Theorem 3 that C^4 does not satisfy resource monotonicity.

Examples 2, 3 and 4 show that the conditions characterizing Suppes-Sen and Pigou-Dalton—namely that $g^t(x)$ and $g^t(x) - g^{t+1}(x)/(1+r)$ are monotone with respect to t—are independent of the conditions characterizing resource monotonicity—namely that $g^t(x)$ and $g^t(x) - g^{t+1}(x)/(1+r)$ are monotone with respect to x.

We conclude with an example showing that condition (G5) is not necessary for an infinite-horizon choice function to satisfy time consistency and resource monotonicity, as long as efficiency is not imposed.

Example 5. The infinite-horizon choice function C^5 of this example is obtained by setting $r = 1$, so that $1 + r = 2$, and by letting g be given by:

$$g(x) = \begin{cases} 2x & \text{if } \quad 0 \leq x \leq 1, \\ 2(x - \tfrac{1}{2}) & \text{if } \quad x > 1. \end{cases}$$

Clearly (G1) is satisfied, while condition (G5) is not satisfied, since

$$g(1) = 2 > \tfrac{3}{2} = g\big(\tfrac{5}{4}\big).$$

Resource monotonicity still holds since, by substituting into (CG′), it follows that

$$C^5(x) = \begin{cases} (0, 0, \dots) & \text{if } \quad x = 0, \\ \big(\underbrace{0, \dots, 0}_{n+1 \text{ times}}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\big) & \text{if } \quad x \in \big((\tfrac{1}{2})^{n+1}, (\tfrac{1}{2})^n\big] \quad \text{for } n \in \mathbb{Z}_+, \\ \big(\tfrac{1}{2}, \tfrac{1}{2}, \dots\big) & \text{if } \quad x > 1. \end{cases}$$

It is straightforward to verify that C^5 does not satisfy efficiency; in particular, increasing the initial resource stock beyond x does not lead to increased consumption for any generation, provided that $x > 1$.

Examples 1 and 2 provide infinite-horizon choice functions that are continuous in the initial endowment, even though there are no continuous orderings satisfying strong Pareto and finite anonymity that rationalize them. This observation serves to further underline the gains that are possible from adopting a choice-theoretic approach.

6 Concluding remarks

We conclude this paper with some thoughts on possible directions where the approach of this paper might be taken in future work. An issue that suggests itself naturally when considering a choice function is its *rationalizability* by a relation defined on the objects of choice—in our case, infinite consumption streams. The rationalizability of choice functions with arbitrary domains has been examined thoroughly in contributions such as Richter (1966) and Hansson (1968) and, more recently, Bossert, Sprumont and Suzumura (2005) and Bossert and Suzumura (2006). While the generality of the results obtained in these papers allows for their application in our intergenerational setting, it might be possible to obtain new observations due to the specific structure of the domain considered here. Note that the existence of a rationalizing ordering does *not* conflict with the impossibility results established for such orderings in the earlier literature: the existence of a rationalization of an infinite-horizon choice function satisfying requirements such as Suppes-Sen does not imply that the choice function is rationalizable by an ordering that possesses properties such as the Suppes-Sen principle formulated for binary relations.

An interesting difference emerges when, in the linear case, the technology parameter r is equal to zero instead of positive. For $r = 0$, Suppes-Sen and the conjunction of efficiency and Pigou-Dalton no longer are equivalent—in fact, their implications are strikingly different. In this case, the Pigou-Dalton principle rules out the choice of any unequal stream. Thereby the principle becomes incompatible with efficiency because, for any finite initial endowment x, the only possible equitable choice is zero consumption in every period, which clearly violates efficiency if x is positive. On the other hand, Suppes-Sen reduces to efficiency because no stream that is not dominated according to the efficiency criterion is dominated by a permutation of any feasible stream.

As mentioned earlier, we made the conscious choice to work with a simple model in order to emphasize the novel aspect of the paper—the choice-theoretic approach in an infinite-horizon setting. It might turn out to be of interest to explore possible generalizations in future work.

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