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Rational Choice on Arbitrary Domains: A Comprehensive Treatment*

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Abstract

The rationalizability of a choice function on arbitrary domains by means of a transitive relation has been analyzed thoroughly in the literature. Moreover, characterizations of various versions of consistent rationalizability have appeared in recent contributions. However, not much seems to be known when the coherence property of quasi-transitivity or that of P-acyclicity is imposed on a rationalization. The purpose of this paper is to fill this significant gap. We provide a unified approach in order to characterize all forms of rationalizability on arbitrary domains. *Journal of Economic Literature* Classification No.: D11.

Keywords: Rational Choice, Quasi-Transitivity, P-Acyclicity.

1 Introduction

The question whether observed (individual or social) choice behaviour can be generated by some notion of optimization is one of the most fundamental issues in the analysis of economic decisions. The basic question to be addressed is the following. Given some observed (or, at least, observable) choices from feasible sets, does there exist a preference relation (possibly with some additional properties) such that, for each choice situation under consideration, the set of chosen objects is given by some set of 'best' elements according to this relation? There are two basic forms of rationalizability, namely, *greatest-element rationalizability* and *maximal-element rationalizability.* Greatest-element rationalizability requires the existence of a relation such that, for any feasible set in the domain of a choice function, the set of chosen elements coincides with those elements of the feasible set that are at least as good as all feasible alternatives. Maximal-element rationalizability, on the other hand, demands the existence of a relation such that the set of chosen objects consists of all undominated elements in the feasible set, that is, all elements such that there exists no feasible alternative that is strictly preferred.

In addition to the basic type of rationalizability (in terms of greatest elements or in terms of maximal elements), we can require the rationalizing relation to be endowed with certain fundamental properties. Two standard requirements are *reflexivity* and *completeness,* which we refer to as *richness* properties because they require a relation to contain at least certain pairs of alternatives. Furthermore, it is customary to impose *coherence* properties such as *transitivity, consistency, quasi-transitivity* or *P-acyclicity;* see the following section for formal definitions.

Revealed preference theory has its origins in the theory of consumer demand, where the choices to be analyzed are those of a competitive consumer from budget sets. This area of research has been developed in contributions such as those of Samuelson (1938; 1947, Chapter V; 1948; 1950) and Houthakker (1950).

Uzawa (1957) and Arrow (1959) considered alternative choice situations by introducing the general concept of a choice function defined on the domain of all subsets of a universal set of alternatives. In this setting, Sen (1971), Schwartz (1976), Bandyopadhyay and Sengupta (1991), to name but a few, characterized notions of rational choice under various coherence conditions imposed on rationalizing relations. Most notably, the theory of rational choice on such rich domains was greatly simplified by the equivalence results between several revealed preference axioms, for example, the weak axiom of revealed preference and the strong axiom of revealed preference, whose subtle difference had been regarded as lying at the heart of the integrability problem for a competitive consumer.

Clearly, the above-described assumptions regarding the class of possible choice situations to be considered restrict the applicability of the results obtained. Thus, it is of great interest to examine what the logic of rational choice—and nothing else—entails in general, irrespective of the domain of a choice function. A crucial step along this line was taken by Richter (1966; 1971), Hansson (1968) and Suzumura (1976; 1977; 1983, Chapter 2) who assumed the domain of a choice function to be an arbitrary family of non-empty subsets of an arbitrary non-empty universal set of alternatives without any further structural assumptions.

While the theory of rational choice on arbitrary domains is well-developed if a rationalizing relation is assumed to be transitive, much less is known when weaker coherence properties are imposed. The case of consistent rationalizability has been addressed recently in Bossert, Sprumont and Suzumura (2005a). Furthermore, some versions of maximal-element rationalizability (and, thus, those versions of greatest-element rationalizability that are equivalent to them) have been characterized in Bossert, Sprumont and Suzumura (2005c) but a comprehensive treatment of rationalizability on arbitrary domains in the presence of quasi-transitivity or P-acyclicity is still missing. An analysis of some conditions that are necessary and others that are sufficient for some forms of quasi-transitive or P-acyclical rationalizability can be found in Suzumura (1983) and Bossert, Sprumont and Suzumura (2005b) but full characterizations of most of these concepts have not yet been provided.

The purpose of this paper is to fill the above-mentioned gap in the literature, thereby providing characterizations of all relevant notions of rationalizability on arbitrary domains. Thus, the results of this paper provide systematic answers to some important open questions in the literature on rational choice and revealed preference. It may be worth pointing out that answering these open questions have more substantial relevance than simply filling in logical lacunae in the literature. Recollect that assuming the rationalizing weak preference relation to be transitive entices strong empirical criticism. There are many experimental studies that suggest that the imperfect discriminatory power of human beings leads to non-transitive indifference and Armstrong (1948, p.3) went as far as to assert: "That indifference is not transitive is indisputable, and the world in which it were transitive is indeed unthinkable." Thus, to liberate the theory of rationalizable choice functions from the unsustainable assumption of transitive indifference seems to be an important step for the sake of making the theoretical edifice more of empirical relevance than otherwise.

Our basic definitions and some preliminary observations are collected in Section 2. Section 3 introduces our different notions of rationalizability and examines their logical relationships on arbitrary domains. Section 4 is devoted to characterizations of all versions of rationalizability defined in Section 3. Section 5 concludes.

2 Preliminaries

We consider a non-empty (but otherwise arbitrary) universal set of alternatives X, and we let $R \subseteq X \times X$ be a (binary) relation on X. The *asymmetric factor* $P(R)$ of R is defined by

$$P(R) = \{(x, y) \in X \times X \mid (x, y) \in R \text{ and } (y, x) \notin R\}.$$

The *symmetric factor* $I(R)$ of R is defined by

$$I(R) = \{(x, y) \in X \times X \mid (x, y) \in R \text{ and } (y, x) \in R\}.$$

If R is interpreted as a *weak preference relation,* that is, $(x, y) \in R$ means that x is considered at least as good as y, then $P(R)$ and $I(R)$ can be interpreted as the *strict preference relation* and the *indifference relation* corresponding to R, respectively. The *diagonal relation* on X is given by $\Delta = \{(x, x) \mid x \in X\}$.

Let $\mathbb{N}$ denote the set of positive integers. The following properties of a binary relation R are of importance in this paper.

Reflexivity. For all $x \in X$,

$$(x, x) \in R.$$

Completeness. For all $x, y \in X$ such that $x \neq y$,

$$(x, y) \in R \text{ or } (y, x) \in R.$$

Transitivity. For all $x, y, z \in X$,

$$[(x, y) \in R \text{ and } (y, z) \in R] \;\Rightarrow\; (x, z) \in R.$$

Quasi-transitivity. For all $x, y, z \in X$,

$$[(x, y) \in P(R) \text{ and } (y, z) \in P(R)] \;\Rightarrow\; (x, z) \in P(R).$$

Consistency. For all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \;\Rightarrow\; (x^K, x^0) \notin P(R).$$

P-acyclicity. For all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in P(R) \text{ for all } k \in \{1, \ldots, K\} \;\Rightarrow\; (x^K, x^0) \notin P(R).$$

A reflexive and transitive relation is called a *quasi-ordering* and a complete quasi-ordering is called an *ordering.*

We refer to reflexivity and completeness as *richness* conditions. This term is motivated by the observation that the properties in this group require that, at least, some pairs must belong to the relation under consideration. In the case of reflexivity, all pairs of the form (x, x) are required to be in the relation, whereas completeness demands that, for any two distinct alternatives x and y, at least one of (x, y) and (y, x) must be in R. Clearly, the reflexivity requirement is equivalent to the set inclusion $\Delta \subseteq R$.

Transitivity, quasi-transitivity, consistency and P-acyclicity are *coherence* properties. They require that if certain pairs belong to R, then certain other pairs must belong to R as well (as is the case for transitivity and for quasi-transitivity) or certain other pairs cannot belong to R (which applies to the cases of consistency and of P-acyclicity). Quasi-transitivity and consistency are independent. A transitive relation is quasi-transitive, and a quasi-transitive relation is P-acyclical. Moreover, a transitive relation is consistent, and a consistent relation is P-acyclical. The reverse implications are not true in general. However, the distinction between transitivity and consistency disappears for a reflexive and complete relation; see Suzumura (1983, p.244). Thus, if a relation R on X is reflexive, complete and consistent, then R is transitive, hence an ordering.

Transitivity is *the* classical coherence requirement on preference relations and its significance in theories of individual and collective choice is obvious. Quasi-transitivity was introduced by Sen (1969; 1970, Chapter 1*), and it has been employed in numerous approaches to the theory of individual and social choice, including issues related to rationalizability. P-acyclicity has the important property that it is not only sufficient for the existence of undominated choices from any arbitrary finite subset of a universal set, but it is also necessary for the existence of such choices from all possible finite subsets of the universal set; see Sen (1970, Chapter 1*).

Violations of transitivity are quite likely to be observed in practical choice situations. For instance, Luce's (1956) well-known coffee-sugar example provides a plausible argument against assuming that indifference is always transitive: the inability of a decision maker to perceive 'small' differences in alternatives is bound to lead to intransitivities. As this example illustrates, transitivity frequently is too strong an assumption to impose in the context of individual choice. In collective choice problems, it is even more evident that the plausibility of transitivity can be questioned. The concept of consistency, which is due to Suzumura (1976), is of particular interest in this context. To underline its importance, note that this property is exactly what is required to prevent the problem of a 'money pump.' If consistency is violated, there exists a preference cycle with at least one strict preference. In this case, an agent with such preferences is willing to trade (where 'willingness to trade' is assumed to require that the alternative acquired in the trade is at least as good as that relinquished) an alternative x^K for another alternative x^{K-1}, x^{K-1} for an alternative x^{K-2} and so on until we reach an alternative x^0 such that the agent *strictly prefers* getting back to x^K to retaining possession of x^0. Thus, at the end of a chain of exchanges, the agent is willing to pay a positive amount in order to get back to the alternative it had in its possession in the first place—a classical example of a money pump.

There is yet another reason for the importance of the concept of consistency. As shown by Suzumura (1976; 1983, Chapter 1), consistency is necessary and sufficient for the existence of an ordering which subsumes all the pairwise information contained in the binary relation. This result is a generalization of Szpilrajn's (1930) classical result on extending quasi-orderings to orderings.

The *transitive closure* $tc(R)$ of a relation R on X is defined by

$$\begin{aligned} tc(R) &= \{(x,y) \in X \times X \mid \exists K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that} \\ &\qquad x = x^0, (x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}. \end{aligned}$$

Clearly, for any relation R, $R \subseteq tc(R)$. Furthermore, R is transitive if and only if $R = tc(R)$. Therefore, $tc(R)$ is a transitive superset of R for any R. The crucial importance of the transitive closure of a relation R lies in its property of being the unique smallest transitive relation containing R.

Analogously, the *consistent closure* $sc(R)$ of a relation R is defined by

$$sc(R) = R \cup \{(x,y) \mid (x,y) \in tc(R) \text{ and } (y,x) \in R\}.$$

We have $R \subseteq sc(R)$ for any relation R, and the set inclusion is satisfied with an equality if and only if R is itself consistent. Moreover, just as $tc(R)$ is the unique smallest transitive relation containing the relation R, $sc(R)$ is the unique smallest consistent relation containing R.

To illustrate the definition of the consistent closure and its relationship to the transitive closure, consider the following examples. Let $X = \{x, y, z\}$, and define two relations R and R' on X by $R = \{(x,x), (x,y), (y,y), (y,z), (z,x), (z,z)\}$ and $R' = \{(x,y), (y,z)\}$. We obtain $sc(R) = tc(R) = X \times X$, $sc(R') = R'$ and $tc(R') = \{(x,y), (y,z), (x,z)\}$. The consistent closure of R coincides with the transitive closure thereof, whereas the consistent closure of R' is a strict subset of the transitive closure of R'. More generally, for any relation R on X, $sc(R)$ is always a subset of $tc(R)$. Thus, for any relation R,

$$R \subseteq sc(R) \subseteq tc(R).$$

Let $\mathcal{X}$ be the set of all non-empty subsets of X. We now introduce the concepts of greatestness and maximality with respect to a relation. Suppose R is a relation on X and $S \in \mathcal{X}$. The set $G(S, R)$ of all *R-greatest elements* of S is defined by

$$G(S, R) = \{x \in S \mid (x, y) \in R \text{ for all } y \in S\} \tag{1}$$

and the set $M(S, R)$ of all *R-maximal elements* of S is defined by

$$M(S, R) = \{x \in S \mid (y, x) \notin P(R) \text{ for all } y \in S\}.$$

As is straightforward to verify, $G(S, R) \subseteq M(S, R)$ for all relations R on X and for all $S \in \mathcal{X}$. Furthermore, if R is reflexive and complete, then $G(S, R) = M(S, R)$; for relations R that are not reflexive or not complete, the set inclusion can be strict.

A *choice function* is a mapping that assigns, to each feasible set in its domain, a subset of this feasible set. This subset is interpreted as the set of chosen alternatives. The domain of the choice function depends on the choice situation to be analyzed, but it will always be a set of subsets of X, that is, a subset of $\mathcal{X}$. We assume this subset of $\mathcal{X}$ to be non-empty to avoid degenerate situations. Thus, letting $\Sigma \subseteq \mathcal{X}$ be a non-empty domain, a choice function defined on that domain is a mapping $C \colon \Sigma \to \mathcal{X}$ such that, for all $S \in \Sigma$, $C(S) \subseteq S$. The *image of* Σ *under* C is given by $C(\Sigma) = \cup_{S \in \Sigma} C(S)$.

The *direct revealed preference relation* R_C of a choice function C with domain Σ is defined as

$$R_C = \{(x, y) \in X \times X \mid \exists S \in \Sigma \text{ such that } x \in C(S) \text{ and } y \in S\}.$$

We conclude this section with the statement of a fundamental result regarding the existence of *ordering extensions.* A relation R' is an *extension* of a relation R if and only if (i) $R \subseteq R'$; and (ii) $P(R) \subseteq P(R')$. Conversely, R is said to be a *subrelation* of R' if and only if R' is an extension of R. The following classical theorem, which is a variant of the basic theorem due to Szpilrajn (1930), specifies a sufficiency condition for the existence of an extension that is an ordering, to be called an *ordering extension.* This convenient variant of Szpilrajn's theorem was stated by Arrow (1951, p.64) without a proof, whereas Hansson (1968) provided a full proof thereof on the basis of Szpilrajn's original theorem. Arrow (1951) and Suzumura (1976; 1983, Chapter 1; 2004) provide generalizations of this result.

Theorem 1 *Any quasi-ordering R on X has an ordering extension.*

3 Definitions of Rationalizability

There are two basic forms of rationalizability properties that are commonly considered in the literature. The first is *greatest-element rationalizability* which requires the existence of a relation such that, for any feasible set, every chosen alternative is at least as good as every alternative in the set. Thus, this notion of rationalizability is based on the view that chosen alternatives should weakly dominate all feasible alternatives. *Maximal-element rationalizability,* on the other hand, demands the existence of a relation such that, for each feasible set, there exists no alternative in this set that is strictly preferred to any one of the chosen alternatives. Hence, this version of rationalizability does not require chosen alternatives to weakly dominate all elements of the feasible set but, instead, demands that they are not strictly dominated by any other feasible alternative.

In addition to one or the other of these two concepts of rationalizability, we have a choice regarding the properties that we require a rationalizing relation to possess. We consider the standard richness requirements of *reflexivity* and *completeness* and, in addition, the coherence properties of *transitivity, quasi-transitivity, consistency* and *P-acyclicity.* By combining each version of rationalizability with one or both (or none) of the richness conditions and with one (or none) of the coherence properties, various definitions of rationalizability are obtained. Some of these definitions are equivalent, others are independent, and some are implied by others. To get an understanding of what each of these definitions entails, we summarize all logical relationships between them in this section.

A choice function C is *greatest-element rationalizable, G-rationalizable* for short, if there exists a relation R on X, to be called a *G-rationalization* of C, such that $C(S) = G(S, R)$ for all $S \in \Sigma$. Analogously, a choice function C is *maximal-element rationalizable, M-rationalizable* for short, if there exists a relation R on X, to be called an *M-rationalization* of C, such that $C(S) = M(S, R)$ for all $S \in \Sigma$. If a rationalization R is required to be reflexive and complete, the notion of greatest-element rationalizability coincides with that of maximal-element rationalizability because, in this case, $G(S, R) = M(S, R)$ for all $S \in \mathcal{X}$. Without these properties, however, this is not necessarily the case. Greatest-element rationalizability is based on the idea of chosen alternatives weakly dominating all alternatives in the feasible set under consideration, whereas maximal-element rationalizability requires chosen elements not to be strictly dominated by any other feasible alternative.

The following theorem presents a fundamental relationship between the direct revealed preference relation and a G-rationalization of a choice function. This observation, which is due to Samuelson (1938; 1948), states that any G-rationalization of a G-rationalizable choice function must respect the direct revealed preference relation of this choice function. This observation follows immediately from combining the definitions of the direct revealed preference relation R_C and of G-rationalizability. Moreover, an analogous result is valid for the relationship between G-rationalizability by a consistent relation and the consistent closure of R_C (see Bossert, Sprumont and Suzumura, 2005a) and for the transitive closure of R_C and G-rationalizability by a transitive relation (see Richter, 1971).

Theorem 2 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$ and R is a relation on X.*

(i) If R is a G-rationalization of C, then $R_C \subseteq R$.

(ii) If R is a consistent G-rationalization of C, then $sc(R_C) \subseteq R$.

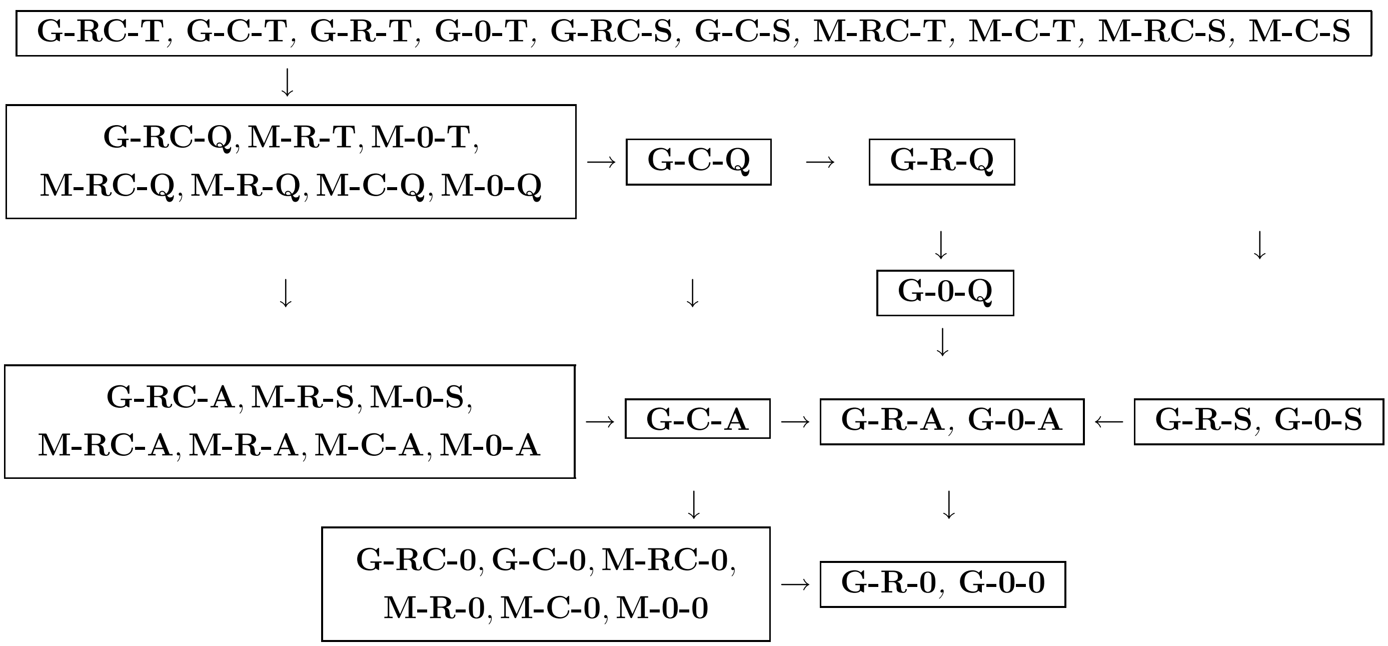
(iii) If R is a transitive G-rationalization of C, then $tc(R_C) \subseteq R$.

Analogous set inclusions are not valid for M-rationalizability: an M-rationalization does not necessarily have to respect the direct revealed preference relation because chosen alternatives merely have to be undominated within the feasible set from which they are chosen.

Depending on the additional properties that we might want to impose on a rationalization (if any), different notions of rationalizability can be defined. For simplicity of presentation, we use the following convention when formulating a rationalizability axiom. We distinguish three groups of properties of a relation, namely, *rationalization* properties, *richness* properties and *coherence* properties. The first group consists of the two rationalizability properties of G-rationalizability and M-rationalizability, the second of the two requirements of reflexivity and completeness and, finally, the third of the axioms of transitivity, quasi-transitivity, consistency and P-acyclicity. Greatest-element rationalizability is abbreviated by **G**, **M** is short for maximal-element rationalizability, **R** stands for reflexivity and **C** is completeness. Transitivity, quasi-transitivity, consistency and P-acyclicity are denoted by **T**, **Q**, **S** and **A**, respectively. We identify the property or properties to be satisfied within each of the three groups and separate the groups with hyphens. If none of the properties within a group is required, this is denoted by using the symbol **0**. Either greatest-element rationalizability or maximal-element rationalizability may be required. In addition to imposing one of the two richness properties only, reflexivity and completeness may be required simultaneously and we may require rationalizability properties without either of the two. We only consider notions of rationalizability involving at most one of the coherence properties at a time. As is the case for the richness properties, imposing none of the coherence properties is a possibility. Formally, a rationalizability property is identified by an expression of the form α-β-γ, where $\alpha \in \{\mathbf{G}, \mathbf{M}\}$, $\beta \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \mathbf{0}\}$ and $\gamma \in \{\mathbf{T}, \mathbf{Q}, \mathbf{S}, \mathbf{A}, \mathbf{0}\}$. For example, greatest-element rationalizability by a reflexive, complete and transitive relation is denoted by **G-RC-T**, maximal-element rationalizability by a complete relation is **M-C-0**, greatest-element rationalizability by a reflexive and consistent relation is **G-R-S** and maximal-element rationalizability without any further properties of a rationalizing relation is **M-0-0**. Clearly, according to this classification, there are $2 \cdot 4 \cdot 5 = 40$ versions of rationalizability.

We now provide a full description of the logical relationships between these different notions of rationalizability. This result synthesizes contributions due to Bossert, Sprumont and Suzumura (2005a; 2005b; 2005c) and, therefore, we do not provide a proof; see the original papers for details. For convenience, a diagrammatic representation is employed. All axioms that are depicted within the same box are equivalent, and an arrow pointing from one box b to another box b' indicates that the axioms in b imply those in b', and the converse implication is not true. In addition, of course, all implications resulting from chains of arrows depicted in the diagram are valid.

Theorem 3 *Suppose $C\colon \Sigma \rightarrow \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. Then*



Although the equivalences established in the above theorem reduce the number of distinct notions of rationalizability from a possible forty to eleven, there remains a relatively rich set of possible definitions. In particular, note that none of the coherence properties of transitivity, quasi-transitivity, consistency and P-acyclicity is redundant: eliminating any one of them reduces the number of distinct definitions, and so does the elimination of the versions not involving any coherence property.

Furthermore, it is worth pointing out an important and remarkable difference between G-rationalizability by a transitive or a consistent relation on the one hand and G-rationalizability by a quasi-transitive or a P-acyclical relation on the other. In the case of G-rationalizability with transitivity or consistency, the reflexivity requirement is redundant in all cases. That is, irrespective of whether or not completeness is imposed as a richness condition, any version of G-rationalizability with transitivity or consistency and without reflexivity is equivalent to the definition that is obtained if reflexivity is added. This observation applies to the case where no coherence property is imposed as well. In contrast, G-rationalizability by a complete and quasi-transitive relation is not equivalent to G-rationalizability by a reflexive, complete and quasi-transitive relation, and the same is true for the relationship between G-rationalizability by a complete and P-acyclical relation and G-rationalizability by a reflexive, complete and P-acyclical relation. In addition, while G-rationalizability by a P-acyclical relation and G-rationalizability by a reflexive and P-acyclical relation are equivalent, there is yet another discrepancy in the quasi-transitive case: G-rationalizability by a quasi-transitive relation is not the same as G-rationalizability by a reflexive and quasi-transitive relation.

In the case of M-rationalizability, only four distinct notions of rationalizability exist although, in principle, there are twenty definitions, as in the case of G-rationalizability. This means that there is a dramatic reduction of possible definitions due to the equivalences established in the theorem. Note that, within the set of definitions of M-rationalizability, there is a substantial degree of redundancy. In particular, it is possible to generate all possible versions of M-rationalizability with merely two coherence properties: any of the combinations of transitivity and consistency, transitivity and P-acyclicity, quasi-transitivity and consistency is sufficient to obtain all four notions of M-rationalizability (provided, of course, that the option of not imposing any coherence property is retained). Moreover, reflexivity is redundant in all forms of M-rationalizability, irrespective of the coherence property imposed (if any), including quasi-transitivity and P-acyclicity: any version of M-rationalizability without reflexivity is equivalent to the version obtained by adding this richness property.

There is an interesting feature that distinguishes the notions of transitive or consistent M-rationalizability from those involving quasi-transitivity, P-acyclicity or none of the coherence properties. All four M-rationalizability properties involving quasi-transitivity are equivalent, and so are all four notions involving P-acyclicity as well as the four versions without any coherence property. In contrast, there are two distinct notions of transitive M-rationalizability and two distinct notions of consistent M-rationalizability.

As is apparent from Theorem 3, M-rationalizability does not add any new versions of rationalizability, provided that all definitions of G-rationalizability involving all of the four combinations of coherence properties are present. Therefore, we can, without loss of generality, restrict attention to G-rationalizability in the characterization results stated in the following section.

4 Characterizations

This section constitutes the main contribution of the paper. We present necessary and sufficient conditions for all notions of G-rationalizability. Some of these forms of G-rationalizability have been characterized before; see, for instance, Richter (1966), Hansson (1968) and Suzumura (1977) for the case of **G-RC-T** and its equivalents, Richter (1971) for the case of **G-R-0** and **G-0-0**, and Bossert, Sprumont and Suzumura (2005a) for the case of **G-R-S** and **G-0-S**. However, for the sake of a comprehensive treatment, we provide new axiomatizations of these notions of rationalizability as well within our unified framework. The results stated in the remainder of the paper are extremely general because they apply to any arbitrary domain, and they serve to close an important gap in the literature: they provide characterizations of all notions of rationalizability, including those that have not been axiomatized before.

The notions of rationalizability considered in this paper involve relatively complex formulations of necessary and sufficient conditions. The reason is that there is no such thing as a unique *smallest* quasi-transitive relation or a unique *smallest* P-acyclical relation containing a given arbitrary relation, and similar ambiguities exist in the absence of coherence properties when completeness is imposed. In contrast, any relation R has a well-defined transitive closure and a well-defined consistent closure as defined in Section 2. Intuitively, when moving from R to its transitive or consistent closure, pairs are added that are *necessarily* in any transitive or consistent relation containing R. As soon as there exist alternatives $x^0, \ldots, x^K$ connecting two alternatives x and y via a chain of weak preferences, transitivity demands that the pair (x, y) is included in any transitive relation that contains R. Analogously, a chain of that nature implies that if, in addition, the pair (y, x) is in R, (x, y) must be added if the resulting relation is to be consistent. In contrast, there are no necessary additions to a relation in order to transform it into a quasi-transitive relation by augmenting it. For instance, suppose we have $(x, y) \in P(R)$, $(y, z) \in P(R)$ and $(z, x) \in P(R)$. In order to define a quasi-transitive relation that contains R, at least two of the three strict preferences must be converted into indifferences but *any two* will do. Thus, there is no unique smallest quasi-transitive relation containing R. Similarly, if we have a P-cycle, a P-acyclical relation containing R merely has to have the property that at least *one* of the pairs along the cycle, representing a strict preference, must be converted into an indifference. But, without further information, there is nothing that forces this indifference on a specific pair along the cycle. As a consequence, there is, in general, no unique smallest P-acyclical relation containing an arbitrary relation R. The same difficulty arises when G-rationalizability involving completeness without any coherence conditions is considered: there exists no unique smallest complete relation containing a given incomplete relation. For that reason, we introduce some further concepts in order to be able to formulate necessary and sufficient conditions for the definitions of rationalizability considered in this paper.

Let $C\colon \Sigma \to \mathcal{X}$ be a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$, and define

$$\mathcal{A}_C = \{(S, y) \mid S \in \Sigma \text{ and } y \in S \setminus C(S)\}.$$

For a choice function C such that $\mathcal{A}_C \neq \emptyset$, let

$$\mathcal{F}_C = \{f\colon \mathcal{A}_C \to X \mid f(S, y) \in S \text{ for all } (S, y) \in \mathcal{A}_C\}.$$

The set $\mathcal{A}_C$ consists of all pairs of a feasible set and an element that belongs to the set but is not chosen by C. If $C(S) = S$ for all $S \in \Sigma$, the set $\mathcal{A}_C$ is empty; in all other cases, $\mathcal{A}_C \neq \emptyset$. The functions in $\mathcal{F}_C$ have an intuitive interpretation. They assign a feasible element to each pair of a feasible set S and an alternative y that is in S, but not chosen from S. Within the framework of G-rationalizability, the intended interpretation is that $f(S, y)$ is an alternative in S that can be used to prevent y from being chosen in the sense that y is not at least as good as $f(S, y)$ according to a G-rationalization. Clearly, the existence of such an alternative for each (S, y) in $\mathcal{A}_C$ is a necessary condition for G-rationalizability.

Our characterizations involving the existence of a function $f \in \mathcal{F}_C$ with suitable properties do not, by any means, constitute obvious results. As will become clear in the proofs of this section, there are substantial steps to be followed in order to deduce the existence and properties of a rationalization from the existence of a function f with the required properties.

We begin with the rationalizability property **G-R-0** (and, of course, its equivalents). To do so, we introduce a crucial property of a function $f \in \mathcal{F}_C$ which proves instrumental in our subsequent axiomatization. It imposes a restriction on the relationship between a choice function C and a function $f \in \mathcal{F}_C$.

Direct exclusion (DRE). For all $(S, y) \in \mathcal{A}_C$, for all $T \in \Sigma$ and for all $x \in T$,

$$f(S, y) = x \Rightarrow y \notin C(T).$$

The interpretation of this condition is very intuitive, given the purpose of a function f as mentioned above. According to the definition of G-rationalizability, if $x = f(S, y) \in S$ is responsible for y being prevented from being chosen in S, then y is not at least as good as x according to a G-rationalization of C. This being the case, y cannot possibly be chosen from any set containing x because, according to G-rationalizability, such a choice would require that y be at least as good as x, which we have just ruled out. Thus, provided that $\mathcal{A}_C$ is non-empty, the existence of a function f satisfying **DRE** clearly is necessary for G-rationalizability even if no richness or coherence properties are imposed on a rationalization. Conversely, this requirement is also sufficient for **G-R-0** and **G-0-0** so that we obtain the following theorem. See Richter (1971) for an alternative characterization of **G-0-0** that is not formulated in terms of the existence of a function $f \in \mathcal{F}_C$.

Theorem 4 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-R-0**, **G-0-0** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE**.

Proof. By Theorem 3, it is sufficient to consider **G-0-0**.

To prove the only if part of the theorem, let R be a G-rationalization of C, and suppose $\mathcal{A}_C \neq \emptyset$. We define a function $f \in \mathcal{F}_C$ as follows. Consider any $(S, y) \in \mathcal{A}_C$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Let $f(S, y) = x$. We show that the function f satisfies **DRE**. Suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

We now prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ clearly is a G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE**. Define

$$R = \{(x, y) \in X \times X \mid \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y\}.$$

To prove that R is a G-rationalization of C, let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exists $y \in S$ such that $(x, y) \notin R$, it follows from the definition of R that there exists $T \in \Sigma$ such that $(T, x) \in \mathcal{A}_C$ and $f(T, x) = y$. But this contradicts the property **DRE** and, therefore, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By definition of R, we obtain $(x, y) \notin R$ and thus $x \notin G(S, R)$. ■

The intuition underlying the definition of R in the above proof is quite transparent. If $x = f(S, y)$, it follows that y cannot be at least as good as x. That R is indeed a G-rationalization of C follows because f satisfies **DRE**.

Next, we examine the consequences of adding completeness as a property of a G-rationalization. The following condition prevents a function $f \in \mathcal{F}_C$ itself from exhibiting incoherent behavior, without reference to its relationship with a choice function.

Direct irreversibility (DRI). For all $(S, y), (T, x) \in \mathcal{A}_C$,

$$[f(S, y) = x \text{ and } x \neq y] \Rightarrow f(T, x) \neq y.$$

The existence of a function f with this property is a consequence of requiring a G-rationalization to be complete, given the interpretation of f alluded to above. Suppose $f(S, y) = x$ and $f(T, x) = y$ with distinct $x, y \in X$. According to the interpretation of f, this means that x is responsible for keeping y out of $C(S)$ and y is responsible for keeping x out of $C(T)$. By definition of G-rationalizability, this means that, according to a G-rationalization, x fails to be at least as good as y and, at the same time, y is not at least as good as x. But this is in conflict with the completeness requirement.

Conversely, the existence of a function f with the two properties **DRE** and **DRI** is sufficient for **G-C-0** and its equivalents. The resulting characterization is a variant of a theorem due to Bossert, Sprumont and Suzumura (2005c).

Theorem 5 *Suppose $C \colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-0**, **G-C-0**, **M-RC-0**, **M-R-0**, **M-C-0**, **M-0-0** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **DRI**.

Proof. Using Theorem 3, it is sufficient to treat the case of **G-C-0**.

To prove the only if part of the theorem, let R be a complete G-rationalization of C, and suppose $\mathcal{A}_C \neq \emptyset$. We define a function $f \in \mathcal{F}_C$ as follows. Consider any $(S, y) \in \mathcal{A}_C$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Let $f(S, y) = x$. We show that the function f has the required properties.

To show that **DRE** is satisfied, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To establish the property **DRI**, let $(S, y), (T, x) \in \mathcal{A}_C$ and suppose $f(S, y) = x$ and $x \neq y$. The definition of f again implies $(y, x) \notin R$. If $f(T, x) = y$, we obtain $(x, y) \notin R$, a contradiction to the completeness of R. Thus, $f(T, x) \neq y$.

We now prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ clearly is a complete G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE** and **DRI**. Define

$$R = \{(x, y) \in X \times X \mid \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y\}.$$

To prove that R is complete, suppose $x, y \in X$ are such that $x \neq y$, $(x, y) \notin R$ and $(y, x) \notin R$. By definition, there exist $S, T \in \Sigma$ such that $(S, x), (T, y) \in \mathcal{A}_C$, $f(S, x) = y$ and $f(T, y) = x$, contradicting the property **DRI**.

It remains to be shown that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exists $y \in S$ such that $(x, y) \notin R$, it follows from the definition of R that there exists $T \in \Sigma$ such that $(T, x) \in \mathcal{A}_C$ and $f(T, x) = y$. But this contradicts the property **DRE** and, therefore, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By definition of R, we obtain $(x, y) \notin R$ and thus $x \notin G(S, R)$. ■

The intuition underlying the definition of R in this result is quite straightforward. If $x = f(S, y)$ for distinct $x, y \in X$, it follows that y cannot be at least as good as x and, because of the completeness requirement, this means that x must be better than y. The completeness of the resulting relation R is a consequence of the assumption that f possesses the property **DRI** and, moreover, R is a G-rationalization of C because f satisfies **DRE**.

Next, we characterize the rationalizability properties that are equivalent to **G-0-A**. As a consequence of adding P-acyclicity as a requirement on a rationalization and removing the completeness condition, the property of direct irreversibility has to be replaced by the following revelation irreversibility axiom.

Revelation irreversibility (RI). For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$\left[f(S^k, x^k) = x^{k-1} \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } (x^K, x^0) \in R_C\right]$$
$$\Rightarrow f(S^0, x^0) \neq x^K.$$

Revelation irreversibility differs from direct irreversibility in two ways. First, its conclusion applies to chains of relationships between alternatives via f and not only to direct instances thereof. Moreover, the axiom is conditional on any two consecutive elements in the chain being related not only through f but also by means of a direct revealed preference according to C. As is straightforward to verify, **DRI** and **RI** are independent.

RI is a consequence of the P-acyclicity of a G-rationalization. To see that this is the case, note first that, according to the interpretation of f, $f(S^k, x^k) = x^{k-1}$ means that x^k cannot be at least as good as x^{k-1} according to a G-rationalization. Furthermore, because the direct revealed preference relation has to be respected by any G-rationalization, x^{k-1} must be at least as good as x^k, thus leading to a strict preference of x^{k-1} over x^k. Thus, a violation of **RI** immediately yields a violation of the P-acyclicity of a G-rationalization. Again, the existence of a function f with the requisite properties not only is necessary but also sufficient for the rationalizability definitions under consideration.

Theorem 6 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-R-A**, **G-0-A** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **RI**.

Proof. By Theorem 3, it is sufficient to treat the case of **G-0-A**.

Suppose C satisfies **G-0-A** and let R be a P-acyclical G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$. The assumption that R G-rationalizes C implies that, for any pair $(S, y) \in \mathcal{A}_C$, there exists $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$. That f satisfies **DRE** follows as in the previous theorem.

To establish the property **RI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and, moreover, $(x^K, x^0) \in R_C$. By the definition of f, we obtain $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$. By Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and, thus, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. If $f(S^0, x^0) = x^K$, it follows that $(x^0, x^K) \notin R$ by definition. Because $(x^K, x^0) \in R_C$ implies $(x^K, x^0) \in R$ by Theorem 2, we obtain $(x^K, x^0) \in P(R)$. If $K = 1$, this contradicts the observation that $(x^K, x^0) \notin R$, which follows from the hypothesis $f(S^1, x^1) = x^0$ and the definition of f. If $K > 1$, we obtain a contradiction to the P-acyclicity of R. Therefore, $f(S^0, x^0) \neq x^K$ and **RI** is satisfied.

We now prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a P-acyclical G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE** and **RI**. Define

$$\begin{aligned} R \;=\; & R_C \cup \{(x, y) \in X \times X \mid (y, x) \in R_C \text{ and} \\ & \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y\}. \end{aligned}$$

To demonstrate that R is P-acyclical, suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. Consider any $k \in \{1, \ldots, K\}$. By definition,

$$\begin{aligned} \{(x^{k-1}, x^k) \in R_C \quad \text{or} \quad & [(x^k, x^{k-1}) \in R_C \text{ and } \nexists T^k \in \Sigma \text{ such that} \\ & (T^k, x^{k-1}) \in \mathcal{A}_C \text{ and } f(T^k, x^{k-1}) = x^k]\} \end{aligned}$$

and

$$\begin{aligned} (x^k, x^{k-1}) \notin R_C \quad \text{and} \quad & [(x^{k-1}, x^k) \notin R_C \text{ or } \exists S^k \in \Sigma \text{ such that} \\ & (S^k, x^k) \in \mathcal{A}_C \text{ and } f(S^k, x^k) = x^{k-1}]. \end{aligned}$$

Because $(x^k, x^{k-1}) \notin R_C$ must be true,

$$(x^k, x^{k-1}) \in R_C \text{ and } \nexists T^k \in \Sigma \text{ such that } (T^k, x^{k-1}) \in \mathcal{A}_C \text{ and } f(T^k, x^{k-1}) = x^k$$

cannot be true. Therefore, $(x^{k-1}, x^k) \in R_C$ must be true, which, in turn, implies that $(x^{k-1}, x^k) \notin R_C$ cannot be true. Therefore, it follows that

$$(x^{k-1}, x^k) \in R_C \text{ and } \exists S^k \in \Sigma \text{ such that } (S^k, x^k) \in \mathcal{A}_C \text{ and } f(S^k, x^k) = x^{k-1}.$$

Using the same argument, it follows that $(x^K, x^0) \in P(R)$ implies that $(x^K, x^0) \in R_C$ and there exists $S^0 \in \Sigma$ such that $(S^0, x^0) \in \mathcal{A}_C$ and $f(S^0, x^0) = x^K$. This contradicts the property **RI** and, thus, $(x^K, x^0) \notin P(R)$ and R is P-acyclical.

We complete the proof by showing that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x, y) \in R_C$ and, by Part (i) of Theorem 2, $(x, y) \in R$ for all $y \in S$. Hence, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Thus, $(S, x) \in \mathcal{A}_C$. Let $y = f(S, x)$. If $(x, y) \in R_C$, there exists $T \in \Sigma$ such that $y \in T$ and $x \in C(T)$. Because $y \in S$, this contradicts the property **DRE**. Therefore, $(x, y) \notin R_C$ and, together with the observations $(S, x) \in \mathcal{A}_C$ and $y = f(S, x)$, it follows that $(x, y) \notin R$ and hence $x \notin G(S, R)$. ■

As usual, any G-rationalization R has to respect the direct revealed preference relation R_C. Furthermore, the construction of R employed in the above theorem converts all strict direct revealed preferences into indifferences whenever this is possible without conflicting with the interpretation of the function f. This is done to reduce the potential for conflicts with P-acyclicity as much as possible. That the resulting relation satisfies the required properties follows from the properties of f.

In order to accommodate completeness as well as P-acyclicity, we replace **RI** with the following property of distinctness irreversibility.

Distinctness irreversibility (DSI). For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$\left[f(S^k, x^k) = x^{k-1} \text{ and } x^{k-1} \neq x^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K \neq x^0\right] \Rightarrow f(S^0, x^0) \neq x^K.$$

Clearly, distinctness irreversibility implies direct irreversibility (set $K = 1$ to verify this claim). Although **RI** and **DSI** by themselves are independent, **DSI** implies **RI** in the presence of direct exclusion. To see that this is the case, suppose f violates **RI**. Then there exist $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$, $(x^K, x^0) \in R_C$ and $f(S^0, x^0) = x^K$. If any two consecutive elements in this cycle are distinct, we immediately obtain a contradiction to **DSI**. If $x^{k-1} = x^k$ for all $k \in \{1, \ldots, K\}$, it follows that $f(S^0, x^0) = x^0$ and $(x^0, x^0) \in R_C$. By definition of R_C, there exists $T \in \Sigma$ such that $x^0 \in C(T) \subseteq T$, contradicting **DRE**.

The property of **DSI** rather than of **RI** must be added to **DRE** if a rationalization is to be complete in addition to being P-acyclical. If **DSI** is violated, the completeness of a G-rationalization and the interpretation of f together imply that a G-rationalization must have a strict preference cycle, which immediately yields a contradiction to the P-acyclicity requirement.

The following theorem establishes that the existence of a function f satisfying **DRE** and **DSI** is necessary and sufficient for the rationalizability properties that are equivalent to **G-C-A**.

Theorem 7 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-C-A** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **DSI**.

Proof. First, suppose C satisfies **G-C-A** and let R be a complete and P-acyclical G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$. The assumption that R G-rationalizes C implies that, for any pair $(S, y) \in \mathcal{A}_C$, there exists $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To establish the property **DSI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$ and, furthermore, $x^K \neq x^0$. By the definition of f, it follows that $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$. Because R is complete and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$ by assumption, it follows that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. If $f(S^0, x^0) = x^K$, it follows that $(x^0, x^K) \notin R$ by definition and, by the assumption $x^K \neq x^0$ and the completeness of R, we obtain $(x^K, x^0) \in P(R)$. If $K = 1$, this contradicts the observation that $(x^K, x^0) \notin R$ which follows from the hypothesis $f(S^1, x^1) = x^0$ and the definition of f. If $K > 1$, we obtain a contradiction to the P-acyclicity of R. Therefore, $f(S^0, x^0) \neq x^K$.

We now prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a complete P-acyclical G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE** and **DSI**. Define

$$R = \{(x, y) \in X \times X \mid \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y\}.$$

We prove that R is complete. By way of contradiction, suppose $x, y \in X$ are such that $x \neq y$, $(x, y) \notin R$ and $(y, x) \notin R$. By the definition of R, this implies that there exist $S, T \in \Sigma$ such that $(S, x), (T, y) \in \mathcal{A}_C$, $f(S, x) = y$ and $f(T, y) = x$. Because $x \neq y$, this contradicts the property **DSI**. Thus, R is complete.

To show that R is P-acyclical, suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. By the definition of R, this implies that there exist $S^1, \ldots, S^K \in \Sigma$ such that $(S^k, x^k) \in \mathcal{A}_C$ and $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$. Moreover, for all $k \in \{1, \ldots, K\}$, there exists no $T^k \in \Sigma$ such that $(T^k, x^{k-1}) \in \mathcal{A}_C$ and $x^k = f(T^k, x^{k-1})$. This implies $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. If $(x^K, x^0) \in P(R)$, there exists $S^0 \in \Sigma$ such that $(S^0, x^0) \in \mathcal{A}_C$ and $x^K = f(S^0, x^0)$. Furthermore, there exists no $T^0 \in \Sigma$ such that $(T^0, x^K) \in \mathcal{A}_C$

and $x^0 = f(T^0, x^K)$. This implies $x^0 \neq x^K$ and we obtain a contradiction to the property **DSI** and, thus, $(x^K, x^0) \notin P(R)$ and R is P-acyclical.

It remains to be shown that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exist $y \in S$ and $T \in \Sigma$ such that $(T, x) \in \mathcal{A}_C$ and $f(T, x) = y$, we obtain a contradiction to the property **DRE**. Thus, by definition, $(x, y) \in R$ for all $y \in S$ and hence $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By the definition of R, this implies $(x, y) \notin R$ and, therefore, $x \notin G(S, R)$. ■

The intuition underlying the definition of R in this result is quite straightforward. If $x = f(S, y)$, it follows that y cannot be at least as good as x and, because of the completeness assumption, this means that x must be better than y whenever $x \neq y$. The resulting relation has all the required properties as a consequence of the properties of f.

Our last set of rationalizability properties involving P-acyclical G-rationalizations is that containing **G-RC-A**. Because reflexivity is added as a requirement, an unconditional version of irreversibility is called for.

Indirect irreversibility (II). For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$f(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \Rightarrow f(S^0, x^0) \neq x^K.$$

Clearly, indirect irreversibility implies all of the irreversibility conditions introduced earlier. The full force of the axiom is needed because, as opposed to the G-rationalizability property **G-C-A**, its conclusion must hold not only for chains of distinct alternatives but, due to the added reflexivity assumption, for any chain. We obtain the following characterization which, with a slightly different proof, can be found in Bossert, Sprumont and Suzumura (2005c).

Theorem 8 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-A**, **M-R-S**, **M-0-S**, **M-RC-A**, **M-R-A**, **M-C-A**, **M-0-A** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **II**.

Proof. Invoking Theorem 3 again, it is sufficient to consider **G-RC-A**.

We first prove the only if part of the theorem. Let R be a reflexive, complete and P-acyclical G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$ and consider an arbitrary pair $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S,y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S,y) = x$. By the definition of f, we obtain $(y,x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To establish the property **II**, suppose $K \in \mathbb{N}$ and $(S^0,x^0),\ldots,(S^K,x^K) \in \mathcal{A}_C$ are such that $f(S^k,x^k) = x^{k-1}$ for all $k \in \{1,\ldots,K\}$. By definition, $(x^k,x^{k-1}) \notin R$ for all $k \in \{1,\ldots,K\}$. Because R is reflexive, we have $x^{k-1} \neq x^k$ for all $k \in \{1,\ldots,K\}$ and, thus, the completeness of R implies $(x^{k-1},x^k) \in P(R)$ for all $k \in \{1,\ldots,K\}$. If $f(S^0,x^0) = x^K$, it follows analogously that $(x^K,x^0) \in P(R)$. If $K = 1$, this contradicts the hypothesis $(x^K,x^0) \notin R$, and if $K > 1$, we obtain a contradiction to the P-acyclicity of R. Therefore, $f(S^0,x^0) \neq x^K$.

Next, we prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ clearly is a reflexive, complete and P-acyclical G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE** and **II**. Define

$$R = \{(x,y) \in X \times X \mid \nexists S \in \Sigma \text{ such that } (S,x) \in \mathcal{A}_C \text{ and } f(S,x) = y\}.$$

To prove that R is reflexive, suppose, by way of contradiction, that there exists $x \in X$ such that $(x,x) \notin R$. By definition, there exists $S \in \Sigma$ such that $(S,x) \in \mathcal{A}_C$ and $f(S,x) = x$. Letting $K = 1$, $S^0 = S^K = S$ and $x^0 = x^K = x$, we obtain a contradiction to the property **II**.

Next, we establish the completeness of R. Suppose $x,y \in X$ are such that $x \neq y$, $(x,y) \notin R$ and $(y,x) \notin R$. By definition, there exist $S,T \in \Sigma$ such that $(S,x),(T,y) \in \mathcal{A}_C$, $f(S,x) = y$ and $f(T,y) = x$, contradicting the property **II** for $K = 1$, $(S^0,x^0) = (S,x)$ and $(S^K,x^K) = (T,y)$.

To show that R is P-acyclical, suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0,\ldots,x^K \in X$ are such that $(x^{k-1},x^k) \in P(R)$ for all $k \in \{1,\ldots,K\}$. By the definition of R, this implies that there exist $S^1,\ldots,S^K \in \Sigma$ such that $(S^k,x^k) \in \mathcal{A}_C$ and $x^{k-1} = f(S^k,x^k)$ for all $k \in \{1,\ldots,K\}$. If $(x^K,x^0) \in P(R)$, there exists $S^0 \in \Sigma$ such that $(S^0,x^0) \in \mathcal{A}_C$ and $x^K = f(S^0,x^0)$. But this contradicts the property **II**. Thus, $(x^K,x^0) \notin P(R)$ and R is P-acyclical.

It remains to be shown that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exists $y \in S$ such that $(x,y) \notin R$, it follows from the definition of R that there exists $T \in \Sigma$ such that $(T,x) \in \mathcal{A}_C$ and $f(T,x) = y$. But this contradicts the property **DRE** and, therefore, $x \in G(S,R)$.

Now suppose $x \notin C(S)$. Let $y = f(S,x)$. By the definition of R, we obtain $(x,y) \notin R$ and thus $x \notin G(S,R)$. ■

The intuition underlying the definition of R in this result is as follows. If $x = f(S,y)$, it follows that y cannot be at least as good as x and, because of reflexivity and completeness, this means that x must be better than y. As opposed to the previous result, f satisfies **II** rather than merely **DSI** and, as a consequence, R is reflexive in addition to being complete and P-acyclical.

We now turn to rationalizability properties involving quasi-transitivity as the coherence property to be satisfied by a G-rationalization. We begin with **G-0-Q**. According to Theorem 6, the existence of a function f satisfying **DRE** and **RI** is necessary and sufficient for **G-0-A**. If P-acyclicity is strengthened to quasi-transitivity, the following additional property of f is required.

Revelation exclusion (RE). For all $K \in \mathbb{N}$, for all $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$\big[f(S^k, x^k) = x^{k-1} \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\}\big] \Rightarrow x^K \notin C(S^0).$$

Revelation exclusion is necessary to ensure that a G-rationalization is quasi-transitive as opposed to merely P-acyclical. As illustrated earlier, the conjunction of $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ implies, given the interpretation of f, that a G-rationalization must exhibit a strict preference. Following the resulting chain of strict preferences, quasi-transitivity demands that x^0 is strictly preferred to x^K according to the rationalization. This is incompatible with $(x^K, x^0) \in R_C$ and, thus, **RE** must be satisfied. We obtain the following characterization.

Theorem 9 *Suppose $C \colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-0-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE**, **RI** *and* **RE**.

Proof. Suppose C satisfies **G-0-Q** and let R be a quasi-transitive G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$ and consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To establish the property **RI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R_C$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Therefore, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and thus $(x^0, x^K) \in R$, which, by the definition of f, implies $f(S^0, x^0) \neq x^K$.

To show that f satisfies **RE**, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Thus, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and the quasi-transitivity of R implies $(x^0, x^K) \in P(R)$. Therefore, $(x^K, x^0) \notin R$ and, because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

Now suppose that there exists $f \in \mathcal{F}_C$ satisfying **DRE**, **RI** and **RE** whenever $\mathcal{A}_C \neq \emptyset$. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying the properties **DRE**, **RI** and **RE**. Define

$$\begin{aligned} R &= R_C \\ &\cup \{(x,y) \in X \times X \mid (y,x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S,x) \in \mathcal{A}_C \text{ and } f(S,x) = y \\ &\qquad \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ &\qquad y = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = x\} \\ &\cup \{(x,y) \in X \times X \mid \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ &\qquad x = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = y\}. \end{aligned}$$

To prove that R is quasi-transitive, suppose $x, y, z \in X$ are such that $(x,y) \in P(R)$ and $(y,z) \in P(R)$. By definition, $(x,y) \in R$ implies

$$(x,y) \in R_C \tag{2}$$

or

$$\begin{gathered} (y,x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S,x) \in \mathcal{A}_C \text{ and } f(S,x) = y \\ \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ y = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = x \end{gathered} \tag{3}$$

or

$$\begin{gathered} \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = y. \end{gathered} \tag{4}$$

Analogously, $(y,x) \notin R$ implies

$$(y,x) \notin R_C \tag{5}$$

and

$$\begin{gathered} (x,y) \notin R_C \text{ or } \exists S \in \Sigma \text{ such that } (S,y) \in \mathcal{A}_C \text{ and } f(S,y) = x \\ \text{or } \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = y \end{gathered} \tag{6}$$

and

$$\begin{gathered} \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ y = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = x. \end{gathered} \tag{7}$$

Because (5) must be true, (3) must be false. Therefore, it follows that (2) or (4) is true and that (6) is true. Because (2) and $(x, y) \notin R_C$ are incompatible, it follows that we must have

$$(2) \text{ and } \exists S \in \Sigma \text{ such that } (S, y) \in \mathcal{A}_C \text{ and } f(S, y) = x \tag{8}$$

or

$$(2) \text{ and } (4) \tag{9}$$

or (4). Clearly, (8) implies (4) and (9) implies (4) trivially. Thus, (4) follows in all possible cases. Analogously, $(y, z) \in P(R)$ implies

$$\begin{gathered} \exists L \in \mathbb{N}, y^0 \in X \text{ and } (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C \text{ such that} \\ y = y^0, y^{\ell-1} = f(T^\ell, y^\ell) \text{ and } (y^{\ell-1}, y^\ell) \in R_C \text{ for all } \ell \in \{1, \ldots, L\} \text{ and } y^L = z. \end{gathered} \tag{10}$$

Letting $M = K + L$, $z^0 = x^0$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K + 1, \ldots, K + L\}$, (4) and (10) together imply

$$x = z^0, z^{m-1} = f(U^m, z^m) \text{ and } (z^{m-1}, z^m) \in R_C \text{ for all } m \in \{1, \ldots, M\} \text{ and } z^M = z. \tag{11}$$

Therefore, by the definition of R, $(x, z) \in R$. Suppose we also have $(z, x) \in R$. This implies

$$(z, x) \in R_C \tag{12}$$

or

$$\begin{gathered} (x, z) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, z) \in \mathcal{A}_C \text{ and } f(S, z) = x \\ \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = z \end{gathered} \tag{13}$$

or

$$\begin{gathered} \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ z = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x. \end{gathered} \tag{14}$$

If (12) is true, (11) yields a contradiction to the property **RE**. (13) immediately contradicts (11). Finally, if (14) applies, combining it with (11), we are led to a contradiction to the property **RI**. Thus, R is quasi-transitive.

To show that R is a G-rationalization of C, let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x, y) \in R_C \subseteq R$ for all $y \in S$ and, therefore, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Thus, $(S, x) \in \mathcal{A}_C$. Let $y = f(S, x)$ and suppose $(x, y) \in R$.

If $(x, y) \in R_C$, there exists $T \in \Sigma$ such that $y \in T$ and $x \in C(T)$. This contradicts the property **DRE**. If (3) applies, it follows that there exists no $S \in \Sigma$ such that $(S, x) \in \mathcal{A}_C$ and $y = f(S, x)$, an immediate contradiction to our hypothesis. Finally, if (4) applies, we obtain a contradiction to the property **RI**. Thus, $(x, y) \notin R$ and hence $x \notin G(S, R)$. ■

In the above proof, the components of R are constructed by including all pairs that are necessarily in this relation and then invoking the properties of f to ensure that R satisfies all of the requirements. In particular, as is the case whenever G-rationalizability is considered, the direct revealed preference relation R_C has to be respected. To avoid as many potential conflicts with quasi-transitivity as possible, any strict revealed preference is converted into an indifference whenever possible without contradiction. Finally, any chain of strict preference imposed by the conjunction of relationships imposed by f and by the direct revealed preference criterion has to be respected due to the quasi-transitivity requirement.

If reflexivity is added to quasi-transitivity as a requirement on a G-rationalization, the function f must possess an additional property as well. This is accomplished by imposing the following axiom.

Self irreversibility (SI). For all $(S, x) \in \mathcal{A}_C$,

$$f(S, x) \neq x.$$

According to the interpretation of f, $f(S, x) = x$ means that x is excluded from $C(S)$ because x fails to be considered at least as good as itself by a G-rationalization. Clearly, this is incompatible with the reflexivity of a G-rationalization and, thus, self irreversibility is an additional necessary requirement to be satisfied by f. This leads to the following theorem.

Theorem 10 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-R-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE**, **RI**, **RE** *and* **SI**.

Proof. Suppose C satisfies **G-R-Q** and let R be a reflexive and quasi-transitive G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$ and consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To establish the property **RI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R_C$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Therefore, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and thus $(x^0, x^K) \in R$ which, by the definition of f, implies $f(S^0, x^0) \neq x^K$.

To show that f satisfies **RE**, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Thus, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and the quasi-transitivity of R implies $(x^0, x^K) \in P(R)$. Therefore, $(x^K, x^0) \notin R$ and, because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

To prove that **SI** is satisfied, suppose there exists $(S, x) \in \mathcal{A}_C$ such that $f(S, x) = x$. By the definition of f, this implies $(x, x) \notin R$, contradicting the reflexivity of R.

Suppose that, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **DRE**, **RI**, **RE** and **SI**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a reflexive and quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE**, **RI**, **RE** and **SI**. Define

$$\begin{aligned} R \;=\; & R_C \cup \Delta \\ \cup\; & \{(x, y) \in X \times X \mid (y, x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y \\ & \quad \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ & \quad y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\} \\ \cup\; & \{(x, y) \in X \times X \mid \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ & \quad x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}. \end{aligned}$$

Clearly, R is reflexive because $\Delta \subseteq R$.

To prove that R is quasi-transitive, suppose $x, y, z \in X$ are such that $(x, y) \in P(R)$ and $(y, z) \in P(R)$. This implies that $x \neq y$ and $y \neq z$ and, thus, $(x, y) \notin \Delta$ and $(y, z) \notin \Delta$. Therefore, $(x, y) \in R$ implies

$$(x, y) \in R_C \tag{15}$$

or

$$\begin{aligned} & (y, x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y \\ & \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ & y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x \end{aligned} \tag{16}$$

or

$$\begin{aligned} &\exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ &x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y. \end{aligned} \tag{17}$$

Analogously, $(y, x) \notin R$ implies

$$(y, x) \notin R_C \tag{18}$$

and

$$\begin{aligned} &(x, y) \notin R_C \text{ or } \exists S \in \Sigma \text{ such that } (S, y) \in \mathcal{A}_C \text{ and } f(S, y) = x \\ &\text{or } \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ &x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ &y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x. \end{aligned} \tag{20}$$

Because (18) must be true, (16) must be false. Therefore, it follows that (15) or (17) is true and that (19) is true. Because (15) and $(x, y) \notin R_C$ are incompatible, it follows that we must have

$$(15) \text{ and } \exists S \in \Sigma \text{ such that } (S, y) \in \mathcal{A}_C \text{ and } f(S, y) = x \tag{21}$$

or

$$(15) \text{ and } (17) \tag{22}$$

or (17). Clearly, (21) implies (17) and (22) implies (17) trivially. Thus, (17) follows in all possible cases. Analogously, $(y, z) \in P(R)$ implies

$$\begin{aligned} &\exists L \in \mathbb{N}, y^0 \in X \text{ and } (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C \text{ such that} \\ &y = y^0, y^{\ell-1} = f(T^\ell, y^\ell) \text{ and } (y^{\ell-1}, y^\ell) \in R_C \text{ for all } \ell \in \{1, \ldots, L\} \text{ and } y^L = z. \end{aligned} \tag{23}$$

Letting $M = K + L$, $z^0 = x^0$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K+1, \ldots, K+L\}$, (17) and (23) together imply

$$x = z^0, z^{m-1} = f(U^m, z^m) \text{ and } (z^{m-1}, z^m) \in R_C \text{ for all } m \in \{1, \ldots, M\} \text{ and } z^M = z. \tag{24}$$

Therefore, by the definition of R, $(x, z) \in R$. Suppose we also have $(z, x) \in R$. This implies

$$(z, x) \in R_C \tag{25}$$

or

$$
\begin{aligned}
(x,z) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S,z) \in \mathcal{A}_C \text{ and } f(S,z) = x \\
\text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\
x = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = z
\end{aligned}
\tag{26}
$$

or

$$
\begin{aligned}
\exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\
z = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = x.
\end{aligned}
\tag{27}
$$

If (25) is true, (24) yields a contradiction to the property **RE**. (26) immediately contradicts (24). Finally, if (27) applies, in combination with (24), it implies a contradiction to the property **RI**. Thus, R is quasi-transitive.

To show that R is a G-rationalization of C, let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x,y) \in R_C \subseteq R$ for all $y \in S$ and, therefore, $x \in G(S,R)$.

Now suppose $x \notin C(S)$. Thus, $(S,x) \in \mathcal{A}_C$. Let $y = f(S,x)$ and suppose $(x,y) \in R$.

If $(x,y) \in R_C$, there exists $T \in \Sigma$ such that $y \in T$ and $x \in C(T)$. This contradicts the property **DRE**. If $(x,y) \in \Delta$, we obtain a contradiction to the property **SI**. If (16) applies, it follows that there exists no $S \in \Sigma$ such that $(S,x) \in \mathcal{A}_C$ and $y = f(S,x)$, an immediate contradiction to our hypothesis. Finally, if (17) applies, we obtain a contradiction to the property **RI**. Thus, $(x,y) \notin R$ and hence $x \notin G(S,R)$. ■

The construction of the G-rationalization in this proof is analogous to that of the previous theorem. The only difference is that diagonal relation Δ appears now. This is necessitated by the addition of reflexivity as a requirement.

If completeness rather than reflexivity is added to quasi-transitivity, we obtain a stronger rationalizability condition; see Theorem 3. As a consequence, the property **RI** of f in Theorem 9 is replaced by **DSI** and, instead of **RE**, the following requirement is imposed.

Distinctness exclusion (DSE). For all $K \in \mathbb{N}$, for all $(S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$
\left[f(S^k,x^k) = x^{k-1} \text{ and } x^{k-1} \neq x^k \text{ for all } k \in \{1,\ldots,K\}\right] \Rightarrow x^K \notin C(S^0).
$$

In the presence of **DRE**, **RE** is implied by **DSE**. Suppose f violates **RE**. If all x^k are identical, we obtain an immediate contradiction to **DRE**. If there exists a $k \in \{1,\ldots,K\}$ such that $x^{k-1} \neq x^0$, we can without loss of generality assume that all of them are pairwise distinct (otherwise, the chain can be reduced to one involving pairwise distinct elements), which leads to a violation of **DSE**.

The strengthening of revelation exclusion to distinctness exclusion is necessary as a consequence of adding completeness to quasi-transitivity. If $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^K$, the interpretation of f and the completeness of a G-rationalization together imply that x^{k-1} is strictly preferred to x^k. Following this chain of strict preferences, quasi-transitivity demands that x^0 is strictly preferred to x^K, which is not compatible with $(x^K, x^0) \in R_C$. The corresponding characterization result is stated in the following theorem.

Theorem 11 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-C-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE**, **DSI** *and* **DSE**.

Proof. We first prove that **G-C-Q** implies the existence of $f \in \mathcal{F}_C$ which satisfies **DRE**, **DSI** and **DSE** whenever $\mathcal{A}_C \neq \emptyset$. Let R be a complete and quasi-transitive G-rationalization of C. Consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To show that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we have $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

Next, we establish **DSI**. Suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$ and, because R is complete, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and hence $(x^0, x^K) \in R$. By the definition of f, this implies $f(S^0, x^0) \neq x^K$.

To prove that f satisfies **DSE**, let $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ be such that $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$ and the completeness of R implies $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and, thus, $(x^K, x^0) \notin R$. Because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

Now suppose that, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **DRE**, **DSI** and **DSE**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a complete and quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE**, **DSI** and **DSE**. Define

$$\begin{aligned} R \;=\; & \{(x, y) \in X \times X \mid \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ & \quad y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } x^{k-1} \neq x^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\} \\ \setminus \; & \{(x, x) \in \Delta \mid \exists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = x\}. \end{aligned}$$

To prove that R is complete, suppose, by way of contradiction, that there exist $x, y \in X$ such that $x \neq y$, $(x,y) \notin R$ and $(y,x) \notin R$. By definition, this implies that there exist $K, L \in \mathbb{N}$, $x^0, y^0 \in X$ and $(S^1,x^1),\ldots,(S^K,x^K),(T^1,y^1),\ldots,(T^L,y^L) \in \mathcal{A}_C$ such that $y = x^0$, $x^{k-1} = f(S^k,x^k)$ and $x^{k-1} \neq x^k$ for all $k \in \{1,\ldots,K\}$, $x^K = x$, $x = y^0$, $y^{\ell-1} = f(T^\ell,y^\ell)$ and $y^{\ell-1} \neq y^\ell$ for all $\ell \in \{1,\ldots,L\}$ and $y^L = y$. Letting $M = K+L-1$, $(U^0,z^0) = (T^L,y^L)$, $(U^m,z^m) = (S^m,x^m)$ for all $m \in \{1,\ldots,K\}$ and $(U^m,z^m) = (T^{m-K},y^{m-K})$ for all $m \in \{K+1,\ldots,K+L\} \setminus \{K+L\}$, it follows that $z^{m-1} = f(U^m,z^m)$ and $z^{m-1} \neq z^m$ for all $m \in \{1,\ldots,M\}$ and $z^M = f(U^0,z^0)$, contradicting the property **DSI**.

Next, we show that R is quasi-transitive. Suppose $x,y,z \in X$ are such that $(x,y) \in P(R)$ and $(y,z) \in P(R)$. This implies that $x \neq y$ and $y \neq z$ and, by the definition of R, there exist $K, L \in \mathbb{N}$, $x^0, y^0 \in X$ and $(S^1,x^1),\ldots,(S^K,x^K),(T^1,y^1),\ldots,(T^L,y^L) \in \mathcal{A}_C$ such that $x = x^0$, $x^{k-1} = f(S^k,x^k)$ and $x^{k-1} \neq x^k$ for all $k \in \{1,\ldots,K\}$, $x^K = y$, $y = y^0$, $y^{\ell-1} = f(T^\ell,y^\ell)$ and $y^{\ell-1} \neq y^\ell$ for all $\ell \in \{1,\ldots,L\}$ and $y^L = z$. Letting $M = K+L$, $z^0 = x^0$, $(U^m,z^m) = (S^m,x^m)$ for all $m \in \{1,\ldots,K\}$ and $(U^m,z^m) = (T^{m-K},y^{m-K})$ for all $m \in \{K+1,\ldots,K+L\}$, it follows that $(z,x) \notin R$. Because R is complete, we obtain $(x,z) \in P(R)$.

Finally, we prove that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$ and, by way of contradiction, that there exists $y \in S$ such that $(x,y) \notin R$. If $x = y$ and there exists $S \in \Sigma$ such that $(S,x) \in \mathcal{A}_C$ and $f(S,x) = x$, we obtain a contradiction to the property **DRE**. If there exist $K \in \mathbb{N}$, $x^0 \in X$ and $(S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C$ such that $y = x^0$, $x^{k-1} = f(S^k,x^k)$ and $x^{k-1} \neq x^k$ for all $k \in \{1,\ldots,K\}$ and $x^K = x$, letting $S^0 = S$ yields a contradiction to the property **DSE**. Therefore, $x \in G(S,R)$. Thus, $C(S) \subseteq G(S,R)$.

Now suppose $x \notin C(S)$. Let $y = f(S,x)$. By definition, this implies $(x,y) \notin R$ and hence $x \notin G(S,R)$. Thus, $G(S,R) \subseteq C(S)$. ■

The G-rationalization R employed in this proof is less complex because of the completeness assumption—an absence of a weak preference for one of two distinct alternatives implies a strict preference for the other. In addition, whenever an element x is not chosen in a set S because, according to f, x is not at least as good as itself, the pair (x,x) cannot be in R. Because R is also required to be quasi-transitive, chains of strict preference have to be respected as well. In order to arrive at a complete and quasi-transitive G-rationalization of C, we define R to be composed of all pairs $(x,y) \in X \times X$ such that y does not have to be strictly preferred to x according to the above-described criterion. The properties of f ensure that R is indeed a complete and quasi-transitive G-rationalization of C.

Now we consider the rationalizability property **G-RC-Q**. Because reflexivity *and* completeness are required, both the exclusion axiom and the irreversibility condition to be employed are unconditional—whenever it is the case that $f(S^k, x^k) = x^{k-1}$, the conjunction of reflexivity and completeness, together with the interpretation of f, implies that x^{k-1} must be strictly preferred to x^k by a G-rationalization. Quasi-transitivity demands that any chain of strict preferences from x^0 to x^K be respected and, thus, x^0 must be strictly preferred to x^K. This immediately rules out $(x^K, x^0) \in R_C$ (as required by the axiom **IE** introduced below) and $f(S^0, x^0) = x^K$ (see **II**).

Indirect exclusion (IE). For all $K \in \mathbb{N}$, for all $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$f(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \Rightarrow x^K \notin C(S^0).$$

Clearly, indirect exclusion implies all of the exclusion properties introduced earlier. We can now state the following result which, with an alternative proof, has been established in Bossert, Sprumont and Suzumura (2005c).

Theorem 12 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-Q**, **M-R-T**, **M-0-T**, **M-RC-Q**, **M-R-Q**, **M-C-Q**, **M-0-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **II** *and* **IE**.

Proof. In view of Theorem 3, it is sufficient to treat the case of **G-RC-Q**.

We first prove that **G-RC-Q** implies the existence of an $f \in \mathcal{F}_C$ which satisfies **II** and **IE** provided that $\mathcal{A}_C \neq \emptyset$. Let R be a reflexive, complete and quasi-transitive G-rationalization of C. Consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that **II** is satisfied, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$. Because R is reflexive, we have $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$ and, because R is complete, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and hence $(x^0, x^K) \in R$. By definition of f, this implies $f(S^0, x^0) \neq x^K$.

To show that any such function f satisfies **IE**, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$. Because R is reflexive, it must be the case that $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. Thus, the completeness of R implies $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and, thus, $(x^K, x^0) \notin R$. Because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

Now suppose that, provided $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **II** and **IE**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a reflexive, complete and quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **II** and **IE**. Define

$$\begin{aligned} R &= \{(x, y) \in X \times X \mid \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ &\qquad y = x^0, x^{k-1} = f(S^k, x^k) \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\}. \end{aligned}$$

To see that R is reflexive, let $x \in X$. If $(x, x) \notin R$, there exist $K \in \mathbb{N}$, $x^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$ such that $x = x^0$, $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$ and $x^K = x$. Letting $S^0 = S^K$, we obtain a contradiction to **II**. Thus, we must have $(x, x) \in R$.

To establish the completeness of R, suppose $x, y \in X$ are such that $x \neq y$, $(x, y) \notin R$ and $(y, x) \notin R$. By definition, this implies that there exist $K, L \in \mathbb{N}$, $x^0, y^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K), (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C$ such that $y = x^0$, $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$, $x^K = x$, $x = y^0$, $y^{\ell-1} = f(T^\ell, y^\ell)$ for all $\ell \in \{1, \ldots, L\}$ and $y^L = y$. Letting $M = K + L - 1$, $(U^0, z^0) = (T^L, y^L)$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K+1, \ldots, K+L\} \setminus \{K+L\}$, it follows that $z^{m-1} = f(U^m, z^m)$ for all $m \in \{1, \ldots, M\}$ and $z^M = f(U^0, z^0)$, contradicting **II**.

Next, we show that R is quasi-transitive. Suppose three alternatives $x, y, z \in X$ are such that $(x, y) \in P(R)$ and $(y, z) \in P(R)$. This implies that there exist $K, L \in \mathbb{N}$, $x^0, y^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K), (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C$ such that $x = x^0$, $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$, $x^K = y$, $y = y^0$, $y^{\ell-1} = f(T^\ell, y^\ell)$ for all $\ell \in \{1, \ldots, L\}$ and $y^L = z$. Letting $M = K + L$, $z^0 = x^0$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K+1, \ldots, K+L\}$, it follows that $(z, x) \notin R$. Because R is complete, we obtain $(x, z) \in P(R)$. Thus, R is quasi-transitive.

It remains to show that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$. If there exists $y \in S$ such that $(x, y) \notin R$, it follows that there exist $K \in \mathbb{N}$, $x^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$ such that $y = x^0$, $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$ and $x^K = x$. Letting $S^0 = S$, we obtain a contradiction to **IE**. Therefore, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By definition, this implies $(x, y) \notin R$ and hence $x \notin G(S, R)$. ■

The definition of the relation R in the above proof is based on the following intuition. Recall that f is intended to identify, for each feasible set S and for each element y of S that is not chosen by C, an alternative x in S such that y is not at least as good as x. Because G-rationalizability by a reflexive and complete relation is considered in the above theorem, the absence of a weak preference of y over x is equivalent to a strict preference of x over y, that is, $(x, y) \in P(R)$. In consequence, we must have a strict preference of an alternative x over an alternative y according to a reflexive and complete G-rationalization whenever x is identified by f to be responsible for keeping y out of the set of chosen alternatives from S. Because R is also required to be quasi-transitive, chains of strict preference have to be respected as well. In order to arrive at a reflexive, complete and quasi-transitive G-rationalization of C, we define R to be composed of all pairs $(x, y) \in X \times X$ such that y does not have to be strictly preferred to x according to the above-described criterion. The properties of f ensure that R indeed is a reflexive, complete and quasi-transitive G-rationalization of C.

We conclude this section with characterizations of **G-R-S** and **G-RC-T** (and, of course, their equivalents). Their axiomatizations are simpler than those analyzed thus far because there are well-defined consistent and transitive closure operations whose existence facilitates the formulation of the requisite properties of f.

In the case of consistent G-rationalizability, the following property of f is relevant.

Consistent-closure irreversibility (CCI). For all $(S, x) \in \mathcal{A}_C$ and for all $y \in S$,

$$(x, y) \in sc(R_C) \Rightarrow f(S, x) \neq y.$$

CCI requires that the consistent closure of the direct revealed preference relation R_C be respected as established in Part (ii) of Theorem 2: if a pair of alternatives (x, y) is in this consistent closure, then (x, y) must be in any consistent G-rationalization of C and, as a consequence, y cannot be the element that keeps x from being chosen from a set in which both are present. The existence of a function f with this property is also sufficient for **G-R-S** and **G-0-S**. An alternative characterization of this rationalizability notion can be found in Bossert, Sprumont and Suzumura (2005a).

Theorem 13 *Suppose $C\colon \Sigma \rightarrow \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-R-S**, **G-0-S** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **CCI**.

Proof. In view of Theorem 3, it is sufficient to treat the case of **G-0-S**.

We first prove that **G-0-S** implies the existence of a function $f \in \mathcal{F}_C$ which satisfies **CCI** provided that $\mathcal{A}_C \neq \emptyset$. Let R be a consistent G-rationalization of C. Consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$. To show that f satisfies **CCI**, suppose $(S, x) \in \mathcal{A}_C$ and $y \in S$ are such that $(x, y) \in sc(R_C)$. By Part (ii) of Theorem 2, it follows that $(x, y) \in R$ and, thus, $f(S, x) \neq y$ by definition of f.

Now suppose that, provided $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **CCI**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a consistent G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **CCI**. Define

$$R = sc(R_C).$$

Clearly, R is consistent. We complete the proof by showing that it is a G-rationalization of C. To that end, suppose $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$. This implies $(x, y) \in R_C \subseteq sc(RC) = R$ for all $y \in S$ and, thus, $x \in G(S, R)$.

Now suppose $x \in G(S, R)$. By definition, $(x, y) \in sc(R_C)$ for all $y \in S$. If $(S, x) \in \mathcal{A}_C$, **CCI** implies $f(S, x) \neq y$ for all $y \in S$, contrary to the existence of f. Thus, $(S, x) \notin \mathcal{A}_C$ which implies $x \in C(S)$ by definition. ■

Our final result provides an analogous characterization of **G-RC-T** and its equivalents. All that needs to be done is to replace the consistent closure with the transitive closure in the relevant property of f.

Transitive-closure irreversibility (TCI). For all $(S, x) \in \mathcal{A}_C$ and for all $y \in S$,

$$(x, y) \in tc(R_C) \Rightarrow f(S, x) \neq y.$$

Analogously to **CCI**, **TCI** requires that the transitive closure of the direct revealed preference relation R_C be respected as established in Part (iii) of Theorem 2: if a pair of alternatives (x, y) is in the transitive closure of R_C, then (x, y) must be in any transitive G-rationalization of C and, as a consequence, y cannot be the element that keeps x from being chosen from a set in which both are present. The existence of a function f with this property is also sufficient for **G-0-T** and all of its equivalent properties. Alternative characterizations of this rationalizability notion can be found in Richter (1966; 1971), Hansson (1968) and Suzumura (1977).

Theorem 14 *Suppose $C\colon \Sigma \rightarrow \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-T**, **G-C-T**, **G-R-T**, **G-0-T**, **G-RC-S**, **G-C-S**, **M-RC-T**, **M-C-T**, **M-RC-S**, **M-C-S** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **TCI**.

Proof. In view of Theorem 3, it is sufficient to treat the case of **G-0-T**.

We begin by proving that **G-0-T** implies the existence of a function $f \in \mathcal{F}_C$ which satisfies **TCI** provided that $\mathcal{A}_C \neq \emptyset$. Let R be a transitive G-rationalization of C. Consider any $(S, y) \in$

$\mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$. To show that f satisfies **TCI**, suppose $(S, x) \in \mathcal{A}_C$ and $y \in S$ are such that $(x, y) \in tc(R_C)$. By Part (iii) of Theorem 2, it follows that $(x, y) \in R$ and, thus, $f(S, x) \neq y$ by definition of f.

Now suppose that, provided $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **TCI**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **TCI**. Define

$$R = tc(R_C).$$

Clearly, R is transitive. We complete the proof by showing that it is a G-rationalization of C. To that end, suppose $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$. This implies $(x, y) \in R_C \subseteq tc(RC) = R$ for all $y \in S$ and, thus, $x \in G(S, R)$.

Now suppose $x \in G(S, R)$. By definition, $(x, y) \in tc(R_C)$ for all $y \in S$. If $(S, x) \in \mathcal{A}_C$, **TCI** implies $f(S, x) \neq y$ for all $y \in S$, contrary to the existence of f. Thus, $(S, x) \notin \mathcal{A}_C$ which implies $x \in C(S)$ by definition. ■

For the sake of easy reference, our characterization theorems are summarized in Table 1. Each row corresponds to a rationalizability property and each column except for the last (which identifies the relevant theorem) represents a property of a function f as defined earlier. An asterisk in a cell means that the corresponding property of f is used in the characterization of the corresponding rationalizability requirement.

Table 1: Rationalizability and Properties of *f*

	DRE	**DRI**	**RI**	**DSI**	**II**	**RE**	**SI**	**DSE**	**IE**	**CCI**	**TCI**	**Theorem**
G-R-0	*											**4**
G-RC-0	*	*										**5**
G-R-A	*		*									**6**
G-C-A	*			*								**7**
G-RC-A	*				*							**8**
G-0-Q	*		*			*						**9**
G-R-Q	*		*			*	*					**10**
G-C-Q	*			*				*				**11**
G-RC-Q					*				*			**12**
G-R-S										*		**13**
G-RC-T											*	**14**

5 Concluding Remarks

The conditions employed in our axiomatizations involve existential clauses. This is sometimes seen as a shortcoming, but this objection, by itself, does not stand on solid ground: there is nothing inherently undesirable in an axiom involving existential clauses. If the argument is that existential clauses are difficult to verify in practice, this is easily countered by the observation that universal quantifiers are no easier to check algorithmically than existential quantifiers. At least, in the case of existential clauses, a search algorithm can terminate once one object with the desired property is found. In this respect, our conditions compare rather favorably with those that are required for many forms of rationalizability where universal quantifiers play a dominant role.

We suspect that a major reason behind the reluctance to accept existential clauses in the context of rational choice may be that conditions involving existential requirements are seen as being 'too close' to the rationalizability property itself, because the desired property is expressed in terms of the existence of a rationalization. This is (except for obvious cases) a matter of judgement, of course. Our view is that the combinations of the axioms employed in the characterizations of the weak forms of rationalizability represent an interesting and insightful way of separating the properties which are necessary and sufficient for each class of weak rationalizability. Moreover, as is apparent from the proofs of our results, there is a substantial amount of work to be done in order to deduce the existence of a rationalization from the mere existence of a function f with the requisite properties. Furthermore, the axioms we use appear to be rather clear and the roles they play in the respective results have very intuitive interpretations. Finally, we should observe that the mathematical structures encountered here are similar to those appearing in dimension theory, which addresses the question of how many orderings are required to express a quasi-ordering as the intersection of those orderings. Consequently, closely related complexities cannot but arise. In fact, existential clauses appear in many of the characterization results in that area; see, for example, Dushnik and Miller (1941).

In concluding this paper, some remarks on further problems to be explored are in order. Because we do not impose any restrictions on the domain of a choice function (other than non-emptiness), our results are extremely general. As a result, our theorems can be of relevance in whatever context of rational choice as purposive behavior we may care to specify, which is an obvious merit of our general approach. Note, however, that this approach may overlook some meaningful further directions to explore by being insensitive to the structural properties of the domain which may make perfect sense in the specific contexts on which we are focusing. Some consequences of an important example of such structural properties, namely, set-theoretic *closedness* assumptions, are examined in Bossert and Suzumura (2005). However, there are many others that one might want to analyze in future work.

References

Armstrong, W.E. (1948): "Uncertainty and the Utility Functions," *Economic Journal,* Vol.58, pp.1–10.

Arrow, K.J. (1951, second ed. 1963): *Social Choice and Individual Values,* New York: Wiley.

Arrow, K.J. (1959): "Rational Choice Functions and Orderings," *Economica,* Vol.26, pp.121–127.

Bandyopadhyay, T. and K. Sengupta (1991): "Revealed Preference Axioms for Rational Choice," *Economic Journal,* Vol.101, pp.202–213.

Bossert, W., Y. Sprumont and K. Suzumura (2005a): “Consistent Rationalizability,” *Economica,* Vol.72, pp.185–200.

Bossert, W., Y. Sprumont and K. Suzumura (2005b): “Rationalizability of Choice Functions on General Domains Without Full Transitivity,” forthcoming in *Social Choice and Welfare.*

Bossert, W., Y. Sprumont and K. Suzumura (2005c): “Maximal-Element Rationalizability,” forthcoming in *Theory and Decision.*

Bossert, W. and K. Suzumura (2005): “Domain Closedness Conditions and Rational Choice,” *Working Paper,* CIREQ, Université de Montréal.

Dushnik, B. and E.W. Miller (1941): “Partially Ordered Sets,” *American Journal of Mathematics,* Vol.63, pp.600–610.

Hansson, B. (1968): “Choice Structures and Preference Relations,” *Synthese,* Vol.18, pp.443–458.

Houthakker, H.S. (1950): “Revealed Preference and the Utility Function,” *Economica,* Vol.17, pp.159–174.

Luce, R.D. (1956): “Semiorders and the Theory of Utility Discrimination,” *Econometrica,* Vol.24, pp.178–191.

Richter, M.K. (1966): “Revealed Preference Theory,” *Econometrica,* Vol.41, pp.1075–1091.

Richter, M.K. (1971): “Rational Choice,” in J.S. Chipman, L. Hurwicz, M.K. Richter and H.F. Sonnenschein, eds., *Preferences, Utility, and Demand,* New York: Harcourt Brace Jovanovich, pp.29–58.

Samuelson, P.A. (1938): “A Note on the Pure Theory of Consumer’s Behaviour,” *Economica,* Vol.5, pp.61–71.

Samuelson, P.A. (1947, second ed. 1983): *Foundations of Economic Analysis,* Cambridge: Harvard University Press.

Samuelson, P.A. (1948): “Consumption Theory in Terms of Revealed Preference,” *Economica,* Vol.15, pp.243–253.

Samuelson, P.A. (1950): “The Problem of Integrability in Utility Theory,” *Economica,* Vol.17, pp.355–385.

Schwartz, T. (1976): "Choice Functions, 'Rationality' Conditions, and Variations of the Weak Axiom of Revealed Preference," *Journal of Economic Theory,* Vol.13, pp.414–427.

Sen, A.K. (1969): "Quasi-Transitivity, Rational Choice and Collective Decisions," *Review of Economic Studies,* Vol.36, pp.381–393.

Sen, A.K. (1970): *Collective Choice and Social Welfare,* San Francisco: Holden-Day.

Sen, A.K. (1971): "Choice Functions and Revealed Preference," *Review of Economic Studies,* Vol.38, pp.307–317.

Suzumura, K. (1976): "Remarks on the Theory of Collective Choice," *Economica,* Vol.43, pp.381–390.

Suzumura, K. (1977): "Houthakker's Axiom in the Theory of Rational Choice," *Journal of Economic Theory,* Vol.14, pp.284–290.

Suzumura, K. (1983): *Rational Choice, Collective Decisions and Social Welfare,* New York: Cambridge University Press.

Suzumura, K. (2004): "An Extension of Arrow's Lemma with Economic Applications," *Working Paper,* Institute of Economic Research, Hitotsubashi University.

Szpilrajn, E. (1930): "Sur l'Extension de l'Ordre Partiel," *Fundamenta Mathematicae,* Vol.16, pp.386–389.

Uzawa, H. (1957): "Notes on Preference and the Axiom of Choice," *Annals of the Institute of Statistical Mathematics,* Vol.8, pp.35–40.