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Generations Economies

Tomoichi Shinotsuka
(Department of Economics, Otaru University of Commerce)
Koichi Suga
(School of Political Science and Economics, Waseda University)
Kotaro Suzumura
(Institute of Economic Research, Hitotsubashi University)
Koichi Tadenuma
(Faculty of Economics, Hitotsubashi University)

Naka 2-1, Kunitachi, Tokyo 186-8603, Japan
Phone: +81-42-580-8350   Fax: +81-42-580-8351
URL: http://wakame.econ.hit-u.ac.jp/~koho/1intro/COE/index.htm
E-mail: COE-RES@econ.hit-u.ac.jp

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Tomoichi Shinotsuka
Department of Economics
Otaru University of Commerce
Hokkaido 470-8501, Japan
e-mail: shino@res.otaru-uc.ac.jp

Koichi Suga
School of Political Science and Economics
Waseda University
Shinjuku-ku, Tokyo 169-8050, Japan
e-mail:ksuga@waseda.jp

Kotaro Suzumura
Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo 186-8603, Japan
e-mail:suzumura@ier.hit-u.ac.jp

Koichi Tadenuma
Faculty of Economics
Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan
e-mail:tadenuma@econ.hit-u.ac.jp

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Abstract

We introduce several distinct notions of equity as no-envy into the overlapping generations economy formulated by Samuelson (1958). No-Envy in Overlapping Consumptions requires that for each time period, no person should prefer the bundle of any other person who lives in the same period to his/her own. No-Envy in Lifetime Consumptions states that no person should prefer the lifetime consumption plan of any other person to his/her own. Equity in Lifetime Rate of Return requires that the lifetime rate of return (Cass and Yaari,1966) be equal for all persons. For each of the three notions, we characterize allocations satisfying it, and clarify logical relations among these conditions. We also examine existence of a stationary allocation that attains maximal utility under No-Envy in Lifetime Consumptions.

1 Introduction

This paper studies equity and efficiency of allocations in the overlapping generations economy formulated by Samuelson (1958). A central notion of distributional equity is *no-envy* (Foley (1967), Kolm (1972)): no person prefers the consumption of any other person to his/her own. Suzumura (2002) proposed three distinct notions of equity based on no-envy in overlapping generations economies. The first one concerns contemporary (overlapping) consumptions in each time period. It requires that for each period, no person should prefer the consumption of any other person in that period to his/her own. In contrast, the second notion is about lifetime consumption plans. It stipulates that no person should prefer the lifetime consumption plan of any other person to his/her own. The third notion is based on the *lifetime rate of return* due to Cass and Yaari (1966). It simply requires an equal lifetime rate of return of all persons.

In a simple model where there is one (composite) commodity with no production, the preferences are identical, and the rate of population growth is constant, we examine the implications of each notion of equity, and clarify the logical relations of the three notions, paying special attention to their relations with stationarity of allocations. We also study the existence and characterizations of allocations attaining maximal utility under no-envy in liftime consumption plans, which are the allocations selected by the *equity-first and efficiency-second principle* due to Tadenuma (2002).

The rest of the paper is organized as follows. The next section formulates the model and the notions of equity. The following three sections examine the implications of no-envy in contemporary (overlapping) consumptions, equality in lifetime rate of return, and no-envy in lifetime consumption plans, respectively. Section 6 studies allocations attaining maximal utility under no-envy in lifetime consumption plans. The final section summarizes the results.

2 The Model and the Notions of Equity

We consider the overlapping generations model formulated by Samuelson (1958). Each person lives for two periods, earning one unit of a perishable good in the first period, and earning nothing in the second period. Let C^i denote an amount of consumption in her i-th period. Preferences are assumed to be identical for all persons. Let $U : \mathbb{R}^2_+ \to \mathbb{R}$ be the common utility function, which is increasing in each argument. The value $U(C^1, C^2)$ represents the utility level when a person consumes C^1 and C^2 in his first period and in his second period, respectively. At the beginning of each period t, $(1+n)^t$ persons are born where $n \geq 0$ is the *rate of population growth*. We call the persons born at t *generation* t.

In order to focus on *inter*-generational distribution problems rather than *intra*-generational problems, we assume that all persons belonging to the same generation consumes the same bundle. Let $\mathbf{C}_t = (C^1_t, C^2_t)$ be a lifetime consumption plan of each person in generation t. Since $(1+n)^t$ persons are born at the beginning of each period t, the aggregate consumption vector of generation t is given by $(1+n)^t\mathbf{C}_t$. An *allocation* is a doubly infinite and nonnegative sequence $\{\mathbf{C}_t\}_{t\in Z}$, where Z denotes the set of all integers. An allocation $\{\mathbf{C}_t\}_{t\in Z}$ is *stationary* if $\mathbf{C}_{t-1} = \mathbf{C}_t$ for all $t \in Z$. An allocation $\{\mathbf{C}_t\}_{t\in Z}$ is *feasible* if for every $t \in Z$,

$$C^1_t + \frac{C^2_{t-1}}{1+n} \leq 1, \tag{1}$$

and it is *exactly feasible* if for every $t \in Z$,

$$C^1_t + \frac{C^2_{t-1}}{1+n} = 1. \tag{2}$$

An allocation $\{\mathbf{C}_t\}_{t\in Z}$ is *Pareto efficient* if it is feasible and there exists no feasible allocation $\{\mathbf{C}'_t\}_{t\in Z}$ such that $U(\mathbf{C}'_t) \geq U(\mathbf{C}_t)$ for all $t \in Z$ with strict inequality for some t.

Following Suzumura (2002), we formulate three distinct notions of equity-as-no-envy. The first one requires that for each time period, no person prefers the consumption of any other person in the same period. Since there is only one (composite) commodity in our model, the condition reduces to the following one:

No-Envy in Overlapping Consumptions (NEOC): For all $t \in Z$, $C^2_{t-1} = C^1_t$.

The second notion means that no person prefers the lifetime consumption plan of any other person to his/her own. Because it is assumed that the preferences are identical, and that all persons belonging to the same generation have the same lifetime consumption, the condition reduces to the following one:

No-Envy in Lifetime Consumptions (NELC): For all $t \in Z$, $U(\mathbf{C}_{t-1}) = U(\mathbf{C}_t)$.

The third notion of equity is based on the *lifetime rate of return* due to Cass and Yaari (1966). Consider allocations $\{\mathbf{C}_t\}_{t \in Z}$ such that $C_t^1 > 0$ and $C_t^2 > 0$ for all t. We call such allocation *positive*. Note that exact feasibility and positivity together imply $C_t^1 < 1$ for all $t \in Z$. Thus, the following notion of lifetime rate of return is well-defined:

$$r_t = \frac{C_t^2 - (1 - C_t^1)}{1 - C_t^1}.$$

Equity in Lifetime Rate of Return (ELRR): For all positive and exactly feasible allocations, and for all $t \in Z$, $r_{t-1} = r_t$.

In the following sections, we investigate implications and logical relations of these equity notions.

3 No-Envy in Overlapping Consumptions

We start with examining implications of No-Envy in Overlapping Consumptions (NEOC). It turns out that this is the strongest requirement of all the equity notions introduced above, and a fundamental trade-off between equity and efficiency emerges.

Proposition 1 *There exists one and only one allocation satisfying exact feasibility and NEOC. It is given by*

$$\hat{\mathbf{C}}_t = \hat{\mathbf{C}} := \left(\frac{n+1}{n+2}, \frac{n+1}{n+2} \right)$$

for all $t \in Z$. If the marginal rate of substitution of consumption in period 2 for consumption in period 1 at $\hat{\mathbf{C}}$ is not equal to $1 + n$, then the allocation is not Pareto efficient.

Proof: Let $\{\hat{\mathbf{C}}_t\}_{t \in Z}$ be an allocation satisfying exact feasibility and NEOC. Let $t \in Z$ be given. By exact feasibility,

$$\hat{C}_t^1 + \frac{\hat{C}_{t-1}^2}{1+n} = 1. \tag{3}$$

By NEOC,

$$\hat{C}^2_{t-1} = \hat{C}^1_t \tag{4}$$

Solving equations (3) and (4), we have

$$(\hat{C}^1_t, \hat{C}^2_{t-1}) = \left(\frac{n+1}{n+2}, \frac{n+1}{n+2}\right).$$

Because the rate of population growth n is constant, the solution does not depend on t. Hence, $\hat{C}^2_{t-1} = \hat{C}^2_t$. Therefore, for all $t \in Z$,

$$\hat{\mathbf{C}}_t := (\hat{C}^1_t, \hat{C}^2_t) = \left(\frac{n+1}{n+2}, \frac{n+1}{n+2}\right) = \hat{\mathbf{C}}.$$

This means that the allocation $\{\hat{\mathbf{C}}_t\}_{t\in Z}$ is stationary.

Define

$$\eta(\hat{\mathbf{C}}) := \frac{\frac{\partial}{\partial C^1}U(\hat{\mathbf{C}})}{\frac{\partial}{\partial C^2}U(\hat{\mathbf{C}})}.$$

$\eta(\hat{\mathbf{C}})$ is the marginal rate of substitution of consumption in period 2 for consumption in period 1 at $\hat{\mathbf{C}}$. Notice that a stationary allocation $\{\mathbf{C}_t\}_{t\in Z}$ is feasible if and only if

$$C^1_t + \frac{C^2_t}{1+n} \leq 1$$

for all $t \in Z$. Hence, if $\eta(\hat{\mathbf{C}}) \neq 1+n$, then there exists a feasible and stationary allocation that Pareto dominates $\{\hat{\mathbf{C}}_t\}_{t\in Z}$ (Figure 1). *Q.E.D.*

Let us call the allocation $\{\hat{\mathbf{C}}_t\}_{t\in Z}$ defined above the *NEOC allocation.*

Corollary 1 *If an exactly feasible allocation satisfies NEOC, then it also satisfies both NELC and ELRR.*

Proof: From Proposition 1, if an exactly feasible allocation satisfies NEOC, then it must be the NEOC allocation. Since the NEOC allocation is stationary, it satisfies both NELC and ELRR. *Q.E.D.*

Corollary 2 *If the marginal rate of substitution of consumption in period 2 for consumption in period 1 at* $\hat{\mathbf{C}}$ *is not equal to* $1+n$*, then there exists no allocation satisfying NEOC and Pareto efficiency together.*

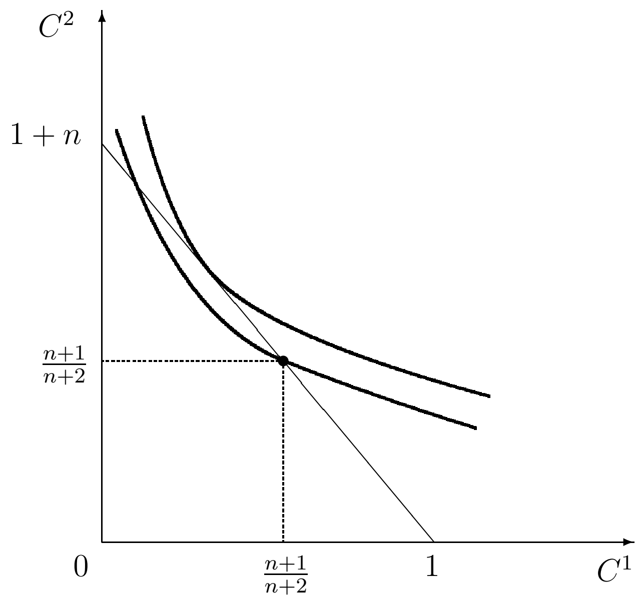

Figure 1: Trade-off between NEOC and Pareto efficiency

Proof: Suppose, on the contrary, that there exists an allocation satisfying NEOC and Pareto efficiency together. Since it is Pareto efficient, it must be exactly feasible. By Proposition 1, exact feasibility and NEOC imply that the allocation is the NEOC allocation. But since $\eta(\hat{\mathbf{C}}) \neq 1+n$, it also follows from Proposition 1 that the allocation cannot be Pareto efficient. This is a contradiction. *Q.E.D.*

Consider the separable utility function U defined by

$$U(C^1, C^2) = u(C^1) + \delta u(C^2) \tag{5}$$

where δ is a positive constant and u is a twice continuously differentiable function with a positive first derivative and a negative second derivative. Then, $\delta^{-1} - 1$ is called the *pure rate of time preference*. In this case,

$$\eta(\hat{\mathbf{C}}) = \frac{\frac{\partial}{\partial C^1} U(\hat{\mathbf{C}})}{\frac{\partial}{\partial C^2} U(\hat{\mathbf{C}})} = \delta^{-1}.$$

Hence, we have the following corollary.

Corollary 3 *In the case of the separable utility function, if the pure rate of time preference is not equal to the rate of population growth, then there exists no allocation satisfying NEOC and Pareto efficiency together.*

Proposition 1 has two interesting implications. First, *growth is compatible with equity.* If there were no growth, each young person would have to split her earnings to give a half to an old person in order to achieve equity in overlapping consumptions. The faster the economy grows, the more each person can receive because she has to give less to older persons in her young age while she can receive more from younger persons in her old age. Indeed, the NEOC allocation is strictly increasing in the rate of population growth n.

The second implication of Proposition 1 is that *there exists almost surely a room for improvement in the welfare of all persons at the NEOC allocation.* In the static model with one commodity, equal split is always Pareto efficient as well as equitable. By contrast, in this simple model with overlapping generations, equal split among contemporary persons at each period does not lead to a Pareto efficient allocation.

4 Equity in Lifetime Rate of Return and the Biological Rate of Interest

In this section, we consider positive and exactly feasible allocations, for which the lifetime rate of return r_t is well-defined.

It is easy to see that the lifetime rate of return associated with the NEOC allocation is equal to n. That is, *NEOC implies that the lifetime rate of return should be equal to the "biological rate of interest".* This welfare implication on the biological rate of interest seems new.

As we saw in the previous section, under exact feasibility, NEOC implies Equity in Lifetime Rate of Return (ELRR), but the converse is not true. Indeed, any stationary allocation satisfies ELRR. Hence, interesting questions would be:

(1) Are there non-stationary allocations satisfying ELRR?

(2) Does ELRR provide some implications on the biological rate of interest?

Our next result answers the above questions.

Proposition 2 *Let $\{\mathbf{C}_t\}_{t\in Z}$ be an exactly feasible, positive allocation satisfying ELRR with the common lifetime rate of return r being nonnegative. Then, $r = n$ and the allocation is stationary.*

Proof: By ELRR, $C_t^2 = (1+r)(1-C_t^1)$ for all t. By exact feasibility, $C_{t-1}^2 = (1+n)(1-C_t^1)$ for all t. Hence, $(1+r)(1-C_t^1) = (1+n)(1-C_{t+1}^1)$ for all t. Let $\lambda = (1+r)/(1+n)$. Since r is nonnegative, λ is positive. Then, $(1-C_{t+1}^1) = \lambda(1-C_t^1)$ for all t. Hence, $(1-C_t^1) = \lambda^t(1-C_0^1)$ for all t. Note that $1-C_t^1$ is positive for all t since $\{\mathbf{C}_t\}_{t\in Z}$ is exactly feasible and positive. If $\lambda > 1$, $\{|1-C_t^1|\}_{t\in Z}$ goes to infinity as t goes to infinity. If $\lambda > 1$, $\{|1-C_t^1|\}_{t\in Z}$ goes to infinity as t goes to *minus infinity*. Hence, exact feasibility is violated unless $\lambda = 1$. Therefore, $r = n$. Then, the allocation is stationary. *Q.E.D.*

Proposition 2 means that, for positive and exactly feasible allocations, *ELRR is equivalent to stationarity*. In other words, an allocation satisfies ELRR if and only if the lifetime rate of return of each generation is equal to the biological rate of interest. This seems to be an interesting characterization of the biological rate of interest.

In Figure 1, the set of all exactly feasible and positive allocations satisfying ELRR are depicted as the line

$$C^1 + \frac{C^2}{1+n} = 1.$$

The point $((n+1)/(n+2), (n+1)/(n+2))$ on the line represents the unique (under exact feasibility) allocation satisfying NEOC. This figure clearly shows that under exact feasibility, NEOC implies ELRR.

5 No-Envy in Lifetime Consumptions

In this section, we investigate implications of requiring No-Envy in Lifetime Consumptions (NELC). Clearly, any stationary allocation satisfies NELC. The question is: *Is there any non-stationary allocation satisfying exact feasibility and NELC?*

A basic observation is that exact feasibility and NELC together generate a dynamical system defined by the following difference equation: for all $t \in Z$,

$$U(C_{t-1}^1, (1+n)(1-C_t^1)) = U(C_t^1, (1+n)(1-C_{t+1}^1)). \tag{6}$$

By nonnegativity of consumption and exact feasibility,

$$0 \leq C_t^1 \leq 1 \tag{7}$$

for all $t \in Z$.

Proposition 3 *Let* $\{\mathbf{C}_t\}_{t \in Z}$ *be an exactly feasible allocation satisfying NELC. If it is not a stationary allocation, then there exist* $\bar{C}^1 \in [0,1]$ *and* $\bar{\bar{C}}^1 \in [0,1]$ *such that* $\lim_{t\to\infty} C_t^1 = \bar{C}^1$, $\lim_{t\to-\infty} C_t^1 = \bar{\bar{C}}^1$, *and for all* $t \in Z$, $U(\mathbf{C}_t) = U(\bar{C}^1, (1+n)(1-\bar{C}^1)) = U(\bar{\bar{C}}^1, (1+n)(1-\bar{\bar{C}}^1))$.

Proof: Assume that $\{\mathbf{C}_t\}_{t \in Z}$ is exactly feasible, satisfies NELC, but is not stationary. Then, there exists $t^* \in Z$ such that $C_{t^*-1}^1 \neq C_{t^*}^1$.
Case 1: $C_{t^*-1}^1 < C_{t^*}^1$.
By equation (6) and the strict monotonicity of U, it must be true that $(1+n)(1-C_{t^*}^1) > (1+n)(1-C_{t^*+1}^1)$. Hence, $C_{t^*}^1 < C_{t^*+1}^1$. Repeating this argument, we have $C_{t^*-1}^1 < C_{t^*}^1 < C_{t^*+1}^1 < C_{t^*+2}^1 \ldots$. That is, C_t^1 is monotonically increasing as t increases. Since C_t^1 is bounded in $[0,1]$, there exists $\bar{C}^1 \in [0,1]$ such that $\lim_{t\to\infty} C_t^1 = \bar{C}^1$.

On the other hand, by $C_{t^*-1}^1 < C_{t^*}^1$, we have $(1+n)(1-C_{t^*-1}^1) > (1+n)(1-C_{t^*}^1)$. It follows from equation (6) and the strict monotonicity of U that $C_{t^*-2}^1 < C_{t^*-1}^1$. Repeating this argument, we can show that C_t^1 is monotonically decreasing as t decreases. Since C_t^1 is bounded in $[0,1]$, there exists $\bar{\bar{C}}^1 \in [0,1]$ such that $\lim_{t\to-\infty} C_t^1 = \bar{\bar{C}}^1$.

By continuity of U and equation (6), we have for all $t \in Z$, $U(\mathbf{C}_t) = U(\bar{C}^1, (1+n)(1-\bar{C}^1)) = U(\bar{\bar{C}}^1, (1+n)(1-\bar{\bar{C}}^1))$.
Case 2: $C_{t^*-1}^1 > C_{t^*}^1$.
It can be shown that C_t^1 is monotonically decreasing as t increases, and monotonically increasing as t decreases. Then, the claim in the proposition follows from a similar argument to case 1. *Q.E.D.*

5.1 Linear Utility Case

In this subsection, we assume that the utility function U is linear:

$$U(C^1, C^2) = aC^1 + C^2 \tag{8}$$

where a is a positive constant.

Proposition 4 *If the utility function U is linear, then there does not exist a non-stationary allocation satisfying exact feasibility and NELC.*

Proof: Substituting (8) into (6), we obtain the following linear, second order difference equation:

$$aC_{t-1}^1 + (1+n)(1-C_t^1) = aC_t^1 + (1+n)(1-C_{t+1}^1). \tag{9}$$

Rearranging the terms gives

$$(1+n)C_{t+1}^1 - (1+n+a)C_t^1 + aC_{t-1}^1 = 0. \tag{10}$$

Consider the corresponding characteristic equation:

$$(1+n)x^2 - (1+n+a)x + a = 0. \tag{11}$$

The solutions to (11) are:

$$x_1 = 1,\ x_2 = \frac{a}{1+n}. \tag{12}$$

Therefore, the solution to the difference equation (6) is as follows:
if $a = 1+n$,

$$C_t^1 = B_1 + B_2 t, \tag{13}$$

and if $a \neq 1+n$,

$$C_t^1 = B_1 + B_2 x_2^t, \tag{14}$$

where B_1 and B_2 are constants depending on the initial condition. Note that an allocation $\{\mathbf{C}_t\}_{t\in Z}$ is stationary if and only if $\{C_t^1\}_{t\in Z}$ is constant over time.

The case of $a = 1+n$ is devided into three subcases. If $B_2 > 0$, then $\{C_t^1\}_{t\in Z}$ goes above 1 as t goes to infinity, and it goes below 0 as t goes to minus infinity. Thus the feasibility condition does not hold for sufficiently large or small t. If $B_2 < 0$, then $\{C_t^1\}_{t\in Z}$ goes below 0 as t goes to infinity, and it goes above 1 as t goes to minus infinity. Thus the feasibility condition does not hold for sufficiently large or small t. When $B_2 = 0$, $\{C_t^1\}_{t\in Z}$ is constant.

Suppose $\frac{a}{1+n} < 1$. If $\{C_t^1\}_{t\in Z}$ is not constant over time, then $\{C_t^1\}_{t\in Z}$ goes to infinity as t goes to minus infinity. Hence, there does not exist a non-stationary allocation satisfying exact feasibility and NELC.

Suppose $\frac{a}{1+n} > 1$. If $\{C_t^1\}_{t\in Z}$ is not constant over time, then $\{C_t^1\}_{t\in Z}$ goes to infinity as t goes to infinity. Hence, there does not exist a non-stationary allocation satisfying exact feasibility and NELC in this case either. *Q.E.D.*

5.2 Quasi-Linear Utility Case

In this subsection, we assume that the utility function U is of the following form:

$$U(C^1, C^2) = v(C^1) + C^2, \tag{15}$$

where $v(C^1)$ is of the class C^2 and has positive first derivative and negative second derivative. Define $\alpha := \min\{v'(c) \mid c \in [0,1]\}$ and $\beta := \max\{v'(c) \mid c \in [0,1]\}$. The parameters α and β measure the magnitudes of marginal utility in the first-period consumption.

Proposition 5 *If either $\alpha > 1+n$ or $\beta < 1+n$, there does not exist a non-stationary allocation satisfying exact feasibility and NELC.*

Proof: Substituting (15) into (6), we obtain a non-linear, second order difference equation:

$$v(C^1_{t-1}) + (1+n)(1 - C^1_t) = v(C^1_t) + (1+n)(1 - C^1_{t+1}). \tag{16}$$

Rearranging the terms, we have

$$C^1_{t+1} - C^1_t = \frac{1}{1+n}[v(C^1_t) - v(C^1_{t-1})]. \tag{17}$$

By the mean value theorem, there exists $c^*_t \in [0,1]$ such that

$$|C^1_{t+1} - C^1_t| = \frac{v'(c^*_t)}{1+n}|C^1_t - C^1_{t-1}|. \tag{18}$$

Assume that $\alpha > 1+n$. For all $t \in Z$, since $v'(c^*_t) \geq \alpha > 1+n$, we have $\frac{v'(c^*_t)}{1+n} > 1$. It then follows from (18) that if $\{C^1_t\}_{t\in Z}$ is not constant, then $\{|C^1_{t+1} - C^1_t|\}_{t\in Z}$ goes to infinity as t goes to infinity. Therefore, $\{C^1_t\}_{t\in Z}$ violates feasibility.

Next, assume that $0 < \beta < 1+n$. Then, $0 < \frac{\beta}{1+n} < 1$. For all $t \in Z$, since $v'(c^*_t) \leq \beta$, it follows from (18) that

$$|C^1_{t+1} - C^1_t| \leq \frac{\beta}{1+n}|C^1_t - C^1_{t-1}|.$$

By iteration, for all positive integers T,

$$|C^1_{t+1} - C^1_t| \leq \left(\frac{\beta}{1+n}\right)^T |C^1_{t-T+1} - C^1_{t-T}|.$$

Letting $t = 0$, for all positive integers T, we obtain

$$\left(\frac{\beta}{1+n}\right)^{-T} |C_1^1 - C_0^1| \leq |C_{-T+1}^1 - C_{-T}^1|.$$

Therefore, if $\{C_t^1\}_{t\in Z}$ is not constant, then $\{|C_{t+1}^1 - C_t^1|\}_{t\in Z}$ goes to infinity as t goes to *minus infinity.* *Q.E.D.*

5.3 Non-Stationary Allocation Satisfying NELC

If the utility function is not linear or quasi-linear, then NELC does not imply stationarity. There exist non-stationary allocations satisfying NELC, as we present an example in what follows.

Let $n > -1$ be the rate of population growth. Let $a > \max\{1+n, \frac{1}{1+n}\}$ be given. The utility function U is defined by:

$$U(C^1, C^2) = \begin{cases} aC^1 + C^2 & \text{if } C^1 \leq C^2 \\ C^1 + aC^2 & \text{if } C^1 > C^2. \end{cases}$$

Since $a > \max\left\{1+n, \frac{1}{1+n}\right\}$, we have

$$\max\left\{\frac{1+n}{1+a}, \frac{1}{1+a}\right\} < \frac{1+n}{2+n}.$$

Choose $\alpha \in \mathbb{R}$ such that

$$\max\left\{\frac{1+n}{1+a}, \frac{1}{1+a}\right\} < \alpha < \frac{1+n}{2+n}. \tag{19}$$

Define $\bar{u} := \alpha(1+a)$. Notice that $\bar{u} = U(\alpha, \alpha)$. It follows from (19) that the indifference curve passing through (α, α) intersects with the line

$$C^1 + \frac{C^2}{1+n} = 1$$

at two distinct points (A and B in Figure 2).

Let Z^- be the set of all negative integers, and Z^+ be the set of all positive integers. Consider the following system of equations:

$$\begin{cases} aC_t^1 + C_t^2 = \bar{u} & \text{for all } t \in Z^- \\ C_t^1 + \frac{C_{t-1}^2}{1+n} = 1 & \text{for all } t \in \{0\} \cup Z^- \\ C_0^1 = \alpha \end{cases}$$

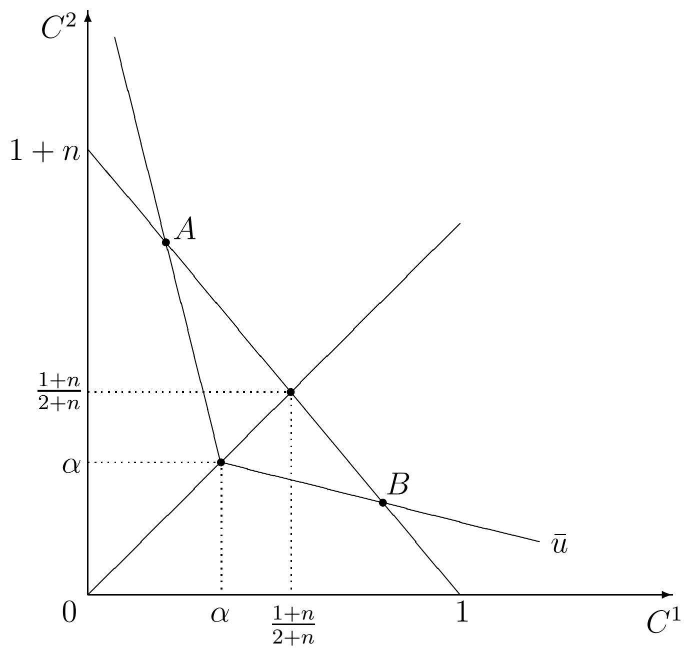

Figure 2: Piecewise-Linear Utility Function

The solution for this system of equations is:

$$C_t^1 = [\bar{u} - (1+n)] \left[\frac{1}{a} + \frac{1+n}{a^2} + \ldots + \frac{(1+n)^{-t-1}}{a^{-t}} \right] + \frac{(1+n)^{-t}\alpha}{a^{-t}} \tag{20}$$

$$C_t^2 = (1+n) - [\bar{u} - (1+n)] \left[\frac{1+n}{a} + \frac{(1+n)^2}{a^2} + \ldots + \frac{(1+n)^{-t-1}}{a^{-t-1}} \right] - \frac{(1+n)^{-t}\alpha}{a^{-t-1}} \tag{21}$$

for all $t \in Z^-$, and $C_0^1 = \alpha$. Because $a > 1+n$, it follows that

$$\lim_{t \to -\infty} C_t^1 = \frac{\bar{u} - (1+n)}{a - (1+n)}$$

$$\lim_{t \to -\infty} C_t^2 = (1+n) \left[1 - \frac{\bar{u} - (1+n)}{a - (1+n)} \right].$$

Let $\bar{C}^1 := \frac{\bar{u}-(1+n)}{a-(1+n)}$ and $\bar{C}^2 := (1+n) \left[1 - \frac{\bar{u}-(1+n)}{a-(1+n)} \right]$. Notice that $\bar{C}^1 + \frac{\bar{C}^2}{1+n} = 1$, and $(\bar{C}^1, \bar{C}^2) = A$ in Figure 2.

Next, consider the following system of equations:

$$\begin{cases} C_t^1 + aC_t^2 = \bar{u} & \text{for all } t \in Z^+ \\ C_t^1 + \frac{C_{t-1}^2}{1+n} = 1 & \text{for all } t \in Z^+ \\ C_0^2 = \alpha \end{cases}$$

The solution for this system of equations is:

$$\begin{aligned} C_t^1 &= 1-(\bar{u}-1)\left[\frac{1}{a(1+n)}+\frac{1}{a^2(1+n)^2}+\ldots+\frac{1}{a^{t-1}(1+n)^{t-1}}\right] \\ &\quad -\frac{\alpha}{a^{t-1}(1+n)^t} \qquad (22) \\ C_t^2 &= (\bar{u}-1)\left[\frac{1}{a}+\frac{1}{a^2(1+n)}+\ldots+\frac{1}{a^t(1+n)^{t-1}}\right]+\frac{\alpha}{a^t(1+n)^t} \qquad (23) \end{aligned}$$

for all $t \in Z^+$, and $C_0^2 = \alpha$. Since $a > \frac{1}{1+n}$, we have

$$\begin{aligned} \lim_{t\to\infty} C_t^1 &= 1-\frac{\bar{u}-1}{a(1+n)-1} \\ \lim_{t\to\infty} C_t^2 &= \frac{(1+n)(\bar{u}-1)}{a(1+n)-1}. \end{aligned}$$

Let $\bar{\bar{C}}^1 := 1-\frac{\bar{u}-1}{a(1+n)-1}$ and $\bar{\bar{C}}^2 := \frac{(1+n)(\bar{u}-1)}{a(1+n)-1}$. Note that $\bar{\bar{C}}^1 + \frac{\bar{\bar{C}}^2}{1+n} = 1$, and $(\bar{\bar{C}}^1, \bar{\bar{C}}^2) = B$ in Figure 2.

The allocation $\{\mathbf{C}_t\}_{t\in Z}$ defined by (20), (21), (22), (23) and $(C_0^1, C_0^2) = (\alpha, \alpha)$ is exactly feasible and satisfies NELC (that is, $U(\mathbf{C}_t) = U(\mathbf{C}_{t'})$ for all $t, t' \in Z$), but it is not stationary. It should be noted, however, that $\mathbf{C}_t$ converges to a consumption bundle at a stationary allocation when $t \to -\infty$, and it also converges to a consumption bundle at another stationary allocation when $t \to +\infty$.

The crucial point in the above example is that there are indifference curves cutting the line $C^1 + \frac{C^2}{1+n} = 1$ twice. Then, we can construct a non-stationary allocation such that the consumption bundle converges to one intersection as t goes to infinity, and to the other intersection as t goes to minus infinity.

In contrast, when the utility function is linear or quasi-linear with $v'(1) > 1+n$ or $v'(0) < 1+n$, then every indifference curve cuts the line $C^1 + \frac{C^2}{1+n} = 1$ at most once. In this case, any non-stationary allocation satisfying NELC must violate feasibility either as t goes to infinity, or as t goes to minus infinity.

6 Pareto Efficiency and Maximal Utility under NELC

In this section, we only assume that $U(C^1, C^2)$ is continuous. Within this most general framework, we settle several problems on the existence of relevant allocations with the purpose of securing the non-emptiness of our analysis on the properties of these allocations.

Proposition 6 *There exists a Pareto efficient allocation.*

Proof: Let $\{\gamma_t\}_{t\in Z}$ be a doubly infinite sequence with $\gamma_t > 0$ for all t and $\sum_{t\in Z}\gamma_t = 1$. Let F be the set of exactly feasible allocations. Clearly, F is non-empty and compact in the product topology. For each $\mathbf{C} \in F$, let $V(\mathbf{C}) = \sum_{t\in Z}\gamma_t U(C_t^1, C_t^2)$. It is easy to see that V is product continuous on F. By the Weierstrass theorem, there exists $\mathbf{C}^*$ in F that attains the maximum value of V over F. Clearly, $\mathbf{C}$ is Pareto efficient. *Q.E.D.*

Next, we consider the following procedure to select allocations. First, we choose all exactly feasible allocations satisfying NELC. Then, we select "optimal" allocations, from an efficiency standpoint, among those allocations. The selection procedure is based on the *equity-first and efficiency-second principle* due to Tadenuma (2002).

Let NE be the set of all exactly feasible allocations satisfying NELC. Clearly, NE is non-empty. Recall that at any allocation satisfying NELC, all the generations t attain the same level of utility. Let

$$u^* := \sup\{\alpha \mid \exists\mathbf{C} \in NE \text{ such that } \forall t \in Z, U(C_t^1, C_t^2) = \alpha\}.$$

We say that an allocation $\mathbf{C} \in NE$ *attains maximal utility under NELC* if $U(C_t^1, C_t^2) = u^*$ for all $t \in Z$.

Proposition 7 *There exists an allocation that attains maximal utility under NELC.*

Proof: Let $\{\{\mathbf{C}_t^n\}_{t\in Z}\}_{n=1}^{\infty}$ be a sequence in NE such that $\{U(C_t^{1n},\ C_t^{2n})\}_{n=1}^{\infty}$ converges to u^*. Since $\{\mathbf{C}_t^n\}_{t\in Z}$ is exactly feasible, there exists a subsequence $\{\{\mathbf{C}_t^{n_q}\}_{t\in Z}\}_{q=1}^{\infty}$ such that for each t $\{\mathbf{C}_t^{n_q}\}_{q=1}^{\infty}$ converges to some $\mathbf{C}_t^*$. Clearly, $\{\mathbf{C}_t^{n*}\}_{t\in Z}$ is exactly feasible. Since $U(C^1, C^2)$ is continuous, $U(C_t^{1*}, C_t^{2*}) = \alpha$. Thus, $\{\mathbf{C}_t^{n*}\}_{t\in Z}$ attains maximal utility under NELC. *Q.E.D.*

Remark: It is an open question whether an allocation that attains maximal utility under NELC is Pareto efficient or not.

If the utility function U is, in addition, quasi-concave, one can prove a stronger result.

Proposition 8 *If U is, in addition, quasi-concave, then there exists a stationary allocation that attains maximal utility under NELC.*

Proof: Let $\{\mathbf{C}_t^{n*}\}_{t\in Z}$ be an allocation that attains maximal utility under NELC. By averaging operations, we construct a sequence of allocations that attain maximal utility under NELC. For each positive integer N and for each $t \in Z$, let

$$\mathbf{C}_t^N = \frac{1}{2N+1}(\mathbf{C}_{t-N}^* + \mathbf{C}_{t-N+1}^* + \cdots + \mathbf{C}_{t-1}^* + \mathbf{C}_t^* + \mathbf{C}_{t+1}^* + \cdots + \mathbf{C}_{t+N}^*).$$

Then, it is easy to see that $\mathbf{C}^N = \{\mathbf{C}_t^N\}_{t\in Z}$ is exactly feasible. By quasi-concavity of U,

$$U(\mathbf{C}_t^N) \geq \min_{t-N\leq n\leq t+N}\{U(\mathbf{C}_n^N)\} = u^*$$

for each $t \in Z$, where the last equality follows from the hypothesis that $\{\mathbf{C}_t^{n*}\}_{t\in Z}$ attains maximal utility under NELC. By exact feasibility, the sequence $\{\mathbf{C}_t^N\}_{N=1}^\infty$ is uniformly bounded for each $t \in Z$. By Cantor's diagonalization process, there exists a subsequence $\{\mathbf{C}^{N_q}\}_{q=1}^\infty = \{\{\mathbf{C}^{N_q}\}_{t\in Z}\}_{q=1}^\infty$ such that for each $t \in Z$, $\{\mathbf{C}^{N_q}\}_{q=1}^\infty$ converges to some $\mathbf{C}_t$. Clearly, $\mathbf{C} = \{\mathbf{C}_t\}_{t\in Z}$ is exactly feasible. By continuity of U, $U(\mathbf{C}_t) \geq u^*$ for each $t \in Z$.

Now, we will show that $\mathbf{C}_t = \mathbf{C}_{t-1}$ for each $t \in Z$. Suppose that $\mathbf{C}_t \neq \mathbf{C}_{t-1}$ for some $t \in Z$. Let $\delta = |\mathbf{C}_t - \mathbf{C}_{t-1}|$. Then, $\delta > 0$. For each $t \in Z$,

$$|\mathbf{C}_t^N - \mathbf{C}_{t-1}^N| = \frac{|\mathbf{C}_{t+N}^* - \mathbf{C}_{t-1-N}^*|}{2N+1} \leq \frac{2}{2N+1}.$$

Hence, there exists a positive integer $N(\delta)$ such that for all $N \geq N(\delta)$, $|\mathbf{C}_t^N - \mathbf{C}_{t-1}^N| < \frac{\delta}{2}$. On the other hand, we have

$$\begin{aligned}\delta = |\mathbf{C}_t - \mathbf{C}_{t-1}| &\leq |\mathbf{C}_t - \mathbf{C}_t^N| + |\mathbf{C}_t^N - \mathbf{C}_{t-1}^N| + |\mathbf{C}_{t-1}^N - \mathbf{C}_{t-1}| \\ &< \frac{\delta}{4} + |\mathbf{C}_t^N - \mathbf{C}_{t-1}^N| + \frac{\delta}{4}\end{aligned}$$

holds for all $N \geq N(\delta)$. Hence, for all $N \geq N(\delta)$,

$$\frac{\delta}{2} < |\mathbf{C}_t^N - \mathbf{C}_{t-1}^N|.$$

This is a contradiction. Thus, $\mathbf{C}_t = \mathbf{C}_{t-1}$ for each $t \in Z$. Let us denote this common vector by $\tilde{\mathbf{C}} = (\tilde{C}^1, \tilde{C}^2)$. By exact feasibility of $\mathbf{C} = \{\mathbf{C}_t\}_{t \in Z}$,

$$\tilde{C}^1 + \frac{\tilde{C}^2}{1+n} = 1.$$

Clearly, $U(\tilde{\mathbf{C}}) \geq u^*$. By maximality under NELC, $U(\tilde{\mathbf{C}}) = u^*$. Therefore, $\tilde{\mathbf{C}} = (\tilde{C}^1, \tilde{C}^2)$ generates a stationary allocation that attains maximal utility under NELC. *Q.E.D.*

Corollary 4 *A consumption vector* $\hat{\mathbf{C}} = (\hat{C}^1, \hat{C}^2)$ *generates a stationary allocation that attains maximal utility under NELC if and only if it maximizes* $U(\mathbf{C})$ *subject to* $C^1 + C^2/(1+n) = 1$.

Proof: Let $\tilde{\mathbf{C}} = (\tilde{C}^1, \tilde{C}^2)$ be a consumption vector generating a stationary allocation that attains maximal utility under NELC. By maximality under NELC, for any consumption vector $\hat{\mathbf{C}} = (\hat{C}^1, \hat{C}^2)$ satisfying $\hat{C}^1 + \hat{C}^2/(1+n) = 1$, $U(\tilde{\mathbf{C}}) \geq U(\hat{\mathbf{C}})$.

Conversely, let $\hat{\mathbf{C}}$ be a consumption vector maximizing $U(\mathbf{C})$ subject to $\hat{C}^1 + \hat{C}^2/(1+n) = 1$. Then, $U(\hat{\mathbf{C}}) \geq U(\tilde{\mathbf{C}})$. Since $\tilde{\mathbf{C}}$ attains maximal utility under NELC, $U(\tilde{\mathbf{C}}) \geq U(\hat{\mathbf{C}})$. Hence, $U(\tilde{\mathbf{C}}) = U(\hat{\mathbf{C}})$. This completes the proof. *Q.E.D.*

7 Conclusion

The main conclusions of our analysis may be summarized in Figure 3, where an arrow represents a logical implication which cannot be reversed in general. An arrow from NELC to Stationarity is valid subject to the condition that the utility function is either linear or quasi-linear.

Recollect that our three alternative concepts of intergenerational equity are not just hairsplitting theoretical toys, but are relevant in many actual disputes in the design and implementation of equitable social security schemes and intergenerational transfer mechanisms. Despite their obvious importance, not many studies have been done on equity and efficiency in overlapping generations economies. It is hoped that our exploration of the fundamental implications of several distinct notions of equity-as-no-envy may be found useful in orienting further studies of this important issue.

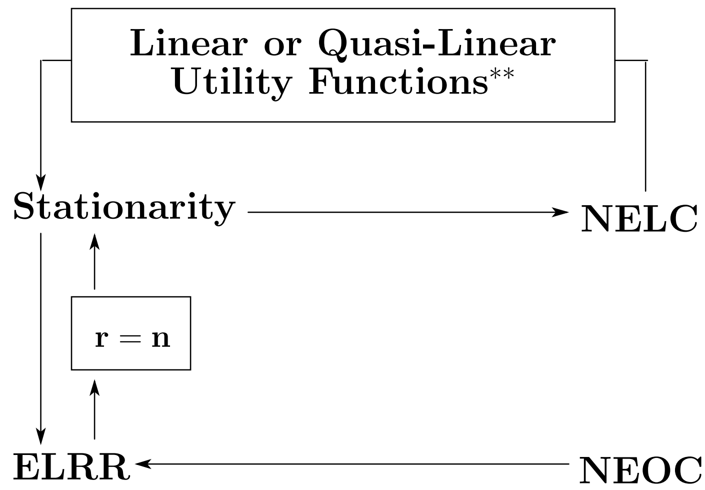

Figure 3: Logical Relations among Equity Concepts*

* Exact feasibility is always assumed.
** For the case of quasi-linear utility functions, we need an assumption that $\min\{v'(c) \mid c \in [0,1]\} > 1+n$ or $\max\{v'(c) \mid c \in [0,1]\} < 1+n$.

References

[1] Cass, D. and M. E. Yaari (1966), "Re-Examination of the Pure Consumption Loans Model," *Journal of Political Economy*, Vol.74, pp.353-367.

[2] Foley, D. (1967), "Resource Allocation and the Public Sector," *Yale Economic Essays*, Vol.7, pp.45-98.

[3] Gale, D. (1973), "Pure Exchange Equilibrium of Dynamic Economic Models," *Journal of Economic Theory*, Vol.6, pp.12-36.

[4] Kolm, S.-C. (1972), *Justice et Equité*, Paris: Ed. du CNRS.

[5] Samuelson, P. A. (1958), "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," *Journal of Political Economy*, Vol.66, pp.467-482.

[6] Suzumura, K. (2002), "On the Concept of Intergenerational Equity," mimeo.

[7] Tadenuma, K. (2002), "Efficiency First or Equity First? Two Principles and Rationality of Social Choice," *Journal of Economic Theory*, Vol.104, pp.462-472.