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Voluntary Participation in a Mechanism Implementing a Public Project

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Summary. In this study, a participation game in a mechanism to implement a public project is considered; in this game, agents decide simultaneously whether they will participate in the mechanism or not. We characterize the sets of participants at strict Nash equilibria, strong equilibria, and coalition-proof equilibria of the participation game. The three sets of equilibria are shown to coincide and exist. All the equilibrium allocations are Pareto efficient at any one of three notions of equilibria. However, if the public good can be provided in multiple units or if there are multiple projects, then these sets may fail to coincide.

Keywords and Phrases: Participation game, Public project, Strong equilibrium, Coalition-proof equilibrium, Multi-unit public good, Multiple projects.

JEL Classification Numbers: C72, D62, D71, H41.

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1 Introduction

This paper studies a participation game in a mechanism to implement a public project. Many interesting mechanisms have been constructed to solve the "free-rider" problem in economies with public goods. In the case of a public project, Bagnoli and Lipman (1989) and Jackson and Moulin (1992) designed mechanisms that implement efficient allocations. However, Palfrey and Rosenthal (1984) pointed out the importance of the strategic behavior of agents as they decide whether or not to participate in the mechanisms. In the real world, as for example the participation problems in international environmental treaties, agents often have the right to make such decisions, and they may have an incentive not to enter the mechanism, hoping that other agents will participate in the mechanism and provide a public good. This will generate another kind of a free-rider problem.

Palfrey and Rosenthal (1984) formulated a participation game in a mechanism to implement a public project with identical agents. In this game, each agent simultaneously chooses either participation or non-participation. If they enter the mechanism, they contribute a fixed amount that is common to every participant. The public good is supplied only if the aggregate contribution of participants outweighs its production cost. Only the participants bear the cost of the public good, while non-participants can benefit from the public good at no cost because the public good is non-excludable. Palfrey and Rosenthal (1984) characterized pure and symmetric mixed Nash equilibria and showed that an efficient allocation is achieved at a Nash equilibrium but multiplicity of equilibria may arise.

In this paper, we examine the participation problem which is similar to Palfrey and Rosenthal (1984). However, there are several differences between our model and that of Palfrey and Rosenthal (1984). First, we consider heterogeneous agents.[1] Secondly,

[1]Maruta and Okada (2001) analyze a different kind of heterogeneity from ours in the group formation

we introduce a mechanism that implements the following allocation rules: (i) the public project is undertaken only if the joint benefit of participants from it is more than its cost, (ii) the sum of payments from participants is equal to the cost of producing the public project, (iii) every participant bears a positive cost share, and (iv) the cost share of each participant is less than his willingness to pay for the public project. This kind of allocation rule includes many cost-sharing rules. A proportional cost-sharing rule is an example of such cost-sharing rules. Thirdly, we focus on not only Nash equilibria but also strong equilibria (Aumann, 1959) and coalition-proof equilibria (Bernheim, Peleg, and Whinston, 1987).

Our results are summarized as follows. We first characterize the set of participants at strict Nash equilibria. We show that there exists a strict Nash equilibrium and that every strict Nash equilibrium supports an efficient allocation in the participation game. Secondly, we characterize strong equilibria and show that there is a strong equilibrium in the participation game. Our main result is that the set of strict Nash equilibria, that of strong equilibria, and that of coalition-proof equilibria coincide and that the sets of these three equilibria are not empty. Moreover, there are efficient allocations that are supportable as the three notions of equilibria, and all the equilibrium allocations are Pareto efficient.

We also extend our model to the case with a multi-unit public good and that with multiple projects. In these cases, the set of strict Nash equilibria, and that of strong equilibria, and that of coalition-proof equilibria do not necessarily coincide.

Before the model is introduced, let us discuss the relationship between our work and other work. First, we consider the possibility that agents form a coalition and coordinate the participation decisions. We analyze the effect of such coalitional behavior on the participation decision. Earlier literature on participation games has focused solely

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game.

on Nash equilibria, disregarding the effects (see, for example, Cavaliere (2001), Dixt and Olson (2000), Palfrey and Rosenthal (1984), and Saijo and Yamato (1999)). In this paper, analyses are presented of strong and Nash equilibria in the participation game. A strong equilibrium is a strategy profile that is immune to all possible coalitional deviations. This is a very demanding equilibrium concept, and many games that are of interest to economists do not have a strong equilibrium. However, the equilibrium exists in the participation game studied in this paper. Furthermore, the set of strong equilibria coincides with that of strict Nash equilibria and that of coalition-proof equilibria. This is an interesting respect of our model, since strict Nash equilibria, strong equilibria, and coalition-proof equilibria are based on different concepts of stability and, in particular, the set of strict Nash equilibria and that of strong equilibria are not in general related by inclusion.

The second interesting point is relevant to participation games with local public goods. Konishi, Le Breton, and Weber (1997a) established sufficient conditions for equivalence between coalition-proof equilibria and strong equilibria. Applying their sufficient conditions, it is straightforward to show that the two sets coincide in the participation game with *excludable* public goods. However, we show that the equivalence between the two sets of equilibria is also established even for the case with a *non-excludable* public project. The existence of strong equilibria has been studied in the context of *congestion games*, which can be interpreted as a sort of a participation game in mechanisms providing local public goods with congestion effects. The congestion games satisfying some conditions have a strong equilibrium (Holzman and Yone, 1997; Konishi, Le Breton, and Weber, 1997c). Although the participation game studied in this paper is not a congestion game, it has a strong equilibrium.

Thirdly, we mention the relationship between the participation game with a public project and the other models in the context of the provision of a pure public good. In a participation game in a mechanism producing a perfectly divisible public good, there exists a case in which strict Nash and coalition-proof equilibria exist but strong equilibria do not. (Saijo and Yamato, 1999; Shinohara, 2003). Similar phenomena have also been observed in standard games of the voluntary contribution of a perfectly divisible public good. Therefore, in the games of the provision of perfectly divisible public goods, the set of strict Nash equilibria, that of strong equilibria, and that of coalition-proof equilibria are not necessarily equal. This paper shows that the equivalence of the three sets of equilibria does not necessarily hold in a participation game with a multi-unit public good and that with multiple projects, which are generalizations of the participation game with a public project. However, in the participation game with a public project, there is a strong equilibrium, and all three sets of equilibria coincide. This means that the existence and equivalence results depend on the setting with one and only one public project. In other words, it hardly holds in the context of the provision of public goods that a strong equilibrium exists and the sets of strict Nash, strong, and coalition-proof equilibria all coincide.

2 Participation game in a mechanism implementing a public project

We consider the problem of undertaking a (pure) public project and distributing its cost. Let n be the number of agents. We denote the set of agents by $N = \{1, \ldots, n\}$. Let $y \in \{0, 1\}$ be the public project. If the project is undertaken, then $y = 1$, and $y = 0$ if not. Let $\theta_i > 0$ denote agent i's willingness to pay for the project. Let $x_i \geq 0$ denote a transfer from agent i. Each agent i has a preference relation which is represented by the quasi-linear utility function $V_i(y, x_i) = \theta_i y - x_i$. The cost of the project is $c > 0$.

In this paper, we assume that there exists a mechanism that implements a Pareto efficient and individually rational allocation rule. We consider a two-stage game. In the first stage, each agent simultaneously decides whether or not he participates in the mechanism. In the second stage, following the rule of the mechanism, only the agents who selected participation in the first stage decide the implementation of the project and the distribution of its cost. First, we formally define the outcome of the second stage. Let P be a set of participants, and let $(y^P, (x_j^P)_{j\in N})$ be the outcome of the second stage when P is the set of participants. We denote $\theta_P = \sum_{j\in P}\theta_j$ for all sets of participants P: θ_P is the sum that agents in P are willing to pay for the public project. For each subset P of N, $\#P$ means the cardinality of the set P.

Assumption 1 For every set of participants P, the allocation to the participants $(y^P, (x_j^P)_{j\in P})$ satisfies

(i) $\theta_P > c$ if and only if $y^P = 1$,

(ii) if $y^P = 1$, then $\sum_{i\in P} x_i^P = c$,

(iii) $\theta_i > x_i^P$ for every $i \in P$, and

(iv) $x_i^P > 0$ for every $i \in P$ if and only if $y^P = 1$.

Condition (i) means that the public project is undertaken if and only if the sum that the participants are willing to pay for the project exceeds the project cost. Condition (ii) requires that the expenses paid by the participants be equal to the project cost when the project is undertaken. This is called the *budget balance* condition. Clearly, conditions (i) and (ii) imply that $(y^P, (x_j^P)_{j\in P})$ is a Pareto efficient allocation only for the preferences of agents in P. Item (iii) is the *individual rationality* condition, which means that the payoff of every participant after entering the mechanism is greater than 0, when the project is undertaken. Condition (iv) requires that every participant bear a positive cost share if and only if the public project is undertaken.

Several desirable allocation rules satisfy the conditions. The proportional cost-sharing rule under condition (i) is such an example: for all sets of participants P and for all i in P,

$$x_i^P = \begin{cases} \dfrac{\theta_i}{\theta_P} c & \text{if } y^P = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we are not concerned with the implementation problem of an allocation rule that satisfies (i), (ii), (iii), and (iv) in Assumption 1. However, there is a mechanism in which the above allocation rule is attainable in equilibria. For example, Jackson and Moulin (1992) constructed mechanisms which implement a class of cost-sharing rules satisfying all the above conditions in subgame perfect equilibria and undominated Nash equilibria.

Assumption 2 Let $P \subseteq N$ be a set of participants. We assume $x_i^P = 0$ for all $i \notin P$, and every non-participant can also consume y^P.

This assumption expresses the non-excludability of the project. In this assumption, participants bear the cost share for the project, but non-participants do not. In spite of this, non-participants can benefit from the project.

Given the outcome of the second stage, the participation-decision stage can be reduced to the following simultaneous game. In the game, each agent i simultaneously chooses either $s_i = I$ (participation) or $s_i = O$ (non-participation), and then the set of participants is determined. Let P^s be the set of participants at an action profile $s = (s_1, \ldots, s_n)$. Then, each agent i obtains the utility $V_i(y^{P^s}, x_i^{P^s})$ at the action profile s. That is, if the public project is undertaken, then participants share the cost of it as defined in Assumption 1. Each non-participant can benefit from the public project at no cost. On the other hand, if the project is not carried out, then the payoffs for both participants and non-participants are zero. We call this reduced game a *participation game* and formally define it as follows.

Definition 1 (Participation game) A *participation game* is represented by $G = \left[N,\, S^n = \{I,\, O\}^n,\, (U_i)_{i\in N}\right]$, where U_i is the payoff function of i, which associates a real number $U_i(s)$ with each strategy profile $s \in S^n$: if P^s designates the set of participants at s, then $U_i(s) = V_i(y^{P^s}, x_i^{P^s})$ for all i.

Our attention is limited to the pure strategy profiles.

The notions of equilibria of the participation game are defined as follows. The Nash equilibria of the participation game are defined as usual. First, a definition is given for a strict Nash equilibrium.

Definition 2 (Strict Nash equilibrium) A strategy profile $s^* \in S^n$ is a *strict Nash equilibrium* if, for all $i \in N$ and for all $\widehat{s}_i \in S \setminus \{s_i^*\}$, $U_i(s_i^*, s_{-i}^*) > U_i(\widehat{s}_i, s_{-i}^*)$.

Before defining strong equilibria, some notation is presented. For all $D \subseteq N$, denote the complement of D by $-D$. For all coalitions D, $s_D \in S^{\#D}$ denotes a strategy profile for D. For all $s_N \in S^n$, denote s_N by s.

Definition 3 (Strong equilibrium) A strategy profile $s^* \in S^n$ is a *strong equilibrium of G* if there exist no coalition $T \subseteq N$ and its strategy profile $\widetilde{s}_T \in S^{\#T}$ such that $U_i(\widetilde{s}_T, s_{-T}^*) \geq U_i(s^*)$ for all $i \in T$ with strict inequality for at least one $i \in T$.

A strong equilibrium is a strategy profile at which no subset of agents, taking the strategies of others as given, can jointly deviate in a way in which all members are at least as well off and at least one of its members is strictly better off. Obviously, all strict Nash equilibria and all strong equilibria are Nash equilibria. However, the set of strict Nash equilibria and that of strong equilibria are not necessarily related by inclusion.

Example 1 Let $N = \{1, 2, 3\}$, $\theta_1 = \theta_2 = \theta_3 = 3/4$, and $c = 1$. The cost is distributed among participants in proportion to their willingness to pay for the project. The payoff matrix of this example is depicted in Table 1, where agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There are two types of Nash equilibria. One is the Nash equilibrium with two participants, and the other is the Nash equilibrium with no participants. Only the Nash equilibria with participation of two agents are strict and strong.

⟨Insert Table 1 here.⟩

3 Strict Nash equilibria of the participation game

In this section, we characterize the sets of participants attained at strict Nash equilibria. Since the payoffs to agents depend on the sets of participants, we introduce the following notations for the sake of convenience.

Definition 4 A payoff function of i, $u_i\colon 2^N \to \mathbb{R}_+$, is defined as follows:

$$\text{For all sets of participants } P \in 2^N,\ u_i(P) = \begin{cases} (\theta_i - x_i^P)y^P & \text{if } i \in P, \\ \theta_i y^P & \text{otherwise.} \end{cases}$$

The set of feasible allocations of the economy is defined as A:

$$A = \left\{ (y, (x_j)_{j\in N}) \mid y \in \{0,1\},\ x_i \geq 0 \text{ for all } i \in N,\ \text{and } \sum_{i\in N} x_i \geq cy \right\}.$$

Assumption 3 $\theta_N > c$.

Definition 5 An allocation $(y, (x_j)_{j\in N})$ is called *Pareto efficient* if there is no allocation $(\widehat{y}, (\widehat{x}_j)_{j\in N}) \in A$ such that $V_i(\widehat{y}, \widehat{x}_i) \geq V_i(y, x_i)$ for all $i \in N$ and $V_i(\widehat{y}, \widehat{x}_i) > V_i(y, x_i)$ for some $i \in N$.

We, hereafter, consider a case in which Assumption 3 holds. By Assumption 3, the public project is undertaken at all Pareto efficient allocations. In the next Lemma, we characterize the sets of participants supported as strict Nash equilibria.

Lemma 1 A set of participants P is supported as a strict Nash equilibrium of the participation game if and only if $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$.

Proof. Let P be a set of participants that satisfies $\theta_P > c$ and $\theta_P - \theta_i \leq c$ for all $i \in P$, and let $(y^P, (x_j^P)_{j \in N})$ denote the allocation when P is the set of participants. Then, the following conditions are satisfied:

$$u_i(P) = \theta_i - x_i^P \ > \ 0 = u_i(P \setminus \{i\}) \text{ for all } i \in P, \text{ and}$$
$$u_i(P) = \theta_i \ > \ \theta_i - x_i^{P \cup \{i\}} = u_i(P \cup \{i\}) \text{ for all } i \notin P.$$

Therefore, P can be supported as a strict Nash equilibrium.

Secondly, we suppose that P is a set of participants at a strict Nash equilibrium. Then, we have $u_i(P) > u_i(P \setminus \{i\})$ for all $i \in P$ and $u_i(P) > u_i(P \cup \{i\})$ for all $i \notin P$. If $\theta_P \leq c$, then we have $u_i(P) = u_i(P \setminus \{i\}) = 0$ for all $i \in P$, which is a contradiction. Thus, it must be satisfied that $\theta_P > c$. Since $\theta_P > c$, $u_i(P) = \theta_i - x_i^P$ for all $i \in P$. If $\theta_P - \theta_j > c$ for some $j \in P$, then the agent j has an incentive to deviate from I to O because $u_j(P \setminus \{j\}) = \theta_j > \theta_j - x_i^P = u_j(P)$. This is a contradiction. Therefore, we must have $\theta_P - \theta_i \leq c$ for all $i \in P$. ■

In the following lemma, we verify that there is a strict Nash equilibrium in the participation game.

Lemma 2 There exists a strict Nash equilibrium in the game G under Assumption 3.

Proof. By Lemma 1, we show the existence of a set of participants $P \subseteq N$ that satisfies the condition

$$\theta_P > c \text{ and } \theta_P - \theta_i \leq c \text{ for all } i \in P, \tag{1}$$

in order to prove this statement. Let T be a set of participants such that:

$$T \in \arg\min_{Q \subseteq N} \ \theta_Q \text{ such that } \theta_Q > c. \tag{2}$$

Note that there is at least one set of participants R satisfying $\theta_R > c$ by Assumption 3. Now, suppose that $\theta_T - \theta_i > c$ for some $i \in T$. Since $\theta_T > \theta_{T\setminus\{i\}} > c$, θ_T is not the minimal number, which contradicts (2). Therefore, it holds true that $\theta_T - \theta_i \leq c$ for all $i \in T$. ■

In the participation game, there may be a non-strict Nash equilibrium. For example, a Nash equilibrium at which no agents choose I is obviously not strict in Example 1. Note that, if non-strict Nash equilibria exist, then the project is not done in the equilibrium, and the allocations supported as the non-strict Nash equilibria are Pareto-dominated by that attained at a strict Nash equilibrium. The following proposition shows that the set of strict Nash equilibria coincides with the set of Nash equilibria that support efficient allocations.

Proposition 1 In the participation game, a strategy profile is a strict Nash equilibrium if and only if it is a Nash equilibrium at which an efficient allocation is attained.

Proof. First, we prove that every strict Nash equilibrium is a Nash equilibrium that supports an efficient allocation. Obviously, every strict Nash equilibrium is a Nash equilibrium. Hence, we need to show that every allocation achieved at a strict Nash equilibrium is Pareto efficient. Assume that $(y^P, (x_j^P)_{j\in N})$ is the allocation attained at a strict Nash equilibrium. Note that $V_i(y^P, x_i^P) = \theta_i - x_i^P$ for all $i \in P$ and $V_i(y^P, x_i^P) = \theta_i$ for all $i \notin P$. Suppose, on the contrary, a feasible allocation $(\widehat{y}, (\widehat{x}_j)_{j\in N})$ Pareto dominates $(y^P, (x_j^P)_{j\in N})$. It must be satisfied that $V_i(\widehat{y}, \widehat{x}_i) = \theta_i$ for all $i \notin P$ because θ_i is the greatest payoff of agent i in A. Hence, there is at least one participant $j \in P$ such that $V_j(\widehat{y}, \widehat{x}_j) > V_j(y^P, x_j^P)$. Let $J \subseteq P$ be a set of such participants and let $\varepsilon_j = V_j(\widehat{y}, \widehat{x}_j) - V_j(y^P, x_j^P) > 0$ for all $j \in J$. Since $V_j(y^P, x_j^P) = \theta_j - x_j^P > 0$ for every $j \in J$, we must have $\widehat{y} = 1$: otherwise, $V_j(\widehat{y}, \widehat{x}_j) = 0$. Then, we learn that

$V_j(\widehat{y}, \widehat{x}_j) = \theta_j - x_j^P + \varepsilon_j$ for all $j \in J$. By the argument above,

$$\widehat{x}_j = 0 \text{ for all } j \notin P,$$

$$\widehat{x}_j = x_j^P - \varepsilon_j \text{ for all } j \in J, \text{ and}$$

$$\widehat{x}_j = x_j^P \text{ for all } j \in P \setminus J.$$

Summing up $\widehat{x}_j$ for all $j \in N$ yields $\sum_{j \in N} \widehat{x}_j = \sum_{j \in P} x_j^P - \sum_{j \in J} \varepsilon_j = c - \sum_{j \in J} \varepsilon_j < c$, which contradicts the feasibility of $(\widehat{y}, (\widehat{x}_j)_{j \in N})$. Hence, $(y^P, (x_j^P)_{j \in N})$ is Pareto efficient.

Secondly, each Nash equilibrium that supports an efficient allocation is a strict Nash equilibrium. Let $s \in S^n$ be a Nash equilibrium that attains an efficient allocation. Denote the set of participants at s by P^s. Since the project is done at efficient allocations, we have $\theta_{P^s} > c$. Furthermore, it is satisfied that $\theta_{P^s} - \theta_i \leq c$ for all $i \in P^s$: if there is an agent $j \in P^s$ such that $\theta_{P^s} - \theta_j > c$, then agent j has an incentive to deviate from s because $u_j(P^s \setminus \{j\}) = \theta_j > \theta_j - x_j^{P^s} = u_j(P^s)$. This contradicts the idea that s is a Nash equilibrium. It follows from Lemma 1 that s is a strict Nash equilibrium. ■

4 Strong equilibria in the participation game

4.1 Equivalence between strict Nash equilibrium and strong equilibrium

First, we show that the set of strong equilibria coincides with that of strict Nash equilibria.

Proposition 2 In the participation game with a public project, a strategy profile is a strong equilibrium if and only if it is a strict Nash equilibrium.

Proof. ($\Leftarrow$) Let $s^* \in S^n$ denote a strict Nash equilibrium. Let P^* be the set of participants at s^*. Let $T \subseteq N$ be a coalition and $s_T \in S^{\#T}$ be a strategy profile of T.

We show that some members of T are worse off by jointly deviating from s_T^* to s_T.

We take a partition of T consisting of four sets: $T_I^* \cap T_I$, $T_I^* \backslash T_I$, $T_I \backslash T_I^*$, and $T \backslash (T_I^* \cup T_I)$, where $T_I^* \equiv \{i \in T | s_i^* = I\}$ and $T_I \equiv \{i \in T | s_i = I\}$. The set of participants in (s_T, s_{-T}^*) is $(P^* \setminus (T_I^* \backslash T_I)) \cup (T_I \backslash T_I^*)$. We denote this set by $\widetilde{P}$. In the strict Nash equilibrium s^*,

$$u_i(P^*) = \theta_i - x_i^{P^*} > 0$$

for all $i \in P^*$, and

$$u_i(P^*) = \theta_i > 0$$

for all $i \notin P^*$. We calculate the payoffs of the members of T after the deviation. To do so, we need to consider the following two cases: $\theta_{\widetilde{P}} \leq c$, and $\theta_{\widetilde{P}} > c$.

First, consider the case in which $\theta_{\widetilde{P}} \leq c$. In this case, the public project is not undertaken at (s_T, s_{-T}^*). Since the payoffs of the members of T at (s_T, s_{-T}^*) are given by $u_i(\widetilde{P}) = 0$ for all $i \in T$, we obtain the following four inequalities:

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T_I^* \cap T_I,$$

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T_I^* \backslash T_I,$$

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T_I \backslash T_I^*, \text{ and}$$

$$u_i(P^*) > u_i(\widetilde{P}) \text{ for all } i \in T \backslash (T_I^* \cup T_I).$$

Therefore, the deviation cannot raise the members' payoffs.

Next, let us consider the case in which $\theta_{\widetilde{P}} > c$. Note that the public project is undertaken at (s_T, s_{-T}^*). If $T_I^* \backslash T_I$ is not empty, then it follows from Lemma 1 that $\theta_{P^*} - \theta_i \leq c$ for all $i \in T_I^* \setminus T_I$. Thus, we have $\theta_{P^*} - \theta_{T_I^* \backslash T_I} \leq c$. Because $\theta_{\widetilde{P}} = \theta_{P^*} - \theta_{T_I^* \backslash T_I} + \theta_{T_I \backslash T_I^*} > c$, we must obtain $\theta_{T_I \backslash T_I^*} > 0$. This implies that $T_I \backslash T_I^*$ is non-empty. It is satisfied that $u_i(P^*) > u_i(\widetilde{P})$ for all $i \in T_I \backslash T_I^*$ because $u_i(\widetilde{P}) = \theta_i - x_i^{\widetilde{P}}$ for every $i \in T_I \backslash T_I^*$. Therefore, if $T_I^* \backslash T_I$ is not empty, the deviation does not improve the members' payoffs. If $T_I^* \backslash T_I$ and $T_I \backslash T_I^*$ are empty sets, then $P^* = \widetilde{P}$ holds. Clearly, no member of T is better off by the deviation. If $T_I^*\backslash T_I$ is empty and $T_I\backslash T_I^*$ is non-empty, then none of the agents in $T_I\backslash T_I^*$ can improve their payoffs by the deviation since $u_i(P^*) = \theta_i > \theta_i - x_i^{\widetilde{P}} = u_i(\widetilde{P})$ for all $i \in T_I\backslash T_I^*$. Consequently, s^* is a strong equilibrium of G.

($\Rightarrow$) Let s^* be a strong equilibrium, and let P^* be the set of participants at s^*. If $\theta_{P^*} \leq c$ holds, then we have $u_i(P^*) = 0$ for all $i \in N$. When all agents jointly choose I, then every agent i has the payoff $u_i(N) = \theta_i - x_i^N$, which is positive by Assumption 1 and 3. This is a contradiction. Hence, we have $\theta_{P^*} > c$. It also holds that $\theta_{P^*} - \theta_i \leq c$ for all $i \in P^*$: if there exists an agent $j \in P^*$ such that $\theta_{P^*} - \theta_j > c$, then agent j has an incentive to deviate from s because $u_j(P^* \setminus \{j\}) = \theta_j > \theta_j - x_j^{P^*} = u_j(P^*)$. This contradicts the idea that s^* is a strong equilibrium. Therefore, s^* is a strict Nash equilibrium. ■

Although the sets of strict Nash equilibria and strong equilibria are subsets of that of Nash equilibria, it is not evident whether the two sets coincide. The two-player game depicted in Table 2 shows that the set of strong equilibria does not necessarily coincide with that of strict Nash equilibria. In this game, (B_1, B_2) is the only strict Nash equilibrium, and a strong equilibrium is uniquely determined by (A_1, A_2). Hence, the two sets have an empty intersection, and both of them exist. However, from Proposition 2, the set of strict Nash equilibria coincides with that of the strong equilibria in the participation game. An implication of Proposition 2 is that the two non-cooperative equilibrium concepts based on different types of stability coincide in the participation game with a public project.

Note that a weakly dominated strategy may be used at a strong equilibrium.[2] In the example in Table 2, A_1 is weakly dominated by B_1, and so is A_2 by B_2. However,

[2]For every agent i, a strategy $s_i \in S$ is *weakly dominated* in the game G if there exists another strategy $s_i' \in S$ such that $U_i(s_i', s_{-i}) \geq U_i(s_i, s_{-i})$ for all s_{-i} with strict inequality for some s_{-i}.

(A_1, A_2) is a strong equilibrium of the game. In the participation game with a public project, every strong equilibrium is a strict Nash equilibrium, which implies that the strong equilibrium does not consist of weakly dominated strategies in the participation game.

⟨Insert Table 2 here.⟩

By Lemma 2, Proposition 1, and Proposition 2, the set of strong equilibria and the set of Nash equilibria that support efficient allocations coincide, and a strong equilibrium exists in the participation game.

Corollary 1 The set of strong equilibria coincides with the set of Nash equilibria that support an efficient allocation in the participation game.

Corollary 2 The participation game has a strong equilibrium.

These results contrast with those of a participation game with a perfectly divisible public good. Saijo and Yamato (1999) introduced a model of voluntary participation in a mechanism to provide a perfectly divisible public good. We find from their results that the Nash equilibria of the game are not always Pareto efficient. Hence, if agents have the right to decide either participation or non-participation in the mechanism, then efficient allocations are not necessarily attained even if the mechanism is constructed to implement efficient allocations in its equilibrium. It was also proven by Shinohara (2003) that the game does not always have a strong equilibrium.[3] In contrast, in a participation game with a public project, there exist strong equilibria, and an efficient allocation of the economy can be supported as the equilibrium.

[3]In the participation game with a perfectly divisible public good, the provision of the public good increases as the number of participants rises. In the case in which the number of participants at a Nash equilibrium is sufficiently small, if all non-participants at the Nash equilibrium jointly switch to participation, then each of them can be better off. For details, refer to Shinohara (2003).

4.2 Coalition-proof equilibria and strong equilibria

The notion of a coalition-proof equilibrium was introduced by Bernheim, Peleg, and Whinston (1987) and is known as a refinement of Nash equilibria based on the stability against self-enforcing coalitional deviations. It is defined by using the notion of *restricted games.* A restricted game is a game in which a subset of agents play the game G, taking strategy profiles of agents outside the subset as given. We formally define it as follows. Let $T \subsetneq N$ and $t = \#T$. Let $\bar{s}_{N\setminus T} \in S^{n-t}$. A *restricted game* $G|\bar{s}_{N\setminus T}$ is a game in which the set of agents is T, the set of strategy profiles is S^t, and the payoff function for each $i \in T$ is the function $U_i(\cdot, \bar{s}_{N\setminus T})$ that associates a real value $U_i(s_T,\, \bar{s}_{N\setminus T})$ with each element s_T in S^t such that: $U_i(s_T,\, \bar{s}_{N\setminus T}) = V_i(y, x_i)$, where $(y, (x_j)_{j\in N})$ is the allocation when agents play $(s_T,\, \bar{s}_{N\setminus T})$ in G.

Definition 6 A *coalition-proof equilibrium* $(s_1^*,\, \ldots,\, s_n^*)$ is defined inductively with respect to the number of agents t:

- When $t = 1$, for all $i \in N$, s_i^* is a coalition-proof equilibrium for $G|s^*_{N\setminus\{i\}}$ if $s_i^* \in \arg\max\ U_i(s_i, s^*_{N\setminus\{i\}})$ s.t. $s_i \in S$.
- Let $T \subseteq N$ with $t = \#T \geq 2$. Assume that coalition-proof equilibria have been defined for all normal form games with fewer agents than t.
- Consider the restricted game $G|s^*_{N\setminus T}$ with t agents.
  - A strategy profile $s_T^* \in S^t$ is called *self-enforcing* if, for all $Q \subsetneq T$, s_Q^* is a coalition-proof equilibrium of $G|s^*_{N\setminus Q}$.
  - A strategy profile s_T^* is a coalition-proof equilibrium of $G|s^*_{N\setminus T}$ if it is a self-enforcing strategy profile and there is no other self-enforcing strategy profile $\widehat{s}_T \in S^t$ such that $U_i(\widehat{s}_T,\, s^*_{N\setminus T}) \geq U_i(s^*_T,\, s^*_{N\setminus T})$ for all $i \in T$ and $U_i(\widehat{s}_T,\, s^*_{N\setminus T}) > U_i(s^*_T,\, s^*_{N\setminus T})$ for some $i \in T$.

Coalition-proof equilibria are defined as the Pareto efficient frontier within the set of self-enforcing strategy profiles. The self-enforcing strategy profiles are recursively defined with respect to the number of agents in coalitions. At a self-enforcing strategy profile of N, no proper coalition of N can coordinate its members' strategies in a way in which all members of the coalition are at least as well off and at least one of them is strictly better off, and no proper subsets of the coalition further deviate in a self-enforcing way. Note that every strong equilibrium is a coalition-proof equilibrium and every coalition-proof equilibrium is a Nash equilibrium, but a coalition-proof equilibrium is not always a strong equilibrium. However, in the participation game with a public project, every coalition-proof equilibrium is a strong equilibrium.

Proposition 3 In the participation game with a public project, a strategy profile is a strong equilibrium if and only if it is a coalition-proof equilibrium.

Proof. By the definitions of coalition-proof equilibria and strong equilibria, every strong equilibrium is a coalition-proof equilibrium. We show that a coalition-proof equilibrium $s \in S^n$ is a strong equilibrium. Suppose, on the contrary, that s is not a strong equilibrium. Since the profile s is a coalition-proof equilibrium, it must be a Nash equilibrium. If s is a strict Nash equilibrium, then it is also a strong equilibrium by Proposition 2. Therefore, s must be a non-strict Nash equilibrium. By Proposition 1, s does not support an efficient allocation. Because of this, we have $U_i(s) = 0$ for all $i \in N$. By Lemma 2, there is at least one strict Nash equilibrium in this game. Denote a strict Nash equilibrium by s^*. Note that s^* must be a coalition-proof equilibrium; hence, it must also be a self-enforcing strategy profile. By Proposition 2, s^* is a strong equilibrium, and we have $U_i(s^*) > 0$ for every $i \in N$. Since s is Pareto-dominated by the self-enforcing strategy profile s^*, s is not coalition-proof, which is a contradiction. Therefore, s is a strong equilibrium. ■

Konishi, Le Breton, and Weber (1997a, 1997b, 1997c) studied the *no-spillover game*, in which the strategy spaces of all players are common. In the no-spillover game, for each player i, his payoff is not affected by the choices of those players who choose strategies different from i.[4] These authors established sufficient conditions for the existence of strong equilibria and the equivalence between coalition-proof equilibria and strong equilibria in the game. One of the sufficient conditions is the condition of *positive population monotonicity*: the payoff of every player i increases if more players choose the same strategy as players i.[5] Konishi, Le Breton, and Weber (1997a) proved that, if the population monotonicity condition is satisfied, the set of coalition-proof equilibria coincides with that of strong equilibria in every no-spillover game. Konishi, Le Breton, and Weber (1997b) also showed that strong equilibria exist in games in which the set of pure strategies for each player consists of two alternatives. Although the participation game is a no-spillover game, it does not satisfy positive population monotonicity because the payoffs of non-participants decrease when a participant switches to non-participation and the project is then not undertaken. It was also proven by Konishi, Le Breton, and Weber (1997c) that, if a no-spillover game satisfies *negative population monotonicity*[6] and *anonymity*[7], then the game has a strong equilibrium. The participation game with a public project does not satisfy negative population monotonicity. Furthermore, the participation game is

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[4]The no-spillover game is formally defined as follows: a game is called a *no-spillover game* if, for all pairs of agents $i,\ j \in N$, for all strategy profiles $s \in S^n$, and for all strategies for i, $\widehat{s}_i$, if $s_j \neq s_i$ and $s_j \neq \widehat{s}_i$, then $U_j\left(s_i,\, s_j,\, s_{N\setminus\{i,j\}}\right) = U_j\left(\widehat{s}_i,\, s_j,\, s_{N\setminus\{i,j\}}\right)$.

[5]The game satisfies *positive population monotonicity* if, for all $i, j \in N$ and for all $s \in S^n$, if $s_i \neq s_j$, then $U_j(s_i, s_j, s_{N\setminus\{i,j\}}) \leq U_j(s_j, s_j, s_{N\setminus\{i,j\}})$.

[6]The game satisfies *negative population monotonicity* if, for all $i, j \in N$, for all $s \in S^n$, if $s_i \neq s_j$, then $U_j(s_i, s_j, s_{N\setminus\{i,j\}}) \geq U_j(s_j, s_j, s_{N\setminus\{i,j\}})$.

[7]The condition of anonymity requires that the payoff of a player depend only on the number of players who choose the same strategy. The formal definition is as follows: a game is *anonymous* if, for all s, $\widehat{s} \in S^n$ and all $i \in N$, if $s_i = \widehat{s}_i$ and $\#\{j \in N | s_j = \bar{s}\} = \#\{j \in N | \widehat{s}_j = \bar{s}\}$ for all $\bar{s} \in S$, then $U_i(s) = U_i(\widehat{s})$.

not anonymous because agents are heterogeneous and the payoffs of participants depend not on the number of participants but on their composition in our model. Although the conditions of Konishi, Le Breton, and Weber are not sufficiently met in our game, the set of strong equilibria coincides with that of coalition-proof equilibria and is not empty.

The following theorem summarizes the results that have been obtained so far.

Theorem In the participation game, the set of strict Nash equilibria, that of strong equilibria, that of coalition-proof equilibria, and the set of Nash equilibria that support efficient allocations coincide.

Remark 1 Let us consider an allocation rule that satisfies (ii), (iv), and the following conditions:

(i)$'$ For all sets of participants P, $\theta_P \geq c$ if and only if $y^P = 1$.

(iii)$'$ For all $P \subseteq N$ and for all $i \in P$, $\theta_i \geq x_i$. (weakly individual rationality)

In the participation game in a mechanism to implement this allocation rule, the set of strong equilibria contains that of strict Nash equilibria, and they do not always coincide. Furthermore, a strict Nash equilibrium does not necessarily exist in the game. However, the game has a Nash equilibrium at which efficient allocations are attained, and every set of participants at Nash equilibria that support efficient allocations is characterized as $P \subseteq N$ with $\theta_P \geq c$ and $\theta_P - \theta_i < c$ for all $i \in P$. We can show that the set of Nash equilibria that support efficient allocations, that of strong equilibria, and that of coalition-proof equilibria coincide in a similar way to Propositions 2 and 3. Therefore, the equivalence between a strong equilibrium and a coalition-proof equilibrium can be obtained in a case in which the allocation rule satisfies (i)$'$ and (iii)$'$ instead of (i) and (iii).

5 More general participation games: examples

In Section 4, we prove that the set of strong, strict Nash, and coalition-proof equilibria coincide in the participation game with a public project. In this section, we consider two natural generalizations of the participation game with a public project: *participation games with a multi-unit public good* and *participation games with multiple public projects.* The purpose of this section is to investigate whether or not the results in Section 4 can be extended to the more general participation games.

5.1 Participation games with a multi-unit public good

There is one private and one public good. We assume that the public good is produced in the units of integers only. Let $l > 1$ be a natural number. Let Y be a subset of $\mathbb{R}^l_+$ such that $Y = \{(y_1, \ldots, y_l) \in \{0,1\}^l | \, y_1 \geq y_2 \geq \cdots \geq y_l\}$: in this model, at most l units of the public good can be produced. Let $c > 0$ be the cost of producing one unit of the pubic good. Each agent i has a preference relation that is represented by the utility function $V_i : Y \times \mathbb{R}_+ \to \mathbb{R}_+$, which associates a real value $V_i(y, x_i) = \sum_{k \in \{1,2,\ldots,l\}} \theta_i^k y_k - x_i$ with each element (y, x_i) in $Y \times \mathbb{R}_+$, where $\theta_i^k > 0$ denotes agent i's willingness to pay for the k-th unit of the public good.

Example 2 Let $N = \{1, 2, 3, 4\}$. Let $l = 2$. Suppose that $\theta_i^1 = 2$ and $\theta_i^2 = 0.8$ for all $i \in N$ and $c = 1$. Assume that a mechanism implements the equal cost-sharing rule. Let P be a set of participants. Note that one unit of the public good is produced if $\#P = 1$, and two units of the public good are provided if $\#P \geq 2$. Table 3 shows the payoffs to participants and non-participants in this example. From the table, we can easily find that one and only one agent enters the mechanism at all strict Nash and coalition-proof equilibria. However, these Nash equilibria are not strong equilibria, since three non-participants at the Nash equilibrium can gain higher payoffs if all of them jointly deviate from non-participation to participation; thus, a strong equilibrium does not exist in this example. Therefore, the set of strict Nash equilibria and that of strong equilibria do not necessarily coincide in the participation game with a multi-unit public good.

⟨Insert Table 3 here.⟩

5.2 Participation games with multiple public projects

Let us consider an economy with two public projects (A and B) and their corresponding mechanisms. The set of strategies of every agent is denoted by $S = \{A, B, O\}$: A means participation in the mechanism undertaking the public project A, B designates participation in the mechanism implementing the public project B, and O represents participation in neither mechanism. The public project A is produced from c units of the private good, and B is produced from αc units of the private good, where $\alpha > 0$. The production costs of public projects A and B are shared by participants equally. Every agent i has a preference relation that is represented by the quasi-linear utility function $\theta_i^A y_A + \theta_i^B y_B - x_i$, where $y_A \in \{0, 1\}$ and $y_B \in \{0, 1\}$ represent the public projects A and B, and θ_i^A and θ_i^B denote the willingness to pay for public projects A and B, respectively.

Example 3 Assume that $\theta_1^A = \theta_1^B = \theta_2^A = \theta_2^B = \theta > 0$, $2\theta > \alpha c > \theta > c$, and $1 < \alpha < 2$, say $\alpha = 1.5$, $c = 1$, and $\theta = 1.25$. The payoff matrix is depicted in Table 4. In this example, the cost of project B is higher that that of project A. Project A is undertaken if one or two agents choose A, and project B is undertaken only if two agents choose B. Thus, it is a Nash equilibrium for the two agents to select B. This strategy profile is also coalition-proof, because (A, A) is the only deviation that improves payoffs of the two agents, but the deviation is not self-enforcing. However, strategy profile (B, B) is not strong since the deviation from (B, B) to (A, A) is profitable. Hence, in the participation game with two projects, there may be a coalitional deviation that increases payoffs of its members but is not self-enforcing. Therefore, the set of strong equilibria does not always coincide with that of coalition-proof equilibria.

⟨Insert Table 4 here.⟩

The above examples indicate that the equivalence among the three sets of equilibria does not always hold in the games with a discrete public good and multiple public projects. Therefore, it is an essential assumption to the equivalence result that there is one and only one public project in the economy.

Remark 2 Konishi, Le Breton, and Weber (1997a) showed that the set of coalition-proof equilibria and that of strong equilibria coincide in many games of the provision of local public goods. (Refer to Greenberg and Weber (1993) and Konishi, Le Breton, and Weber (1998) for games of the provision of local public goods.) However, in games of the provision of non-excludable public goods, the equivalence rarely holds. The above results show that the two equilibrium sets coincide in the participation game with a public project, while they may fail to coincide if the public good can be provided in multiple units or if there are multiple projects.

6 Conclusion

We have investigated a participation game in a mechanism providing a public project. We characterized the strict Nash, strong, and coalition-proof equilibria of the participation game. We showed that the set of strict Nash, strong, and coalition-proof equilibria coincide and that all of the equilibria exist. We find from the result that the participation in a public project is in a class of games in which the three different non-cooperative equilibria coincide. Furthermore, an efficient allocation of the economy can be achieved as various notions of equilibria, and only the efficient allocations are supportable as the equilibria. These results are contrasted with those in the models of providing a perfectly divisible public good, such as a participation game with a perfectly divisible public good and the voluntary contribution of a perfectly divisible public good. The equivalence between the sets of coalition-proof and strong equilibria is established, although the conditions of the earlier literature have not been sufficiently satisfied in our model. This paper clarified the conditions that the set of coalition-proof equilibria and that of strong equilibria coincide in the game of the provision of non-excludable public goods.

Although efficient allocations are attained at the equilibria, the allocations are less desirable from the viewpoint of equity. In Example 1 on page 8, there exist strict Nash equilibria at which two agents enter the mechanism. Obviously, these Nash equilibria support efficient allocations. However, in these equilibria, only two agents bear the cost for the public project, and the other agent enjoys the project at no cost. To achieve more equitable allocations, it is desirable that all agents participate in the mechanism. It is left for future researches to study the possibility of constructing the mechanism, in which all agents participate at equilibria.

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	I	O
I	$\frac{5}{12}$, $\frac{5}{12}$, $\frac{5}{12}$	$\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{4}$
O	$\frac{3}{4}$, $\frac{1}{4}$, $\frac{1}{4}$	0, 0, 0

I

	I	O
I	$\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$	0, 0, 0
O	0, 0, 0	0, 0, 0

O

Table 1: Payoff matrix of Example 1

1\2	A_2	B_2
A_1	2, 2	0, 2
B_1	2, 0	1, 1

Table 2: An example in which the set of strong equilibria and that of strong equilibria are disjoint.

The number of participants	Payoffs to participants	Payoffs to non-participants
0	-	0
1	1	2
2	1.8	2.8
3	$\frac{32}{15}$	2.8
4	2.3	-

Table 3: Payoffs of Example 2

$1\backslash 2$	A	B	O
A	0.75, 0.75	0.25, 1.25	0.25, 1.25
B	1.25, 0.25	0.5, 0.5	0, 0
O	1.25, 0.25	0, 0	0, 0

Table 4: A participation game with two public projects