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On the General Existence of Party-Unanimity Nash Equilibria in Multi-dimensional Political Competition Games

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Abstract

In this paper, we consider the general existence problem of *party-unanimity Nash equilibria* (**PUNE**s) [Roemer (2001)] in multi-dimensional political competition games. We consider the case in which the membership of a given political party is endogenously formed, and a given faction of each party does not necessarily have dictatorial power. We then provide a few general existence theorems for **PUNE**s in this class of games.

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1 Introduction

Until recently, most formal political analyses of party competition have assumed that both parties are Downsian [Downs (1957)]—that is, their objective is to maximize the probability of winning office. As an alternative, there is a growing body of literature concerning the competition between *partisan* parties (those that have policy preferences)—see, e.g., Wittman (1973) and Roemer (1997). Almost all of these analyses, however, have assumed that the policy space is uni-dimensional. Moreover, neither Downsian nor partisan party models can reliably produce a Nash equilibrium (in pure strategies) whenever, as discussed in Roemer (2001), the policy space is multi-dimensional. Importantly, in many real political competition contexts, we may naturally assume more than a single policy issue, and need to address the issue of non-existence. Given this, there are two possible ways forward: the first is to allow mixed strategies, and the second is to change the game into a stage game and use some variant of a subgame perfect equilibrium. However, in the case of competing political parties, playing mixed strategies is difficult to interpret, and it is not always the case that a policy contest takes place as a stage game between a challenger and an incumbent. Thus, it is still important to investigate another solution for the non-existence of pure strategy equilibrium in multi-dimensional political games with simultaneous moves.

It is Roemer (1998; 1999; 2001) who proposed a new equilibrium concept, known as *party-unanimity Nash equilibrium* (**PUNE**) for these political games. This is where the notion of a Nash equilibrium in a simultaneous move game between the parties is retained, but their preferences are replaced with *incomplete preferences*: put differently, each party's preference is a quasi-ordering. The model introduces the idea that the decision makers in parties have different interests. In this approach, the activists in each party are divided into one of three *factions*: the *Opportunist*, the *Militant*, and the *Reformist*. The Opportunist is solely concerned with winning office, the Militant is only concerned with publicizing the party's view, and the Reformist is concerned with the expected welfare of the party's members. Given the structure of the three factions within a party, how does the party make policy decisions in the electoral context? Roemer (1998; 1999; 2001) proposed the following scenario. The three factions of each party should bargain on the policy proposal, given a policy proposal by its opponent party, and if a policy proposal agreed on in this party is Pareto efficient for the three factions, this is the solution for the bargaining problem within the party. A **PUNE** is then a pair of policy proposals, each component of which is the result of intra party bargaining when facing the other party's proposal.

In this paper, we consider a general existence problem of **PUNE**s in multi-dimensional political competition games. It is worth noting that there are a few studies, such as Roemer (1998; 1999), that show the existence of **PUNE**s in some specific types of multi-dimensional political games. Moreover, Roemer (2001; Section 13.7) discussed the existence of **PUNE**s in general multi-dimensional political games. However, this existence theorem refers only to a specific type of **PUNE** in which the membership of both parties is exogenously given, and the Militants are assumed to have dictatorial powers in both parties. It is easy to see the general existence of this sort of **PUNE**, because the preference of each party is exogenously given, so that the pair of each Militant's *ideal policy* is exogenously given; this constitutes a **PUNE**.

Thus, we still have the following open question concerning the general existence of **PUNE**s: If the membership of both parties is endogenously formed, and if the Militants of both parties are not assumed to have dictatorial powers, under what general conditions is the existence of **PUNE**s guaranteed? This problem is worth investigating, because these two premises appear to be more natural and general as far as real politics is concerned. At the same time, however, these two premises make the problem more difficult. One reason is that, under endogenous party formation, we cannot identify the ideal policy of each party as a primitive datum of the model. Thus, even the existence of **PUNE** with the assumption of dictatorial Militants is difficult to show. Moreover, if the Militants of both parties are not assumed to be dictators, it implies that each party's best response strategy should be a compromise among the factions, particularly reflecting the Opportunists' objective. However, because the objective function of the Opportunist—the probability of winning the election—is neither generally continuous nor quasi-concave, we cannot adopt the strategy of finding a sufficient condition of the model parameters to straightforwardly apply Kakutani's fixed point theorem.[1]

[1] In fact, Roemer (2001; Section 13.7, pp. 277-279) also wrote: "is there an interesting general existence theorem for party-unanimity Nash equilibrium? I conjecture there is not. $\cdots$ What we really desire is a theorem asserting the existence of a PUNE in which no faction is at its ideal point. But that appears to be hard to come by. $\cdots$ It is probably very difficult to find interesting sufficient conditions for the existence of (non-trivial) PUNEs."

In this paper, we provide sufficient conditions for the existence of **PUNE**s in the case in which the party membership is endogenously formed and the Militants of both parties are not assumed to be dictators. In such an existence problem of **PUNE**, we apply the *Urai-Hayashi fixed point theorem* [Urai and Hayashi (2000)], as it does not require the convex valuedness of correspondences. The sufficient condition contains the following: Each voter's preference is represented by a continuous and strictly concave function, and aggregate uncertainty over voters' behavior is sufficiently large. Such a condition appears natural and would be satisfied in many political environments.

The paper is structured as follows. Section 2 defines a basic model of multi-dimensional political games, and introduces the **PUNE** with endogenous party formation and its refinements. Section 3 discusses the existence of **PUNE**s. Section 4 provides some concluding remarks.

2 Model

Let the set of voter types be H, the policy space be T, a probability distribution of voter types in the polity be $\mathbb{F}$ on H, and the utility function of type $h \in H$ over policies be $v(\cdot, h)$. Let $v(\cdot, h)$ be a non-negative real valued function for any $h \in H$. Let $(t^1, t^2) \in T \times T$ be a pair of policies. The set of voters who prefer t^1 to t^2 is denoted by $\Omega(t^1, t^2) \equiv \{h \in H \mid v(t^1, h) > v(t^2, h)\}$. Now, we impose the following assumption:

Assumption 1 (A1): *For any $t, t' \in T$ with $t \neq t'$, the set of voters who are indifferent between t and t' is of $\mathbb{F}$-measure zero.*

Following Roemer (2001; Section 2.3), the *fraction* of the vote going to policy t^1 would be $\mathbb{F}\left(\Omega(t^1, t^2)\right)$. We also assume that there is some *aggregate uncertainty* in how people will vote, so that the true fraction of the vote for t^1 will be $\mathbb{F}\left(\Omega(t^1, t^2)\right) \pm \beta$, where $\beta > 0$ is an error term. Thus, the probability that t^1 defeats t^2 is:

$$\pi(t^1, t^2) = \begin{cases} 0 & \text{if } \mathbb{F}\left(\Omega(t^1, t^2)\right) + \beta \leq \frac{1}{2} \\ \frac{\mathbb{F}\left(\Omega(t^1, t^2)\right) + \beta - \frac{1}{2}}{2\beta} & \text{if } \frac{1}{2} \in \left(\mathbb{F}\left(\Omega(t^1, t^2)\right) - \beta, \mathbb{F}\left(\Omega(t^1, t^2)\right) + \beta\right) \\ 1 & \text{if } \mathbb{F}\left(\Omega(t^1, t^2)\right) - \beta \geq \frac{1}{2} \end{cases},$$

whenever $t^1 \neq t^2$, and $\pi(t^1, t^2) = \frac{1}{2}$ whenever $t^1 = t^2$. Then, one *political environment* is specified by a tuple $\langle H, \mathbb{F}, T, v, \beta \rangle$.

Let us suppose that exactly two parties will form. The two parties will each represent a coalition of voter types: thus, there will be a partition of the set of voter types

$$H = A \cup B,\ A \cap B = \varnothing,$$

which are the parties called A and B, respectively. Each party will represent its members, in the sense that the party's preferences will be the average preferences of its members; that is, we define the parties' utility functions on T by:

$$V^A(t) = \int_{h \in A} v(t,h)\mathrm{d}\mathbb{F}(h), \text{ and } V^B(t) = \int_{h \in B} v(t,h)\mathrm{d}\mathbb{F}(h).$$

We are now ready to define an equilibrium notion of this multi-dimensional political competition game, *party-unanimity Nash equilibria with endogenous parties* (**PUNEEPs**), as introduced by Roemer (2001; Chapter 13).

Definition 1: *A partition of voter types A, B, and a pair of policies $(t^A, t^B) \in T \times T$ constitutes a* party-unanimity Nash equilibrium with endogenous parties (**PUNEEP**) *if*:
(1) $H = A \cup B$ *and* $A \cap B = \varnothing$;
(2) (t^A, t^B) *satisfies the following*:
(a) *given* t^B*, there is no policy* $t \in T$ *such that*

$\pi(t, t^B) \geq \pi(t^A, t^B)$ *and* $V^A(t) \geq V^A(t^A)$, with at least one strict inequality;

(b) *given* t^A*, there is no policy* $t \in T$ *such that*

$\pi(t^A, t) \leq \pi(t^A, t^B)$ *and* $V^B(t) \geq V^B(t^B)$, with at least one strict inequality;

(3) *for all* $h \in A$, $v(t^A, h) > v(t^B, h)$ *and for all* $h \in B$, $v(t^A, h) \leq v(t^B, h)$.

In Definition 1, condition (2a) states that, facing the opponent's proposal t^B, there is no policy in T that can improve the payoffs of all three factions in party A, and condition (2b) makes an analogous statement for the factions of party B. Condition (3) states that party membership is stable in the sense that every party member prefers his or her party's policy to the opponent's policy. By this condition, the coalition of those who vote for a party and the coalition that the party represents are identical. Such a condition was used in Baron (1993) in the context of endogenous party formation, and treated more generally in Caplin and Nalebuff (1997). Note that in this definition, there is no statement for the Reformists' payoffs, because (2a) and (2b) describe the conditions for the Opportunists' payoffs, $\pi(\cdot,\cdot)$ and $1-\pi(\cdot,\cdot)$, and the Militants' payoffs, $V^A(\cdot)$ and $V^B(\cdot)$, only. However, as Roemer (2001: Chapter 8; Theorem 8.1(3)) showed, the equilibrium set corresponding to this simpler definition of **PUNEEP** is equivalent to that of the original definition of **PUNEEP** given in Roemer (2001: Chapter 8; Definition 8.1).

This general definition admits the case in which $t^A = t^B$. Regarding this type of **PUNEEP**, we obtain the following characterization:

Proposition 1: *If* $\left(t^A, t^B; A, B\right)$ *with* $t^A = t^B$ *is a* **PUNEEP**, *then it implies that* $t^B = \arg\max_{t\in T} \int_H v(t,h)\mathrm{d}\mathbb{F}(h)$ s.t. $\pi\left(t^A, t\right) = \frac{1}{2}$. *Conversely, if* $\left(t^A, t^B; A, B\right)$ *with* $t^A = t^B$ *meets* $t^B = \arg\max_{t\in T} \int_H v(t,h)\mathrm{d}\mathbb{F}(h)$ s.t. $\pi\left(t^A, t\right) = \frac{1}{2}$ *and* $\pi(t, t^B) \leq \frac{1}{2}$ *for any* $t \in T$, *then it constitutes a* **PUNEEP**.

Proof. If $t^A = t^B$ constitutes a **PUNEEP** $\left(t^A, t^B; A, B\right)$, then $\Omega(t^A, t^B) = \varnothing$ and $\int_A v(t^A, h)\mathrm{d}\mathbb{F}(h) = 0$ hold. Since t^A is a best response to t^B, $\pi\left(t, t^B\right) \leq \frac{1}{2}$ holds for any $t \neq t^B$. This also implies $\pi\left(t^A, t\right) \geq \frac{1}{2}$ for any $t \neq t^B$. Thus, if $\left(t^A, t^B; A, B\right)$ with $t^A = t^B$ is a **PUNEEP**, then it implies that $t^B = \arg\max_{t\in T} \int_H v(t,h)\mathrm{d}\mathbb{F}(h)$ s.t. $\pi\left(t^A, t\right) = \frac{1}{2}$. The converse is obvious from Definition 1. ■

Thus, Proposition 1 implies that if the ideal policy for the Militant in party B is also the dominant strategy for the Opportunist in party B, then there generally exists a **PUNEEP**. However, such a **PUNEEP** is unlikely, and if this is the sole type of **PUNEEP** that exists in general, then the equilibrium notion of **PUNE** might not be so appealing. This is because a more interesting and realistic **PUNEEP** is the case in which $\widehat{t}^A \neq \widehat{t}^B$ holds and the $\mathbb{F}$-measure of $A = \Omega(\widehat{t}^A, \widehat{t}^B)$ is positive.

Among the various **PUNEEP**s, a polar case is where both parties only care about satisfying their own preferences, such that both parties never care about their probability of winning an election. In other words, the Militants are assumed to be dictators in both parties. Such a specific **PUNEEP** is given by the following:

Definition 2: *A partition of voter types A, B, and a pair of policies* $(t^A, t^B) \in$

$T \times T$ *is a* Militant-dictatorial **PUNEEP** (M-**PUNEEP**) *if this* $\left(A, B; t^A, t^B\right)$ *is a* **PUNEEP** *such that* $t^A = \arg\max_{t \in T} V^A(t)$ *and* $t^B = \arg\max_{t \in T} V^B(t)$.

Note that this definition admits the case of $t^A = t^B$. Thus, we would like to discuss the following refinement:

Definition 3: *A partition of voter types* A, B, *and a pair of policies* $(t^A, t^B) \in T \times T$ *is a* non-trivial M-**PUNEEP** *if this* $\left(A, B; t^A, t^B\right)$ *is a* M-**PUNEEP** *such that* $t^A \neq t^B$.

Finally, the most realistic and interesting type of **PUNEEP** is the case where both parties offer different policies, and where no faction in either party is assumed to have a dictatorial power. Such a **PUNEEP** is given by the following:

Definition 4: *A partition of voter types* A, B, *and a pair of policies* $(t^A, t^B) \in T \times T$ *is a* pure-compromise **PUNEEP** (C-**PUNEEP**) *if this* $\left(A, B; t^A, t^B\right)$ *is a* **PUNEEP** *with (i)* $t^A \neq t^B$, *(ii)* $t^A = \arg\max_{t \in T} \pi\left(t, t^B\right)$ s.t. $V^A(t) \geq V^A(t^A)$, *and (iii)* $t^B = \arg\min_{t \in T} \pi\left(t^A, t\right)$ s.t. $V^B(t) \geq V^B(t^B)$.

This definition assumes the existence of a *real* inter-faction bargaining process within each party, so that the Opportunists in each party have a chance to reflect their own objective in the party's decision making.

Note that, by definition, the whole set of C-**PUNEEP**s contains a non-trivial M-**PUNEEP** in general. Thus, let us call $\left(A, B; t^A, t^B\right)$ a *proper* C-**PUNEEP** if it is a C-**PUNEEP**, but not a non-trivial M-**PUNEEP**.

3 Existence Theorems

This section considers a general existence problem of **PUNEEP**s in multi-dimensional political competition games. In the following discussion, we provide a rather general and reasonable condition under which the existence of M-**PUNEEP**s are shown in multi-dimensional political competition games. Furthermore, in the same type of game, we will also discuss a general condition under which there exists a C-**PUNEEP**.

We begin our discussion by showing the existence of M-**PUNEEP**s:

Theorem 1: *Suppose* T *is a compact and convex subset in a Hausdorff topological vector space, and every voter's utility function* v *is continuous* *and strictly concave on T. Let* **A1** *hold. Then, there exists an* M-**PUNEEP** $\left(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B),\ B = H\backslash\Omega(\widehat{t}^A, \widehat{t}^B)\right)$ *for the environment* $\langle H, \mathbb{F}, T, v, \beta\rangle$.

Proof. For each $(t^A, t^B) \in T \times T$, let us consider the following problems:

$$\max_{\widetilde{t}^A \in T} \int_{\Omega(t^A,t^B)} [v(\widetilde{t}^A, h) - v(t^A, h)]\mathrm{d}\mathbb{F}(h) \quad (1)$$

and

$$\max_{\widetilde{t}^B \in T} \int_{H\backslash\Omega(t^A,t^B)} [v(\widetilde{t}^B, h) - v(t^B, h)]\mathrm{d}\mathbb{F}(h) \quad (2).$$

Let the sets of solutions of the problems (1) and (2) be respectively $\mathcal{B}^A(t^A; t^B)$ and $\mathcal{B}^B(t^B; t^A)$. Both $\mathcal{B}^A(t^A; t^B)$ and $\mathcal{B}^B(t^B; t^A)$ are non-empty, convex, and compact. Moreover, $\mathcal{B}^A(\cdot; t^B) : T \twoheadrightarrow T$ and $\mathcal{B}^B(\cdot; t^A) : T \twoheadrightarrow T$ are upper hemi-continuous. Thus, there exists $\left(\widehat{t}^A, \widehat{t}^B\right) \in T \times T$ such that

$$\left(\widehat{t}^A, \widehat{t}^B\right) \in \mathcal{B}^A\left(\widehat{t}^A; \widehat{t}^B\right) \times \mathcal{B}^B\left(\widehat{t}^B; \widehat{t}^A\right)$$

by the Kakutani fixed point theorem. Then, given the above problems, we can see that:

$$\begin{aligned}
\int_{\Omega\left(\widehat{t}^A,\widehat{t}^B\right)} [v(t^A, h) - v(\widehat{t}^A, h)]\mathrm{d}\mathbb{F}(h) &\leq 0, \text{ for any } t^A \in T,\\
\int_{H\backslash\Omega\left(\widehat{t}^A,\widehat{t}^B\right)} [v(t^B, h) - v(\widehat{t}^B, h)]\mathrm{d}\mathbb{F}(h) &\leq 0, \text{ for any } t^B \in T.
\end{aligned}$$

Let $A \equiv \Omega\left(\widehat{t}^A, \widehat{t}^B\right)$ and $B \equiv H\backslash\Omega\left(\widehat{t}^A, \widehat{t}^B\right)$. Then, $V^A(t) = \int_{\Omega\left(\widehat{t}^A,\widehat{t}^B\right)} v(t, h)\mathrm{d}\mathbb{F}(h)$, and $V^B(t) = \int_{H\backslash\Omega\left(\widehat{t}^A,\widehat{t}^B\right)} v(t, h)\mathrm{d}\mathbb{F}(h)$. Note that, for any $t^A \in T\backslash\{\widehat{t}^A\}$, we have $V^A(t^A) < V^A(\widehat{t}^A)$ by the strict concavity of V^A. Also, for any $t^B \in T\backslash\{\widehat{t}^B\}$, we have $V^B(t^B) < V^B(\widehat{t}^B)$ by the strict concavity of V^B. This implies that $\left(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B),\ B = H\backslash\Omega(\widehat{t}^A, \widehat{t}^B)\right)$ is a **PUNEEP**. ■

This general existence theorem of M-**PUNEEP** unfortunately contains the case in which $\widehat{t}^A = \widehat{t}^B$ with $\widehat{t}^B \equiv \arg\max_{t^B \in T} \int_H v(t^B, h)\mathrm{d}\mathbb{F}(h)$ if $\pi(t, \widehat{t}^B) \leq \frac{1}{2}$ holds for any $t \in T$. In this case, $\pi(\widehat{t}^A, \widehat{t}^B) = \frac{1}{2}$ and $V^A(t) = \int_{\Omega\left(\widehat{t}^A,\widehat{t}^B\right)} v(t, h)\mathrm{d}\mathbb{F}(h) = 0$ for any $t \in T$. However, if there exists only such a *trivial* type of M-**PUNEEP** in general, it is not so appealing. A more interesting subject is to show the general existence of a *non-trivial* M-**PUNEEP** that has $\widehat{t}^A \neq \widehat{t}^B$ with $A = \Omega(\widehat{t}^A, \widehat{t}^B)$ of positive $\mathbb{F}$-measure. Because Theorem 1 is unlike this, we need a general existence theorem of this refinement of M-**PUNEEP**.

As a preliminary step, let us introduce a generalization of the Kakutani fixed point theorem, which was first discussed by Urai and Hayashi (2000), and is useful where the correspondence is nonconvex-valued.

Lemma 1 (**Urai and Hayashi fixed point theorem:** Urai and Hayashi (2000)): *Let X be a compact convex subset of $\mathbb{R}^n$, and ϕ be a non-empty valued correspondence on X to X satisfying one of the following conditions*:

(**LDV1**) *For each $x \in X$ such that $x \notin \phi(x)$, there exists a vector $p(x) \in \mathbb{R}^n$ and an open neighborhood $N(x)$ of x such that $p(x) \cdot (w - z) > 0$ for all $z \in N(x)$ and $w \in \phi(z)$.*

(**LDV2**) *For each $x \in X$ such that $x \notin \phi(x)$, there exists a vector $y(x) \in X$ and an open neighborhood $N(x)$ of x such that $(y(x) - z) \cdot (w - z) > 0$ for all $z \in N(x)$ and $w \in \phi(z)$.*

Then, ϕ has a fixed point $x^* \in \phi(x^*)$.

Note that in the Urai and Hayashi fixed point theorem, the correspondence ϕ need not be convex-valued. In the existence problems of our **PUNEEP** discussed below, the best response correspondence will not be necessarily convex-valued. Thus, **Lemma 1** will play a crucial role.

Let us also introduce the following assumption:

Assumption 2 (A2): *For any $t^B \in T$, there exists $t' \in T$ such that $\Omega(t', t^B)$ is non-empty and $\mathbb{F}\left(\Omega(t', t^B)\right) > 0$. Also, for any $t^A \in T$, there exists $t'' \in T$ with $t'' \neq t^A$ such that $H \backslash \Omega(t^A, t'')$ is non-empty and $\mathbb{F}\left(H \backslash \Omega(t^A, t'')\right) > 0$.*

This assumption implies that a trivial strategy profile, (t^A, t^B) with $t^A = t^B$, is not necessarily the winning strategy profile. This is reasonable, because if it fails, then only the trivial M-**PUNEEP** may exist for this political competition game.

For each $(t^A, t^B) \in T \times T$, let

$$U^A(t^A; t^B) \equiv \left\{ \widetilde{t}^A \in T \mid \int_{\Omega(t^A, t^B)} [v(\widetilde{t}^A, h) - v(t^A, h)] \mathrm{d}\mathbb{F}(h) \geq 0 \right\}$$

and

$$U^B(t^B;t^A) \equiv \left\{ \widetilde{t}^B \in T \mid \int_{H\backslash\Omega(t^A,t^B)} [v(\widetilde{t}^B,h) - v(t^B,h)]\mathrm{d}\mathbb{F}(h) \geq 0 \right\}.$$

Then:

Theorem 2: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and every voter's utility function v is continuous and strictly concave on T. Let* **A1** *and* **A2** *hold. Then, there exists a* non-trivial M-**PUNEEP** $(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B), B = H\backslash\Omega(\widehat{t}^A, \widehat{t}^B))$ *for the environment* $\langle H, \mathbb{F}, T, v, \beta\rangle$.

Proof. For each $(t^A, t^B) \in T \times T$, let us consider the following problems:

$$\max_{\widetilde{t}^A \in U^A(t^A;t^B)} \int_{\Omega(\widetilde{t}^A,t^B)} v(\widetilde{t}^A,h)\mathrm{d}\mathbb{F}(h), \quad (3)$$

and

$$\max_{\widetilde{t}^B \in U^B(t^B;t^A)} \int_{H\backslash\Omega(t^A,\widetilde{t}^B)} v(\widetilde{t}^B,h)\mathrm{d}\mathbb{F}(h). \quad (4)$$

Let the sets of solutions of problems (3) and (4) be respectively denoted by $\mathcal{B}^A(t^A;t^B)$ and $\mathcal{B}^B(t^B;t^A)$.

Define a correspondence $\varphi : T \times T \twoheadrightarrow T \times T$ as follows: for each $(t^A, t^B) \in T \times T$, $\varphi(t^A, t^B) = \mathcal{B}^A(t^A;t^B) \times \mathcal{B}^B(t^B;t^A)$. It is clear that φ is non-empty valued, and upper hemi-continuous. Moreover, as we show below, φ also satisfies **LDV1**.

Let $(t^A, t^B) \notin \varphi(t^A, t^B)$. This implies

$$\varphi(t^A, t^B) \subseteq \left(U^A(t^A;t^B) \times U^B(t^B;t^A)\right) \setminus \left\{(t^A, t^B)\right\}.$$

Let *co* $\left[\varphi(t^A, t^B)\right]$ be the convex hull of $\varphi(t^A, t^B)$.[2] Since $U^A(t^A;t^B) \times U^B(t^B;t^A)$ is convex, *co* $\left[\varphi(t^A, t^B)\right] \subseteq U^A(t^A;t^B) \times U^B(t^B;t^A)$. Moreover, since $U^A(t^A;t^B) \times U^B(t^B;t^A)$ is strictly convex by the strict concavity of v, we have $(t^A, t^B) \notin$ *co* $\left[\varphi(t^A, t^B)\right]$. Thus, by the separation theorem, there exists $p(t^A, t^B) \equiv \left(p(t^A), p(t^B)\right) \in \mathbb{R}^{|T| \times |T|}$ such that for any $(t'^A, t'^B) \in \varphi(t^A, t^B)$, $p(t^A, t^B) \cdot \left[(t'^A, t'^B) - (t^A, t^B)\right] > 0$.

[2] In the following discussion, we use the notation coX as the convex hull of X for any set X.

Consider a neighborhood $N\left(\varphi(t^A,t^B)\right)$ of $\varphi(t^A,t^B)$, which is small enough and satisfies

$$co\left[\varphi(t^A,t^B)\right] \subseteq N\left(\varphi(t^A,t^B)\right) \text{ and } (t^A,t^B) \notin N\left(\varphi(t^A,t^B)\right).$$

Since φ is upper hemi-continuous, there is a neighborhood $N(t^A,t^B)$ of (t^A,t^B) such that, for this $N\left(\varphi(t^A,t^B)\right)$, we have $\varphi(t^{*A},t^{*B}) \subseteq N\left(\varphi(t^A,t^B)\right)$ for any $(t^{*A},t^{*B}) \in N(t^A,t^B)$. Consider $\overline{N}(t^A,t^B) \subsetneq N(t^A,t^B)$, which is small enough, so that $\overline{N}(t^A,t^B) \cap N\left(\varphi(t^A,t^B)\right) = \varnothing$. Since $N\left(\varphi(t^A,t^B)\right) \supseteq co\left[\varphi(t^A,t^B)\right]$, and $(t^A,t^B) \notin co\left[\varphi(t^A,t^B)\right]$, we can find such a small set $\overline{N}(t^A,t^B)$. Thus, since φ is upper hemi-continuous, we have $\varphi(t^{*A},t^{*B}) \subseteq N\left(\varphi(t^A,t^B)\right)$ for any $(t^{*A},t^{*B}) \in \overline{N}(t^A,t^B)$.

By the construction of $N\left(\varphi(t^A,t^B)\right)$ and $\overline{N}(t^A,t^B)$, we can see that for any $(t^{*A},t^{*B}) \in \overline{N}(t^A,t^B)$ and any $(t'^A,t'^B) \in N\left(\varphi(t^A,t^B)\right)$, $p(t^A,t^B) \cdot \left[(t'^A,t'^B) - (t^{*A},t^{*B})\right] > 0$. This implies that for any $(t^{*A},t^{*B}) \in \overline{N}(t^A,t^B)$ and any $(t'^A,t'^B) \in \varphi(t^{*A},t^{*B})$, we have

$$p(t^A,t^B) \cdot \left[(t'^A,t'^B) - (t^{*A},t^{*B})\right] > 0.$$

The last argument implies that φ satisfies **LDV1**. Thus, by **Lemma 1**, φ has a fixed point $(\widehat{t}^A,\widehat{t}^B) \in \varphi(\widehat{t}^A,\widehat{t}^B)$.

By the definition of problems (3) and (4),

$$\int_{\Omega(\widehat{t}^A,\widehat{t}^B)} v(\widehat{t}^A,h)\mathrm{d}\mathbb{F}(h) \geq \int_{\Omega(t^A,\widehat{t}^B)} v(t^A,h)\mathrm{d}\mathbb{F}(h) \text{ for any } t^A \in U^A(\widehat{t}^A;\widehat{t}^B), \quad (5)$$

$$\int_{H\backslash\Omega(\widehat{t}^A,\widehat{t}^B)} v(\widehat{t}^B,h)\mathrm{d}\mathbb{F}(h) \geq \int_{H\backslash\Omega(\widehat{t}^A,t^B)} v(t^B,h)\mathrm{d}\mathbb{F}(h) \text{ for any } t^B \in U^B(\widehat{t}^B;\widehat{t}^A). \quad (6)$$

By **A2**, $\Omega(\widehat{t}^A,\widehat{t}^B) \neq \varnothing$ and $\mathbb{F}(\Omega(\widehat{t}^A,\widehat{t}^B)) > 0$. Also, $H\backslash\Omega(\widehat{t}^A,\widehat{t}^B) \neq \varnothing$ and $\mathbb{F}(H\backslash\Omega(\widehat{t}^A,\widehat{t}^B)) > 0$. In fact, if $\mathbb{F}(\Omega(\widehat{t}^A,\widehat{t}^B)) = 0$, then $\int_{\Omega(\widehat{t}^A,\widehat{t}^B)} v(\widehat{t}^A,h)\mathrm{d}\mathbb{F}(h) = 0$, while by **A2**, there exists $t^{*A} \in T$ such that $\mathbb{F}(\Omega(t^{*A},\widehat{t}^B)) > 0$. Since $\int_{\Omega(\widehat{t}^A,\widehat{t}^B)} v(t^{*A},h)\mathrm{d}\mathbb{F}(h) = 0$, it follows that $t^{*A} \in U^A(\widehat{t}^A;\widehat{t}^B)$. However, $\int_{\Omega(t^{*A},\widehat{t}^B)} v(t^{*A},h)\mathrm{d}\mathbb{F}(h) > 0$ since v is a non-negative real-valued function: this is a contradiction.

Note that, in general, we have

$$\int_{\Omega(t^A,t^B)} v(t^A,h)\mathrm{d}\mathbb{F}(h) \geq \int_{\Omega(t,t^B)} v(t^A,h)\mathrm{d}\mathbb{F}(h) \text{ for any } t \in T, \quad (7)$$

$$\int_{\Omega(t^B,t^A)} v(t^B,h)\mathrm{d}\mathbb{F}(h) \geq \int_{\Omega(t,t^A)} v(t^B,h)\mathrm{d}\mathbb{F}(h) \text{ for any } t \in T, \quad (8)$$

for any $t \in T$, since some $h' \in \Omega(t, t^B)$ may have $v(t^A, h') - v(t^B, h') \leq 0$, and if $v(t^A, h') - v(t^B, h') > 0$ for $h' \in \Omega(t, t^B)$, then $h' \in \Omega(t^A, t^B)$. Thus, combining with **A1**, (5) and (7), and (6) and (8) respectively imply that,

$$\int_{\Omega(\widehat{t}^A, \widehat{t}^B)} v(\widehat{t}^A, h) \mathrm{d}\mathbb{F}(h) \geq \int_{\Omega(\widehat{t}^A, \widehat{t}^B)} v(t^A, h) \mathrm{d}\mathbb{F}(h) \text{ for any } t^A \in T, \quad (9)$$

$$\text{and} \int_{H \backslash \Omega(\widehat{t}^A, \widehat{t}^B)} v(\widehat{t}^B, h) \mathrm{d}\mathbb{F}(h) \geq \int_{H \backslash \Omega(\widehat{t}^A, \widehat{t}^B)} v(t^B, h) \mathrm{d}\mathbb{F}(h) \text{ for any } t^B \in T. \quad (10)$$

Let $A \equiv \Omega\left(\widehat{t}^A, \widehat{t}^B\right)$ and $B \equiv H \backslash \Omega\left(\widehat{t}^A, \widehat{t}^B\right)$. Then, $V^A(t) = \int_{\Omega\left(\widehat{t}^A, \widehat{t}^B\right)} v(t, h) \mathrm{d}\mathbb{F}(h)$, and $V^B(t) = \int_{H \backslash \Omega\left(\widehat{t}^A, \widehat{t}^B\right)} v(t, h) \mathrm{d}\mathbb{F}(h)$. Note that, for any $t^A \in T \backslash \{\widehat{t}^A\}$, we have $V^A(t^A) < V^A(\widehat{t}^A)$ by the strict concavity of V^A. Also, for any $t^B \in T \backslash \{\widehat{t}^B\}$, we have $V^B(t^B) < V^B(\widehat{t}^B)$ by the strict concavity of V^B. This implies that $\left(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B), B = H \backslash \Omega(\widehat{t}^A, \widehat{t}^B)\right)$ is a **PUNEEP**.

Finally, suppose $\widehat{t}^A = \widehat{t}^B$. Then, $\Omega(\widehat{t}^A, \widehat{t}^B) = \varnothing$. However, by **A2**, we have $\widetilde{t}^A \in T$ such that $\Omega(\widetilde{t}^A, \widehat{t}^B) \neq \varnothing$ and $\mathbb{F}\left(\Omega(\widetilde{t}^A, \widehat{t}^B)\right) > 0$, so that:

$$\int_{\Omega(\widetilde{t}^A, \widehat{t}^B)} v(\widetilde{t}^A, h) \mathrm{d}\mathbb{F}(h) > 0 = \int_{\Omega(\widehat{t}^A, \widehat{t}^B)} v(\widehat{t}^A, h) \mathrm{d}\mathbb{F}(h).$$

Since $\int_{\Omega(\widetilde{t}^A, \widehat{t}^B)} v(\widetilde{t}^A, h) \mathrm{d}\mathbb{F}(h) = 0$, so that $\widetilde{t}^A \in U^A(\widehat{t}^A; \widehat{t}^B)$, the last inequality implies a contradiction, because of (5). Hence, $\widehat{t}^A \neq \widehat{t}^B$. ■

The next interesting problem is to show the existence of C-**PUNEEP**s without the presumption of dictatorial Militants. Because the whole set of C-**PUNEEP**s contains non-trivial M-**PUNEEP**s, we may regard Theorem 2 as an existence theorem of C-**PUNEEP**. However, Theorem 2 presumes the existence of dictatorial Militants in both parties. In the following, we discuss the general existence of C-**PUNEEP**s, without relying on the existence of non-trivial M-**PUNEEP**s.

To do this, let us introduce the following additional assumption:

Assumption 3 (A3): *For any $t^B \in T$, there exists $t' \in T$ such that $\Omega(t', t^B) \neq \varnothing$ and $\mathbb{F}\left(\Omega(t', t^B)\right) > \frac{1}{2} - \beta$. Also, for any $t^A \in T$, there exists $t'' \in T$ with $t'' \neq t^A$ such that $H \backslash \Omega(t^A, t'') \neq \varnothing$ and $\mathbb{F}\left(\Omega(t^A, t'')\right) < \frac{1}{2} + \beta$.*

A3 is a slightly stronger version of **A2**. In fact, if $\beta \in \left(0, \frac{1}{2}\right)$ is close enough to $\frac{1}{2}$, then **A3** is almost equivalent to **A2**.

We are now ready to discuss the existence of C-**PUNEEP.**

Theorem 3: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and every voter's utility function v is continuous and strictly concave on T. Let the error term $\beta \in \left(0, \frac{1}{2}\right)$ be close enough to $\frac{1}{2}$, and* **A1** *and* **A2** *hold. Then, there exists a* C-**PUNEEP** *for the environment $\langle H, \mathbb{F}, T, v, \beta \rangle$.*

As a preliminary step to showing Theorem 3, we present the following:

Lemma 2: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and every voter's utility function v is continuous and strictly concave on T. Let* **A1** *and* **A3** *hold. Then, there exists a* C-**PUNEEP** $(\widehat{t}^A, \widehat{t}^B;\ A = \Omega(\widehat{t}^A, \widehat{t}^B),$ $B = H \backslash \Omega(\widehat{t}^A, \widehat{t}^B))$ *for the environment $\langle H, \mathbb{F}, T, v, \beta \rangle$.*

Proof. Let us define a function $\overline{\pi} : T \times T \to [0, 1]$ as follows: for any $(t^A, t^B) \in T \times T$,

$$\overline{\pi}(t^A, t^B) = \begin{cases} \pi\left(t^A, t^B\right) & \text{if } t^A \neq t^B \\ 0 & \text{if } t^A = t^B. \end{cases}$$

We can see that this $\overline{\pi}$ is continuous, because $\pi \circ \mathbb{F}$ is continuous on $T \times T$ except for the case $t^A = t^B$, and $\mathbb{F}\left(\Omega(t^A, t^B)\right) = 0$ when $t^A = t^B$.

Given each $(t^A, t^B) \in T \times T$, consider the following problems:

$$\max_{t \in U^A(t^A; t^B)} \overline{\pi}\left(t, t^B\right), \quad (15)$$

$$\max_{t \in U^B(t^B; t^A)} 1 - \overline{\pi}\left(t^A, t\right). \quad (16)$$

Let the sets of solutions to problems (15) and (16) be respectively denoted by $\mathcal{B}^A(t^A; t^B)$ and $\mathcal{B}^B(t^B; t^A)$. Since $\overline{\pi} \circ \mathbb{F}$ is continuous, and both $U^A(t^A; t^B)$ and $U^B(t^B; t^A)$ are compact, both $\mathcal{B}^A(t^A; t^B)$ and $\mathcal{B}^B(t^B; t^A)$ are non-empty.

Define a correspondence $\varphi : T \times T \twoheadrightarrow T \times T$ as follows: for each $(t^A, t^B) \in T \times T$, $\varphi(t^A, t^B) = \mathcal{B}^A(t^A; t^B) \times \mathcal{B}^B(t^B; t^A)$. It is clear that φ is non-empty valued, and upper hemi-continuous. Moreover, φ also satisfies **LDV1**[Urai and Hayashi (2000; Theorem 2; p. 586)], which we will show below.

Let $(t^A, t^B) \notin \varphi(t^A, t^B)$. This implies

$$\varphi(t^A, t^B) \subseteq \left(U^A(t^A; t^B) \times U^B(t^B; t^A)\right) \setminus \left\{(t^A, t^B)\right\}.$$

Figure 1 around here.

Take $co\left[\varphi(t^A,t^B)\right]$. Since $U^A(t^A;t^B)\times U^B(t^B;t^A)$ is convex, $co\left[\varphi(t^A,t^B)\right]\subseteq U^A(t^A;t^B)\times U^B(t^B;t^A)$. Moreover, since $U^A(t^A;t^B)\times U^B(t^B;t^A)$ is strictly convex by the strict concavity of v, we can guarantee that $(t^A,t^B)\notin co\left[\varphi(t^A,t^B)\right]$ holds. Thus, by the separation theorem, there exists $p(t^A,t^B)\equiv\left(p(t^A),p(t^B)\right)\in\mathbb{R}^{|T|\times|T|}$ such that for any $(t'^A,t'^B)\in\varphi(t^A,t^B)$, $p(t^A,t^B)\cdot\left[(t'^A,t'^B)-(t^A,t^B)\right]>0$.

Consider a neighborhood $N\left(\varphi(t^A,t^B)\right)$ of $\varphi(t^A,t^B)$, which is small enough and satisfies

$$co\left[\varphi(t^A,t^B)\right]\subseteq N\left(\varphi(t^A,t^B)\right) \text{ and } (t^A,t^B)\notin N\left(\varphi(t^A,t^B)\right).$$

Since φ is upper hemi-continuous, there is a neighborhood $N(t^A,t^B)$ of (t^A,t^B) such that, for this $N\left(\varphi(t^A,t^B)\right)$, we have $\varphi(t^{*A},t^{*B})\subseteq N\left(\varphi(t^A,t^B)\right)$ for any $(t^{*A},t^{*B})\in N(t^A,t^B)$. Consider $\overline{N}(t^A,t^B)\subsetneq N(t^A,t^B)$, which is small enough, so that $\overline{N}(t^A,t^B)\cap N\left(\varphi(t^A,t^B)\right)=\varnothing$. Since $N\left(\varphi(t^A,t^B)\right)\supseteq co\left[\varphi(t^A,t^B)\right]$, and $(t^A,t^B)\notin co\left[\varphi(t^A,t^B)\right]$, we can find such a small set $\overline{N}(t^A,t^B)$. Thus, since φ is upper hemi-continuous, we have $\varphi(t^{*A},t^{*B})\subseteq N\left(\varphi(t^A,t^B)\right)$ for any $(t^{*A},t^{*B})\in\overline{N}(t^A,t^B)$.

By the construction of $N\left(\varphi(t^A,t^B)\right)$ and $\overline{N}(t^A,t^B)$, we can see that for any $(t^{*A},t^{*B})\in\overline{N}(t^A,t^B)$ and any $(t'^A,t'^B)\in N\left(\varphi(t^A,t^B)\right)$, $p(t^A,t^B)\cdot\left[(t'^A,t'^B)-(t^{*A},t^{*B})\right]>0$. This implies that, for any $(t^{*A},t^{*B})\in\overline{N}(t^A,t^B)$ and any $(t'^A,t'^B)\in\varphi(t^{*A},t^{*B})$, we have

$$p(t^A,t^B)\cdot\left[(t'^A,t'^B)-(t^{*A},t^{*B})\right]>0.$$

This final argument implies that φ satisfies **LDV1**. Thus, by **Lemma 1**, φ has a fixed point $(\widehat{t}^A,\widehat{t}^B)\in\varphi(\widehat{t}^A,\widehat{t}^B)$.

Figure 2 around here.

By the definition of problems (15) and (16), if $\widehat{t}^A\neq\widehat{t}^B$ holds, then

$$\begin{aligned}\pi\left(\widehat{t}^A;\widehat{t}^B\right) &\geq \pi\left(t^A,\widehat{t}^B\right) \text{ for any } t^A\in U^A(\widehat{t}^A;\widehat{t}^B), \quad (17)\\ \pi\left(\widehat{t}^A;t^B\right) &\geq \pi\left(\widehat{t}^A;\widehat{t}^B\right) \text{ for any } t^B\in U^B(\widehat{t}^B;\widehat{t}^A). \quad (18)\end{aligned}$$

This is because $\widehat{t}^B\notin U^A(\widehat{t}^A;\widehat{t}^B)$ and $\widehat{t}^A\notin U^B(\widehat{t}^B;\widehat{t}^A)$ for $\widehat{t}^A\neq\widehat{t}^B$ under **A1**. Thus, if $\widehat{t}^A\neq\widehat{t}^B$ holds, then $\left(\widehat{t}^A,\widehat{t}^B;A=\Omega(\widehat{t}^A,\widehat{t}^B),\ B=H\backslash\Omega(\widehat{t}^A,\widehat{t}^B)\right)$ is a **PUNEEP** such that Definition 4 (*ii*) and (*iii*) hold.

If $\widehat{t}^A = \widehat{t}^B$, then $\Omega(\widehat{t}^A, \widehat{t}^B) = \varnothing$, so that $\mathbb{F}\left(\Omega(\widehat{t}^A, \widehat{t}^B)\right) = 0$. Thus, $\int_{\Omega(\widehat{t}^A, \widehat{t}^B)} v(\widehat{t}^A, h)\mathrm{d}\mathbb{F}(h) = 0$, which implies that $U^A(\widehat{t}^A; \widehat{t}^B) = T$. Thus, $\widehat{t}^A = \arg\max_{t \in T} \pi\left(t, \widehat{t}^B\right)$. However, by **A3**, there exists $t' \in T$ such that $t' \neq \widehat{t}^A = \widehat{t}^B$ and $\mathbb{F}\left(\Omega(t', \widehat{t}^B)\right) > \frac{1}{2} - \beta$. Hence, $\pi\left(t', \widehat{t}^B\right) > \pi\left(\widehat{t}^A, \widehat{t}^B\right) = 0$, which is a contradiction. This implies that $\widehat{t}^A \neq \widehat{t}^B$ holds, so that $(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B)$, $B = H \backslash \Omega(\widehat{t}^A, \widehat{t}^B))$ is a C-**PUNEEP**. ■

Proof of Theorem 3: Theorem 3 holds from **Lemma 2** and the fact that **A2** is almost equivalent to **A3** if β is close enough to $\frac{1}{2}$. ■

Since **A2** is a reasonable condition, we can see from **Theorem 3** that in almost any political environment $\langle H, \mathbb{F}, T, v, \beta \rangle$ where the uncertainty of voters' behavior is large enough, there always exists a C-**PUNEEP.**

Thus far, in the previous theorems, we have discussed the existence of non-trivial M-**PUNEEP** and C-**PUNEEP**. However, Theorem 3 does not necessarily imply the existence theorem of *proper* C-**PUNEEP**s. This is because the premise of Theorem 3 is stronger than that of Theorem 2, which implies that the non-empty set of C-**PUNEEP**s in Theorem 3 always contains non-trivial M-**PUNEEP**s.

In the following discussion, we show that under a slightly stronger condition than that of Theorem 3, proper C-**PUNEEP**s exist. To do this, let us introduce the following:

Assumption 4 (A4): *For any non-trivial* M-***PUNEEP*** $(t^{*A}, t^{*B}; A = \Omega(t^{*A}, t^{*B}),\ B = H \backslash \Omega(t^{*A}, t^{*B}))$, *we have, for any* $t \in co\{t^{*A}, t^{*B}\}$ *with* $t \neq t^{*A}, t^{*B}$, $\Omega(t, t^{*B}) \supsetneq \Omega(t^{*A}, t^{*B})$ *and* $\pi\left(t, t^{*B}\right) > \pi\left(t^{*A}, t^{*B}\right)$.

A4 is reasonable, if every voter's utility function is strictly concave. In the first place, for any $t, t' \in T$ and any $t'' \in co\{t, t'\}$ with $t'' \neq t$ and $t'' \neq t'$, it follows that $\Omega(t, t') \subseteq \Omega(t'', t')$, since any $h \in \Omega(t, t')$ has $v(t, h) > v(t', h)$, and also $v(t'', h) > v(t', h)$ by the strict concavity of v. Nevertheless, there may be another type of voter who has the following property: $v(t, h) < v(t', h)$ and $v(t'', h) > v(t', h)$. These two inequalities are compatible by the strict concavity of v. **A4** only requires that there are voters whose utility functions meet these two inequalities if $t = t^{*A}$ and $t' = t^{*B}$, where (t^{*A}, t^{*B}) is a non-trivial M-**PUNEEP**, and the measure of such voters is positive. In other words, **A4** requires a variety of voter types.

Then:

Theorem 4: *Suppose T is a compact and convex subset in $\mathbb{R}^n$, and every voter's utility function v is continuous and strictly concave on T. Let the error term $\beta \in \left(0, \frac{1}{2}\right)$ be close enough to $\frac{1}{2}$, and* **A1**, **A2**, *and* **A4** *hold. Then, there exists a proper* C-**PUNEEP** *for the environment $\langle H, \mathbb{F}, T, v, \beta \rangle$.*

Proof. For each $(t^A, t^B) \in T \times T$, and for any positive real number $\varepsilon > 0$, let

$$U_\varepsilon^A(t^A; t^B) \equiv \left\{ t \in T \mid \exists \widetilde{t}^A \in U^A(t^A; t^B) : \| t, \widetilde{t}^A \| \leq \varepsilon \right\}$$

and

$$U_\varepsilon^B(t^B; t^A) \equiv \left\{ t \in T \mid \exists \widetilde{t}^B \in U^B(t^B; t^A) : \| t, \widetilde{t}^B \| \leq \varepsilon \right\}.$$

By definition, $U_\varepsilon^A(t^A; t^B)$ (*resp.* $U_\varepsilon^B(t^B; t^A)$) is a *closed* neighborhood of $U^A(t^A; t^B)$ (*resp.* $U^B(t^B; t^A)$). Also, $U_\varepsilon^A(t^A; t^B)$ (*resp.* $U_\varepsilon^B(t^B; t^A)$) is convex, since $U^A(t^A; t^B)$ (*resp.* $U^B(t^B; t^A)$) is convex.

Define a function $\overline{\pi} : T \times T \to [0, 1]$, as in the proof of **Lemma 2**. Then, given small enough numbers $\varepsilon > 0$, $\omega^A > 0$, and $\omega^B > 0$, for each $\mathbf{t} = (t^A, t^B) \in T \times T$, let $t'^A \in U_\varepsilon^A(t^A; t^B)$ have the following property:

There exists $\alpha^A \in \left[\omega^A, 1\right]$ such that t'^A is a solution of the following problem:

$$\max_{t \in U_\varepsilon^A(t^A; t^B)} \left(\overline{\pi}\left(t, t^B\right)\right)^{\alpha^A} \cdot \left(V^A\left(t\right) - V^A\left(t^B\right)\right)^{1-\alpha^A}. \quad (19)$$

Denote the set of such t'^A by $G^A\left(t^A, t^B\right)$. In the same way, let $t'^B \in U_\varepsilon^B(t^B; t^A)$ have the following property:

There exists $\alpha^B \in \left[\omega^B, 1\right]$ such that t'^B is a solution of the following problem:

$$\max_{t \in U_\varepsilon^B(t^B; t^A)} \left(1 - \overline{\pi}\left(t^A, t\right)\right)^{\alpha^B} \cdot \left(V^B\left(t\right) - V^B\left(t^A\right)\right)^{1-\alpha^B}. \quad (20)$$

Denote the set of such t'^B by $G^B\left(t^A, t^B\right)$. We can check that both $G^A\left(t^A, t^B\right)$ and $G^B\left(t^A, t^B\right)$ are non-empty and compact. This is because, for each $\alpha^A \in \left[\omega^A, 1\right]$, the solution set of:

$$\max \left(\overline{\pi}\left(t, t^B\right)\right)^{\alpha^A} \cdot \left(V^A\left(t\right) - V^A\left(t^B\right)\right)^{1-\alpha^A}$$

is non-empty and compact. Then, the union of these solution sets over $\left[\omega^A, 1\right]$ is also non-empty and compact by Border (1985; Proposition 11.16). Because

$\omega^A > 0$ is small enough, the intersection of this union of the solution-sets with $U_\varepsilon^A(t^A;t^B)$ constitutes a non-empty and compact set, $G^A\left(t^A,t^B\right)$.

Define a correspondence $\varphi : T\times T \twoheadrightarrow T\times T$ as follows: for each $(t^A,t^B)\in T\times T$, $\varphi(t^A,t^B) = G^A\left(t^A,t^B\right)\times G^B\left(t^A,t^B\right)$. It is clear that φ is non-empty valued. Moreover, φ is upper hemi-continuous by Berge's maximum theorem and Border (1985; Proposition 11.23). We show below that φ also satisfies **LDV2** of **Lemma 1**.

Let $(t^A,t^B)\notin\varphi(t^A,t^B)$. This implies

$$\varphi(t^A,t^B)\subseteq\left(U_\varepsilon^A(t^A;t^B)\times U_\varepsilon^B(t^B;t^A)\right)\setminus\left\{(t^A,t^B)\right\}.$$

Take $co\left[\varphi(t^A,t^B)\right]$. Since $U_\varepsilon^A(t^A;t^B)\times U_\varepsilon^B(t^B;t^A)$ is convex, $co\left[\varphi(t^A,t^B)\right]\subseteq U_\varepsilon^A(t^A;t^B)\times U_\varepsilon^B(t^B;t^A)$. Moreover, since $U_\varepsilon^A(t^A;t^B)\times U_\varepsilon^B(t^B;t^A)$ is strictly convex by the strict concavity of v, and $\varepsilon > 0$ is small enough, we can guarantee that $(t^A,t^B)\notin co\left[\varphi(t^A,t^B)\right]$ holds. This implies that there exists $\left(y_A\left(t^A\right),y_B\left(t^B\right)\right)\in T\times T$ such that $\left(y_A\left(t^A\right)-t^A\right)\cdot\left(t'^A-t^A\right) > 0$ and $\left(y_B\left(t^B\right)-t^B\right)\cdot\left(t'^B-t^B\right) > 0$ hold for any $t'^A\in G^A\left(t^A,t^B\right)$ and any $t'^B\in G^B\left(t^A,t^B\right)$.

Insert Figure 3 around here.

Consider a neighborhood $N\left(\varphi(t^A,t^B)\right)$ of $\varphi(t^A,t^B)$, which is small enough and satisfies $co\left[\varphi(t^A,t^B)\right]\subseteq N\left(\varphi(t^A,t^B)\right)$ and $(t^A,t^B)\notin N\left(\varphi(t^A,t^B)\right)$. Since φ is upper hemi-continuous, there is a neighborhood $N(t^A,t^B)$ of (t^A,t^B) such that, for this $N\left(\varphi(t^A,t^B)\right)$, we have $\varphi(t^{*A},t^{*B})\subseteq N\left(\varphi(t^A,t^B)\right)$ for any $(t^{*A},t^{*B})\in N(t^A,t^B)$. Consider $\overline{N}(t^A,t^B)\subsetneq N(t^A,t^B)$, which is small enough, so that $\overline{N}(t^A,t^B)\cap N\left(\varphi(t^A,t^B)\right)=\varnothing$. Since $N\left(\varphi(t^A,t^B)\right)\supseteq co\left[\varphi(t^A,t^B)\right]$, and $(t^A,t^B)\notin co\left[\varphi(t^A,t^B)\right]$, we can find such a small set $\overline{N}(t^A,t^B)$. Thus, since φ is upper hemi-continuous, we have $\varphi(t^{*A},t^{*B})\subseteq N\left(\varphi(t^A,t^B)\right)$ for any $(t^{*A},t^{*B})\in\overline{N}(t^A,t^B)$.

By the construction of $N\left(\varphi(t^A,t^B)\right)$ and $\overline{N}(t^A,t^B)$, we can see that for any $(t^{*A},t^{*B})\in\overline{N}(t^A,t^B)$ and any $(t'^A,t'^B)\in N\left(\varphi(t^A,t^B)\right)$, $\left(y_A\left(t^A\right)-t^{*A}\right)\cdot\left(t'^A-t^{*A}\right) > 0$ and $\left(y_B\left(t^B\right)-t^{*B}\right)\cdot\left(t'^B-t^{*B}\right) > 0$. This implies that for any $(t^{*A},t^{*B})\in\overline{N}(t^A,t^B)$ and any $(t'^A,t'^B)\in\varphi(t^{*A},t^{*B})$, we have $\left(y_A\left(t^A\right)-t^{*A}\right)\cdot\left(t'^A-t^{*A}\right) > 0$ and $\left(y_B\left(t^B\right)-t^{*B}\right)\cdot\left(t'^B-t^{*B}\right) > 0$. This last argument implies that φ satisfies **LDV2**. Thus, by **Lemma 1**, φ has a fixed point $(\widehat{t^A},\widehat{t^B})\in\varphi(\widehat{t^A},\widehat{t^B})$.

By the definition of problems (19) and (20), if $\widehat{t^A}\neq\widehat{t^B}$ holds, then

$$\pi\left(\widehat{t^A};\widehat{t^B}\right)\quad\geq\quad\pi\left(t^A,\widehat{t^B}\right)\text{ for any }t^A\in U^A(\widehat{t^A};\widehat{t^B}),\quad(21)$$

$$\pi\left(\widehat{t}^A; t^B\right) \quad \geq \quad \pi\left(\widehat{t}^A; \widehat{t}^B\right) \text{ for any } t^B \in U^B(\widehat{t}^B; \widehat{t}^A). \quad (22)$$

This is because $\widehat{t}^B \notin U^A(\widehat{t}^A; \widehat{t}^B)$ and $\widehat{t}^A \notin U^B(\widehat{t}^B; \widehat{t}^A)$ for $\widehat{t}^A \neq \widehat{t}^B$ under **A1**. Thus, if $\widehat{t}^A \neq \widehat{t}^B$ holds, then $\left(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B), B = H \backslash \Omega(\widehat{t}^A, \widehat{t}^B)\right)$ is a **PUNEEP** such that conditions (i) and (ii) hold. Moreover, through an argument similar to the proofs of **Lemma 2** and **Theorem 3**, we can show that $\widehat{t}^A \neq \widehat{t}^B$ holds by **A1**, **A2**, and β close enough to $\frac{1}{2}$.

Let $\left(t^{*A}, t^{*B}; A = \Omega(t^{*A}, t^{*B}), B = H \backslash \Omega(t^{*A}, t^{*B})\right)$ be a non-trivial M-**PUNEEP**. Thus, $U^A(t^{*A}; t^{*B}) = \left\{t^{*A}\right\}$ and $U^B(t^{*B}; t^{*A}) = \left\{t^{*B}\right\}$. However, $U^A_\varepsilon(t^{*A}; t^{*B}) \supsetneq \left\{t^{*A}\right\}$ and $U^B_\varepsilon(t^{*B}; t^{*A}) \supsetneq \left\{t^{*B}\right\}$. Hence, if $t^{*A} \neq \arg\max_{t \in T} \pi\left(t, t^{*B}\right)$, there exists $t^A \in U^A_\varepsilon(t^{*A}; t^{*B})$ such that $\pi\left(t^A, t^{*B}\right) > \pi\left(t^{*A}, t^{*B}\right)$ by **A4**. Note that given $U^A_\varepsilon(t^{*A}; t^{*B})$ (resp. $U^B_\varepsilon(t^{*A}; t^{*B})$), if t^{*A} (*resp.* t^{*B}) is a solution of the problem (19) (*resp.* (20)), then it is the case that α^A (*resp.* α^B) is equal to *zero*. However, this is a contradiction, since α^A (*resp.* α^B) is in $\left[\omega^A, 1\right]$ (*resp.* $\left[\omega^B, 1\right]$). Thus, the above fixed point $(\widehat{t}^A, \widehat{t}^B)$ does not coincide with $\left(t^{*A}, t^{*B}\right)$. This implies that $(\widehat{t}^A, \widehat{t}^B; A = \Omega(\widehat{t}^A, \widehat{t}^B), B = H \backslash \Omega(\widehat{t}^A, \widehat{t}^B))$ is a proper C-**PUNEEP**. ■

Note that the combination of **A1**, **A3**, and **A4** (*resp.* **A1**, **A2**, and **A4** with β close enough to $\frac{1}{2}$) is nonvacuous: there are examples of multi-dimensional political games that meet these three assumptions. For instance, consider the *Euclidean model with a two-dimensional policy space* given by Roemer (2001; Section 8.7), where the two-dimensional policy space is a *disc* and the probability distribution defined on the disc is the *uniform* distribution; we can see that the model meets **A1**, **A3**, and **A4** (*resp.* **A1**, **A2**, and **A4** with β close enough to $\frac{1}{2}$). In this Euclidean model, we can see that a (proper) C-**PUNEEP** exists.

There is yet another example of multi-dimensional political games, which meets **A1**, **A2**, and **A4** with β close enough to $\frac{1}{2}$. This model is based on Roemer (1998), though the modeling of party uncertainty differs from Roemer (1998), which is given as follows:

Example 1: Consider a political environment $\langle H, \mathbb{F}, T, v, \beta \rangle$ such that $H = \{(w,a) \in W \times \mathcal{A} \mid W \equiv [\underline{w}, \overline{w}] \subsetneq \mathbb{R}_+ \ \& \ \mathcal{A} \equiv [\underline{a}, \overline{a}] \subsetneq \mathbb{R}\}$, where W is the set of income levels, and $\mathcal{A}$ is the set of religious views, $T = \{(\tau, z) \mid \tau \in [0,1] \text{ and } z \in \mathcal{A}\}$, where τ is a uniform tax rate on income, and z is a religious position of the government, and $v(\tau, z; w, a) = (1-\gamma)\left[(1-\tau)w + \tau\mu\right] - \frac{\gamma}{2}(z-a)^2$, where μ is the mean income of this society. Moreover, $\mathbb{F}$ has its associated density function $f(w,a) = g(w)\, r(a;w)$ such that $F(a') \equiv \int_W \int_{\underline{a}}^{a'} g(w)\, r(a;w) \mathrm{d}a \mathrm{d}w$ is strictly increasing at every $a' \in \mathcal{A}$. Finally, $\mathbb{F}$ is assumed to satisfy **A1**, and β is close enough to $\frac{1}{2}$.

Note that for any $h \in H$, if his or her income $w_h > \mu$, then $\tau = 0$ is the ideal tax rate for him or her, whereas if $w_h \leq \mu$, then $\tau = 1$ is the ideal tax rate for him or her. Let $L \equiv \{h \in H \mid w_h \leq \mu\}$ and $R \equiv \{h \in H \mid w_h > \mu\}$. Let z_L be the median religious view over L and z_R be the median religious view over R. Moreover, let z_H be the median religious view over H. Assume $z_L \neq z_R$. Then, $\mathbb{F}$ also satisfies **A2**. Thus, since v is continuous and strictly concave, **Theorem 2** tells us that there exists a non-trivial M-**PUNEEP** $\left(t^{*A}(\gamma), t^{*B}(\gamma); A(\gamma), B(\gamma)\right)$ for each $\gamma \in [0,1]$.

If $\gamma = 1$, then a non-trivial M-**PUNEEP** for $\gamma = 1$ implies $z^{*A}(1) \neq z^{*B}(1)$. Thus, if $\gamma < 1$ is close to one, then $z^{*A}(\gamma) \neq z^{*B}(\gamma)$ holds. Then, for such $\gamma < 1$ close to one, **A4** holds. This is because for any $a \in \left[z^{*A}(\gamma), z^{*B}(\gamma)\right]$, there are some voters whose ideal religious policies are identical to a, and the $\mathbb{F}$-measure of those voters is positive. The last condition follows from the strictly increasing $F(a')$. Note in the case in which $\gamma < 1$ is sufficiently close to one, the effect of the tax policy τ on the voters' welfare is negligible relative to that of the religious policy z.

Thus, when β is close enough to $\frac{1}{2}$, for any $\gamma < 1$ close to one, **Theorem 4** tells us that there exists a proper C-**PUNEEP**. In fact, for $\gamma = 1$, there exists a non-trivial M-**PUNEEP** $\left(t^{*A}(1), t^{*B}(1); A(1), B(1)\right)$ such that $t^{*A}(1) = (0, z_A^*)$ and $t^{*B}(1) = (0, z_B^*)$, where, for some $a^* \in \mathcal{A}$,
$z_A^* = \arg\max_{z \in \mathcal{A}} \int_W \int_{\underline{a}}^{a^*} -\frac{1}{2}(z-a)^2 g(w)\, r(a;w) \mathrm{d}a \mathrm{d}w$,
$z_B^* = \arg\max_{z \in \mathcal{A}} \int_W \int_{a^*}^{\overline{a}} -\frac{1}{2}(z-a)^2 g(w)\, r(a;w) \mathrm{d}a \mathrm{d}w$, and
$A(1) = \{(w,a) \in H \mid a \in [\underline{a}, a^*)\} = \Omega\left(t^{*A}(1), t^{*B}(1)\right)$ and $B(1) = H \backslash A(1)$.
Then, for $\gamma = 1$, consider any profile

$$\left(\widehat{t}^A(1), \widehat{t}^B(1); A(1), B(1)\right) = \left((0, \widehat{z}_A), (0, \widehat{z}_B); H(a^*), H \backslash H(a^*)\right)$$

such that $z_A^* < \widehat{z}_A < a^* < \widehat{z}_B < z_B^*$ with $\frac{\widehat{z}_A + \widehat{z}_B}{2} = a^*$ and $H(a^*) \equiv \{(w,a) \in H \mid a \in [\underline{a}, a^*)\}$. This profile constitutes a proper C-**PUNEEP** when β is close enough to $\frac{1}{2}$. Moreover, assume that the mean income of the cohort of voters with the median religious view z_H is higher than the mean income, μ, of the population. Then, for any $\gamma < 1$ close enough to one, any profile $\left(\widehat{t}^A(\alpha), \widehat{t}^B(\alpha); A(\alpha), B(\alpha)\right) = \left(\widehat{t}^A(1), \widehat{t}^B(1); A(1), B(1)\right)$ still constitutes a proper C-**PUNEEP** when β is close enough to $\frac{1}{2}$. ■

Note that the condition, β close enough to $\frac{1}{2}$, is indispensable for the existence of proper C-**PUNEEP**. In fact, without this condition, we can find an example of political games in which there only exist one non-trivial M-**PUNEEP** and one trivial **PUNEEP** under β close enough to 0. Such an example is illustrated as follows:

Example 2 [Yoshihara (2007; Theorems 1 and 4)]**:** Consider a political environment $\langle H, \mathbb{F}, T, v, \beta \rangle$ such that $H = W \equiv [\underline{w}, \overline{w}]$, where W is the set of income levels, $T = \{(\tau, \alpha) \mid \tau \in [0,1] \text{ and } \alpha \in [0,1]\}$, where τ is a uniform tax rate on income, and α is the ratio of defense expenditure over tax revenue, and $v(\tau, \alpha; w) = [(1-\tau) w + (1-\alpha) \tau\mu] + \sigma(\alpha\tau\mu)$, where μ is the mean income of this society. In this environment, if the society chooses (τ, α), then its tax revenue is $\tau\mu$ per capita, and its defense expenditure becomes $\alpha\tau\mu$ per capita. Then, $(1-\alpha)\tau\mu$ is the subsidy that every citizen receives through the income redistribution policy. Thus, the choice of (τ, α) implies the choice of redistribution and military forces in this society. In every citizen's utility function v, the term $(1-\tau) w + (1-\alpha)\tau\mu$ represents the voter's after-tax income when the policy (τ, α) is implemented; the term $\sigma(\alpha\tau\mu)$ represents the citizen's benefit from the national security supplied by the military forces. Finally, $\mathbb{F}$ is assumed to satisfy **A1**, which is characterized by a density function $g(w)$ over W.

Assume that $\lim_{\lambda \to 0} \frac{\partial \sigma(\lambda\mu)}{\partial \lambda\mu} = +\infty$, and for some $\lambda^* \in (0,1)$, $\frac{\partial \sigma(\lambda^*\mu)}{\partial \lambda^*\mu} = 1$. Also, assume that $G(\mu) > \frac{1}{2}$. Then, it can be shown that there exists $h^* \in H$ with $w_{h^*} > \mu$ such that $L = \{h \in H \mid w_h < w_{h^*}\}$ and $R = \{h \in H \mid w_h \geq w_{h^*}\}$ with $w_R = \int_{h \in R} w_h \mathrm{d}\mathbb{F}(h) > \mu$, so that $w_L = \int_{h \in L} w_h \mathrm{d}\mathbb{F}(h) < \mu$. Moreover, there are $\alpha^* \in (0,1)$ and $\tau^* \in (0,1)$ such that $\alpha^* > \tau^*$ and $(1, \alpha^*)$ is the ideal policy for any citizen in L, while $(t^*, 1)$ is the ideal policy for any citizen in R. Thus, $\mathbb{F}$ satisfies **A2**. Then, if β is close enough to 0, any **PUNEEP** $\left(t^A, t^B; A, B\right)$ is characterized by either of the following types: $\left(t^A, t^B; A, B\right) = ((1, \alpha^*), (\tau^*, 1); L, R)$ with $\pi((1, \alpha^*), (\tau^*, 1)) = 1$ and $\left(t^A, t^B; A, B\right) = ((1, \alpha^*), (1, \alpha^*); L, R)$.[3] The former type is a non-trivial M-**PUNEEP**, while the latter is a trivial **PUNEEP**. Thus, there is no proper C-**PUNEEP** in this policy game under β close enough to 0. ■

[3] See Yoshihara (2007) for the detailed proof of this.

4 Concluding Remarks

In this paper, we discussed the existence of **PUNEEP**s in general multi-dimensional political games. In particular, we introduced two refinements of **PUNEEP**s and provided sufficient conditions for the existence of the two refinements of **PUNEEP**s. The sufficient conditions appear natural and plausible, and this implies that we can have many reasonable models of multi-dimensional political games in which the two refinements exist.

In this paper, we focus on a specific modeling of party uncertainty, which Roemer (2001; Chapter 2) called the *Error-Distribution Model of Uncertainty.* However, Roemer (2001; Chapter 2) also proposed other types of uncertainty model, including the *State-Space Approach to Uncertainty.* We have not yet considered the existence problem using this alternative uncertainty model.

The existence theorems in this paper depend on the assumption of strictly concave utility functions of voters. However, there are some examples of political games with only weakly concave utility functions, such as set out by Roemer (1999). The existence of the refined **PUNEEP**s in political games with only weakly concave utility functions remains an open question.

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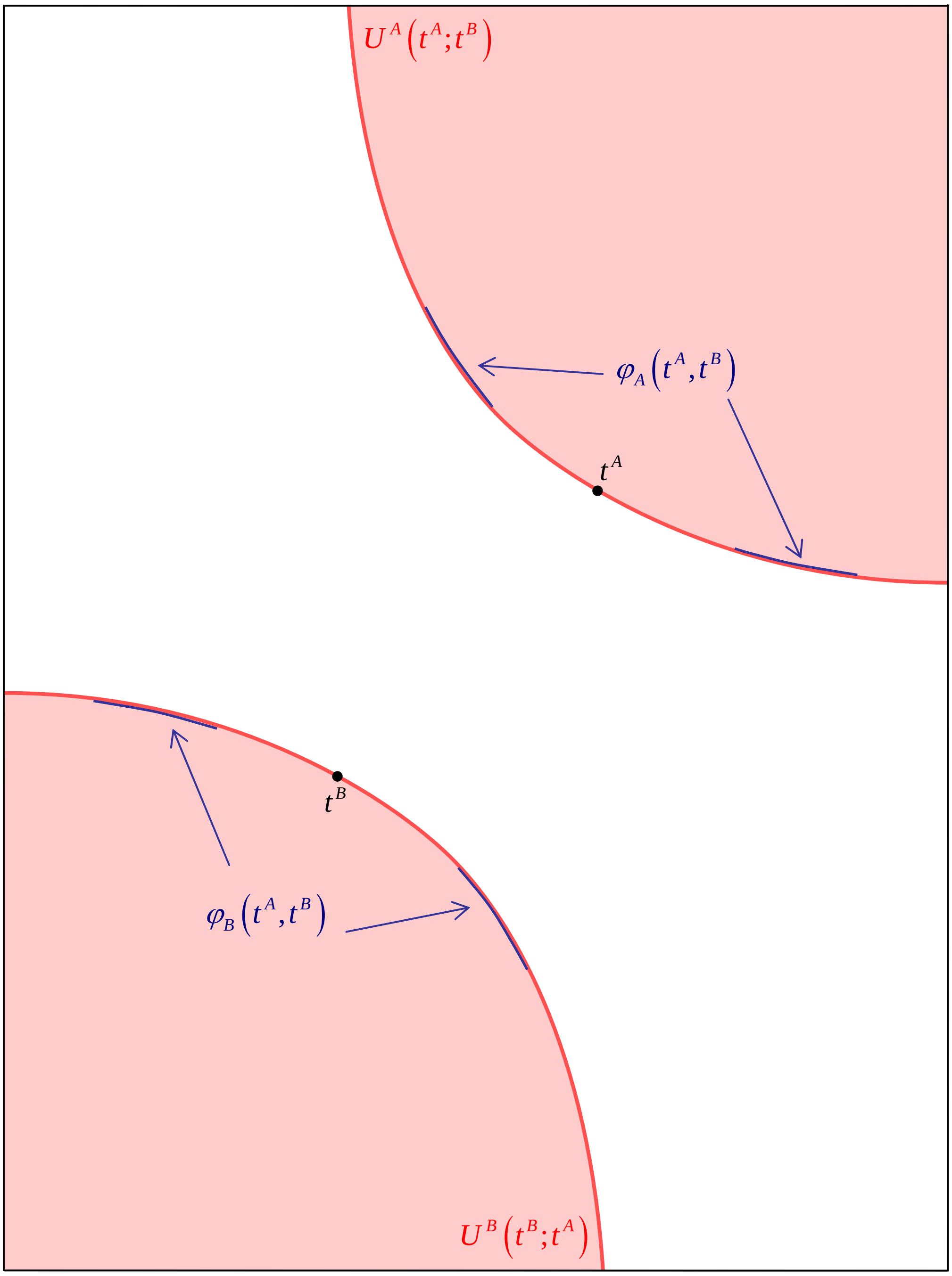

Figure1: $\left(t^A,t^B\right) \notin \varphi\left(t^A,t^B\right)$

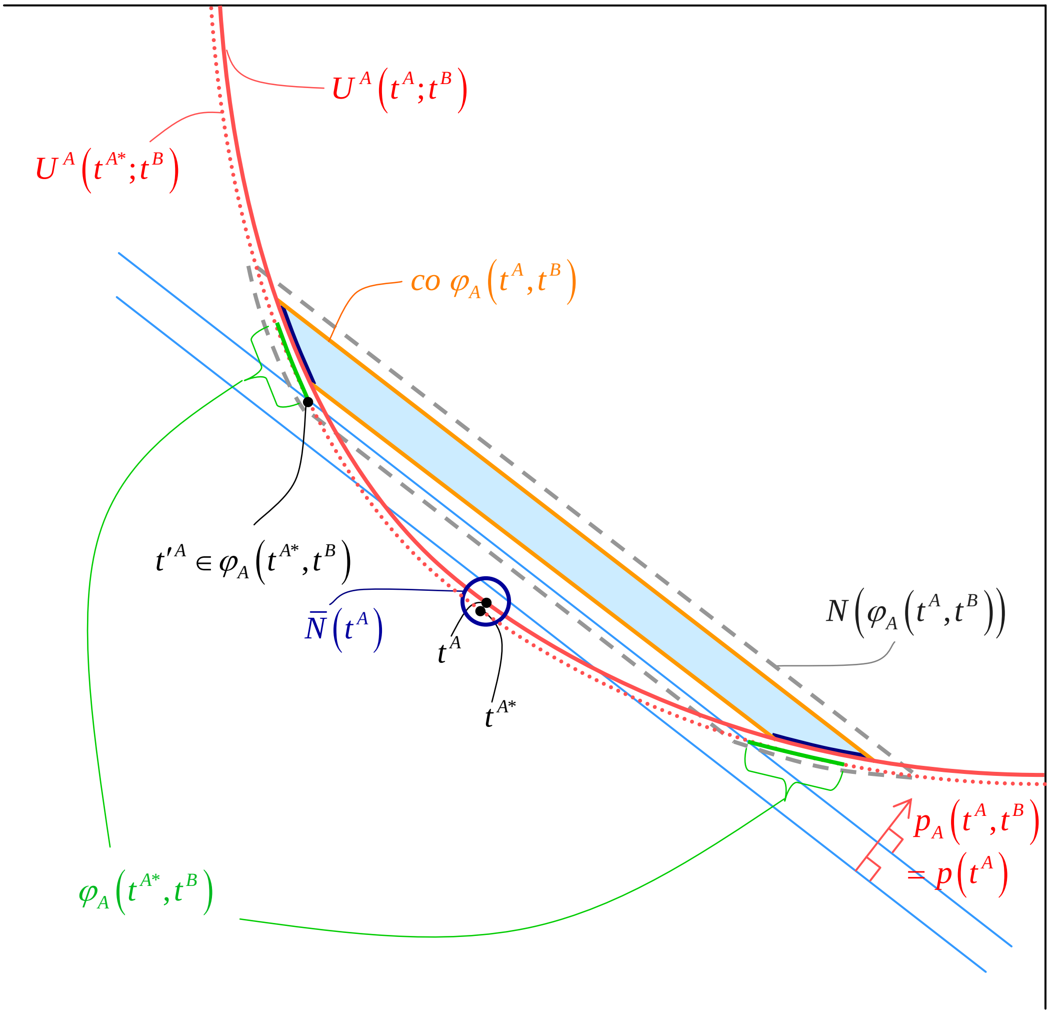

Figure 2: φ satisfies **LDV1**

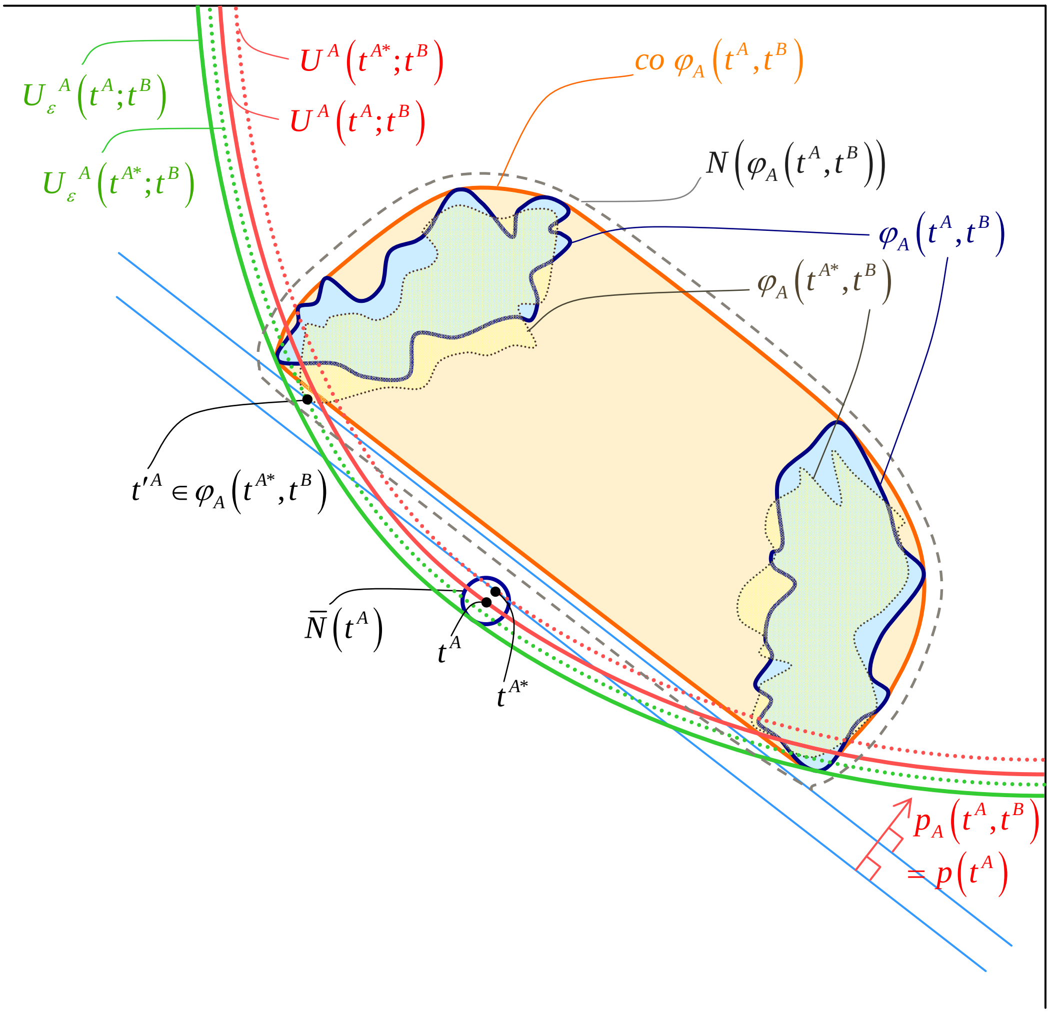

Figure 3: φ satisfies **LDV2**