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Abstract

In two-sided matching problems, we formulate (i) a concept of equity of matchings based on envy minimization, and (ii) a solidarity property of matching rules under "natural" and "simple" changes of preferences which represent enhancement of partnership of the pairs. We show that there exists no rule that selects an envy-minimizing matching in the set of stable matchings, and that also satisfies the solidarity property. In contrast, any rule with a certain separability condition that selects an envy-minimizing matching in the set of individually rational and Pareto efficient matchings satisfies solidarity.

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1 Introduction

Consider a firm which has several different factories. The firm must assign workers to these factories. Each factory manager has preferences over the workers, while each worker has preferences over which factory he works at. What is a desirable rule to match workers to factories?

Allocation problems such as the above example are called *two-sided matching problems*. Gale and Shapley (1962) first formulated these matching problems, and defined *stable* matchings: a matching is *stable* if no pair of a worker and a factory manager can be both better off by becoming new partners. They also presented an algorithm to find a stable matching.[1] However, the matching is only optimal *to one side* among all the stable matchings.

In this paper, we search for rules that select a matching in a socially desirable way. A *matching rule* is a mapping that associates with each preference profile of agents a matching. We approach desirable matching rules in two aspects. First, for each *given* preference profile, a rule should recommend an equitable matching. Here, our notion of equity is the concept of *no-envy*, which has been playing a central role in fair allocation theory since it is introduced and first studied by Foley (1967) and Kolm (1972). However, a difficulty immediately comes up. Unless there exists a matching at which every agent is matched to his first choice, someone must envy another agent. Hence, in general there are no envy-free matchings, and we could at most seek matchings at which envy is "minimized." Several social measures of envy have been proposed in social choice theory. Feldman and Kirman (1974) adopted the total number of instances of envy as the measure of envy. Counting the instances of envy of each agent, Suzumura (1983) proposed to minimize the maximal number over all agents. In this paper, we consider envy minimization according to each of the two social measures of envy, namely, *total envy minimization* and *maximal envy minimization*.

Another aspect of socially desirable matching rules concerns situations where preferences of agents may change. In the foregoing example, workers usually acquire factory-specific skills in the long-run, and thereby become ranked higher in the preference orders of the currently matched factory managers. At the same time, workers also prefer working at the currently matched factories to working at other factories after obtaining such skills. Thus, it is

[1] A good reference to the subsequent extensive analyses of this subject is Roth and Sotomayor (1990).

natural to consider preference changes such that for each agent i, his current partner at a matching chosen by the rule is now preferred to more agents in agent i's new preferences. We call such a transformation of preferences *rank-enhancement of the partners*.

The *solidarity principle*, a fundamental principle in normative economics, requires that, when some data in the problem (preferences, the amount of resources, and so on) change, all the agents in the same situation under the change should be affected in the same direction: they are all better-off, or they are all worse-off at the new allocation chosen by the rule. A form of the principle is *solidarity under preference changes*: when the preferences of some agents change, the agents *whose preferences are fixed* should be affected in the same direction. This version was first studied by Moulin (1987) in the context of quasi-linear binary social choice. Thomson (1993, 97, 98) extensively analyzed the property in classes of resource allocations problems with single-peaked preferences, and with indivisible goods. Sprumont (1996) considered a class of general choice problems and formulated a solidarity property when the feasibility constraints and the preferences change possibly jointly. All these authors have considered *arbitrary changes* of preferences. However, to require solidarity under arbitrary changes of preferences is often too much demanding.

Here we restrict our attention to "natural" or "simple" changes of preferences, namely rank-enhancement of the partners. We look for matching rules that satisfy *solidarity under rank-enhancement of the partners*.

We consider two weak versions of the solidarity property. The first version says that when *only one* agent increases the rank of his/her original partner, the other agents should be all better-off, or all worse-off at the new matching chosen by the rule.

The second version considers situations where several agents enhance their rankings of their initial partners. Then, we divide the set of agents who do not change their preferences into two groups. One is the group of agents whose ranks become strictly higher in the preference orders of their initial partners, and the other is the group of agents whose ranks are unchanged. Then, we require that for either one of the groups, all the agents in the group should be affected in the same direction at the new matching.

We examine the existence of a matching rule (i) that selects an envy-minimizing matchings in the set of stable matchings, and (ii) that satisfies *solidarity under rank-enhancement of the partners*. Unfortunately, our first result is an impossibility theorem. If a rule always selects an envy-minimizing matching in the set of stable matchings, then it cannot satisfy either weak version of *solidarity under rank-enhancement of the partners*.

Faced with the impossibility results, we weaken the requirement of stability on matching rules to individual rationality and Pareto efficiency. This weakening drastically changes the result: there exist a rule that selects an envy-minimizing matching in the set of *individually rational and Pareto efficient* matchings, and that satisfies *solidarity under rank-enhancement of the partners*. Moreover, we show that any such selection rule meeting a certain separability condition satisfies the solidarity property.

The organization of the paper is as follows. The next section gives basic definitions and notation. Section 3 introduces our concept of equity as envy minimization. Section 4 presents the solidarity properties and the impossibility and possibility results on their compatibility with envy minimization. The final section contains some concluding remarks.

2 Basic Definitions and Notation

Let $F = \{f_1, f_2, \cdots, f_n\}$ and $W = \{w_1, w_2, \cdots, w_n\}$ be given two disjoint finite sets such that $|F| = |W| = n$. We call F *the set of factory managers*, and W *the set of workers*. For each $i \in F \cup W$, let $X_i \in \{F, W\}$ be the set with $i \notin X_i$, and $Y_i \in \{F, W\}$ the set with $i \in Y_i$. We call X_i *the set of possible partners for agent* i. For each $i \in F \cup W$, a *preference relation of agent* i, denoted by R_i, is a linear order on $X_i \cup \{i\}$.[2] An alternative $j \in X_i$ indicates that agent i is matched to agent j in X_i, and the alternative i that agent i is not matched to any agent in X_i (i.e., he is "matched to himself"). Let $\mathcal{R}_i$ be the set of all possible preference relations of agent i. Given $R_i \in \mathcal{R}_i$, we define the relation P_i on $X_i \cup \{i\}$ as follows: for all $x, x' \in X \cup \{i\}$, $x \; P_i \; x'$ if and only if $x \; R_i \; x'$ holds but $x' R_i x$ does not hold.[3]

To concisely express a preference relation $R_i \in \mathcal{R}_i$, we represent it, as in Roth and Sotomayor (1990), by an ordered list of the members of $X \cup \{i\}$. For example, the list

$$R_{f_2} \quad = \quad w_3 \quad w_1 \quad f_2 \quad w_2 \quad \cdots$$

indicates that factory manager f_2 prefers being matched to worker w_3 to

[2] Note that we exclude indifference between any two distinct elements in $X_i \cup \{i\}$.

[3] Since R_i is a linear order, $x \; P_i \; x'$ if and only if $x \; R_i \; x'$ and $x' \neq x$.

being matched to w_1, and prefers being matched to w_1 to being unmatched, and so on.

A *preference profile* is a list $R = (R_i)_{i \in F \cup W}$. Let $\mathcal{R} = \prod_{i \in F \cup W} \mathcal{R}_i$ be the class of all preference profiles. We also consider the subclass $\mathcal{R}^*$ of preference profiles such that being unmatched is the worst for every agent: $\mathcal{R}^* = \{R \in \mathcal{R} \mid \forall i \in F \cup W, \forall j \in X_i,\ j\ P_i\ i\}$. A *matching* μ is a one-to-one function from $F \cup W$ onto itself such that for all $i \in F \cup W, \mu^2(i) = i$, and if $\mu(i) \notin X_i$, then $\mu(i) = i$. Let $\mathcal{M}$ be the set of all matchings.

Following Roth and Sotomayor (1990), we represent a matching as a list of matched pairs. For example, the matching

$$\mu = \begin{matrix} f_1 & f_2 & f_3 & (w_2) \\ w_3 & w_1 & (f_3) & w_2 \end{matrix}$$

has two matched pairs (f_1, w_3) and (f_2, w_1), and f_3 and w_2 remaining unmatched.

Let $R \in \mathcal{R}$ be given. A matching $\mu \in \mathcal{M}$ is *individually rational for* R if for all $i \in F \cup W$, $\mu(i)\ R_i\ i$. It is *Pareto efficient for* R if there is no $\mu' \in \mathcal{M}$ such that for all $i \in F \cup W$, $\mu'(i)\ R_i\ \mu(i)$, and for some $i \in F \cup W$, $\mu'(i)\ P_i\ \mu(i)$. It is *stable for* R if it is individually rational for R, and there is no pair $(f, w) \in F \times W$ such that $w\ P_f\ \mu(f)$ and $f\ P_w\ \mu(w)$. Let $I(R), P(R)$ and $S(R)$ be the set of individually rational matchings for R, the set of Pareto efficient matchings for R, and the set of stable matchings for R, respectively. Let $IP(R) = I(R) \cap P(R)$.

Let $\mathcal{R}_0 \subseteq \mathcal{R}$. A *matching rule* (or simply a *rule*) *on* $\mathcal{R}_0$, denoted by φ, is a function from $\mathcal{R}_0$ to $\mathcal{M}$. For each $R \in \mathcal{R}_0, \varphi(R)$ is interpreted as a desirable matching for the preference profile R. If $\varphi(R) = \mu$, we write $\varphi_i(R) = \mu(i)$ for each $i \in F \cup W$. Given a correspondence Ψ from $\mathcal{R}$ to $\mathcal{M}$, we say a matching rule φ on $\mathcal{R}_0$ is a *selection rule from* Ψ if for all $R \in \mathcal{R}_0$, $\varphi(R) \in \Psi(R)$.

3 Envy Minimization

A fundamental notion of equity in fair allocation theory is *no-envy.* In our model, a matching is *envy-free* if no agent prefers being matched to another agent's partner to being matched to his present partner. However, except for the rare case where every agent can be matched to his first choice, there exists no envy-free matching. Thus, following Feldman and Kirman (1974) and Suzumura (1983), we introduce a social measure of envy, and look for matchings at which the measure of envy is minimized.

Let a preference profile $R \in \mathcal{R}$ and a matching $\mu \in \mathcal{M}$ be given. For each agent $i \in F \cup W$, define[4]

$$r_i(R_i, \mu) = \#\{j \in X_i \mid j = \mu(k) \text{ for some } k \in Y_i \text{ and } j \; P_i \; \mu(i)\}.$$

The integer $r_i(R_i, \mu)$ is the number of instances of envy that agent i has at μ. Let $r(R, \mu) = (r_i(R_i, \mu))_{i \in F \cup W} \in \mathbb{R}^{2n}$.[5] Summing up the $2n$ numbers $r_i(R_i, \mu)$ over all the agents, we obtain the number of total instances of envy at the matching. Define

$$t(R, \mu) = \sum_{i \in F \cup W} r_i(R_i, \mu)$$

This is the "social measure of envy" due to Feldman and Kirman (1974), and they proposed to minimize the number.

Let Ψ be a correspondence from $\mathcal{R}$ to $\mathcal{M}$. For each $R \in \mathcal{R}$, let

$$T^{\Psi}(R) = \{\mu \in \Psi(R) \mid \forall \mu' \in \Psi(R), t(R, \mu') \geq t(R, \mu)\}$$

The set $T^{\Psi}(R)$ is the set of matchings that minimize the total instances of envy in $\Psi(R)$.

Suzumura (1983) considered an agent with the largest $r_i(R, \mu)$ the "worst off" agent at μ, and proposed to minimize the maximal element in $r(R, \mu) = (r_i(R_i, \mu))_{i \in F \cup W}$. This idea is based on the maximin principle due to Rawls (1971). For each $R \in \mathcal{R}$, let

$$E^{\Psi}(R) = \{\mu \in \Psi(R) \mid \forall \mu' \in \Psi(R), \max_{i \in F \cup W}(r_i(R, \mu')) \geq \max_{i \in F \cup W}(r_i(R, \mu))\}$$

This is the set of matchings that minimize the maximal individual instances of envy in $\Psi(R)$.

Further refinements can be obtained by using the lexicographic order (Schmeidler, 1969, and Sen, 1970). Let $\theta : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the function that rearranges the coordinates of each vector in $\mathbb{R}^{2n}$ in decreasing order. We denote by $\geq_L$ the lexicographic order on $\mathbb{R}^{2n}$.[6] For each $R \in \mathcal{R}$, let

$$L^{\Psi}(R) = \{\mu \in \Psi(R) \mid \forall \mu' \in \Psi(R), \; \theta(r(R, \mu')) \geq_L \theta(r(R, \mu))\}$$

[4]Given a set A, we denote by $\#A$ the cardinality of A.

[5]We denote by $\mathbb{R}$ the set of real numbers.

[6]For all $x, y \in \mathbb{R}^{2n}$, $x >_L y$ if and only if there is $k \in \{1, \cdots, 2n\}$ such that for all $i < k, x_i = y_i$, and $x_k > y_k$. For all $x, y \in \mathbb{R}^{2n}$, $x \geq_L y$ if and only if $x >_L y$ or $x = y$.

It is the set of matchings that lexicographically minimize the maximal individual instances of envy in $\Psi(R)$.

Note that all the social measures of envy defined above do not use any cardinal "utilities" of agents or the "intensity of preferences," but depend only on ordinal preferences (rankings).

4 Solidarity

In this section, we formulate the solidarity properties of matching rules under certain "natural" changes of preferences. In the following definitions of the properties of rules, the symbol $\mathcal{R}_0$ denotes the domain of a matching rule φ.

Let an agent $i \in F \cup W$, a preference relation $R_i \in \mathcal{R}_i$, and a matching $\mu \in \mathcal{M}$ be given. We say that *a preference relation $R'_i \in \mathcal{R}_i$ is obtained from R_i by rank-enhancement of the partner at μ* if (i) for all $j \in X_i \cup \{i\}$, $\mu(i)\ R_i\ j$ implies $\mu(i)\ R'_i\ j$, and (ii) for all $j, k \in X_i \cup \{i\}$ with $j, k \neq \mu(i)$, $j\ R'_i\ k$ if and only if $j\ R_i\ k$. Let $Q(R_i, \mu)$ be the set of preference relations that are obtained from R_i by rank-enhancement of the parters at μ. Given $R \in \mathcal{R}$ and $\mu \in \mathcal{M}$, let $Q(R, \mu) = \{R' \in \mathcal{R} \mid \forall i \in F \cup W,\ R'_i \in Q(R_i, \mu)\}$.

Under rank-enhancement of the partners, only the current partners at the matching μ is preferred to more agents, while the preferences over any other agents are unchanged. See the Introduction for motivations to consider this kind of changes in preferences,

The solidarity principle requires that when some agents change their preferences, then all the other agents whose preferences are unchanged should be affected in the same direction. Given $R, R' \in \mathcal{R}$, let $K(R, R') = \{i \in F \cup W \mid R_i = R'_i\}$ be the set of agents whose preferences are the same in the profiles R and R'.

Solidarity under Rank-Enhancement of the Partners: For all $R, R' \in \mathcal{R}_0$, if $R' \in Q(R, \varphi(R))$, then either $\varphi_i(R')\ R_i\ \varphi_i(R)$ for all $i \in K(R, R')$, or $\varphi_i(R)\ R_i\ \varphi_i(R')$ for all $i \in K(R, R')$.

A weaker version of the above propery is obtained if we apply the requirement only when *only one* agent changes his/her preferences.

Solidarity under Single Rank-Enhancement of the Partner: For all $R, R' \in \mathcal{R}_0$, if if $R'_i \in Q(R_i, \varphi(R))$ for some $i \in F \cup W$ and $R'_k = R_k$ for all $k \in F \cup W$ with $k \neq i$, then either $\varphi_k(R')\ R_k\ \varphi_k(R)$ for all $k \in K(R, R')$, or $\varphi_k(R)\ R_k\ \varphi_k(R')$ for all $k \in K(R, R')$.

We examine whether there exists rules that always select an envy-minimizing matching in the set of stable matchings, and that satisfy *solidarity under single rank-enhancement of the partner*. Our first result is an impossibility: even on the restricted domain $\mathcal{R}^*$ (the class of preference profiles such that for each agent, being unmatched is the worst alternative), if a rule always selects an envy-minimizing matching in the set of stable matchings, then it cannot satisfy *solidarity under single rank-enhancement of the partner*. The result holds for both maximin envy minimization and total envy minimization.

Theorem 1 *Suppose $n \geq 3$.*
(i) There exists no selection rule from the E^S on $\mathcal{R}^*$ satisfying solidarity under single rank-enhancement of the partner.
(ii) There exists no selection rule from T^S on $\mathcal{R}^*$ satisfying solidarity under single rank-enhancement of the partner.

Proof. (i) Let φ be a selection rule from E^S. Let $R \in \mathcal{R}^*$ be a preference profile such that

$$\begin{array}{lclllll} R_{f_1} & = & w_1 & w_4 & w_5 & w_2 & \cdots \\ R_{f_2} & = & w_2 & w_4 & w_5 & w_1 & \cdots \\ R_{f_3} & = & w_1 & w_3 & \cdots & & \\ R_{w_1} & = & f_2 & f_3 & f_1 & \cdots & \\ R_{w_2} & = & f_1 & f_2 & \cdots & & \\ R_{w_3} & = & f_3 & \cdots & & & \end{array}$$

and for all $i > 3$, $w_i P_{f_i} w$ for all $w \in W$, and $f_i P_{w_i} f$ for all $f \in F$. Then, the unique stable matching for R is

$$\mu = \begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & \cdots & f_n \\ w_2 & w_1 & w_3 & w_4 & \cdots & w_n \end{array}$$

Hence, $\varphi(R) = \mu$.

Let $R' \in \mathcal{R}^*$ be such that

$$R'_{f_3} \;=\; w_3 \;\; w_1 \;\; \cdots$$

and for all $i \in F \cup W$ with $i \neq f_3, R'_i = R_i$. There are exactly two stable matchings for R': μ and

$$\mu' = \begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & \cdots & f_n \\ w_1 & w_2 & w_3 & w_4 & \cdots & w_n \end{array}$$

Since $\max_{i \in F \cup W}(r_i(R'_i, \mu')) = 2 < 3 = \max_{i \in F \cup W}(r_i(R'_i, \mu))$, $E^S(R') = \{\mu'\}$. Hence, $\varphi(R') = \mu'$. Observe that $R' \in Q(R, \mu)$ and $f_1, w_1 \in K(R, R')$. But $\mu'(f_1) = w_1 \ P_{f_1} \ w_2 = \mu(f_1)$ while $\mu(w_1) = f_2 \ P_{w_1} \ f_1 = \mu'(w_1)$. Thus, the rule φ violates *solidarity under single rank-enhancement of the partner.*
(ii) Let φ be a selection rule from T^S. Let $R, R' \in \mathcal{R}^*$ be the preference profiles as defined above. Notice that $t(R', \mu') = 3 < 6 = t(R', \mu)$ and hence $\varphi(R') = \mu'$. Then, by the same argument as above, we can show that the rule φ violates *solidarity under single rank-enhancement of the partner.* ■

To introduce our second version of the solidarity principle, which is also weaker than original *solidarity under rank-enhancement of the partners*, let us divide the agents whose preferences are unchanged into two groups. One is the group of agents whose ranks become strictly higher in the preference orders of their initial partners, and the other is the group of agents whose ranks are unchanged. Then, we require that for either one of the groups, all the agents in the group should be affected in the same direction at the new matching chosen by the rule compared with the initial matching.

Notice that if *only one* agent changes his preferences, then the above condition is automatically met. Hence, the property concerns situations where two or more agents changes their preferences.[7]

Given $R \in \mathcal{R}, \mu \in \mathcal{M}$ and $R' \in Q(R, \mu)$, let $N_1(R, R', \mu) = \{i \in F \cup W \mid R'_{\mu(i)} \neq R_{\mu(i)}\}$, and $N_2(R, R', \mu) = \{i \in F \cup W \mid R'_{\mu(i)} = R_{\mu(i)}\}$.

Partial Solidarity under Rank-Enhancement of the Partners: For all $R, R' \in \mathcal{R}_0$, if $R' \in Q(R, \varphi(R))$, then for some $h \in \{1, 2\}$, either $\varphi_i(R') \ R_i \ \varphi(R)$ for all $i \in K(R, R') \cap N_h(R, R', \varphi(R))$ or $\varphi_i(R) \ R_i \ \varphi(R')$ for all $i \in K(R, R') \cap N_h(R, R', \varphi(R))$.

Unfortunately, with this second weaker version of solidarity, we again reach an impossibility.

Theorem 2 *Suppose $n \geq 5$.*
(i) *There exists no selection rule from E^S on $\mathcal{R}^*$ satisfying* partial solidarity under rank-enhancement of the partners*.*
(ii) *There exists no selection rule from T^S on $\mathcal{R}^*$ satisfying* partial solidarity under rank-enhancement of the partners*.*

[7] Notice that there is no logical relation between our two versions of the solidarity principle.

Proof. (i) Let φ be a selection rule from E^S. Let $R \in \mathcal{R}^*$ be the preference profile such that:

$$\begin{array}{lcllllll}
R_{f_1} & = & w_1 & w_2 & w_5 & w_3 & \cdots \\
R_{f_2} & = & w_2 & w_3 & w_5 & w_1 & \cdots \\
R_{f_3} & = & w_3 & w_1 & w_2 & \cdots \\
R_{f_4} & = & w_1 & w_4 & \cdots \\
\\
R_{w_1} & = & f_2 & f_4 & f_1 & f_3 & \cdots \\
R_{w_2} & = & f_3 & f_2 & f_1 & \cdots \\
R_{w_3} & = & f_4 & f_1 & f_3 & f_2 & \cdots \\
R_{w_4} & = & f_4 & \cdots
\end{array}$$

and for all $i \geq 5$, $w_i \; P_{f_i} \; w$ for all $w \in W$, and $f_i \; P_{w_i} \; f$ for all $f \in F$. Then, the unique stable matching for R is

$$\mu = \begin{array}{lllllll} f_1 & f_2 & f_3 & f_4 & f_5 & \cdots & f_n \\ w_3 & w_1 & w_2 & w_4 & w_5 & \cdots & w_n \end{array}$$

Hence, $\varphi(R) = \mu$.

Let $R' \in \mathcal{R}^*$ be such that

$$\begin{array}{lcllllll}
R'_{f_2} & = & w_2 & w_3 & w_1 & w_5 & \cdots \\
R'_{f_4} & = & w_4 & w_1 & w_3 & w_1 & \cdots \\
\\
R'_{w_3} & = & f_1 & f_4 & f_3 & \cdots
\end{array}$$

and for all $i \in F \cup W$ with $i \neq f_2, f_4, w_3$, $R'_i = R_i$. It is clear that $R' \in Q(R, \mu)$. There are exactly two stable matchings for R': μ and

$$\mu' = \begin{array}{lllllll} f_1 & f_2 & f_3 & f_4 & f_5 & \cdots & f_n \\ w_1 & w_2 & w_3 & w_4 & w_5 & \cdots & w_n \end{array}$$

Since $\max_{i \in F \cup W}(r_i(\mu')) = r_{w_1} = 2 < 3 = r_{f_1}(\mu) = \max_{i \in F \cup W}(r_i(\mu))$, $E^S(R') = \{\mu'\}$. Hence, $\varphi(R') = \mu'$. Then, $w_1, f_1 \in K(R, R') \cap N^+(R, R', \varphi(R))$, and $\mu(w_1) = f_2 \; P_{w_1} \; f_1 = \mu'(w_1)$ while $\mu'(f_1) = w_1 \; P_{f_1} \; w_3 = \mu(f_1)$. Also, $w_2, f_3 \in K(R, R') \cap N^=(R, R', \varphi(R))$, and $\mu'(f_3) = w_3 \; P_{f_3} \; w_2 = \mu(f_3)$ whereas $\mu(w_2) = f_3 \; P_{w_2} \; f_2 = \mu(w_2)$. Thus, the rule φ violates *partial solidarity under rank-enhancement of the partners.*

(ii) Let φ be a selection rule from T^S. Let $R, R' \in \mathcal{R}^*$ be the preference profiles as defined above. Then, $t(R', \mu') = 5 < 7 = t(R', \mu)$, and hence $\varphi(R') = \mu'$. Then, the same observations as above imply that the rule φ violates *solidarity under rank-enhancement of the partners.* ■

Theorems 1 and 2 show a trade-off, under the requirement of stability, between envy minimization and solidarity. On the one hand, a rule should always choose an equitable matching in the set of stable matchings. On the other hand, a rule should satisfy an appealing solidarity property under a most natural class of preference changes. But these requirements are incompatible, and we have to give up some property of matching rules.

Next, we weaken the requirement of stability on matchings, and consider selection rules from *individually rational and Pareto efficient* matchings. However, among envy-minimizing selection rules from individual rational and Pareto efficient matchings, there still exist rules that violate *solidarity under rank-enhancement of the partners.* Consider the following example. Let $n = 4$ and $R \in \mathcal{R}$ be such that

$$\begin{array}{lcllll} R_{f_1} & = & w_1 & w_2 & w_3 & w_4 \\ R_{f_2} & = & w_1 & w_2 & w_3 & w_4 \\ R_{f_3} & = & w_4 & w_3 & w_1 & w_2 \\ R_{f_4} & = & w_4 & w_3 & w_1 & w_2 \\ \\ R_{w_1} & = & f_1 & f_2 & f_3 & f_4 \\ R_{w_2} & = & f_1 & f_2 & f_3 & f_4 \\ R_{w_3} & = & f_3 & f_4 & f_1 & f_2 \\ R_{w_4} & = & f_4 & f_3 & f_1 & f_2. \end{array}$$

Assume that $\varphi(R) = \mu$ where

$$\mu = \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_2 & w_3 & w_4. \end{array}$$

Note that $\mu \in T^{IP}(R)$ and $\mu \in L^{IP}(R)$, that is, μ is an envy-minimizing matching in the set of individually rational and Pareto efficient matchings. Let $R' \in \mathcal{R}$ be such that

$$R'_{f_3} \;=\; w_3 \quad w_4 \quad w_1 \quad w_2$$

and for all $i \neq f_3$, $R'_i = R_i$. Notice that $R' \in Q(R, \mu)$. Now assume that $\varphi(R') = \mu'$ where

$$\mu' = \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ w_2 & w_1 & w_3 & w_4. \end{array}$$

We have $\mu' \in T^{IP}(R')$ and $\mu' \in L^{IP}(R')$. However, because $\mu(f_1)\ P_{f_1}\ \mu'(f_1)$ whereas $\mu'(f_2)\ P_{f_2}\ \mu(f_2)$, the rule φ violates *solidarity under rank-enhancement of the partners*.

The rule φ in the above example looks peculiar. Under the matching $\varphi(R) = \mu$, the agents are separated into two groups, $N_1 = \{f_1, f_2, w_1, w_2\}$ and $N_2 = \{f_3, f_4, w_3, w_4\}$, where every agent is matched to another agent in the same group as herself. Then, only agent f_3 increases the rank of his partner at μ, namely agent w_3, at R'. This change in the preferences of f_3 should be "irrelevant" to group N_1, and it should not affect the matching of the members within N_1.

The following property formalizes the above idea.

Separability: For all $R, R' \in \mathcal{R}_0$, and all $S \subseteq F \cup W$, if $\varphi(R) = \mu$, $\mu(S) = S$, $R' \in Q(R, \mu)$ and $R'_i = R_i$ for all $i \in S$, then $\varphi_i(R') = \mu(i)$ for all $i \in S$.

The next result shows that *any* separable rule that minimizes envy (in the sense of leximin or total number) in the set of individually rational and Pareto efficient matchings satisfies *solidarity under rank-enhancement of the partners*.

Theorem 3 *(i) Any* separable *selection rule from L^{IP} on $\mathcal{R}$ satisfies* solidarity under rank-enhancement of the partners.
(ii) Any separable *selection rule from T^{IP} on $\mathcal{R}$ satisfies* solidarity under rank-enhancement of the partners.

Proof. (i) Assume that φ is a selection rule from L^{IP}. Let $R \in \mathcal{R}$ and $\varphi(R) = \mu$. Then, μ lexicographically minimizes envy in the set $I(R) \cap P(R)$. Let $R' \in Q(R, \mu)$.

We will first show that μ lexicographically minimizes envy in the set $I(R') \cap P(R')$ as well. It is clear that μ is individually rational and Pareto efficient for R' since $R' \in Q(R, \mu)$. It remains to show that for all $\mu' \in I(R') \cap P(R')$, $\theta(r(R', \mu)) \leq_L \theta(r(R', \mu'))$. Notice that if $\mu \in I(R)$ and $R' \in Q(R, \mu)$, then $I(R') \subseteq I(R)$ holds true. However, in general, there is no inclusion relation between $P(R')$ and $P(R)$ even if $\mu \in P(R)$ and $R' \in Q(R, \mu)$. Hence, we distinguish two cases.

Case 1: $\mu' \in I(R) \cap P(R)$.
Since μ lexicographically minimizes envy in $I(R) \cap P(R)$, we have (1) $\theta(r(R, \mu)) \leq_L \theta(r(R, \mu'))$. Since $R' \in Q(R, \mu)$, it holds true that (2) $r_i(R', \mu) - r_i(R, \mu) \leq r_i(R', \mu') - r_i(R, \mu')$ for all $i \in F \cup W$. It follows from (1) and (2) that $\theta(r(R', \mu)) \leq_L \theta(r(R', \mu'))$.

Case 2: $\mu' \notin I(R) \cap P(R)$.
Since $\mu' \in I(R') \subseteq I(R)$, we have $\mu' \notin P(R)$. Thus, there exists $\mu'' \in I(R) \cap P(R)$ that Pareto dominates μ' at R. Then, (3) $\theta(r(R, \mu'')) \leq_L \theta(r(R, \mu'))$ because at μ'', no agent is matched to an agent with lower rank in his/her preference order than at μ', and hence no agent has more instances of envy at μ'' than at μ'. Since μ lexicographically minimizes envy in $I(R) \cap P(R)$, it follows that (4) $\theta(r(R, \mu)) \leq_L \theta(r(R, \mu''))$. By (3) and (4), we have $\theta(r(R, \mu)) \leq_L \theta(r(R, \mu'))$. The rest of the argument is the same as Case 1.

We have shown that μ lexicographically minimizes envy in the set $I(R') \cap P(R')$. Let $\varphi(R') = \mu^*$. We will show that $\mu^* = \mu$. Let $J(R, R') := \{i \in F \cup W \mid R'_i \neq R_i\}$. Then, for all $i \in J(R, R') \cup \mu(J(R, R'))$, $\mu^*(i) = \mu(i)$, for otherwise, μ^* cannot lexicographically minimize envy in the set $I(R') \cap P(R')$. By separability, for all $i \in F \cup W$ with $i \notin J(R, R') \cup \mu(J(R, R'))$, $\mu^*(i) = \mu(i)$. Thus, $\varphi(R') = \mu^* = \mu = \varphi(R)$, and $\varphi_i(R')\ R_i\ \varphi_i(R)$ for all $i \in K(R, R') = \{i \in F \cup W \mid R'_i = R_i\}$. Therefore, φ satisfies *solidarity under rank-enhancement of the partners*.

(ii) The proof of (ii) is essentially the same as that of (i). Simply replace $\theta(r(\cdot, \cdot))$ with $t(\cdot, \cdot)$, as well as $\leq_L$ with $\leq$. ■

An example of a separable selection rule from L^{IP} (or from T^{IP}) is as follows. For all $R \in \mathcal{R}$, all $i \in F \cup W$, and all $Z \subseteq \mathcal{M}$, let $G_i^R(Z) = \{\mu \in Z \mid \forall \mu' \in Z,\ \mu(i)\ R_i\ \mu'(i)\}$. Define the rule φ^s as $\varphi^s(R) = G_{f_n}^R \circ G_{f_{n-1}}^R \cdots \circ G_{f_1}^R(L^{IP}(R))$ for all $R \in \mathcal{R}$. It can be checked that $\#\varphi^s(R) = 1$ for all $R \in \mathcal{R}^*$. Indeed, if $\mu, \mu' \in \varphi^s(R)$, then $\mu(i)\ I_i\ \mu'(i)$ for all $i \in F$. Since any R_i is a linear order on $W \cup \{i\}$, we have $\mu(i) = \mu'(i)$ for all $i \in F$, and hence $\mu = \mu'$. It is clear that the rule φ^s satisfies separability, and by Theorem 3, it satisfies *solidarity under rank-enhancement of the partners*.

Note that the rule φ^s violates *anonymity*, requiring that the rule should not depend on the "names" (f_1, f_2, $\cdots$) of agents in each group, nor on the "names" (F and W) of the groups. However, searching for a rule satisfying *anonymity* would lead us to a dead end. As Masarani and Gokturk (1989) showed, there exists no rule that selects a Pareto efficient matching, and that satisfies *anonymity*.[8]

[8] Actually, Masarani and Gokturk (1989) listed four axioms as the requirements on a "fair" matching rule: two of them are *anonymity*, one is *stability*, and the other is *maximin optimality*. They showed that there exists no rule satisfying the four axioms together. In order to establish the impossibility result, however, the last axiom is not necessary, and *stability* can be weakened to *Pareto efficiency*.

5 Conclusion

In this paper, we have formulated a principle of equity as envy minimization on the one hand, and a principle of solidarity on the other hand, in the class of two-sided matching problems. The former requires that a rule should always select an envy-minimizing matching in the set of matchings meeting some basic conditions. The latter requires that when preferences of some agents change naturally, all the agents whose preferences are fixed should be affected in the same direction between the old and new matchings chosen by the rule. We have shown that under the requirement of stability of matchings, the two principles are incompatible, but under the weaker requirement of individual rationality and Pareto efficiency, they are fully compatible. What we have done in this paper is, along all the literature in social choice theory, to draw a line between the cases when we can obtain a rule satisfying the desirable properties together and the cases when we cannot.

Our analysis has been confined to the case of one-to-one matchings. However, impossibility results straightforwardly extend to the more general class of many-to-one matching problems if we do not impose any constraints on the number of workers that each factory should accommodate, since the class of one-to-one matching problems is a subclass of this general class. It may be of interest to examine whether the impossibility and possibility results extend to the case of many-to-one matchings with some constraints such that there is a minimum number of workers that each factory must have. To consider other desirable properties of matching rules under some "natural" changes of the data and examine their compatibility may also be an interesting topic of future research.

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