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Ordering Infinite Utility Streams

Walter Bossert
(Département de Sciences Economiques and CIREQ, Université de Montréal)
Yves Sprumont
(Département de Sciences Economiques and CIREQ, Université de Montréal)
Kotaro Suzumura
(Institute of Economic Research, Hitotsubashi University)

Naka 2-1, Kunitachi, Tokyo 186-8603, Japan
Phone: +81-42-580-8350 Fax: +81-42-580-8351
URL: http://wakame.econ.hit-u.ac.jp/~koho/1intro/COE/index.htm
E-mail: COE-RES@econ.hit-u.ac.jp

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WALTER BOSSERT
Département de Sciences Economiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada
FAX: (+1 514) 343 7221
e-mail: walter.bossert@umontreal.ca

YVES SPRUMONT
Département de Sciences Economiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada
FAX: (+1 514) 343 7221
e-mail: yves.sprumont@umontreal.ca

KOTARO SUZUMURA
Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo 186-8603
Japan
FAX: (+81 42) 580 8353
e-mail: suzumura@ier.hit-u.ac.jp

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Abstract

This paper reconsiders the problem of ordering infinite utility streams. As has been established in earlier contributions, if no representability condition is imposed, there do exist strongly Paretian and finitely anonymous orderings of intertemporal utility streams. We examine the possibility of adding suitably formulated versions of classical equity conditions to these two requirements. In particular, we provide a characterization of all strongly Paretian and finitely anonymous rankings satisfying the strict transfer principle. We also offer a characterization of an infinite-horizon extension of leximin obtained by adding an equity-preference axiom to strong Pareto and finite anonymity. *Journal of Economic Literature* Classification Nos.: D63, D71.

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1 Introduction

Treating generations equally is one of the basic principles in the utilitarian tradition of moral philosophy. As Sidgwick (1907, p. 414) observes, "the time at which a man exists cannot affect the value of his happiness from a universal point of view; and [...] the interests of posterity must concern a Utilitarian as much as those of his contemporaries". This view, which is formally expressed by the anonymity condition, is also strongly endorsed by Ramsey (1928).

Following Koopmans (1960), Diamond (1965) establishes that anonymity is incompatible with the strong Pareto principle when ordering *infinite* utility streams. Moreover, he shows that if anonymity is weakened to *finite* anonymity—which restricts the application of the standard anonymity requirement to situations where utility streams differ in at most a finite number of components—and a continuity requirement is added, an impossibility results again. Suzumura and Shinotsuka (2003) adapt the well-known strict transfer principle due to Pigou (1912) and Dalton (1920) to the infinite-horizon context. They show that this principle is incompatible with strong Pareto and continuity even if the social preference is merely required to be acyclical. Basu and Mitra (2003a) show that strong Pareto, finite anonymity and representability by a real-valued function are incompatible.

Faced with these impossibilities, it seems to us that the most natural assumption to drop is that of continuity or representability. We view the strong Pareto principle and finite anonymity as being on much more solid ground than axioms such as continuity or representability, especially in the context of the ranking of infinite utility streams where these conditions may be considered to be overly demanding. Svensson (1980) proves that strong Pareto and finite anonymity are compatible by showing that any ordering extension of an infinite-horizon variant of Suppes' (1966) grading principle satisfies the required axioms. The Suppes grading principle is a quasi-ordering that combines the Pareto quasi-ordering and finite anonymity. Given Arrow's (1951) version of Szpilrajn's (1930) extension theorem, this establishes the compatibility result. As noted by Asheim, Buchholz and Tungodden (2001), Svensson's possibility result is easily converted into a characterization: ordering extensions of the Suppes grading principles are *the only* orderings satisfying strong Pareto and finite anonymity.

Once the possibility of satisfying these two fundamental axioms is established, another natural question to ask is what orderings satisfy additional desirable properties. Asheim and Tungodden (2004) provide a characterization of an infinite-horizon version of the leximin principle by adding an equity-preference condition (the infinite-horizon equivalent of Hammond equity; see Hammond, 1976) and a preference-continuity property to strong Pareto and finite anonymity. An infinite-horizon version of utilitarianism is characterized by Basu and Mitra (2003b) by adding an information-invariance condition to the two fundamental axioms. Furthermore, they narrow down the class of infinite-horizon utilitarian orderings to those resulting from the overtaking criterion (von Weizsäcker, 1965). This is accomplished by using a consistency condition in addition to the three axioms characterizing their utilitarian orderings.

In this paper, we focus on equity properties. One of the most fundamental equity properties (if not *the* most fundamental) is the Pigou-Dalton transfer principle, adapted to the infinite-horizon framework by Suzumura and Shinotsuka (2003). Our first result characterizes all orderings that satisfy strong Pareto, anonymity and the strict transfer principle.

In the presence of strong Pareto, the axiom of equity preference (the infinite-horizon version of Hammond equity) is a strengthening of the strict transfer principle. We use it to identify a subclass of the class of orderings satisfying the three axioms just mentioned. These orderings are extensions of a particular infinite-horizon incomplete version of leximin. This second result leaves a larger class of orderings than that identified by Asheim and Tungodden (2004) because they employ an additional axiom. The relationship between our leximin characterization and that of Asheim and Tungodden is analogous to the relationship between Basu and Mitra's (2003b) characterizations of infinite-horizon leximin and of the overtaking criterion.

2 Basic definitions

The set of infinite utility streams is $X = \mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ denotes the set of all natural numbers. A typical element of X is an infinite-dimensional vector $x = (x_1, x_2, \ldots, x_n, \ldots)$ and, for $n \in \mathbb{N}$, we write $x_{-n} = (x_1, \ldots, x_n)$ and $x_{+n} = (x_{n+1}, x_{n+2}, \ldots)$. The standard interpretation of $x \in X$ is that of a countably infinite utility stream where x_n is the utility experienced in period $n \in \mathbb{N}$. Of course, other interpretations are possible—for example, x_n could be the utility of an individual in a countably infinite population.

Our notation for vector inequalities on X is as follows. For all $x, y \in X$, (i) $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$; (ii) $x > y$ if $x \geq y$ and $x \neq y$; (iii) $x \gg y$ if $x_n > y_n$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $x \in X$, $(x_{(1)}, \ldots, x_{(n)})$ is a rank-ordered permutation of x_{-n} such that $x_{(1)} \leq \ldots \leq x_{(n)}$, ties being broken arbitrarily.

$R \subseteq X \times X$ is a weak preference relation on X with strict preference $P(R)$ and indifference relation $I(R)$. A quasi-ordering is a reflexive and transitive relation, and an ordering is a complete quasi-ordering. Analogously, a partial order is an asymmetric and transitive relation, and a linear order is a complete partial order. Let R and R' be relations on X. R' is an extension of R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. If an extension R' of R is an ordering, we call it an ordering extension of R, and if R' is an extension of R that is a linear order, we refer to it as a linear order extension of R. The transitive closure of a relation R is denoted by $\overline{R}$, that is, for all $x, y \in X$, $(x, y) \in \overline{R}$ if there exist $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x = z^0$, $(z^{k-1}, z^k) \in R$ for all $k \in \{1, \ldots, K\}$ and $z^K = y$.

A finite permutation of $\mathbb{N}$ is a bijection $\rho: \mathbb{N} \to \mathbb{N}$ such that there exists $m \in \mathbb{N}$ with $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. The corresponding finite permutation matrix $B^\rho = (b^\rho_{ij})_{i,j \in \mathbb{N}}$ is defined by letting, for all $i \in \mathbb{N}$, $b^\rho_{i\rho(i)} = 1$ and $b^\rho_{ij} = 0$ for all $j \in \mathbb{N} \setminus \{\rho(i)\}$.

Two of the most fundamental axioms in this area are the strong Pareto principle and finite anonymity, defined as follows.

Strong Pareto: For all $x, y \in X$, if $x > y$, then $(x, y) \in P(R)$.

Finite anonymity: For all $x \in X$ and for all finite permutations ρ of $\mathbb{N}$,

$$(B^\rho x, x) \in I(R).$$

Szpilrajn's (1930) fundamental result establishes that every partial order has a linear order extension. Arrow (1951, p. 64) presents a variant of Szpilrajn's theorem stating that every quasi-ordering has an ordering extension; see also Hansson (1968). This implies that the sets of orderings characterized in the theorems of the following sections are non-empty.

The Suppes (1966) grading principle combines the requirements of strong Pareto and anonymity into a criterion for establishing a partial social ranking. Adapted to the multi-period framework, the Suppes quasi-ordering R_S on X is defined as follows. For all $x, y \in X$, $(x, y) \in R_S$ if there exists a finite permutation ρ of $\mathbb{N}$ such that $x \geq B^\rho y$. Svensson (1980, Theorem 2) shows that any ordering extension of R_S satisfies strong Pareto and finite anonymity and Asheim, Buchholz and Tungodden (2001, Proposition 1) extend this result to a characterization: an ordering R on X satisfies strong Pareto and finite anonymity if and only if R is an ordering extension of R_S.

3 Transfer-sensitive infinite-horizon orderings

Now we examine the consequences of adding the strict transfer principle to strong Pareto and finite anonymity. In order to define the strict transfer principle, we introduce the notion of a finite bistochastic matrix. A finite bistochastic matrix is a matrix $B = (b_{ij})_{i,j\in\mathbb{N}}$ such that there exists $m \in \mathbb{N}$ such that $b_{ij} \geq 0$ for all $i, j \in \{1, \ldots, m\}$, $\sum_{i=1}^{m} b_{ij} = 1$ for all $j \in \{1, \ldots, m\}$, $\sum_{j=1}^{m} b_{ij} = 1$ for all $i \in \{1, \ldots, m\}$, $b_{ii} = 1$ for all $i \in \mathbb{N} \setminus \{1, \ldots, m\}$ and $b_{ij} = 0$ for all $i, j \in \mathbb{N} \setminus \{1, \ldots, m\}$ with $i \neq j$. Clearly, finite permutation matrices are special cases of finite bistochastic matrices.

Strict transfer principle: For all $x, y \in X$ and for all $n, m \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$, then $(x, y) \in P(R)$.

The strict transfer principle is the natural analogue of the corresponding condition for finite streams; see also Suzumura and Shinotsuka (2003).

To define the class of orderings satisfying the three axioms introduced thus far, consider first the following relation R_T. For all $x, y \in X$, $(x, y) \in R_T$ if there exist $n, m \in \mathbb{N}$ such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$. This relation captures the requirements imposed by the strict transfer principle. Clearly, $P(R_T) = R_T$. Note that if $(x, y) \in R_T$, then there exists a finite bistochastic matrix B such that $x = By$. This matrix is obtained by letting $b_{nm} = b_{mn} = (x_n - y_n)/(y_m - y_n)$, $b_{nn} = b_{mm} = (x_m - y_n)/(y_m - y_n)$, $b_{ni} = b_{in} = b_{m_i} = b_{im} = 0$ for all $i \in \mathbb{N} \setminus \{n, m\}$, $b_{ii} = 1$ for all $i \in \mathbb{N} \setminus \{n, m\}$ and $b_{ij} = 0$ for all $i, j \in \mathbb{N} \setminus \{n, m\}$ with $i \neq j$.

Because, in addition, we want our ordering to satisfy strong Pareto and finite anonymity, the relation R_S must be respected as well. Finally, because we only consider transitive relations, the transitive closure of the union of these two relations appears in the definition of the relevant class of orderings. Clearly, the transitive closure $\overline{R_S \cup R_T}$ of $R_S \cup R_T$ is a quasi-ordering: reflexivity follows from the reflexivity of R_S and transitivity is satisfied by definition. We obtain the following characterization of the class of all ordering extensions of $\overline{R_S \cup R_T}$.

Theorem 1 *An ordering R on X satisfies strong Pareto, finite anonymity and the strict transfer principle if and only if R is an ordering extension of $\overline{R_S \cup R_T}$.*

Proof. 'If.' We first prove that $\overline{R_S \cup R_T}$ is an extension of both R_S and R_T. It is immediate that $R_S \subseteq \overline{R_S \cup R_T}$ and $R_T \subseteq \overline{R_S \cup R_T}$, so we only need to establish the set inclusions

$$P(R_S) \subseteq P(\overline{R_S \cup R_T}) \tag{1}$$

and

$$P(R_T) \subseteq P(\overline{R_S \cup R_T}). \tag{2}$$

To prove (1), suppose that $(x, y) \in P(R_S)$. This implies $(x, y) \in \overline{R_S \cup R_T}$. By way of contradiction, suppose that $(y, x) \in \overline{R_S \cup R_T}$. Thus, there exist a finite permutation ρ of $\mathbb{N}$, $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x > B^\rho y$, $y = z^0$, $(z^{k-1}, z^k) \in R_S \cup R_T$ for all $k \in \{1, \ldots, K\}$ and $z^K = x$. Let $k \in \{1, \ldots, K\}$. If $(z^{k-1}, z^k) \in R_S$, it follows that there exists a finite permutation ρ^k of $\mathbb{N}$ such that $z^{k-1} \geq B^{\rho^k} z^k$. If $(z^{k-1}, z^k) \in R_T$, it follows that there exists a finite bistochastic matrix B such that $z^{k-1} = Bz^k$. Suppose first that, whenever $(z^{k-1}, z^k) \in R_S$, we have $z^{k-1} = B^{\rho^k} z^k$ for some finite permutation ρ^k. Because the set of finite bistochastic matrices is closed under matrix multiplication, it follows that $y = B^0 x$ for some finite bistochastic matrix B^0. Let $m \in \mathbb{N}$ be such that $b^0_{ii} = b^\rho_{ii} = 1$ for all $i \in \mathbb{N} \setminus \{1, \ldots, m\}$. Because $y = B^0 x$, it follows that $\sum_{i=1}^m y_i = \sum_{i=1}^m x_i$. But $x > B^\rho y$ implies $\sum_{i=1}^m x_i > \sum_{i=1}^m y_i$, a contradiction. If some of the inequalities are strict, an analogous contradiction emerges. Therefore, $(y, x) \in \overline{R_S \cup R_T}$ is impossible and (1) follows. The proof of (2) is analogous.

Next, we prove that any ordering extension of $\overline{R_S \cup R_T}$ satisfies the required axioms. Suppose R is such an ordering extension.

We begin with strong Pareto. Suppose that $x > y$ for some $x, y \in X$. This implies $(x, y) \in P(R_S)$ and, by (1), $(x, y) \in P(\overline{R_S \cup R_T})$. Because R is an ordering extension of $\overline{R_S \cup R_T}$, it follows that $(x, y) \in P(R)$ and strong Pareto is satisfied.

To establish finite anonymity, let $x \in X$ and let ρ be any finite permutation of $\mathbb{N}$. This implies $(B^\rho x, x) \in I(R_S)$ and, because $R_S \subseteq \overline{R_S \cup R_T} \subseteq R$, we obtain $(B^\rho x, x) \in I(R)$.

Finally, we prove that the strict transfer principle is satisfied. Suppose $x, y \in X$ and $n, m \in \mathbb{N}$ are such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$. This implies $(x, y) \in P(R_T)$ and, by (2) and the assumption that R is an ordering extension of $\overline{R_S \cup R_T}$, we obtain $(x, y) \in P(R)$.

'Only if.' Suppose R satisfies the three axioms of the theorem statement. To prove that R is an ordering extension of $\overline{R_S \cup R_T}$, suppose first that $(x, y) \in \overline{R_S \cup R_T}$. By definition, there exist $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x = z^0$, $(z^{k-1}, z^k) \in R_S \cup R_T$ for all $k \in \{1, \ldots, K\}$ and $z^K = y$. As a consequence of the Svensson-Asheim-Buchholz-Tungodden characterization of the Suppes grading principle mentioned at the end of the previous section, $(z^{k-1}, z^k) \in R$ follows whenever $(z^{k-1}, z^k) \in R_S$ and, by the strict transfer principle, $(z^{k-1}, z^k) \in R$ follows whenever $(z^{k-1}, z^k) \in R_T$. Because R is transitive, it follows that $(x, y) \in R$.

Now suppose that $(x, y) \in P(\overline{R_S \cup R_T})$. By definition, there exist $K \in \mathbb{N}$ and

$z^0, \ldots, z^K \in X$ such that $x = z^0$, $(z^{k-1}, z^k) \in R_S \cup R_T$ for all $k \in \{1, \ldots, K\}$ and $z^K = y$. Moreover, at least one of these preferences must be strict because otherwise we would have $(y, x) \in \overline{R_S \cup R_T}$, contradicting $(x, y) \in P(\overline{R_S \cup R_T})$. If the strict preference is such that $(z^{k-1}, z^k) \in P(R_S)$, $(z^{k-1}, z^k) \in P(R)$ follows from the characterization of the Suppes grading principle. If the strict preference is such that $(z^{k-1}, z^k) \in P(R_T)$, $(z^{k-1}, z^k) \in P(R)$ follows immediately from the strict transfer principle. Therefore, in either case, the transitivity of R implies $(x, y) \in P(R)$. This completes the proof that R is an ordering extension of $\overline{R_S \cup R_T}$. ■

4 Infinite-horizon leximin

An equity property that has received a considerable amount of attention in finite settings is Hammond equity and some of its variations. The infinite-horizon version we use is defined as follows.

Equity preference: For all $x, y \in X$ and for all $n, m \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m > x_n > y_n$, then $(x, y) \in R$.

Equity preference is the extension of Hammond's (1976) equity axiom to the infinite-horizon environment. The axiom is used in Asheim and Tungodden (2004); see also Asheim and Tungodden (2005) for an alternative version which they call Hammond equity for the future. d'Aspremont and Gevers (1977) use a stronger condition by requiring $(x, y) \in P(R)$ rather than merely $(x, y) \in R$ in the conclusion of the axiom. In the presence of strong Pareto, the two axioms are equivalent. Moreover, strong Pareto and equity preference together imply the following property which, in turn, obviously implies the strict transfer principle.

Strict equity preference: For all $x, y \in X$ and for all $n, m \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m \geq x_n > y_n$, then $(x, y) \in P(R)$.

To see that strict equity preference is implied by strong Pareto and equity preference, suppose that R satisfies the first two axioms, and let $x, y \in X$ and $n, m \in \mathbb{N}$ be such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m \geq x_n > y_n$. Let $z \in X$ be such that $z_k = x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $x_n > z_m > z_n > y_n$. By strong Pareto, $(x, z) \in P(R)$ and by equity preference, $(z, y) \in R$. Thus, transitivity implies $(x, y) \in P(R)$ and strict equity preference is satisfied.

If the strict transfer principle is replaced by equity preference (which, in the presence of strong Pareto, is a strengthening), the only remaining orderings are infinite-horizon versions of the leximin criterion. Let $n \in \mathbb{N}$. We denote the usual leximin ordering on $\mathbb{R}^n$ by R_ℓ^n, that is, for all $x, y \in X$,

$$(x_{-n}, y_{-n}) \in R_\ell^n \quad \Leftrightarrow \quad \begin{array}{l} x_{-n} \text{ is a permutation of } y_{-n} \text{ or there exists } m \in \{1, \ldots, n\} \text{ such that} \\ x_{(k)} = y_{(k)} \text{ for all } k \in \{1, \ldots, n\} \setminus \{m, \ldots, n\} \text{ and } x_{(m)} > y_{(m)}. \end{array}$$

Again, let $n \in \mathbb{N}$ and define a relation $R_L^n \subseteq X \times X$ by letting, for all $x, y \in X$, $(x, y) \in R_L^n$ if $(x_{-n}, y_{-n}) \in R_\ell^n$ and $x_{+n} \geq y_{+n}$. It is straightforward to verify that R_L^n is a quasi-ordering for all $n \in \mathbb{N}$. Finally, let $R_L = \cup_{n \in \mathbb{N}} R_L^n$. This relation is a quasi-ordering but it is not complete—some infinite utility streams are not ranked by R_L. Our next result characterizes all ordering extensions of R_L.

Theorem 2 *An ordering R on X satisfies strong Pareto, finite anonymity and equity preference if and only if R is an ordering extension of R_L.*

Proof. 'If.' First, we prove that, for all $n, m \in \mathbb{N}$ such that $m > n$,

$$R_L^n \subseteq R_L^m \tag{3}$$

and

$$P(R_L^n) \subseteq P(R_L^m). \tag{4}$$

Let $n, m \in \mathbb{N}$ be such that $m > n$.

To prove (3), suppose that $(x, y) \in R_L^n$. By definition, $(x_{-n}, y_{-n}) \in R_\ell^n$ and $x_{+n} \geq y_{+n}$. Hence $(x_{-m}, y_{-m}) \in R_\ell^m$ and $x_{+m} \geq y_{+m}$, that is, $(x, y) \in R_L^m$.

To establish (4), suppose that $(x, y) \in P(R_L^n)$. By definition, at least one of the following two statements is true:

$$(x_{-n}, y_{-n}) \in P(R_\ell^n) \quad \text{and} \quad x_{+n} \geq y_{+n}; \tag{5}$$

$$(x_{-n}, y_{-n}) \in R_\ell^n \quad \text{and} \quad x_{+n} > y_{+n}. \tag{6}$$

By (3), it follows that $(x, y) \in R_L^m$. To prove that $(x, y) \in P(R_L^m)$, suppose, by way of contradiction, that $(y, x) \in R_L^m$. Then, by definition,

$$(x_{-n}, y_{-n}) \in I(R_\ell^n) \text{ and } x_{+n} = y_{+n},$$

contradicting (5) and (6).

Next, we prove that R_L is a quasi-ordering. Reflexivity is immediate because, for all $x \in X$, $(x, x) \in R_L^n$ for all $n \in \mathbb{N}$ and hence $(x, x) \in R_L$. To prove that R_L is transitive, suppose that $(x, y), (y, z) \in R_L$. By definition, there exist $n, m \in \mathbb{N}$ such that $(x, y) \in R_L^n$ and $(y, z) \in R_L^m$. Let $k = \max\{n, m\}$. By (3), $(x, y), (y, z) \in R_L^k$ and by the transitivity of R_L^k, $(x, z) \in R_L^k$ which, in turn, implies $(x, z) \in R_L$.

We now show that, for all $x, y \in X$,

$$(x, y) \in P(R_L) \Leftrightarrow \exists n \in \mathbb{N} \text{ such that } (x, y) \in P(R_L^n). \tag{7}$$

Suppose first that $(x, y) \in P(R_L)$. By definition, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_L^n$. Moreover, $(y, x) \notin R_L^n$ because otherwise we obtain $(y, x) \in R_L$ by definition and thus a contradiction to our hypothesis that $(x, y) \in P(R_L)$. Hence $(x, y) \in P(R_L^n)$.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_L^n)$. Suppose there exists $m \in \mathbb{N}$ such that $(y, x) \in R_L^m$. Because $(x, y) \in P(R_L^n)$, (4) implies $n > m$. But then (3) implies $(y, x) \in R_L^n$, a contradiction. We conclude that $(x, y) \in R_L^n$ and $(y, x) \notin R_L^m$ for all $m \in \mathbb{N}$. By definition, this implies $(x, y) \in P(R_L)$.

Now let R be an ordering extension of R_L. We complete the proof of the 'if' part by showing that R satisfies the required axioms.

To establish that strong Pareto is satisfied, suppose that $x, y \in X$ are such that $x > y$. Let $n = \min\{m \in \mathbb{N} \mid x_m > y_m\}$. By definition, $(x, y) \in P(R_L^n)$. By (7), $(x, y) \in P(R_L)$ and, because R is an ordering extension of R_L, we obtain $(x, y) \in P(R)$.

Next, we show that finite anonymity is satisfied. Let $x \in X$ and let ρ be a finite permutation of $\mathbb{N}$. By definition, there exists $m \in \mathbb{N}$ such that $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. By definition of R_L^m, $(B^\rho x, x) \in I(R_L^m)$. By definition of R_L, this implies $(B^\rho x, x) \in I(R_L)$. Because R is an ordering extension of R_L, we obtain $(B^\rho x, x) \in I(R)$.

Finally, we show that equity preference is satisfied. Consider $x, y \in X$ and $n, m \in \mathbb{N}$ such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m > x_n > y_n$. Let $j = \max\{n, m\}$. By definition of R_L^j, we obtain $(x, y) \in R_L^j$. By (7), $(x, y) \in R_L$ and, because R is an ordering extension of R_L, $(x, y) \in R$.

'Only if.' Suppose R is an ordering on X satisfying the three axioms of the theorem statement. Fix $n \in \mathbb{N}$ and $z \in X$ and define the relation $Q(n, z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ as follows. For all $x, y \in X$,

$$(x_{-n}, y_{-n}) \in Q(n, z) \Leftrightarrow ((x_{-n}, z_{+n}), (y_{-n}, z_{+n})) \in R.$$

$Q(n, z)$ is an ordering because R is. Furthermore, it is clear that

$$(x_{-n}, y_{-n}) \in P(Q(n, z)) \Leftrightarrow ((x_{-n}, z_{+n}), (y_{-n}, z_{+n})) \in P(R) \tag{8}$$

for all $x, y \in X$. The three axioms imply that $Q(n, z)$ must satisfy the n-person versions of the axioms and, using Hammond's (1976, Theorem 7.2) characterization of n-person leximin (see also d'Aspremont and Gevers, 1977, Theorem 5), it follows that

$$Q(n, z) = R_\ell^n. \tag{9}$$

Because n and z were chosen arbitrarily, (9) is true for all $n \in \mathbb{N}$ and for any $z \in X$.

By way of contradiction, suppose R is not an ordering extension of R_L. There are two possible cases.

Case 1. There exist $x, y \in X$ such that $(x, y) \in R_L$ and $(y, x) \in P(R)$. By definition of R_L, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_L^n$, that is,

$$(x_{-n}, y_{-n}) \in R_\ell^n \text{ and } x_{+n} \geq y_{+n}.$$

Hence, by (9),

$$(x_{-n}, y_{-n}) \in Q(n, z) \text{ and } x_{+n} \geq y_{+n}$$

for all $z \in X$. Choosing $z = y$ and using the definition of $Q(n, z)$, it follows that $((x_{-n}, y_{+n}), (y_{-n}, y_{+n})) \in R$. Because $x_{+n} \geq y_{+n}$, reflexivity (if $x_{+n} = y_{+n}$) or the conjunction of strong Pareto and transitivity (if $x_{+n} > y_{+n}$) implies $((x_{-n}, x_{+n}), (y_{-n}, y_{+n})) = (x, y) \in R$, a contradiction.

Case 2. There exist $x, y \in X$ such that $(x, y) \in P(R_L)$ and $(y, x) \in R$. By (7), there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_L^n)$. Thus, (5) or (6) is true. If (5) holds, (9) implies

$$(x_{-n}, y_{-n}) \in P(Q(n, z)) \text{ and } x_{+n} \geq y_{+n}$$

for all $z \in X$. Setting $z = y$ and using (8), we obtain $((x_{-n}, y_{+n}), (y_{-n}, y_{+n})) \in P(R)$ and, using reflexivity or strong Pareto and transitivity as in case 1, we obtain $(x, y) \in P(R)$, a contradiction. If (6) holds, we proceed as in case 1. ■

5 Concluding remarks

The results of this paper reinforce the findings of earlier contributions regarding the existence of orderings of infinite utility streams with attractive properties. In particular, we provide characterizations of two classes of such orderings. Given the nature of the proofs, we do not provide explicit constructions of these orderings. However, this feature is by no means unique to our approach. Extending quasi-orderings to orderings often requires non-constructive techniques; see, for example, Richter's (1966) use of Szpilrajn's (1930) extension theorem in the context of rational choice.

A plausible conclusion to be drawn is that impossibility results such as those of Diamond (1965), Basu and Mitra (2003a) and Suzumura and Shinotsuka (2003) can be avoided if continuity or representability assumptions are dispensed with. Because continuity and representability can be considered rather demanding in infinite-horizon settings, this confirms, in our view, that the state of affairs in this area is not as disappointing and negative as has been suggested by the impossibility results of many earlier contributions.

The technique employed to characterize infinite-horizon version of leximin appears to be very powerful and applicable to the extension of other finite-population social-choice rules; see also the characterization of infinite-horizon utilitarianism by Basu and Mitra (2003b). We hope that our approach will stimulate further research in the area of intergenerational social choice by identifying alternative sets of attractive axioms and characterizing the social orderings that satisfy them.

The classes of orderings characterized in this paper are relatively large: there are many comparisons of utility streams that are not determined by the axioms employed. An issue to be addressed in future work is to examine to what extent the ranking of more pairs of streams can be determined by employing plausible additional axioms.

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