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# Immediately Reactive Equilibria in Infinitely Repeated Games with Additively Separable Continuous Payoffs

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# Immediately Reactive Equilibria in Infinitely Repeated Games with Additively Separable Continuous Payoffs*

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**Abstract**

This paper studies a class of infinitely repeated games with two players in which the action space of each player is an interval, and the one-shot payoff of each player is additively separable in their actions. We define an immediately reactive equilibrium (IRE) as a pure-strategy subgame perfect equilibrium such that the action of each player is a stationary function of the last action of the other player. We show that the set of IREs in the simultaneous move game is identical to that in the alternating move game. In both games, IREs are completely characterized in terms of indifference curves associated with what we call effective payoffs. A folk-type theorem using only IREs is established in a special case. Our results are applied to a prisoner's dilemma game with observable mixed strategies and a duopoly game. In the latter game, kinked demand curves with a globally stable steady state are derived.

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# 1 Introduction

The standard folk theorem (Fudenberg and Maskin, 1986) indicates that there are typically numerous subgame perfect equilibria in infinitely repeated games. Facing many possibilities, one often focuses on stationary, or time-invariant, subgame perfect equilibria. However, even such equilibria are often difficult to fully characterize. For example, in games with two players, each player chooses a stationary function as his strategy, taking the stationary function of the other player as given (e.g., Maskin and Tirole, 1988a, 1988b); thus one faces a complicated functional fixed point problem.[1] This paper shows that in a certain class of infinitely repeated games, the problem becomes manageable, and one can obtain a complete and graphical characterization of a specific class of nontrivial stationary equilibria.

Specifically, we consider a class of infinitely repeated games in which the action space of each player is an interval, and the one-shot payoff of each player is additively separable in their actions. These assumptions hold for various games, including a prisoner's dilemma with observable mixed strategies and a two country model of international trade in which each country sets its tariff rate to maximize the sum of tariff revenue, consumer surplus, and producer surplus;[2] these and other examples are discussed in Subsection 2.2. It may also be reasonable to assume additively separable payoffs when the sign of their cross partial derivatives is not clear. In this paper we explore in depth the structure implied by additively separable payoffs for both the simultaneous and the alternating move infinitely repeated games. In addition to additive separability, we assume that each player's one-shot payoff is continuous, monotone in the other player's action, and monotone or unimodal in his own action.

We focus on the class of pure-strategy subgame perfect equilibria in which each player's action is a stationary function of the other player's last action. We call such an equilibrium an immediately reactive equilibrium (IRE). One of the reasons why IREs may be of general interest is that they seem to be the natural choice for alternating move games (e.g., Maskin and Tirole,

[1]For alternating move games with two players and with linear-quadratic payoffs, a closed form solution for a Markov perfect equilibrium is available; see Maskin and Tirole (1987) and Lau (1997).

[2]Furusawa and Kamihigashi (2006) study such a model, focusing on issues specific to international trade. A preliminary version of Furusawa and Kamihigashi (2006) contained some of the arguments in this paper, which now appear exclusively in this paper.

1988a; Bhaskar and Vega-Redondo, 2002). Another reason is that IREs are the simplest possible subgame perfect equilibria in which nontrivial dynamic interactions are possible. The simplest possible type of subgame perfect equilibrium is clearly a sequence of static Nash equilibria, which has no dynamic interaction. The natural next step is to allow the action of each player to depend on the last action of the other player in a stationary manner. This is exactly the idea of IRE.

The concept of IRE is related to a few existing ones. First, in the alternating move case, it is consistent with one definition of Markov perfect equilibrium (Maskin and Tirole, 1988a, 1988b) though it is distinct from another (Maskin and Tirole, 2001). Second, it is a special case of single-period-recall equilibrium (Friedman and Samuelson, 1994a). Third, it is also a special case of reactive equilibrium (Kalai, Samet, and Stanford, 1988). More detailed discussions are given in Subsections 2.4 and 2.5.

Under the assumptions mentioned above, we show that the set of IREs in the simultaneous move game is identical to that in the alternating move game. Therefore, as far as IREs are concerned, the choice between simultaneous and alternating moves does not matter.[3] In both games, we completely characterize IREs in terms of indifference curves associated with what we call effective payoffs. The effective payoff of a player is the part of his discounted sum of payoffs that is directly affected by his current action. By additive separability, the effective payoff consists of only two functions. This structure allows us to reduce each player's dynamic maximization problem to a virtually static one.[4]

We show that given a pair of effective payoffs, or the corresponding indifference curves, there is an associated IRE if and only if two conditions are met. First, the intersection of the areas on or above the indifference curves must be nonempty. Second, the lowest point of each indifference curve must not be too low relative to the other indifference curve. It is also shown that in any IRE, the equilibrium paths stay on the associated indifference curves except for the initial period.

[3]Section 4 discusses the relationship of this result to Lagunoff and Matsui's (1997) anti-folk theorem for alternating move games of pure coordination. See Haller and Lagunoff (2000) and Yoon (2001) for further results on alternating move games.

[4]Effective payoffs are similar to what Kamihigashi and Roy (2006) call partial gains in an optimal growth model with linear utility. Equations (2.19)–(2.21) in this paper are similar to (3.7)–(3.9) in Kamihigashi and Roy (2006), but essentially this is the only similarity in analysis between the two papers.

In a special case in which each player's payoff depends monotonically on his own action, we provide a necessary and sufficient condition for an IRE to be "effective efficient," i.e., Pareto optimal among IREs in terms of effective payoffs. The necessary and sufficient condition is that the intersection of the areas strictly above the associated indifference curves be empty. In the same special case, we also obtain the following folk-type theorem: if the discount factors of both players are sufficiently close to one, then any "strictly individually rational" action profile—or any pair of actions in which each player's payoff is strictly greater than his minimax payoff—can be supported as a steady state of an IRE.

Our folk-type theorem is similar in spirit to those shown by Friedman and Samuelson (1994a, 1994b). Their results show that the main idea of the standard folk theorem (Fudenberg and Maskin, 1986) is valid even if one confines oneself to continuous equilibria. In our case the equilibria that we consider are stationary, continuous, and immediately reactive. In addition, the punishment on a deviator is rather minimal since the IREs used in our folk-type theorem have the property that in each period the players are indifferent between conforming to a given equilibrium path and choosing many off-equilibrium alternatives.

We illustrate our results with a prisoner's dilemma game with observable mixed strategies and a rather specific duopoly game. In the prisoner's dilemma game, we show how the structure of IREs changes as the common discount factor increases. In the duopoly game, we show among other things that kinked demand is a necessary feature of effectively efficient IREs. More precisely, in an effectively efficient IRE, there is a unique steady state, which is globally stable. Each firm has a reaction curve kinked at the steady state. If one of the firms raises its price, the other firm does not follow. If one of the firms lowers its price, so does the other firm, but in the long run the prices rise to the steady state levels. While there are game-theoretic models of kinked demand in the literature,[5] they typically require rather specific assumptions. Though our model also requires specific assumptions,[6] it allows one to derive and visualize kinked demand curves as well as equilibrium dynamics in an extremely simple manner.

The rest of the paper is organized as follows. Section 2 describes the one-

[5]See, for example, Maskin and Tiroel (1988b), Radner (2003), and Sen (2004). See Bhaskar, Machin, and Reid (1991) for a survey of earlier theoretical models.

[6]In particular, the owners of the firms are assumed to be "risk-averse."

shot game and our assumptions, discusses several examples, and introduces the simultaneous and the alternating move games. Section 3 characterizes the best responses of a player given the other player's strategy, developing and utilizing various graphical tools. The main result of Section 3 has some immediate implications on IREs, which are shown in Section 4. Section 5 discusses the dynamics induced by IREs. Section 6 gives a complete characterization of IREs. Section 7 characterizes effectively efficient IREs in a special case and shows a folk-type theorem. Section 8 applies our results to a prisoner's dilemma game and a duopoly model. Section 9 concludes the paper. The appendix contains the proof of our main characterization result.

# 2 The Games

## 2.1 The One-Shot Game

Before introducing repeated games, let us describe the one-shot game. There are two players, 1 and 2. Define

$$Q = \{(1,2),(2,1)\}. \tag{2.1}$$

For $(i,j) \in Q$, let $S_i$ denote player $i$'s action space, $\pi_i : S_i \times S_j \to \mathbb{R}$ player $i$'s payoff. The following assumptions are maintained throughout.

**Assumption 2.1.** For $i = 1,2$, $S_i \subset \mathbb{R}$ is an interval with nonempty interior.

**Assumption 2.2.** For $(i,j) \in Q$, there exist $u_i : S_i \to [-\infty,\infty)$ and $v_i : S_j \to \mathbb{R}$ such that

$$\forall (s_i,s_j) \in S_i \times S_j, \quad \pi_i(s_i,s_j) = u_i(s_i) + v_i(s_j). \tag{2.2}$$

**Assumption 2.3.** $v_1$ and $v_2$ are continuous. Either both are strictly increasing or both are strictly decreasing.

**Assumption 2.4.** For $i = 1,2$, $u_i$ is continuous, and there exists $\hat{s}_i \in S_i$ such that $u_i$ is strictly increasing on $S_i \cap (-\infty,\hat{s}_i)$ provided $S_i \cap (-\infty,\hat{s}_i) \neq \emptyset$, and strictly decreasing on $S_i \cap (\hat{s}_i,\infty)$ provided $S_i \cap (\hat{s}_i,\infty) \neq \emptyset$.[7]

**Assumption 2.5.** For $i = 1,2$, $u_i$ is bounded above, and $v_i$ is bounded.

[7]We follow the convention that if $u_i(r) = -\infty$ for some $r \in S_i$, then $u_i$ is continuous at $r$ if $\lim_{s_i \to r} u_i(s_i) = -\infty$. Such $r$ can only be $\min S_i$ or $\max S_i$ by Assumption 2.4.

Assumption 2.2 is our key assumption. Assumptions 2.4 and 2.5 imply that $(\hat{s}_1, \hat{s}_2)$ is the unique static Nash equilibrium. Assumption 2.2 allows $u_i$ to be unbounded below because such cases are common in economic models.

## 2.2 Examples

Though Assumption 2.2 may appear rather strong as a restriction on general games with two players, it is satisfied in various games. We provide specific examples below. Our intention here is not to claim that our assumptions are general, but to suggest that our framework is useful in analyzing certain types of games as well as special cases of more general games.

### 2.2.1 Tariff War

Consider a two country world in which the payoff of each country is given by the sum of its tariff revenue, consumer surplus, and producer surplus. Each country is better off if the other country reduces its tariff rate, while each country has an incentive to choose the tariff rate that maximizes the sum of its tariff revenue and consumer surplus. To be more specific, let $\hat{s}_i$ be this maximizing tariff rate, and $s_i$ be country $i$'s tariff rate imposed on imports from country $j$. Under standard assumptions, country $i$'s producer surplus is strictly decreasing in $s_j$, while the sum of its tariff revenue and consumer surplus is strictly increasing in $s_i$ for $s_i \leq \hat{s}_i$ and strictly decreasing for $s_i \geq \hat{s}_i$. This game satisfies our assumptions, and is analyzed in detail in Furusawa and Kamihigashi (2006).

### 2.2.2 Aggregative Games

Consider a game in which the payoff of player $i$ can be written as a function of $s_i$ and $s_i + s_j$, i.e., $\pi_i(s_i, s_j) = \tilde{\pi}_i(s_i, s_i + s_j)$ for some $\tilde{\pi}_i$. This type of game is called an aggregative game (Corchon, 1994). For example, $s_i$ can be player $i$'s contribution to a public good, or his pollution emission. If $\tilde{\pi}$ is additively separable and depends linearly on $s_i + s_j$, then there are various cases in which our assumptions are satisfied.

### 2.2.3 Bertrand Competition

Consider a game played by two firms, each producing a differentiated product with a constant marginal cost $c_i$ and no fixed cost. Firm $i$ faces a demand function $D_i(p_i, p_j)$ that depends on the prices $p_i$ and $p_j$ chosen by the two firms. Firm $i$'s profit is $D_i(p_i, p_j)(p_i - c_i)$. Suppose $D_i$ is multiplicatively separable: $D_i(p_i, p_j) = d_i^i(p_i)d_i^j(p_j)$ for some functions $d_i^i$ and $d_i^j$. Then the profit maximization problem of firm $i$ is equivalent to maximizing $u_i(p_i) + v_i(p_j)$, where

$$u_i(p_i) = \ln d_i^i(p_i) + \ln(p_i - c_i), \quad v_i(p_j) = \ln d_i^j(p_j). \tag{2.3}$$

This transformation is innocuous in the one-shot game, and our assumptions are satisfied under reasonable assumptions on $d_i^i$ and $d_i^j$. In repeated games the above transformation may be justified by assuming that the owners of the firms are "risk averse" or, more precisely, prefer stable profit streams to unstable ones.

### 2.2.4 Prisoner's Dilemma

Though the action spaces are assumed to be intervals in this paper, our framework applies to $2\times2$ games with mixed strategies. A case in point is the prisoner's dilemma game in Figure 1 (with $a, c > 0$), which is a parametrized version of the game discussed by Fudenberg and Tirole (1991, p. 10, p. 111). For $i = 1, 2$, let $s_i$ be player $i$'s probability of choosing action C. Let $\pi_i(s_i, s_j)$ be player $i$'s expected payoff:

$$\pi_i(s_i, s_j) = s_i s_j c + s_i(1 - s_j)(-a) + (1 - s_i)s_j(c + a) \tag{2.4}$$
$$= -as_i + (c + a)s_j. \tag{2.5}$$

Let $S_1 = S_2 = [0, 1]$. Then all our assumptions are clearly satisfied with $\hat{s}_1 = \hat{s}_2 = 0$.[8]

### 2.2.5 General $2 \times 2$ Games

The preceding example suggests that our framework applies to more general $2 \times 2$ games. To see this, consider the $2 \times 2$ game in Figure 2. For $i = 1, 2$, let $s_i$ be player $i$'s probability of choosing action 1. Let $\pi_i(s_i, s_j)$ be player

[8] Furusawa and Kawakami (2006) use a payoff function similar to (2.4) to analyze perfect Bayesian equilibria in a model with stochastic outside options.

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | C | D |
| Player 1 | C | $c, c$ | $-a, c+a$ |
|  | D | $c+a, -a$ | $0, 0$ |

Figure 1: Prisoner's dilemma

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | 1 | 2 |
| Player 1 | 1 | $p_1^{11}, p_2^{11}$ | $p_1^{12}, p_2^{12}$ |
|  | 2 | $p_1^{21}, p_2^{21}$ | $p_1^{22}, p_2^{22}$ |

Figure 2: General $2 \times 2$ game

$i$'s expected payoff:

$$\pi_i(s_i, s_j) \tag{2.6}$$
$$= s_i s_j p_i^{11} + s_i(1-s_j)p_i^{12} + (1-s_i)s_j p_i^{21} + (1-s_i)(1-s_j)p_i^{22} \tag{2.7}$$
$$= (p_i^{12} - p_i^{22})s_i + (p_i^{21} - p_i^{22})s_j + (p_i^{11} - p_i^{12} - p_i^{21} + p_i^{22})s_i s_j + p_i^{22}. \tag{2.8}$$

It is easy to see that all our assumptions hold if and only if $p_i^{12} \neq p_i^{22}$, $(p_1^{21} - p_1^{22})(p_2^{21} - p_2^{22}) > 0$, and $p_i^{11} - p_i^{12} - p_i^{21} + p_i^{22} = 0$. The last condition suggests some form of additive separability. For example, it can be written as $p_i^{11} - p_i^{21} = p_i^{12} - p_i^{22}$, i.e., player $i$'s choice has the same effect on his payoff independently of player $j$'s action. Alternatively, it can be written as $p_i^{11} - p_i^{12} = p_i^{21} - p_i^{22}$, i.e., player $j$'s choice has the same effect on player $i$'s payoff independently of player $i$'s action.

## 2.3 Normalizing Assumptions

To simplify the exposition, we introduce some assumptions that can be made without loss of generality.

**Assumption 2.6.** For $i = 1, 2, \inf S_i = 0$ and $\sup S_i = 1$.

This can be assumed without loss of generality since none of our assumptions is affected by strictly increasing, continuous transformations of $S_i$. If $0 \notin S_i$ and/or $1 \notin S_i$, we extend $u_i$ and $v_i$ to 0 and/or 1 as follows:

$$u_i(0) = \lim_{s \downarrow 0} u_i(s), \quad u_i(1) = \lim_{s \uparrow 1} u_i(s), \tag{2.9}$$

$$v_i(0) = \lim_{s \downarrow 0} v_i(s), \quad v_i(1) = \lim_{s \uparrow 1} v_i(s). \tag{2.10}$$

By the above and Assumption 2.5, the following can be assumed without loss of generality.

**Assumption 2.7.** For $i = 1, 2$, $u_i : [0, 1] \to [-\infty, \infty)$ and $v_i : [0, 1] \to \mathbb{R}$ are continuous, and $u_i((0, 1]), v_i([0, 1]) \subset \mathbb{R}$.

Strictly speaking, the next assumption is not a normalization, but it is innocuous and is made merely for notational simplicity.[9]

**Assumption 2.8.** $S_1 = S_2 = [0, 1]$.

The following is our last normalizing assumption.

**Assumption 2.9.** For $i = 1, 2$, $v_i$ is strictly increasing.

To see that this is a normalization, suppose $v_1$ and $v_2$ are both strictly decreasing (recall Assumption 2.3). For $(i, j) \in Q$, define $\tilde{S}_i = [0, 1], \tilde{s}_i = 1 - s_i, \tilde{u}_i(\tilde{s}_j) = u_i(1 - \tilde{s}_j)$, and $\tilde{v}_i(\tilde{s}_i) = v_i(1 - \tilde{s}_i)$. Then $\tilde{v}_1$ and $\tilde{v}_2$ are strictly increasing, and $\tilde{u}_1, \tilde{v}_1, \tilde{S}_1, \tilde{u}_2, \tilde{v}_2$, and $\tilde{S}_2$ satisfy all the other assumptions.

## 2.4 The Repeated Game with Simultaneous Moves

Consider the infinitely repeated game in which the stage game is given by the one-shot game defined above. For $i = 1, 2$, let $\delta_i \in (0, 1)$ be player $i$'s discount factor. We restrict ourselves to pure-strategy subgame perfect equilibria in which player $i$'s action in period $t$, $s_{i,t}$, is a stationary function $f_i$ of player $j$'s action in period $t - 1$, $s_{j,t-1}$. Such strategies are a special case of single-period-recall strategies (Friedman and Samuelson, 1994a) and reactive strategies (Kalai et al., 1988). Single-period-recall strategies depend

[9] It is innocuous because removing 0 and/or 1 from $S_i$ does not affect our analysis.

only on both players' last actions, and reactive strategies depend only on the other player's past actions. We focus on stationary strategies that depend only on the other player's last action.

Let $F$ be the set of all functions from $[0,1]$ to $[0,1]$. Let $(i,j) \in Q$. Taking player $j$'s strategy $f_j \in F$ as given, player $i$ faces the following problem:

$$\max_{\{s_{i,t}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta_i^{t-1}[u_i(s_{i,t}) + v_i(s_{j,t})] \tag{2.11}$$

$$\text{s.t.} \quad \forall t \in \mathbb{N}, \quad s_{j,t} = f_j(s_{i,t-1}), \tag{2.12}$$

$$\forall t \in \mathbb{N}, \quad s_{i,t} \in [0,1]. \tag{2.13}$$

We say that $f_i \in F$ is a *best response to* $f_j$ if for any $(s_{i,0}, s_{j,0}) \in [0,1]^2$, the above maximization problem has a solution $\{s_{i,t}\}_{t=1}^{\infty}$ such that $s_{i,t} = f_i(s_{j,t-1})$ for all $t \in \mathbb{N}$. We call a strategy profile $(f_1, f_2) \in F^2$ an *immediately reactive equilibrium (IRE)* if $f_1$ is a best response to $f_2$ and vice versa. Note that $f_1$ and $f_2$ are not required to be continuous or even measurable, but the maximization problem (2.11)–(2.13) is required to be well defined given $f_j$.[10]

## 2.5 The Repeated Game with Alternating Moves

Now consider the case of alternating moves. Player 1 updates his action in odd periods, while player 2 updates his action in even periods.[11] Define

$$T_1 = \{1, 3, 5, \cdots\}, \quad T_2 = \{2, 4, 6, \cdots\}. \tag{2.14}$$

As in the simultaneous move case, we restrict ourselves to subgame perfect equilibria in which in each period $t \in T_i$, player $i$ chooses an action $s_{i,t}$ according to a stationary function $f_i$ of player $j$'s last (or equivalently current) action $s_{j,t-1}$.

Let $(i,j) \in Q$. Given player $j$'s strategy $f_j \in F$, player $i$ faces the

[10] Our results are unaffected even if $f_1$ and $f_2$ are required to be continuous or upper semi-continuous. The same remark applies to the alternating move game.

[11] In alternating move games, it is often assumed that the players play simultaneously in the initial period and take turns afterwards. Such an assumption does not affect our analysis, which is concerned only with stationary subgame perfect equilibria.

following problem:

$$\max_{\{s_{i,t}\}_{t=1}^{\infty}} \sum_{t=i}^{\infty} \delta_i^{t-i}[u_i(s_{i,t}) + v_i(s_{j,t})] \tag{2.15}$$

$$\text{s.t.} \quad \forall t \in T_j, \quad s_{j,t} = f_j(s_{i,t-1}),\ s_{i,t} = s_{i,t-1}, \tag{2.16}$$

$$\forall t \in T_i, \quad s_{i,t} \in [0,1],\ s_{j,t} = s_{j,t-1}. \tag{2.17}$$

We say that $f_i \in F$ is a *best response to* $f_j$ if for any $s_{j,i-1} \in [0,1]$,[12] the above maximization problem has a solution $\{s_{i,t}\}_{t=1}^{\infty}$ such that $s_{i,t} = f_i(s_{j,t-1})$ for all $t \in T_i$. We call a strategy profile $(f_1, f_2) \in F^2$ an *immediately reactive equilibrium (IRE)* if $f_1$ is a best response to $f_2$ and vice versa. This equilibrium concept is consistent with one definition of Markov perfect equilibrium (Maskin and Tirole, 1988b, Section 2), but distinct from another (Maskin and Tirole, 2001) due to additive separability of payoffs.

## 2.6 Effective Payoffs

We now introduce a function that plays a central role in our analysis. For $(i,j) \in Q$, define $w_i : [0,1]^2 \to \mathbb{R}_+$ by

$$w_i(s_i, s_j) = u_i(s_i) + \delta_i v_i(s_j). \tag{2.18}$$

We call this function player $i$'s *effective payoff* since in both repeated games, player $i$ in effect seeks to maximize the discounted sum of effective payoffs. Indeed, in both games, player $i$'s discounted sum of payoffs from period 1 onward is written as

$$\sum_{t=1}^{\infty} \delta_i^{t-1}[v_i(s_{j,t}) + u_i(s_{i,t})] \tag{2.19}$$

$$= v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1}[u_i(s_{i,t}) + \delta_i v_i(s_{j,t+1})] \tag{2.20}$$

$$= v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} w_i(s_{i,t}, s_{j,t+1}). \tag{2.21}$$

In both games, player $i$ has no influence on $s_{j,1}$, so that player $i$'s problem is equivalent to maximizing the discounted sum of effective payoffs.

[12]Notice that for $i = 1, 2$, the first period in which player $i$ plays is period $i$.

# 3 Characterizing Best Responses

Let $(i, j) \in Q$. This section takes player $j$'s strategy $f_j \in F$ as given, and studies player $i$'s best responses. We show first a simple result that characterizes them. The purpose of this section is to reexpress the result in terms of indifference curves associated with effective payoffs so as to obtain a graphical understanding of player $i$'s problem. The following result characterizes player $i$'s best responses in both the simultaneous and alternating move games.

**Proposition 3.1.** *In both the simultaneous and the alternating move games, $f_i \in F$ is a best response to $f_j$ if and only if*

$$\forall s_j \in [0,1], \quad f_i(s_j) \in \operatorname*{argmax}_{s_i \in [0,1]} w_i(s_i, f_j(s_i)) \equiv M(f_j). \tag{3.1}$$

*Proof.* Consider the simultaneous move game. From (2.19)–(2.21) and (2.12), player $i$'s discounted sum of payoffs is written as

$$\sum_{t=1}^{\infty} \delta_i^{t-1}[u_i(s_{i,t}) + v_i(s_{j,t})] = v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} w_i(s_{i,t}, f_j(s_{i,t})). \tag{3.2}$$

Thus the maximization problem (2.11)–(2.13) is equivalent to maximizing the right-hand side of (3.2), which is maximized if and only if $s_{i,t} \in M(f_j)$ for all $t \in \mathbb{N}$. Therefore, if $f_i \in F$ is a best response, then $f_i(s_{j,0}) \in M(f_j)$ for all $s_{j,0} \in [0,1]$; thus (3.1) holds. Conversely, if $f_i \in F$ satisfies (3.1), then it is a best response since $s_{i,t} = f_i(s_{j,t-1}) \in M(f_j)$ for all $t \in \mathbb{N}$.

Now consider the alternating move game. From (2.19)–(2.21), (2.16), and (2.17), player $i$'s discounted sum of payoffs from period $i$ onward is written as

$$\sum_{t=i}^{\infty} \delta_i^{t-i}[u_i(s_{i,t}) + v_i(s_{j,t})] \tag{3.3}$$

$$= v_i(s_{j,i}) + \sum_{t \in T_i} \delta_i^{t-i}(1 + \delta_i) w_i(s_{i,t}, s_{j,t+1}) \tag{3.4}$$

$$= v_i(s_{j,i}) + (1 + \delta_i) \sum_{t \in T_i} \delta_i^{t-1} w_i(s_{i,t}, f_j(s_{i,t})). \tag{3.5}$$

Thus the maximization problem (2.15)–(2.17) is equivalent to maximizing the right-hand side of (3.5), which is maximized if and only if $s_{i,t} \in M(f_j)$ for all $t \in T_i$. Hence the proposition follows as in the simultaneous move case. □

To translate the above result into more usable forms, define

$$R(f_i) = \{f_i(s_j) \mid s_j \in [0,1]\}. \tag{3.6}$$

Note that $R(f_i)$ is the range of $f_i$. The following is a simple restatement of Proposition 3.1.

**Corollary 3.1.** *$f_i \in F$ is a best response to $f_j$ if and only if $R(f_i) \subset M_i(f_j)$.*

This result can be better understood in terms of indifference curves associated with effective payoffs. Since $v_i$ is strictly increasing by Assumption 2.9, each indifference curve $w_i(s_i, s_j) = \omega$ can be expressed as the graph of a function from $s_i$ to $s_j$. We denote this function by $g_j^\omega$, i.e.,

$$\omega = w_i(s_i, g_j^\omega(s_i)) = u_i(s_i) + \delta_i v_i(g_j^\omega(s_i)). \tag{3.7}$$

Depending on $s_i$ and $\omega$, however, $g_j^\omega(s_i)$ may or may not be defined. We specify the domain of $g_j^\omega$, denoted $D(g_j^\omega)$, as follows:

$$D(g_j^\omega) = \{s_i \in [0,1] \mid \exists s_j \in [0,1], u_i(s_i) + \delta_i v_i(s_j) = \omega\} \tag{3.8}$$
$$= \{s_i \in [0,1] \mid \omega - \delta_i v_i(1) \le u_i(s_i) \le \omega - \delta_i v_i(0)\}. \tag{3.9}$$

See Figure 3. It follows from (3.7) that

$$\forall s_i \in D(g_j^\omega), \quad g_j^\omega(s_i) = v_i^{-1}\left(\frac{\omega - u_i(s_i)}{\delta_i}\right). \tag{3.10}$$

The following lemma collects useful observations on $g_j^\omega$.

**Lemma 3.1.** *Let $\Omega = [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1)]$.* (i) *For $\omega \in \Omega, g_j^\omega(\cdot)$ is continuous on $D(g_j^\omega)$, $\hat{s}_i \in D(g_j^\omega)$, and $D(g_j^\omega)$ is a nonempty closed interval.* (ii) *If $\omega, \omega' \in \Omega$ with $\omega < \omega'$, then $D(g_j^\omega) \subset D(g_j^{\omega'})$ and*

$$\forall s_i \in D(g_j^{\omega'}), \quad g_j^\omega(s_i) < g_j^{\omega'}(s_i). \tag{3.11}$$

(iii) *Let $\omega \in [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1))$. Then $D(g_j^\omega)$ is a closed interval with nonempty interior. Furthermore, $g_j^\omega(\cdot)$ is strictly decreasing on $D(g_j^\omega) \cap [0, \hat{s}_i]$ provided $\hat{s}_i > 0$, and strictly increasing on $D(g_j^\omega) \cap [\hat{s}_i, 1]$ provided $\hat{s}_i < 1$.*

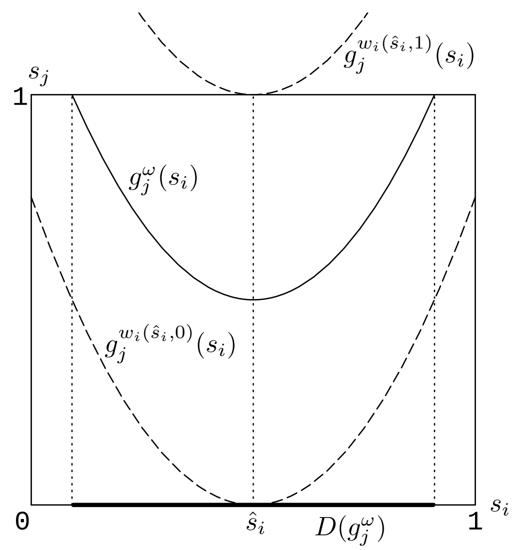



Figure 3: Indifference curve $g_j^{\omega}$ and $D(g_j^{\omega})$

*Proof.* Let $\omega \in \Omega$. The continuity of $g_j^{\omega}$ is obvious. Both inequalities in (3.9) hold with $s_i = \hat{s}_i$ since

$$\omega \leq w_i(\hat{s}_i, 1) = u_i(\hat{s}_i) + \delta_i v_i(1), \tag{3.12}$$

$$\omega \geq w_i(\hat{s}_i, 0) = u_i(\hat{s}_i) + \delta_i v_i(0). \tag{3.13}$$

Hence $\hat{s}_i \in D(g_j^{\omega})$. Note from Assumption 2.4 and (3.13) that

$$\forall s_i \in [0,1], \quad u_i(s_i) \leq u_i(\hat{s}_i) \leq \omega - \delta_i v_i(0). \tag{3.14}$$

Thus by (3.9),

$$D(g_j^{\omega}) = \{s_i \in [0,1] \,|\, \omega - \delta_i v_i(1) \leq u_i(s_i)\}. \tag{3.15}$$

It follows by Assumption 2.4 and (3.12) that $D(g_j^{\omega})$ is a nonempty closed interval. We have verified (i).

To see (ii), note that the inequality in (3.11) is immediate from (3.10) for $s_i \in D(g_j^{\omega}) \cap D(g_j^{\omega'})$. Thus it suffices to show $D(g_j^{\omega'}) \subset D(g_j^{\omega})$. Let $s_i \in D(g_j^{\omega'})$. Then the inequality in (3.15) holds with $\omega = \omega'$, so it holds for any $\omega \leq \omega'$. It follows that $D(g_j^{\omega'}) \subset D(g_j^{\omega})$.

To see (iii), let $\omega \in [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1))$. By (i), $D(g_j^{\omega})$ is a nonempty closed interval. Since $\omega < w_i(\hat{s}_i, 1) = u_i(\hat{s}_i) + \delta_i v_i(1)$, i.e., $\omega - \delta_i v_i(1) < u_i(\hat{s}_i)$, it follows by (3.15) that $s_i \in D(g_j^\omega)$ for $s_i$ sufficiently close to $\hat{s}_i$. Thus the first conclusion in (i) holds. The second conclusion is immediate from (3.10) and Assumptions 2.9 and 2.4. □

To understand Corollary 3.1 in terms of indifference curves $g_j^\omega$, define

$$w_i^*(f_j) = \sup_{s_i \in [0,1]} w_i(s_i, f_j(s_i)). \tag{3.16}$$

Since $v_i$ is strictly increasing by Assumption 2.9,

$$w_i(\hat{s}_i, 0) \leq w_i^*(f_j) \leq w_i(\hat{s}_i, 1). \tag{3.17}$$

By Lemma 3.1(i), a higher indifference curve is associated with a higher effective payoff. Thus by (3.17),

$$g_j^{w_i(\hat{s}_i,0)}(\hat{s}_i) \leq g_j^{w_i^*(f_j)}(\hat{s}_i) \leq g_j^{w_i(\hat{s}_i,1)}(\hat{s}_i). \tag{3.18}$$

See Figure 3 (with $\omega = w_i^*(f_j)$).

Now consider the maximization problem associated with (3.1) (or (3.16)), which can equivalently be expressed as

$$\max_{s_i, s_j \in [0,1]} w_i(s_i, s_j) \quad \text{s.t. } s_j = f_j(s_i). \tag{3.19}$$

The graph $s_j = f_j(s_i)$ represents the set of feasible pairs $(s_i, s_j)$ for player $i$, who takes $s_j = f_j(s_i)$ as a constraint. Since the highest feasible indifference curve is given by $s_j = g_j^{w_i^*(f_j)}(s_i)$,

$$\forall s_i \in D(g_j^{w_i^*(f_j)}), \quad f_j(s_i) \leq g_j^{w_i^*(f_j)}(s_i). \tag{3.20}$$

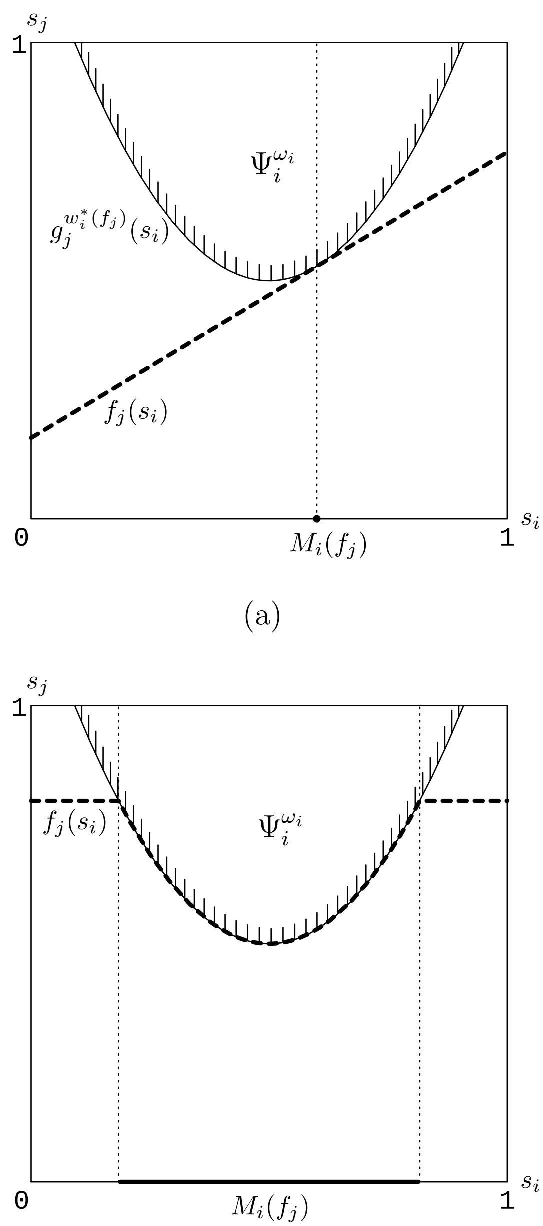
See Figure 4, which shows two ad hoc examples (recall that $f_j$ is arbitrary here). It follows that the solution to (3.19) is to choose any pair $(s_i, s_j)$ satisfying $s_j = f_j(s_i)$ and $s_j = g_j^{w_i^*(f_j)}(s_i)$. More precisely,

$$M_i(f_j) = \{s_i \in D(g_j^{w_i^*(f_j)}) \mid f_j(s_i) = g_j^{w_i^*(f_j)}(s_i)\}. \tag{3.21}$$

See Figure 4 again. By Corollary 3.1 and (3.21),

$$R(f_i) \subset M_i(f_j) \subset D(g_j^{w_i^*(f_j)}), \tag{3.22}$$

whenever $f_i$ is a best response to $f_j$. We are ready to restate Corollary 3.1 in terms of indifference curves $g_j^\omega$.




(a)



(b)

Figure 4: $M_i(f_j)$ and $\Psi_i^{\omega_i}$ (defined in (6.1)) with $\omega_i = w_i^*(f_j)$

**Proposition 3.2.** *$f_i \in F$ is a best response to $f_j$ if and only if*

$$\forall s_i \in R(f_i), \quad s_i \in D(g_j^{w_i^*(f_j)}),\ f_j(s_i) = g_j^{w_i^*(f_j)}(s_i). \tag{3.23}$$

*Proof.* This holds since (3.23) is equivalent to $R(f_i) \subset M_i(f_j)$ by (3.21). $\square$

In Figure 4(a), $M_i(f_j)$ is a singleton, so by Corollary 3.1, there is a unique best response, which is the constant function from $s_j$ to $s_i$ corresponding to the dotted vertical line. This function trivially satisfies (3.23). In Figure 4(b), $M_i(f_j)$ is an interval, so all functions from $s_j$ to $s_i$ whose ranges are confined to that interval are best responses. Notice that all those functions satisfy (3.23).

# 4 Immediate Implications on IREs

The following result is immediate from Corollary 3.1, Proposition 3.2, and the definitions of IRE in Subsections 2.4 and 2.5.

**Theorem 4.1.** *In both the simultaneous and the alternating move games, a strategy profile $(f_1, f_2) \in F^2$ is an IRE if and only if*

$$\forall (i,j) \in Q, \quad R(f_i) \subset M_i(f_j), \tag{4.1}$$

*or, equivalently,*

$$\forall (i,j) \in Q, \forall s_i \in R(f_i), \quad s_i \in D(g_j^{w_i^*(f_j)}),\ f_j(s_i) = g_j^{w_i^*(f_j)}(s_i). \tag{4.2}$$

An important implication of this result is that the simultaneous and the alternating move games are equivalent as far as IREs are concerned. In other words, the choice between simultaneous and alternating moves is unimportant when one restricts oneself to IREs.

This would appear in sharp contrast to the anti-folk theorem of Lagunoff and Matsui (1997) for alternating move games of pure coordination. They showed that there is a considerable difference between the simultaneous and the alternating move games in the case of pure coordination. If $u_i(s_i) = v_j(s_i)$ and $v_i(s_j) = u_j(s_j)$ for all $s_i, s_j \in [0,1]$ and $(i,j) \in Q$, then the one-shot game described in Subsection 2.1 becomes a pure coordination game. Theorem 4.1 of course applies to this case (which is consistent with our assumptions), but does not contradict Lagunoff and Matsui's result. This is because their result

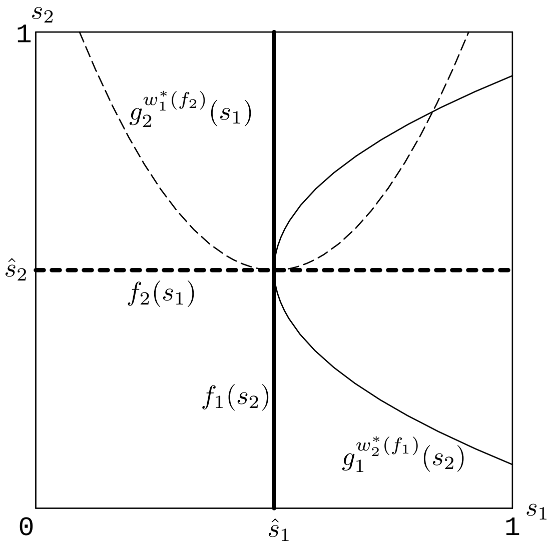



Figure 5: Static Nash equilibrium

deals with all subgame perfect equilibria, while Theorem 4.1 deals only with IREs.[13]

For the remainder of the paper, we do not distinguish between the two games except when we explicitly consider dynamics. The differences in dynamics between the two games are discussed in Section 5.

To illustrate Theorem 4.1, let $(f_1, f_2)$ be given by $f_i(s_j) = \hat{s}_i$ for $s_j \in [0, 1]$ and $(i, j) \in Q$. This strategy profile corresponds to the static Nash equilibrium. Figure 5 shows that $f_2$ and $g_2^{w_1^*(f_2)}$ coincide on $R(f_1)$, which consists only of $\hat{s}_1$, and that likewise $f_1$ and $g_1^{w_2^*(f_1)}$ coincide on $R(f_2)$. Thus $(f_1, f_2)$ satisfies (4.2), so it is an IRE by Theorem 4.1.

To consider less trivial IREs, we define an *IRE associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ as an IRE $(f_1, f_2)$ such that

$$\forall (i, j) \in Q, \quad \omega_i = w_i^*(f_j). \tag{4.3}$$

[13]Theorem 4.1 suggests that the concept of IRE has some resemblance to that of conjectural variation equilibrium. The main difference between the two concepts is that while a conjectural variation equilibrium consists of an equilibrium point and supporting conjectures that are typically required to satisfy certain local properties, an IRE consists of two functions that represent the players' actual reactions. See Tidball et al. (2000) for a survey on conjectural variations.

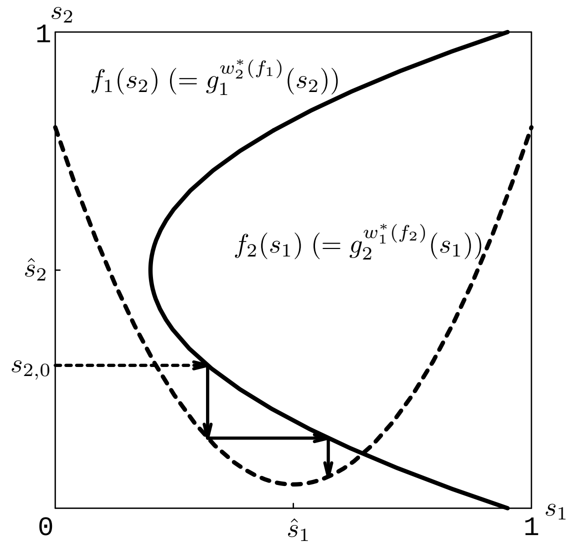



Figure 6: Example of IRE satisfying (4.4) and (4.3).

Notice that any IRE $(f_1, f_2)$ is associated with $(w_1^*(f_2), w_2^*(f_1))$.

**Proposition 4.1.** *Let $\omega_1, \omega_2 \in \mathbb{R}$. Suppose*

$$\forall (i,j) \in Q, \quad D(g_j^{\omega_i}) = [0,1]. \tag{4.4}$$

*Then there exists an IRE associated with $(\omega_1, \omega_2)$. In particular, $(g_1^{\omega_2}, g_2^{\omega_1})$ is such an IRE.*

*Proof.* Let $f_j = g_j^{\omega_i}$ for $(i,j) \in Q$. For $(i,j) \in Q$ and $s_i \in [0,1]$, we have $w_i(s_i, f_j(s_i)) = w_i(s_i, g_j^{\omega_i}(s_i)) = \omega_i$, so $\omega_i = w_i^*(f_j)$. To verify (4.2), let $(i,j) \in Q$ and $s_i \in R(f_j)$. Then $s_i \in [0,1] = D(g_j^{w_i^*(f_j)})$ and $f_j(s_i) = g_j^{w_i^*(f_j)}(s_i)$. Thus (4.2) holds, so $(f_1, f_2)$ is an IRE by Theorem 4.1. □

See Figure 6 for an example of an IRE satisfying (4.4) and (4.3). Since $f_1 = g_1^{w_2^*(f_1)}$ and $f_2 = g_2^{w_1^*(f_2)}$, the example trivially satisfies (4.2).

# 5 Dynamics

Before we turn to a detailed characterization of IREs, it is useful to have a basic understanding of their dynamics. This section takes an IRE $(f_1, f_2) \in F^2$ as given and studies its dynamic properties.

Consider first the alternating move game. Recall that in each period $t \in \mathbb{N}$, player $i$ with $t \in T_i$ updates his action as a function of player $j$'s last (or current) action. So the "state variable" in each period $t \in T_i$ is player $j$'s last action $s_{j,t-1} \in [0,1]$. Given an initial condition $s_{2,0} \in [0,1]$, the entire path $\{s_{1,t}, s_{2,t}\}_{t=1}^{\infty}$ of the game is uniquely determined by

$$\forall (i,j) \in Q, \forall t \in T_i, \quad s_{i,t+1} = s_{i,t} = f_i(s_{j,t-1}). \tag{5.1}$$

In step-by-step form,

$$s_{1,1} = f_1(s_{2,0}),\ s_{2,2} = f_2(s_{1,1}),\ s_{1,3} = f_1(s_{2,2}),\ \cdots . \tag{5.2}$$

For the alternating move game, we define an *IRE path associated with* $(f_1, f_2)$ as a sequence $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ satisfying (5.1). See Figure 6 for an example an IRE path.

Now consider the simultaneous move game. The state variable in each period $t \in \mathbb{N}$ is the pair of both players' last actions $(s_{1,t-1}, s_{2,t-1}) \in [0,1]^2$. Given an initial condition $(s_{1,0}, s_{2,0}) \in [0,1]^2$, the entire path $\{s_{1,t}, s_{2,t}\}_{t=1}^{\infty}$ of the game is uniquely determined by

$$\forall (i,j) \in Q, \forall t \in \mathbb{N}, \quad s_{i,t} = f_i(s_{j,t-1}). \tag{5.3}$$

For the simultaneous move game, we define an *IRE path associated with* $(f_1, f_2)$ as a sequence $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ satisfying (5.3). Any IRE path can be decoupled into two sequences, one originating from $s_{2,0}$, the other from $s_{1,0}$:

$$s_{1,1} = f_1(s_{2,0}),\ s_{2,2} = f_2(s_{1,1}),\ s_{1,3} = f_1(s_{2,2}),\ \cdots , \tag{5.4}$$

$$s_{2,1} = f_2(s_{1,0}),\ s_{1,2} = f_1(s_{2,1}),\ s_{2,3} = f_2(s_{1,2}),\ \cdots . \tag{5.5}$$

Obviously, given $s_{2,0} \in [0,1]$, the sequences given by (5.2) and (5.4) are identical. The sequence given by (5.5) can be viewed as an IRE path for the alternating move game in which player 2 moves first. Hence an IRE path for the simultaneous move game is equivalent to a pair of IRE paths for the two alternating move games in one of which player 1 moves first and in the other of which player 2 moves first.

The following result is a simple consequence of Theorem 4.1.

**Theorem 5.1.** *Any IRE path $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ associated with $(f_1, f_2)$ for the simultaneous move game satisfies*

$$\forall t \geq 2, \forall (i,j) \in Q, \quad s_{i,t} = g_i^{w_j^*(f_i)}(s_{j,t-1}). \tag{5.6}$$

*Furthermore, any IRE path* $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ *associated with* $(f_1, f_2)$ *for the alternating move game satisfies*

$$\forall t \geq 2, \forall (i,j) \in Q, \quad t \in T_i \ \Rightarrow\ s_{i,t} = g_i^{w_j^*(f_i)}(s_{j,t-1}). \tag{5.7}$$

*Proof.* Consider the simultaneous move game. Let $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ be an IRE path associated with $(f_1, f_2)$. Let $(i,j) \in Q$ and $t \geq 2$. Then $s_{j,t-1} \in R(f_j)$. Hence $s_{i,t} = g_i^{w_j^*(f_i)}(s_{j,t-1})$ by (5.3) and (4.2). Thus (5.6) follows. The proof for the alternating move game is similar. □

The above result shows that any IRE path is characterized by the corresponding pair of indifference curves $(g_1^{w_2^*(f_1)}, g_2^{w_1^*(f_2)})$ except for the initial period. To better understand this result, consider the alternating move game. The initial period must be excluded in (5.7) because $s_{2,0}$ is an arbitrary initial condition that need not be optimal for player 2 given $f_1$, i.e., it need not satisfy $s_{1,1} = g_1^{w_2^*(f_1)}(s_{2,0})$. Since all subsequent actions must be individually optimal, they must be on the optimal indifference curves. In Figure 6, any IRE path satisfies the equality in (5.7) for all $t \geq 1$. In Figure 5, by contrast, an IRE path (not shown in the figure) violates the equality for $t = 1$ unless $s_{2,0} = \hat{s}_2$, but trivially satisfies it for $t \geq 2$.

Theorem 5.1 also shows that in both cases the dynamics of an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ are essentially characterized by the same dynamical system:

$$\forall t \in T_1, \quad s_{1,t+2} = g_1^{\omega_2}(g_2^{\omega_1}(s_{1,t})). \tag{5.8}$$

To be precise, the simultaneous move game has another equation, $s_{2,t+2} = g_2^{\omega_1}(g_1^{\omega_2}(s_{2,t}))$ for $t \in T_2$, but this system is equivalent to (5.8) in terms of dynamics. Hence one can obtain conditions for dynamic properties such as monotonicity and chaos by applying numerous results available on one-dimensional dynamical systems (e.g., Devaney, 1989).[14]

# 6 Characterizing IREs

Now we seek to characterize all IREs in terms of effective payoffs, or associated indifference curves. We need additional notation. For $(i,j) \in Q$ and

[14]See Rand (1978) for an early example of complex dynamics in an "adaptive" dynamic model that has a structure similar to Figure 6. See Rosser (2002) for a recent survey of adaptive duopoly/oligopoly models that generate complex dynamics. This paper does not consider complex dynamics, which should be left to more specialized studies.

$\omega_i, \omega_j \in \mathbb{R}$, define

$$\Psi_i^{\omega_i} = \{(s_i, s_j) \in [0,1]^2 \mid w_i(s_i, s_j) \geq \omega_i\} \tag{6.1}$$
$$= \{(s_i, s_j) \in [0,1]^2 \mid s_i \in D(g_j^{\omega_i}), s_j \geq g_j^{\omega_i}(s_i)\}. \tag{6.2}$$

The set $\Psi_i^{\omega_i}$ is the collection of all pairs $(s_i, s_j)$ with player $i$'s effective payoff at least as large as $\omega_i$. In the $(s_i, s_j)$ space, it is the area on or above the graph $s_j = g_j^{\omega_i}(s_i)$; see Figure 4.

Provided $\Psi_i^{\omega_i} \cap \Psi_j^{\omega_j} \neq \emptyset$,[15] define

$$\overline{s}_i^{(\omega_i,\omega_j)} = \max\{s_i \in [0,1] \mid \exists s_j \in [0,1], (s_i, s_j) \in \Psi_i^{\omega_i} \cap \Psi_j^{\omega_j}\}, \tag{6.3}$$
$$\underline{s}_i^{(\omega_i,\omega_j)} = \min\{s_i \in D(g_j^{\omega_i}) \mid g_j^{\omega_i}(s_i) \leq \overline{s}_j^{(\omega_j,\omega_i)}\}. \tag{6.4}$$

By (6.2) and continuity, $g_j^{\omega_i}(\overline{s}_i^{(\omega_i,\omega_j)}) \leq \overline{s}_j^{(\omega_j,\omega_i)}$. Hence $\underline{s}_i^{(\omega_i,\omega_j)}$ exists as long as $\Psi_i^{\omega_i} \cap \Psi_j^{\omega_j} \neq \emptyset$. See Figure 7. In the case of Figure 6, $\overline{s}_1^{(\omega_1,\omega_2)} = \overline{s}_2^{(\omega_2,\omega_1)} = 1$ and $\underline{s}_1^{(\omega_1,\omega_2)} = \underline{s}_2^{(\omega_2,\omega_1)} = 0$. It follows from Lemma 3.1(iii) that

$$\forall (i,j) \in Q, \quad \underline{s}_i^{(\omega_i,\omega_j)} \leq \hat{s}_i \leq \overline{s}_i^{(\omega_i,\omega_j)}. \tag{6.5}$$

See Figure 7 again.[16] The following result characterizes all IREs in terms of effective payoffs.

**Theorem 6.1.** *There exists an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ if and only if*

$$\Psi_1^{\omega_1} \cap \Psi_2^{\omega_2} \neq \emptyset, \tag{6.6}$$
$$\forall (i,j) \in Q, \quad \hat{s}_j \in D(g_i^{\omega_j}),\ \underline{s}_i^{(\omega_i,\omega_j)} \leq g_i^{\omega_j}(\hat{s}_j). \tag{6.7}$$

*In particular, under (6.6) and (6.7), $(f_1, f_2) \in F^2$ is an IRE associated with $(\omega_1, \omega_2)$ if for $(i,j) \in Q$,*

$$f_i(s_j) = \begin{cases} \min\{g_i^{\omega_j}(s_j), \overline{s}_i^{(\omega_i,\omega_j)}\} & \text{if } s_j \in D(g_i^{\omega_j}), \\ \overline{s}_i^{(\omega_i,\omega_j)} & \text{otherwise.} \end{cases} \tag{6.8}$$

[15] Here it is understood that the coordinates of $\Psi_j^{\omega_j}$ (or $\Psi_i^{\omega_i}$) are interchanged so that $\Psi_i^{\omega_i}$ and $\Psi_j^{\omega_j}$ have the same order of the coordinates. Similar comments apply to similar expressions below.

[16] The first inequality in (6.5) is immediate from (6.4) and Lemma 3.1(iii). To formally verify the second inequality, let $(i,j) \in Q$ and suppose $\overline{s}_i < \hat{s}_i$. (We omit superscripts here.) By (6.2) and (6.3), $\overline{s}_j \geq g_j(\overline{s}_i)$ and $\overline{s}_i \geq g_i(\overline{s}_j)$. Since $g_j$ is strictly decreasing at $\overline{s}_i$ by Lemma 3.1(iii), both inequalities continue to hold even if $\overline{s}_i$ is slightly increased, contradicting (6.3).

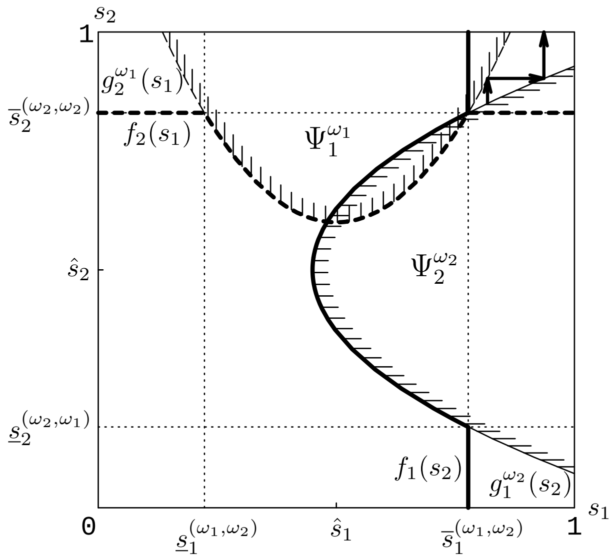


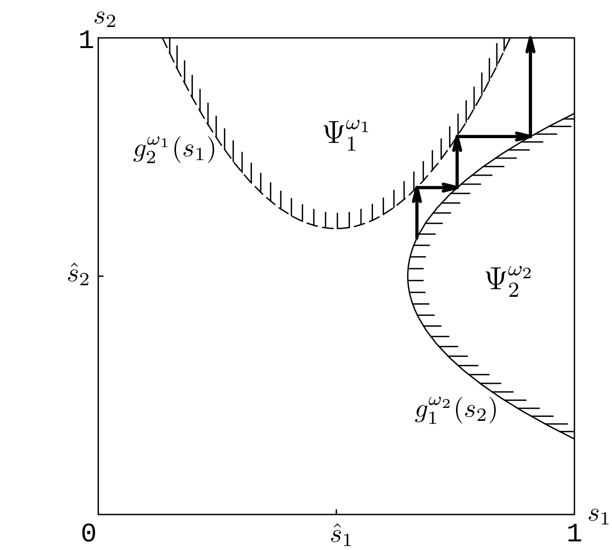

Figure 7: $\underline{s}_i$, $\overline{s}_i$, and IRE given by (6.8)

*Proof.* See Appendix A. □

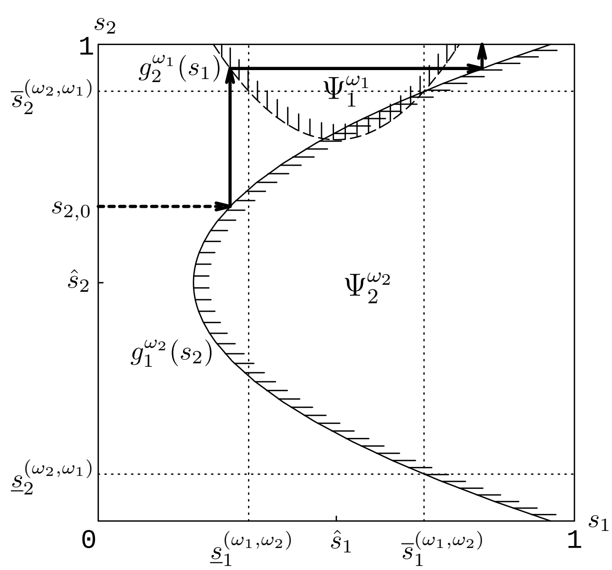
See Figure 7 for an example of an IRE given by (6.8). One can easily see that the example satisfies (4.2). We call (6.6) the *nonemptiness condition*, and (6.7) the *no-sticking-out condition*. The nonemptiness condition says that the intersection of the two sets $\Psi_1^{\omega_1}$ and $\Psi_2^{\omega_2}$ must be nonempty. The no-sticking-out condition says that the graph of $g_i^{\omega_j}$ must not "stick out" of the straight line $s_i = \underline{s}_i$.

These conditions can be better understood by considering examples in which they are violated. In Figure 8(a), the nonemptiness condition (6.6) is violated. In this case, if an IRE exists, any IRE path for the alternating move game must behave like the path depicted in the figure (except for the initial period) by Theorem 5.1. But since such a path cannot stay on the indifference curves forever, it cannot be an IRE path, a contradiction. In Figure 8(b), the no-sticking-out condition (6.7) is violated for $(i, j) = (1, 2)$. In this case, if an IRE exists, there is $s_{2,0}$ such that $f_1(s_{2,0}) \leq g_1^{\omega_2}(s_{2,0}) < \underline{s}_1^{(\omega_1,\omega_2)}$.[17] As shown in the figure, the IRE path from such $s_{2,0}$ cannot stay on the indifference curves forever, contradicting Theorem 5.1.

[17]The first inequality holds by (3.20). In Figure 8(b), $f_1(s_{2,0}) = g_1^{\omega_2}(s_{2,0})$.




(a)




(b)

Figure 8: Examples with no IRE

We should mention that the strategy profile given by (6.8) is not the only IRE under (6.6) and (6.7). In fact, for $s_j \notin [\underline{s}_j^{(\omega_j,\omega_i)}, \overline{s}_j^{(\omega_j,\omega_i)}]$, $f_i(s_j)$ is arbitrary as long as it does not affect $R(f_i)$. However, any IRE satisfies one restriction:

**Proposition 6.1.** *Let $(f_1, f_2)$ be an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$. Then*

$$\forall (i,j) \in Q, \forall s_j \in [0,1], \quad f_i(s_j) \leq \overline{s}_i^{(\omega_i,\omega_j)}. \tag{6.9}$$

*Proof.* Immediate from (A.2), (A.12), and (6.3). □

To see the idea of this result, suppose the inequality in (6.9) is violated for $(i,j) = (1,2)$. Consider the alternating move game. Then for some $s_{2,0}$, $s_{1,1} = f_1(s_{2,0}) > \overline{s}_1^{(\omega_1,\omega_2)}$. If this path is continued, it behaves like the one depicted in Figure 7 by Theorem 5.1. But such a path cannot be an IRE path since it cannot stay on the indifference curves forever.

In what follows, we say that an IRE $(f_1, f_2)$ is *effectively efficient* if there is no IRE $(\tilde{f}_1, \tilde{f}_2)$ such that $w_1^*(f_2) \leq w_1^*(\tilde{f}_2)$ and $w_2^*(f_1) \leq w_2^*(\tilde{f}_1)$ with at least one of them holding strictly. That is, $(f_1, f_2)$ is effectively efficient if it is not Pareto dominated by any other IRE in terms of effective payoffs. For $(i,j) \in Q$ and $\omega_i, \omega_j \in \mathbb{R}$, define

$$\tilde{\Psi}_i^{\omega_i} = \{(s_i, s_j) \in [0,1]^2 \mid w_i(s_i, s_j) > \omega_i\}. \tag{6.10}$$

It is clear from Theorem 6.1 and Lemma 3.1(ii) that an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ is effectively efficient if

$$\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} = \emptyset. \tag{6.11}$$

See Figures 9 and 8(a).

One might conjecture that (6.11) is also necessary for effective efficiency. Unfortunately it is not the case. This is because the no-sticking-out condition (6.7), a necessary condition for an IRE, is not stable under small perturbations to $(\omega_i, \omega_j)$. In other words, even when (6.11) does not hold, (6.7) can be violated if either $\omega_i$ or $\omega_j$ is increased. For example, when (6.7) holds with equality for $(i,j) = (1,2)$, it can be violated after $\omega_2$ is slightly increased, depending on how fast the two sides of the inequality in (6.7) vary with $\omega_2$.

Even if (6.7) holds with strict inequality, (6.7) can be violated after small perturbations to $(\omega_i, \omega_j)$, since $\underline{s}_i^{(\omega_i,\omega_j)}$ need not be continuous in $(\omega_i, \omega_j)$. Figure 10 illustrates this point. There is an IRE in Figure 10(a), but there

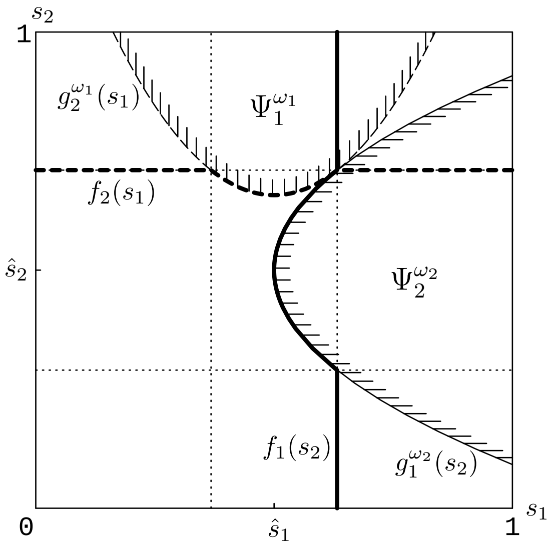



Figure 9: Effectively efficient IRE

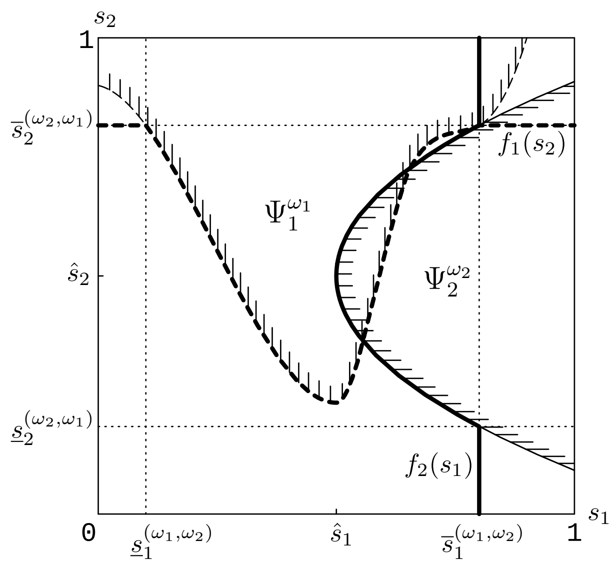
is no IRE in Figure 10(b) due to violation of (6.7). Note that both $\underline{s}_1^{(\omega_1,\omega_2)}$ and $\underline{s}_2^{(\omega_2,\omega_1)}$ are discontinuous in this example. Though in fact Figure 10 shows that the IRE in (a) is only "locally" effectively efficient, it should be clear that one can easily construct a fully specified example of an effectively efficient IRE that violates (6.11).

# 7 Effective Efficiency and a Folk-Type Theorem: A Special Case

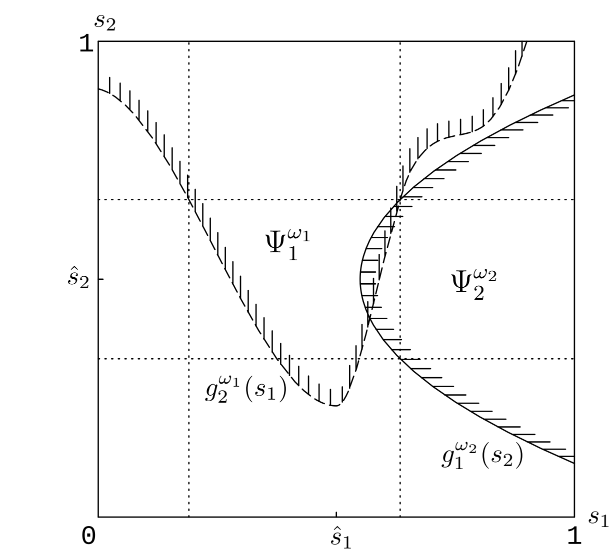
The anomaly in Figure 10 is largely due to the fact that the indifferent curves are unimodal there. The purpose of this section to provide a complete characterization of effective efficiency and to show a folk-type theorem under the assumption that both indifference curves are "upward sloping." More precisely, we focus on the case in which the following assumption holds.

**Assumption 7.1.** For $i = 1, 2$, $\hat{s}_i = 0$ or, equivalently, $u_i$ is strictly decreasing.[18]

[18] If one chooses to normalize $v_i$ in the opposite direction, i.e., if one chooses to assume that $v_i$ is strictly decreasing for $i = 1, 2$ in Assumption 2.9, then the case considered here




(a)




(b)

Figure 10: Effectively efficient IRE violating (6.11)

This assumption holds, for example, in the prisonner's dilemma game in Subsection 2.2.4. More generally, it holds whenever an increase in $s_i$ is costly to player $i$ but beneficial to player $j$. Assumption 7.1 is maintained throughout this section. The following result simplifies Theorem 6.1 and facilitates subsequent analysis.

**Proposition 7.1.** *There exists an IRE associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ *if and only if the nonemptiness condition (6.6) holds and*

$$\forall (i,j) \in Q, \quad 0 \in D(g_i^{\omega_j}). \tag{7.1}$$

*Proof.* By Theorem 6.1, it suffices to show that (7.1) is equivalent to the no-sticking-out condition (6.7) under (6.6). By Assumption 7.1, (6.7) implies (7.1). Conversely, assume (6.6) and (7.1). Let $(i,j) \in Q$. By Assumption 7.1 and (6.5), $\underline{s}_i^{(\omega_i,\omega_j)} = 0$. Since $0 \in D(g_i^{\omega_j})$, we have $\underline{s}_i^{(\omega_i,\omega_j)} = 0 \leq g_i^{\omega_j}(0) = g_i^{\omega_j}(\hat{s}_j)$ by Assumption 7.1. Now (6.7) follows. □

The above proof shows that under (6.6) and (7.1), the inequality in the no-sticking-out condition (6.7) automatically holds. This implies that if an IRE exists such that $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset$, then an IRE continues to exist when both indifference curves are slightly shifted upward. Therefore an IRE cannot be effectively efficient if $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset$. This is the idea of the following result.

**Theorem 7.1.** *Suppose an IRE associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ *exists. Then it is effectively efficient if and only if (6.11) holds, i.e.,* $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} = \emptyset$.

*Proof.* The "if" part is obvious, as mentioned earlier. To see the "only if" part, suppose there is an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$ that is effectively efficient. Suppose (6.11) does not hold, i.e.,

$$\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset. \tag{7.2}$$

Since for $(i,j) \in Q$, $g_j^{\omega_i}(\cdot)$ is strictly increasing by (3.10) and Assumption 7.1, (7.1) and (7.2) imply $0 \leq g_j^{\omega_i}(0) < 1$ for $(i,j) \in Q$. Since $g_j^{\omega_i}$ is continuous and strictly increasing in $\omega_i$ by (3.10), it follows that there is $(\tilde{\omega}_1, \tilde{\omega}_2) \gg (\omega_1, \omega_2)$ such that $0 < g_j^{\tilde{\omega}_i}(0) < 1$ and $\tilde{\Psi}_1^{\tilde{\omega}_1} \cap \tilde{\Psi}_2^{\tilde{\omega}_2} \neq \emptyset$ for $(i,j) \in Q$. Hence (6.6) and (7.1) hold with $\tilde{\omega}_1$ and $\tilde{\omega}_2$ replacing $\omega_1$ and $\omega_2$. But this implies that the given IRE cannot be effectively efficient, a contradiction. □

corresponds to the case $\hat{s}_i = 1$ for $i = 1, 2$.

Let us now develop a folk-type theorem. The following result gives an alternative characterization of IREs that proves useful.

**Proposition 7.2.** *There exists an IRE associated with* $(\omega_1, \omega_2) \in \mathbb{R}^2$ *if and only if there exists* $(s_1, s_2) \in [0, 1]^2$ *such that*

$$\forall (i, j) \in Q, \quad \omega_i = w_i(s_i, s_j), \tag{7.3}$$

$$(s_1, s_2) \in \Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}. \tag{7.4}$$

*Proof.* First we observe from (3.7), (6.1), and Lemma 3.1(ii) that

$$(7.3) \quad \Leftrightarrow \quad \forall (i, j) \in Q, \;\; s_i \in D(g_j^{\omega_i}), s_j = g_j^{\omega_i}(s_i), \tag{7.5}$$

$$(7.3) \quad \Rightarrow \quad (s_1, s_2) \in \Psi_1^{\omega_1} \cap \Psi_2^{\omega_2}. \tag{7.6}$$

*If:* Let $(\omega_1, \omega_2) \in \mathbb{R}^2$. Suppose there exists $(s_1, s_2) \in [0, 1]^2$ satisfying (7.3) and (7.4). Then by (7.6), the nonemptiness condition (6.6) holds. By Proposition 7.1, it suffices to verify (7.1). Let $(i, j) \in Q$. By (7.4) and (6.1), $w_i(s_i, s_j) \geq w_i(0, 0)$. Since $u_i$ is strictly decreasing by Assumption 7.1, $\omega_i = w_i(s_i, s_j) \leq w_i(0, 1)$. It follows that

$$u_i(0) + \delta_i v_i(0) \leq \omega_i \leq u_i(0) + \delta_i v_i(1), \tag{7.7}$$

which can be written as

$$\omega_i - \delta_i v_i(1) \leq u_i(0) \leq \omega_i - \delta_i v_i(0). \tag{7.8}$$

Hence (7.1) holds by (3.9).

*Only if:* Let there be an IRE associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$. By (7.1), $g_j^{\omega_i}(0) \geq 0$ for $(i, j) \in Q$. Thus if the graphs of $g_2^{\omega_1}$ and $g_1^{\omega_2}$ have no intersection, then the nonemptiness condition (6.6) does not hold, a contradiction. Hence the graphs of $g_2^{\omega_1}$ and $g_1^{\omega_2}$ have an intersection $(s_1, s_2) \in [0, 1]^2$, i.e., $s_j = g_j^{\omega_i}(s_i)$ for $(i, j) \in Q$. Thus (7.3) holds by (7.5). We have (7.4) by (7.3), (7.6), (4.3), (3.17), Assumption 7.1, and (6.1). □

See Figure 11 for an illustration of $\Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}$. Note that both indifference curves $g_1^{w_2(0,0)}$ and $g_2^{w_1(0,0)}$ emanate from the origin because they correspond to the effective payoffs associated with the action profile $(0, 0)$.

Given an IRE $(f_1, f_2)$, we say that $(s_1, s_2) \in [0, 1]^2$ is a *steady state* if $s_1 = f_1(s_2)$ and $s_2 = f_2(s_1)$. In other words, any intersection of $f_1$ and $f_2$ is a steady state. Needless to say, the IRE path starting from a steady state remains there forever.

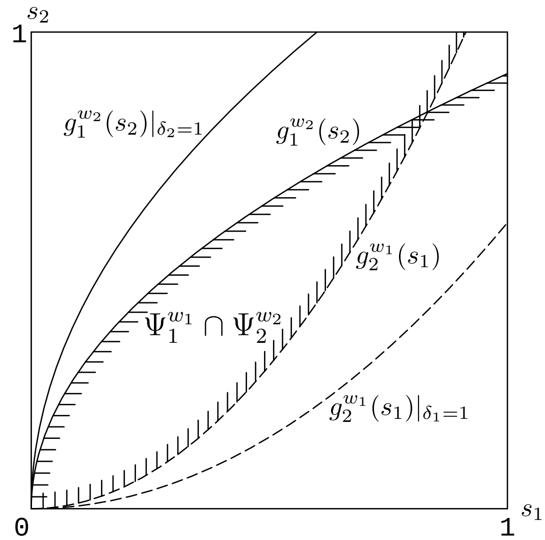



Figure 11: $\Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}$ and lower bounds of $\tilde{\Psi}_i^*$
($w_i = w_i(0,0)$ in this figure)

**Proposition 7.3.** *There exists an IRE such that $(s_1, s_2) \in [0,1]^2$ is a steady state if and only if (7.4) holds.*

*Proof.* Let there be an IRE such that $(s_1, s_2) \in [0,1]^2$ is a steady state. Define $\omega_1$ and $\omega_2$ by (7.3). Then (7.4) holds as in the proof of Proposition 7.2. Conversely, let $(s_1, s_2) \in \mathbb{R}^2$ satisfy (7.4). Define $\omega_1$ and $\omega_2$ by (7.3). By Proposition 7.2, an IRE associated with $(\omega_1, \omega_2)$ exists. Thus (6.6) and (6.7) hold by Theorem 6.1. Let $(f_1, f_2)$ be the IRE given by (6.8). By (7.5), (a) $s_j = g_j^{\omega_i}(s_i)$ for $(i,j) \in Q$. Since $(s_1, s_2) \in \Psi_1^{\omega_1} \cap \Psi_2^{\omega_2}$ by (7.6), (b) $s_i \leq \overline{s}_i^{(\omega_i,\omega_j)}$ for $(i,j) \in Q$. Thus by (a) and (b), for $(i,j) \in Q$, $g_j^{\omega_i}(s_i) = s_j \leq \overline{s}_j^{(\omega_j,\omega_i)}$, so $f_j(s_i) = g_j^{\omega_i}(s_i)$ by (6.8). This together with (a) shows that $(s_1, s_2)$ is a steady state. □

By this result, the set $\Psi_1^{w_1(0,0)} \cap \Psi_2^{w_2(0,0)}$ can be viewed as the collection of all steady states supported by IREs. For $(i,j) \in Q$, define

$$\tilde{\Psi}_i^* = \{(s_i, s_j) \in [0,1]^2 \mid u_i(s_i) + v_i(s_j) > u_i(0) + v_i(0)\}. \qquad (7.9)$$

Since $u_i(0) + v_i(0)$ is player $i$'s minimax payoff in the one-shot game, $\tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$ may be called the set of "strictly individually rational" action profiles without randomization.

Note that the definition of $\tilde{\Psi}_i^{\omega_i}$ in (6.10) becomes identical to (7.9) if $\delta_i = 1$ and $\omega_i = u_i(0) + v_i(0)$. It follows from (3.10) that for $(i, j) \in Q$,

$$\forall s_i \in D(g_j^{w_i(0,0)}), \quad g_j^{w_i(0,0)}(s_i) = v_i^{-1}\left(v_i(0) + \frac{u_i(0) - u_i(s_i)}{\delta_i}\right). \tag{7.10}$$

Thus the graph of $g_j^{w_i(0,0)}$ shifts downward as $\delta_i$ increases; see Figure 11. The idea of our folk-type theorem is that as $\delta_i \uparrow 1$, $\tilde{\Psi}_1^{w_1(0,0)} \cap \tilde{\Psi}_2^{w_2(0,0)}$ "converges" to $\tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$, so by Proposition 7.3, any point in $\tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$ can be supported as a steady state of an IRE for $(\delta_1, \delta_2)$ sufficiently close to $(1, 1)$.

**Theorem 7.2.** *Let*

$$(s_1, s_2) \in \tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*. \tag{7.11}$$

*Then for* $(\delta_1, \delta_2) \ll (1, 1)$ *sufficiently close to* $(1, 1)$*, there exists an IRE such that* $(s_1, s_2)$ *is a steady state.*

*Proof.* Assume (7.11). By (7.9), for $(\delta_1, \delta_2)$ close enough to $(1, 1)$,

$$\forall (i, j) \in Q, \quad u_i(s_i) + \delta_i v_i(s_j) > u_i(0) + \delta_i v_i(0), \tag{7.12}$$

so (7.4) holds by (6.1). Now the theorem follows by Proposition 7.3. □

This result is similar in spirit to the folk-type theorems shown by Friedman and Samuelson (1994a, 1994b). Their results show that the main idea of the standard folk theorem (Fudenberg and Maskin, 1986) is valid even if one confines oneself to continuous equilibria with additional restrictions. Since the IREs given by (6.8) are continuous, any $(s_1, s_2) \in \tilde{\Psi}_1^* \cap \tilde{\Psi}_2^*$ is in fact supported as a steady state of a continuous IRE for $(\delta_1, \delta_2)$ sufficiently close to $(1, 1)$ here. In addition the punishment on a deviator imposed by an IRE satisfying (6.8) is rather minimal since in each period, player $i$ is indifferent between conforming to a given IRE path and choosing any $s_i \in [0, \overline{s}_i^{(\omega_i, \omega_j)}]$ (recall that $\underline{s}_i^{(\omega_i, \omega_j)} = 0$ here by the proof of Proposition 7.1).

# 8 Applications

## 8.1 Prisoner's Dilemma

Consider the alternating move game associated with the prisoner's dilemma game in Subsection 2.2.4.[19] For simplicity, we assume directly that the one-

[19]The simultaneous move game can be analyzed similarly; recall Proposition 6.1 and Section 5.

shot payoff of player $i$ is given by (2.5),[20] and that both players have the same discount factor: $\delta_1 = \delta_2 = \delta \in (0, 1)$. The effective payoff of player $i$ is given by

$$w_i(s_i, s_j) = -as_i + \delta e s_j, \tag{8.1}$$

where $e = c + a$. Replacing $w_i(s_i, s_j)$ with $\omega_i$ and solving for $s_j$, we see that the indifference curve associated with $\omega^i \in \mathbb{R}$, or $g_j^{\omega_i}$, is linear:

$$g_j^{\omega_i}(s_i) = \frac{\omega_i}{\delta e} + \frac{a}{\delta e} s_i. \tag{8.2}$$

Since Assumption 7.1 holds here, all the results in Section 7 apply.

We consider three cases separately. First suppose $\delta < a/e$, i.e., the slope of $g_j^{\omega_i}$ is strictly greater than one. By (7.1), $g_i^{\omega_j}(0) \geq 0$ for $(i, j) \in Q$ in any IRE. Thus if $g_i^{\omega_j}(0) > 0$ for either $(i, j) \in Q$, the nonempiness condition (6.6) will be violated; see Figure 12. Hence in any IRE, $g_i^{\omega_j}(0) = 0$ for $(i, j) \in Q$. It follows that there is a unique IRE, which corresponds to the static Nash equilibrium, i.e., $f_i(s_j) = 0$ for all $s_j \in [0, 1]$ and $(i, j) \in Q$. This is because by Proposition 6.1, $f_i(s_j) \leq \overline{s}_i^{(0,0)} = 0$ for all $s_j \in [0, 1]$ and $(i, j) \in Q$. See Figure 12 again. This IRE is effectively efficient by Theorem 7.1 (or simply by uniqueness).

Now suppose $\delta = a/e$, i.e., the slope of $g_j^{\omega_i}$ is equal to one. In this knife edge case, the two indifference curves emanating from the origin coincide. The above argument still shows $g_i^{\omega_j}(0) = 0$ for $(i, j) \in Q$. Though, as in the previous case, there is an IRE corresponding to the static Nash equilibrium, there are many other IREs here. See Figure 13 for an example.

Finally, suppose $\delta > a/e$, i.e., the slope of $g_j^{\omega_i}$ is strictly less than one. In this case, there are many pairs of effective payoffs supported by IREs. A "typical" case is depicted in Figure 14, where there is a unique and globally stable steady state. The existence of a unique and globally stable steady state is a general property of this case by (5.8) and (8.2).

Figure 15 shows a symmetric IRE that is effectively efficient. In this case the effective payoff of each player corresponds to the action profile $(1, 1)$, and

[20] Alternatively one may assume that player $i$'s mixed action in period $t \in T_i$ is observable to player $j$ at the beginning of period $t + 1$. In this case, the expected one-shot payoff of player $i$ in period $t$ is $-as_{i,t} + (c + a)r_{j,t-1}$, where $s_{i,t}$ is player $i$'s probability of choosing C, and $r_{j,t-1}$ is player $j$'s realized action in period $t - 1$. Since $r_{j,t-1}$ does not affect player $i$'s preferences over his actions from period $t$ onward, all our results hold even in this case. This argument is unnecessary for the simultaneous move game, where $r_{j,t-1}$ must be replaced by $s_{j,t-1}$.

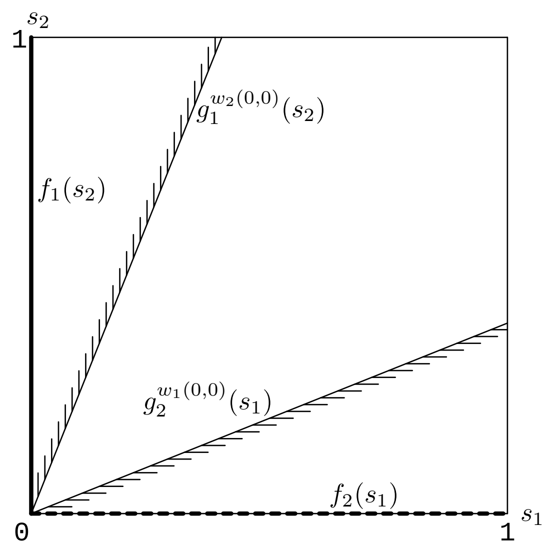



Figure 12: Prisoner's dilemma with $\delta < a/e$

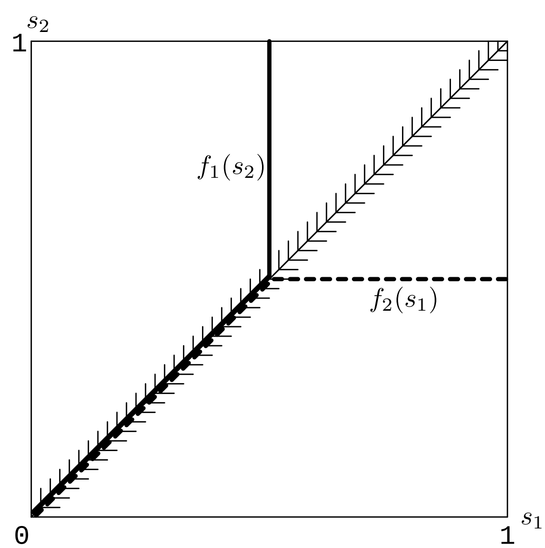



Figure 13: Prisoner's dilemma with $\delta = a/e$

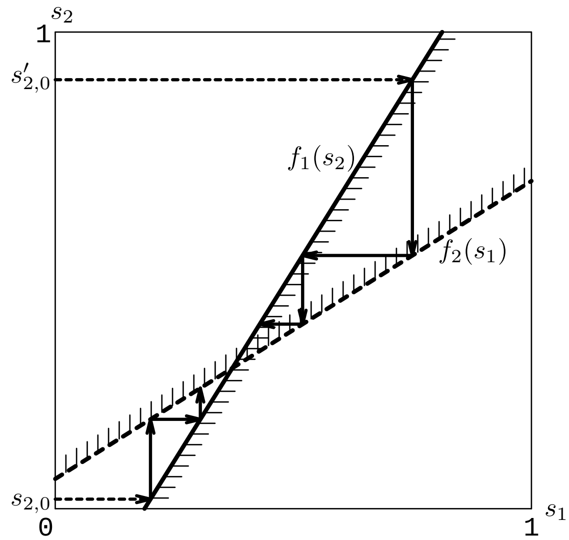



Figure 14: Prisoner's dilemma with $\delta > a/e$: "typical" case

full cooperation is achieved in the long run. Figure 16 shows an effectively efficient IRE in which player 1 receives the highest possible effective payoff, while player 2 receives the minimax effective payoff $w_2(0, 0)$. In this case only player 2 fully cooperates in the long run.

By Theorem 7.2, any strictly individually rational action profile $(s_1, s_2)$, which by definition satisfies $-as_i + es_j > 0$ for $(i, j) \in Q$, is supported as a steady state of an IRE for $\delta$ sufficiently close to one. Notice that the set of payoff profiles supported by strictly individually rational action profiles is convex here, so that this set coincides with the set of "strictly individually rational" payoff profiles (Fudenberge and Maskin, 1986). Thus our folk-type theorem coincides with the standard folk theorem in this example.

## 8.2 Duopoly

Consider the alternating move game associated with the duopoly game of Subsection 2.2.3.[21] For simplicity we assume that the firms are symmetric. Let $c$ and $\delta$ denote their common marginal cost and discount factor. Recall that firm $i$'s one-shot profit is given by $D_i(p_i, p_j)(p_i - c)$. We parametrize $D_i$

[21]Once again, the dynamics of the simultaneous move game can be analyzed similarly.

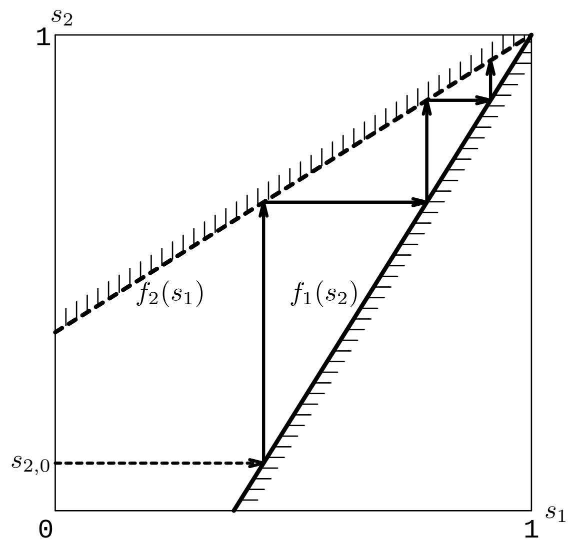



Figure 15: Symmetric effectively efficient IRE

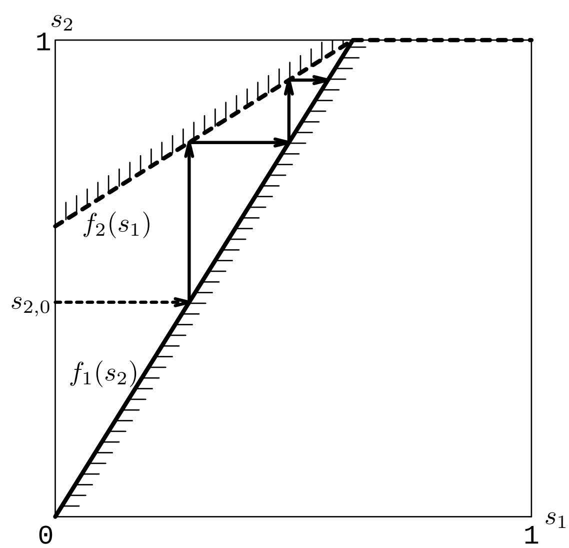



Figure 16: Effectively efficient IRE in which player 2 receives minimax effective payoff

as follows:

$$\forall (i,j) \in Q, \quad D_i(p_i, p_j) = (\overline{p} - p_i)p_j, \tag{8.3}$$

where $\overline{p} > c$. Let $(i,j) \in Q$. Recalling (2.3), we see that the effective payoff of firm $i$ is given by

$$w_i(p_i, p_j) = \ln(\overline{p} - p_i) + \ln(p_i - c) + \delta \ln p_j. \tag{8.4}$$

Replacing $w_i(p_i, p_j)$ with $\omega_i$ and solving for $p_j$, we obtain

$$g_j^{\omega_i}(p_i) = \exp[\{\omega_i - \ln(\overline{p} - p_i) - \ln(p_i - c)\}/\delta]. \tag{8.5}$$

Note that $g_j^{\omega_i}(c) = g_j^{\omega_i}(\overline{p}) = \infty$. Direct calculation of the second derivative shows that $g_j^{\omega_i}$ is strictly convex. It is easy to see that given $p_j$, firm $i$'s one-shot profit, as well as its effective payoff, is maximized at $p_i = \hat{p} \equiv (c + \overline{p})/2$. This is the price charged by both firms in the unique static Nash equilibrium.

Figure 17 illustrates a symmetric IRE in which both firms receive the effective payoff corresponding to the static Nash equilibrium. The indifference curves in this figure are similar to those in Figure 5, which shows the IRE corresponding to the static Nash equilibrium. Figure 17 shows an alternative IRE given by (6.8). In this IRE, there is a steady state in which both firms charge the static Nashu price, as in Figure 5. In Figure 17, however, there is another steady state with a higher symmetric price. At this steady state, each firm faces a "kinked demand curve." If one of the firms raises its price, the other does not follow. Proposition 6.1 implies that this kinked feature is a rather general property in the sense that in any IRE, the firms never charge prices higher than those given by the highest intersection of the two indifference curves. On the other hand, if one of the firms lowers its price, this triggers price war, and the prices converge to the lower steady state. Figure 17 shows an example of an IRE path after a small price cut by firm 2 in period 0 (which is taken as the initial condition of the model).

Clearly the above properties of the two steady states continue to hold even if the firms receive higher effective payoffs, as long as there are two steady states. It is easy to see that there can be at most two steady states by strict concavity of $g_j^{\omega_i}$, provided that the firms receive effective payoffs at least as large as the level associated with the static Nash equilibrium. Note that effective payoffs higher than the static Nash level correspond to indifference curves higher than those depicted in Figure 17.

If there is only one steady state, then the IRE is effectively efficient by (6.11). Figure 18 illustrates a symmetric, effectively efficient IRE. At the

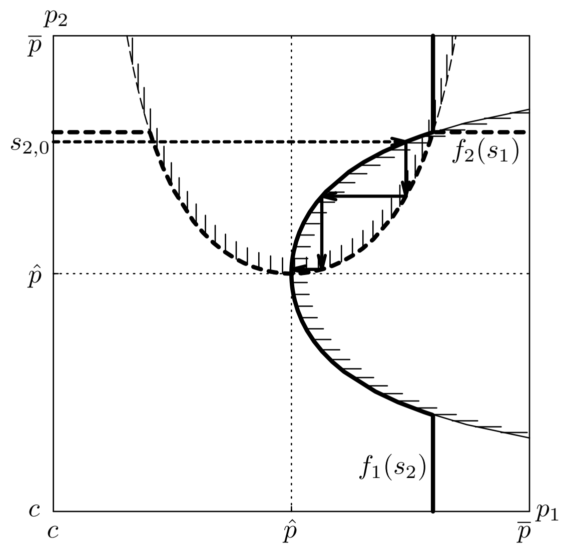



Figure 17: Symmetric IRE in which each firm receives effective payoff equal to static Nash level

unique steady state, each firm faces a kinked demand curve once again. This steady state, however, is globally stable. If one of the firms raises its price, the other does not follow, as in Figure 17. If one of them lowers its price, the other lowers its price too but by a smaller degree. Eventually the prices return to the initial high levels. This process is shown in Figure 18 assuming that firm 2 cuts its price to the static Nash level in period 0. It follows from Theorem 5.1 that the global stability of the unique steady state is a general property of any effectively efficient IRE here.

# 9 Concluding Comments

This paper offers a complete and graphical characterization of immediately reactive equilibria (IREs) for infinitely repeated games with two players in which the action space of each player is an interval, and the one-shot payoff of each player consists of two continuous functions, one unimodal in his own action, the other strictly monotone in the other player's action. IREs are simplest subgame perfect equilibria with nontrivial dynamic interactions. Though IREs constitute only a small subset of the subgame perfect equilibria, the structure of IREs is rich enough to allow us to show a folk-type theorem

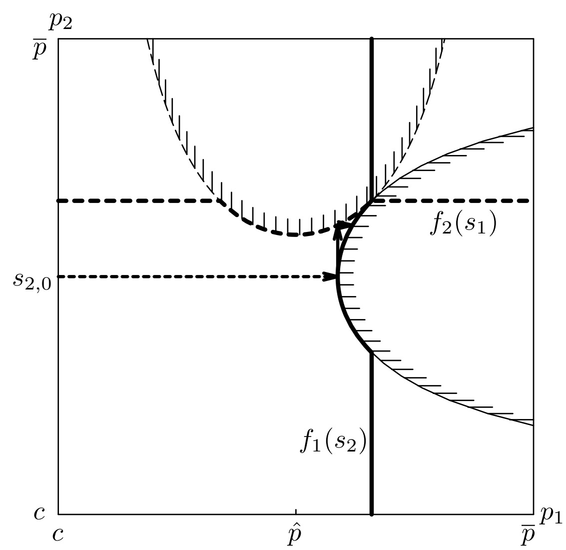



Figure 18: Symmetric effectively efficient IRE

in a special case.

The additive separability of payoffs is critical to our analysis, including the definition of IRE. Although additive separability may appear rather restrictive, there are various interesting games that satisfy it. We have analyzed two such games and characterized their IREs by applying our general results. We believe that our results are useful not only in analyzing games that satisfy our assumptions, but also in constructing completely tractable special cases of more general games. Such special cases, which are fully rational and dynamic, would enhance understanding of various interesting problems.

# Appendix A Proof of Theorem 6.1

Throughout the proof, we omit the superscripts $\omega_i, \omega_j, (\omega_i, \omega_j)$, and $(\omega_j, \omega_i)$.

## A.1 Sufficiency

The "if" part of the proposition follows from the following.

**Lemma A.1.** *Let $\omega_1, \omega_2 \in \mathbb{R}$ satisfy (6.6) and (6.7). Then the strategy profile $(f_1, f_2)$ given by (6.8) is an IRE associated with $(\omega_1, \omega_2)$.*

*Proof.* It suffices to show $R(f_i) \subset M_i(f_j)$ for $(i,j) \in Q$ by Theorem 4.1. Let $(i,j) \in Q$. By (3.21) and (6.8),

$$M_i(f_j) = \{s_i \in D(g_j) \,|\, g_j(s_i) \leq \overline{s}_j\}. \tag{A.1}$$

By (6.3) and (6.2), $g_j(\overline{s}_i) \leq \overline{s}_j$, so $\overline{s}_i \in M_i(f_j)$. By (6.4), $g_j(\underline{s}_i) \leq \overline{s}_j$, so $\underline{s}_i \in M_i(f_j)$. Let $s_i \in R(f_i)$. By (6.7), Lemma 3.1(iii), and (6.8), $\underline{s}_i \leq g_i(\hat{s}_j) \leq s_i \leq \overline{s}_i$. Thus $s_i \in M_i(f_j)$ since $\underline{s}_i, \overline{s}_i \in M_i(f_j)$ and $M_i(f_j)$ is an interval by (A.1) and Lemma 3.1(iii). It follows that $R(f_i) \subset M_i(f_j)$. □

## A.2 Necessity

We show the "only if" part in a few steps. Throughout we take as given an IRE $(f_1, f_2)$ associated with $(\omega_1, \omega_2) \in \mathbb{R}^2$. For $(i,j) \in Q$, define

$$\underline{r}_i = \inf R(f_i), \quad \overline{r}_i = \sup R(f_i), \tag{A.2}$$

$$\tilde{r}_i = \min\{s_i \in D(g_j) \,|\, g_j(s_i) \leq \overline{r}_j\}. \tag{A.3}$$

**Lemma A.2.** *For* $(i,j) \in Q$,

$$\underline{r}_i, \overline{r}_i \in D(g_j), \tag{A.4}$$

$$g_j(\underline{r}_i), g_j(\overline{r}_i) \in D(g_i), \tag{A.5}$$

$$\underline{r}_i \leq g_i(g_j(\underline{r}_i)), \quad \overline{r}_i \geq g_i(g_j(\overline{r}_j)). \tag{A.6}$$

*Proof.* Let $(i,j) \in Q$. Recall from (3.22) that

$$R(f_i) \subset M_i(f_j) \subset D(g_j). \tag{A.7}$$

Since $D(g_j)$ is closed by (3.17) and Lemma 3.1(i), (A.4) follows from (A.7) and (A.2). Note from (4.2) and (A.7) that

$$\forall s_j \in R(f_j), \quad f_i(s_j) = g_i(s_j). \tag{A.8}$$

Hence

$$\underline{r}_i \leq \inf_{s_j \in R(f_j)} g_i(s_j), \quad \overline{r}_i \geq \sup_{s_j \in R(f_j)} g_i(s_j). \tag{A.9}$$

By (A.8) and (A.7) (with $i$ and $j$ interchanged),

$$\forall s_i \in R(f_i), \quad g_j(s_i) = f_j(s_i) \in R(f_j) \subset D(g_i). \tag{A.10}$$

Thus (A.5) follows by continuity of $g_j$ and closeness of $D(g_i)$.

To see (A.6), let $s_i \in R(f_i)$. By (A.10), $g_j(s_i) \in R(f_j)$. Hence by (A.9), $\underline{r}_i \leq g_i(g_j(s_i))$ and $\overline{r}_i \geq g_i(g_j(s_i))$. Thus (A.6) follows by continuity of $g_i$ and $g_j$. □

When $(i,j) \in Q$ is given, we interchange the coordinates in $\Psi_j$ so that $\Psi_i$ and $\Psi_j$ have the same coordinates, i.e., we redefine

$$\Psi_j = \{(s_i, s_j) \in [0,1]^2 \,|\, s_j \in D(g_i), s_i \geq g_i(s_j)\}. \tag{A.11}$$

This is identical to (6.2) with $i$ and $j$ interchanged, except for the order of the coordinates; recall footnote 15.

**Lemma A.3.** *For $(i,j) \in Q$,*

$$(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_i \cap \Psi_j, \tag{A.12}$$

$$\text{(a)} \;\; \hat{s}_j \in D(g_i), \quad \text{(b)} \;\; \tilde{r}_i \leq g_i(\hat{s}_j). \tag{A.13}$$

*Proof.* Let $(i,j) \in Q$. Since $\overline{r}_i \in D(g_j)$ by (A.4), we have $(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_i$ by (6.2) (with $s_j = g_j(\overline{r}_i)$ and $s_i = \overline{r}_i$). By (A.5) and (A.6), $g_j(\overline{r}_i) \in D(g_i)$ and $\overline{r}_i \geq g_i(g_j(\overline{r}_i))$. So by (A.11), $(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_j$. Thus (A.12) follows.[22]

It remains to show (A.13). Note that (A.13)(a) is immediate from (3.17) and Lemma 3.1(i). By (A.2), $f_j(s_i) \leq \overline{r}_j$ for $s_i \in [0,1]$. Thus by (3.21),

$$\forall s_i \in M_i(f_j), \quad g_j(s_i) = f_j(s_i) \leq \overline{r}_j. \tag{A.14}$$

Hence $M_i(f_j) \subset \{s_i \in D(g_j) \,|\, g_j(s_i) \leq \overline{r}_j\} \equiv B$. Thus by (3.20) and (4.1),

$$g_i(\hat{s}_j) \geq f_i(\hat{s}_j) \in M_i(f_j) \subset B. \tag{A.15}$$

Since $\tilde{r}_i = \min B$ by (A.3), (A.13)(b) follows. □

Let us now complete the "only if" part of the proof. We have (6.6) by (A.12). Let $(i,j) \in Q$. By (A.12) and (6.3), $\overline{r}_j \leq \overline{s}_j$. Thus the set in (6.4) includes the set in (A.3), so $\underline{s}_j \leq \tilde{r}_j$ . This together with (A.13) shows (6.7).

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[22] The corresponding result for $\underline{r}_i$ does not hold since $\overline{r}_i$ cannot be replaced by $\underline{r}_i$ in $(\overline{r}_i, g_j(\overline{r}_i)) \in \Psi_j$ unless $\underline{r}_i = g_i(g_j(\underline{r}_i))$; recall (A.6).

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