Lexicographic Compositions of Multiple Criteria for Decision Making*

Nicolas Houy† Koichi Tadenuma‡

November 2007

Abstract

This paper considers two distinct procedures to lexicographically compose multiple criteria for social or individual decision making. The first procedure composes M binary relations into one, and then selects its maximal elements. The second procedure first selects the set of maximal elements of the first binary relation, and then within that set, chooses the maximal elements of the second binary relation, and iterates the procedure until the Mth binary relation. We show several distinct sets of conditions for the choice functions representing these two procedures to satisfy non-emptiness and choice-consistency conditions such as contraction consistency (Chernoff, 1954) and path independence (Arrow, 1963). We also examine the relationships between the outcomes of the two procedures. Then, we investigate under what conditions the outcomes of each procedure are independent of the order of lexicographic application of the criteria. Examples for applications of the results in the economic environments are also presented.

Keywords: multiple criteria for choice, lexicographic application, choice consistency, path independence, order independence

JEL Codes: D01, D63, D71, D74

*The authors are grateful to M. Fleurbaey, M. Mariotti, M. Salles, K. Suzumura, and N. Yoshihara for valuable comments and suggestions which improved the content of the paper. Earlier versions of this paper were presented at the International Conference on Rational Choice, Individual Rights and Non-Welfaristic Normative Economics, Hitotsubashi University, March 2006, the 5th International Conference on Logic, Game Theory and Social Choice, Bilbao, June 2007, and the Japanese-French Seminar on Social Choice Theory and Welfare Economics, Caen, October 2007. K. Tadenuma gratefully acknowledges financial support from the Japan Society for the Promotion of Science through the grant for the 21st Century Center of Excellence Program on the Normative Evaluation and Social Choice of Contemporary Economic Systems and the Grant-in-Aid for Scientific Research (B) No. 18330036.

†CNRS-Laboratoire d'Econométrie, Ecole Polytechnique, France. nhouy@free.fr

‡Faculty of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan. tadenuma@econ.hit-u.ac.jp

1 Introduction

In the process of social decision making, people often advocate multiple criteria on which the desirability of alternatives should be judged. A typical example is the equity-efficiency trade-off. People say that economic growth is desirable because the welfare of most individuals increases, while at the same time they insist that an equitable distribution is essential for social stability. As often argued, however, economic growth may give rise to an inequitable distribution of income and wealth.

Even a single individual's decision may be based upon multiple criteria. As Sen (1985) argues, an individual has not only material preferences over his own consumptions but also has value judgments based on, for instance, the sense of obligation, which may contradict his material preferences. A family's decision also typically involves multiple criteria. Parents' interest often conflicts with children's interest on, for example, video games.

When multiple criteria, each regarded as reasonable for itself, are in contradiction with each other, one resolution would be to make a priority order for application of the criteria. For such lexicographic applications of multiple criteria, however, we can consider two distinct procedures of choice, which are described in the following. Let us first postulate that each criterion is expressed by a binary relation on the set X of all alternatives. In the first procedure, which we call procedure α, we first compose lexicographically multiple binary relations $R^1, \ldots, R^M$ into one binary relation $P(R^1, \ldots, R^M)$ in the following way: an alternative x is better than an alternative y for $P(R^1, \ldots, R^M)$ if and only if (i) x is superior to y for R^1 or (ii) y is not superior to x for R^1 and x is superior to y for R^2, or $\cdots$ or (M) y is not superior to x for $R^1, \ldots, R^{M-1}$ and x is superior to y for R^M. Then, for each subset S of alternatives, we select the set $C_{P(R^1,,\ldots,R^M)}(S)$ of maximal elements for $P(R^1,, \ldots, R^M)$.

By contrast, in the second procedure, which we call procedure β, for each subset S of alternatives, we first choose the set $C_{P(R^1)}(S)$ of maximal elements in S for the first criterion R^1, and then select within the set $C_{P(R^1)}(S)$, its subset $C_{P(R^2)}[C_{P(R^1)}(S)]$ of maximal elements for the second criterion R^2, and iterate the procedure until the Mth binary relation.

Indeed, the above two procedures provide different choices for many cases. As a simple example, consider $S = \{x, y, z\}$, $R^1 = \{(x, z)\}$, and $R^2 = \{(z, y)\}$. Then, since $(x, z), (z, y) \in P(R^1, R^2)$, procedure α chooses $\{x\}$ from S. However, because the set of maximal elements in S for R^1 is $\{x, y\}$, and neither x nor y strictly dominates the other according to R^2, procedure β selects $\{x, y\}$ from S. Procedure α has been introduced and examined by Tadenuma (2002, 2005), while procedure β has been introduced by Suzumura (1983b), Aizerman (1985) and Aizerman and Aleskerov (1995), and studied more recently by Manzini and Mariotti (2005), Tadenuma (2005) and Houy (2007).

When a decision-maker has multiple criteria, his behavior becomes much different from a simple maximizer of a single binary relation. It is more difficult to have consistent choices under multiple criteria than under a single criterion. In this paper, we study under what conditions the choice correspondence derived from each procedure to lexicographically compose multiple criteria satisfy non-emptiness and various properties of choice-consistency such as contraction consistency (Chernoff, 1954) and path independence (Arrow, 1963). We also examine relationships between the outcomes of procedures α and β.

Another interesting question would be whether the final outcome depends on the order of application of the multiple criteria. When we evaluate allocations, which criterion should we apply first, the efficiency criterion or the equity criterion? Such a question is important if the order of application of the multiple criteria affects the final outcome. But if the order is irrelevant, then we do not have to be concerned about which criterion we should take first. We investigate under what conditions the outcomes of the choice correspondence of each procedure are independent of the order of lexicographic application of the multiple criteria.

All the results in this paper are derived without specific restrictions on the set of alternatives, but we present applications of the results in the classical division problem of infinitely divisible commodities.

There are many examples in which multiple criteria, each of which seems reasonable for itself, contradict each other. In economics and social choice theory, the social preference relation that has been most widely accepted is the Pareto domination. However, the Pareto criterion is silent about the distributional equity of allocations but concerns only efficient use of resources. On the other hand, several interesting concepts of distributional equity have been introduced and extensively studied in economics. Two of them are central: *no-envy* and *egalitarian-equivalence*.[1] It was Kolm (1972) and Feldman and Kirman (1974) who pointed out that there is a fundamental conflict between the Pareto criterion and the equity-as-no-envy criterion: there often exist two allocations x and y such that x Pareto dominates y whereas x is not envy-

[1] The concept of no-envy was introduced by Foley (1967) and Kolm (1972), and that of egalitarian-equivalence by Pazner and Schmeidler (1978).

free but y is. The same kind of conflict also arises between the Pareto criterion and the equity-as-egalitarian-equivalence criterion.

Social choice theory on abstract domains has also been extended to take account of intersituational comparisons of individuals.[2] In this "extended sympathy" approach, Suzumura (1981a, b) studied choice-consistency of social choice functions satisfying some conditions concerning Pareto efficiency and equity-as-no-envy in the framework of abstract social choice. Tadenuma (2002, 2005) introduced various lexicographic compositions of the Pareto criterion and the no-envy criterion, and of Pareto and egalitarian-equivalence, respectively, in the classical division problem, and examined rationality of the social preference relations. Tadenuma (2005) also showed that the set of allocations selected by procedure α with the Pareto criterion and the egalitarian-equivalence criterion from the set of all feasible allocations is independent of the order of lexicographic application of the two criteria, and that the essential reason for this independence is because the set of allocation selected by procedure β is also independent of the order of application.

The present paper generalizes the results in these works by showing general conditions for non-emptiness, contraction consistency, and path independence of choice functions representing procedures α and β, clarifying their relationships, and also deriving conditions for independence of the order of application of multiple criteria.

The next section defines the basic notions and notation, and Section 3 introduces the choice-consistency properties. In Sections 4 and 5, we investigate conditions for non-emptiness and choice-consistency of procedure α and procedure β, respectively. Section 6 examines order-independence of each of the two procedures. The final section contains some concluding remarks.

2 Basic Definitions and Notation

Let X be a (finite or infinite) set of alternatives with $|X| \geq 3$. Let $\mathcal{X}$ denote the set of all finite subsets of X. A binary relation on X is a set $R \subseteq X \times X$. The set of all binary relations on X is denoted $\mathcal{R}$. Given $R \in \mathcal{R}$, define $P(R) \in \mathcal{R}$ by $(x, y) \in P(R) \Leftrightarrow [(x, y) \in R$ and $(y, x) \notin R]$, and $I(R) \in \mathcal{R}$ by $(x, y) \in I(R) \Leftrightarrow [(x, y) \in R$ and $(y, x) \in R]$. Given $R \in \mathcal{R}$, a sequence $(x^1, \ldots, x^K) \subseteq X$, $K \geq 2$, is a *cycle* for R if $(x^1, x^2), (x^2, x^3), \ldots, (x^{K-1}, x^K), (x^K, x^1) \in R$. A binary relation $R \in \mathcal{R}$ is

[2] Notable earlier contributions in this line of research are Harsanyi (1955), Suppes (1966), Pattanaik (1968), Sen (1970), Hammond (1976) and Arrow (1977).

- *complete* if for all $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$;
- *transitive* if for all $x, y, z \in X$, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$;
- *quasi-transitive* if for all $x, y, z \in X$, $(x, y) \in P(R)$ and $(y, z) \in P(R)$ imply $(x, z) \in P(R)$;
- *asymmetric* if for all $x, y \in X$, $(x, y) \in R$ implies $(y, x) \notin R$;
- *acyclic* if there exists no cycle for R.

Note that acyclicity implies asymmetry by the above definitions.

In the rest of the paper, if $(x^1, \ldots, x^K) \subseteq X$ is a cycle, we abuse notation by letting $(K+1) := 1$ in order to simplify presentation of the results.

A choice function is a function $C : \mathcal{X} \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \mathcal{X}$. Given $R \in \mathcal{R}$, we define the choice function $C_{P(R)}$ as the one selecting the set of maximal elements for every $S \in \mathcal{X}$, that is,

$$\forall S \in \mathcal{X},\ C_{P(R)}(S) = \{x \in X \mid \forall y \in X,\ (y, x) \notin P(R)\}.$$

We say that a choice function C is *rationalizable by a binary relation* $R \in \mathcal{R}$ if $C = C_{P(R)}$.

In the following, we often consider the classical division problem with n agents and m infinitely divisible commodities defined as follows. Let $N = \{1, \ldots, n\}$ be the set of agents. The consumption set of each agent is $\mathbb{R}^m_+$. Let $\mathcal{R}_E$ be the set of complete, transitive and strictly monotonic[3] relations on $\mathbb{R}^m_+$. Each agent $i \in N$ is endowed with a preference relation $\succsim_i \in \mathcal{R}_E$. The associated strict preference relation and the indifference relation are defined as above, and denoted $\succ_i$ and $\sim_i$, respectively. An allocation is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{mn}_+$ where each $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m_+$ is the consumption bundle of agent $i \in N$. The set of alternatives in this division problem is defined as $X = \mathbb{R}^{mn}_+$.

3 Choice-Consistency Properties

In this section, we introduce some desirable properties of choice functions. A very basic requirement is that at least one alternative should be chosen from any set.

Non-Emptiness: For all $S \in \mathcal{X}$, $C(S) \neq \emptyset$.

[3] A preference relation $\succsim$ is strictly monotonic if for all $a, b \in \mathbb{R}^m_+$, $a > b$ implies $a \succ b$, where $a > b$ is defined as $a \geq b$ and $a \neq b$.

Our next three properties require "consistency" of choices in related situations. The first choice-consistency property means that if the set of available alternatives "shrinks" but previously chosen alternatives are still available, then those alternatives should remain chosen. This is a fundamental requirement of choice-consistency, and it is satisfied by any choice function that is rationalizable by some binary relation.

Contraction Consistency (Chernoff, 1954): For all $S, T \in \mathcal{X}$ with $T \subseteq S$, $T \cap C(S) \subseteq C(T)$.

The second property requires "the independence of the final choice from the path to it" (Arrow, 1963, p.120). In real choice situations, we often divide the set of alternatives into several parts in the first round, and make final choices from the alternatives that have survived in the first round. This property requires that the final choices should not depend on the way we divide the set of alternatives in the first round. It is an important property especially for social choice rules. Were it violated, some arbitrary agenda controls could affect the final choice, which is clearly undesirable.

Path Independence: For all $S, T \in \mathcal{X}$, $C(C(S) \cup C(T)) = C(S \cup T)$.

It is well-known that Path Independence implies Contraction Consistency, but not vice versa.

The third choice-consistency property says that if an alternative is chosen from every pair containing it in the set S, then it should be chosen from S.

Condorcet Consistency: For all $S \in \mathcal{X}$ and all $x \in S$, if $x \in C(\{x, y\})$ for all $y \in S$, then $x \in C(S)$.

The choice-consistency properties are related with rationalizability of the choice functions. The following results are well-established.[4]

Proposition 1 (Blair et al., 1976) *A choice function C satisfies Non-Emptiness, Contraction Consistency and Condorcet Consistency if and only if it is rationalizable by a binary relation R such that $P(R)$ is acyclic.*

Given a binary relation $R \in \mathcal{R}$, $C_{P(R)}$ satisfies Condorcet Consistency by definition. As we have noted, any choice function that is rationalizable by a binary relation satisfies Contraction Consistency. Hence, we have the following corollary.

[4]A good reference for these results is Suzumura (1983a, Ch.2).

Corollary 1 *Let $R \in \mathcal{R}$ be given. The choice function $C_{P(R)}$ satisfies Non-Emptiness if and only if $P(R)$ is acyclic.*

Similar relations hold for Path Independence and rationalizability by a quasi-transitive binary relation.

Proposition 2 (Plott, 1973) *A choice function C satisfies Non-Emptiness, Path Independence and Condorcet Consistency if and only if it is rationalizable by a quasi-transitive binary relation.*

Corollary 2 *Let $R \in \mathcal{R}$ be given. The choice function $C_{P(R)}$ satisfies Non-Emptiness and Path Independence if and only if R is quasi-transitive.*

4 Lexicographic Composition of Multiple Binary Relations

As we have mentioned in the introduction, we consider two distinct procedures to compose multiple criteria for decision making. This section focuses on procedure α in which we first compose M binary relations $R^1, \ldots, R^M$ into one, and then choose its maximal elements. Formally, for all $R^1, \ldots, R^M \in \mathcal{R}$, we define $P(R^1, \ldots, R^M) \in \mathcal{R}$ by

$$\begin{aligned} P(R^1, \ldots, R^M) \;\; = \;\; & \{(x, y) \in X \times X \mid [(x, y) \in P(R^1)] \text{ or} \\ & [(y, x) \notin P(R^1) \text{ and } (x, y) \in P(R^2)] \text{ or } \cdots \\ & [(y, x) \notin P(R^1) \cup \cdots \cup P(R^M) \text{ and } (x, y) \in P(R^M)]\}. \end{aligned}$$

We call $P(R^1, \ldots, R^M)$ the *lexicographic composition of* $R^1, \ldots, R^M$. Notice that $P(R^1, \ldots, R^M)$ is asymmetric and hence $P(P(R^1, \ldots, R^M)) = P(R^1, \ldots, R^M)$.

We examine under what conditions the choice function $C_{P(R^1,\ldots,R^M)}$ satisfies Non-Emptiness and Path Independence. By Corollaries 1 and 2, our examination reduces to checking acyclicity and quasi-transitivity of $P(R^1, \ldots, R^M)$. We also present examples for applications of the results in economic environments.

Our first result gives a necessary and sufficient condition for $P(R^1, \ldots, R^M)$ to be acyclic, and equivalently, for $C_{P(R^1,\ldots,R^M)}$ to be non-empty.

Proposition 3 *Let $R^1, \ldots, R^M \in \mathcal{R}$. The lexicographic composition $P(R^1, \ldots, R^M)$ is acyclic if and only if for every cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^1) \cup \cdots \cup P(R^M)$, there exists $k \in \{1, \ldots, K\}$ such that for every $m \in \{1, \ldots, M\}$ with $(x^k, x^{k+1}) \in P(R^m)$, there exists $m' \in \{1, \ldots, M\}$, $m' < m$, such that $(x^{k+1}, x^k) \in P(R^{m'})$.*

Proof. Necessity. Assume that there exists a cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^1) \cup \cdots \cup P(R^M)$ such that for every $k \in \{1, \ldots, K\}$, there exists $m \in \{1, \ldots, M\}$ such that $(x^k, x^{k+1}) \in P(R^m)$ and $(x^{k+1}, x^k) \notin P(R^{m'})$ for all $m' < m$. Then, by definition, $(x^k, x^{k+1}) \in P(R^1, \cdots, R^M)$ for every $k \in \{1, \ldots, K\}$, and $(x^1, \ldots, x^K) \subseteq X$ is a cycle for $P(R^1, \cdots, R^M)$.

Sufficiency. Assume that for every cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^1) \cup \cdots \cup P(R^M)$, there exists $k \in \{1, \ldots, K\}$ such that for every $m \in \{1, \ldots, M\}$ with $(x^k, x^{k+1}) \in P(R^m)$, there exists $m' \in \{1, \ldots, M\}$, $m' < m$ such that $(x^{k+1}, x^k) \in P(R^{m'})$. Suppose, on the contrary, that $P(R^1, \ldots, R^M)$ has a cycle $(y^1, \ldots, y^L) \subseteq X$. Then, $(y^1, \ldots, y^L)$ is also a cycle for $P(R^1) \cup \cdots \cup P(R^M)$. By the assumption, there exists $\ell \in \{1, \ldots, L\}$ such that for every $m \in \{1, \ldots, M\}$ with $(y^\ell, y^{\ell+1}) \in P(R^m)$, there exists $m' \in \{1, \ldots, M\}$, $m' < m$ such that $(y^{\ell+1}, y^\ell) \in P(R^{m'})$. Then, by definition, $(y^\ell, y^{\ell+1}) \notin P(R^1, \ldots, R^M)$, contradicting the fact that $(y^1, \ldots, y^L) \subseteq X$ is a cycle for $P(R^1, \ldots, R^M)$. ■

In many economic problems, a trade-off arises between two criteria such as efficiency vs. equity, growth vs. environmental quality, efficiency vs. liberty, and so on. In such cases, we can obtain simpler conditions for acyclicity or quasi-transitivity of the lexicographic compositions of two binary relations. Next, we provide several sufficient conditions for $P(R^1, R^2)$ to be acyclic or quasi-transitive, which may be useful in various contexts.

Proposition 4 *Let $R^1, R^2 \in \mathcal{R}$. If R^1 is complete and transitive, and R^2 is quasi-transitive, then $P(R^1, R^2)$ is quasi-transitive.*

Proof. Let $x, y, z \in X$. Assume that $(x, y) \in P(R^1, R^2)$ and $(y, z) \in P(R^1, R^2)$. From $(x, y) \in P(R^1, R^2)$, we have (1) $(x, y) \in P(R^1)$ or (2) $(x, y) \notin P(R^1)$, $(y, x) \notin P(R^1)$ and $(x, y) \in P(R^2)$. By completeness of R^1, $(x, y) \notin P(R^1)$ and $(y, x) \notin P(R^1)$ if and only if $(x, y) \in I(R^1)$. Similarly, it follows from $(y, z) \in P(R^1, R^2)$ that (3) $(y, z) \in P(R^1)$ or (4) $(y, z) \in I(R^1)$ and $(y, z) \in P(R^2)$. If (1) and [(3) or (4)] hold true, then by transitivity of R^1, we have $(x, z) \in P(R^1)$, and hence $(x, z) \in P(R^1, R^2)$. Similarly, (2) and (3) together imply $(x, z) \in P(R^1)$ and $(x, z) \in P(R^1, R^2)$. Finally, if (2) and (4) hold, then $(x, z) \in I(R^1)$ follows from transitivity of R^1, and $(x, z) \in P(R^2)$ from quasi-transitivity of R^2. Hence, we have $(x, z) \in P(R^1, R^2)$. ■

There are many examples in allocation problems to which the above result can be applied.

Example 1 Envy-free allocations. An allocation $x \in \mathbb{R}_+^{mn}$ is *envy-free* if for all $i, j \in N$, $(x_i, x_j) \in \succsim_i$. Let $F \subset \mathbb{R}_+^{mn}$ be the set of envy-free allocations. Define $R^F \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^F$ if and only if $x \in F$ or $y \notin F$. Define $R^P \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^P$ if and only if for all $i \in N$, $(x_i, y_i) \in \succsim_i$. The social preference relation R^P is called the *weak Pareto domination*, and the associated strict social preference relation $P(R^P)$ the *Pareto domination*. Since R^F is complete and transitive, and R^P is quasi-transitive, it follows from Proposition 4 that $P(R^F, R^P)$ is quasi-transitive. Hence, the choice function $C_{P(R^F,R^P)}$ is not empty and satisfies the Path Independence condition.

Example 2 Ranking by the number of envy instances. For each $x \in \mathbb{R}_+^{mn}$, define the set $H(x) \subset N \times N$ by

$$H(x) = \{(i, j) \in N \times N \mid (x_j, x_i) \in \succ_i\}.$$

The set $H(x)$ is the set of all *instances of envy* at x. Following Feldman and Kirman (1974), define $R^H \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^H$ if and only if $\#H(x) \leq \#H(y)$. Then, R^H is complete and transitive. By Proposition 4, $P(R^H, R^P)$ is quasi-transitive (Tadenuma, 2002).

Example 3 Egalitarian-equivalent allocations. An allocation $x \in \mathbb{R}_+^{mn}$ is *egalitarian-equivalent* if there exists $a \in \mathbb{R}_+^m$ such that for all $i \in N$, $(x_i, a) \in \sim_i$. Let $E \subset \mathbb{R}_+^{mn}$ be the set of egalitarian-equivalent allocations. Define $R^E \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^E$ if and only if $x \in E$ or $y \notin E$. Then, R^E is complete and transitive. By Proposition 4, $P(R^E, R^P)$ is quasi-transitive (Tadenuma, 2005).

A similar result can be obtained for acyclicity of $P(R^1, R^2)$.

Proposition 5 *Let $R^1, R^2 \in \mathcal{R}$. If R^1 is complete and transitive, and $P(R^2)$ is acyclic, then $P(R^1, R^2)$ is acyclic.*

Proof. Assume that R^1 is complete and transitive, and $P(R^2)$ is acyclic. Suppose, on the contrary, that there exists a cycle $(x^1, \ldots, x^K) \in X$ for $P(R^1, R^2)$. Because R^1 is complete, for all $x, y \in X$, $(x, y) \in P(R^1, R^2)$ implies that $(x, y) \in R^1$. Hence, we have $(x^k, x^{k+1}) \in R^1$ for all $k \in \{1, \ldots, K\}$. Therefore, by transitivity of R^1, for all $k, k' \in \{1, \ldots, K\}$, $(x^k, x^{k'}) \in I(R^1)$. Then, since $(x^1, \ldots, x^K) \in X$ is a cycle for $P(R^1, R^2)$, we must have $(x^k, x^{k+1}) \in P(R^2)$ for all $k \in \{1, \ldots, K\}$. This contradicts the acyclicity of $P(R^2)$. ■

If $R^1 \in \mathcal{R}$ is only quasi-transitive, then even if R^1 is complete and $R^2 \in \mathcal{R}$ is complete and transitive, $P(R^1, R^2)$ may have a cycle.

Example 4 Define $R^{\hat{P}} \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^{\hat{P}}$ if and only if $(y, x) \notin P(R^P)$. Sen (1970) called $R^{\hat{P}}$ the *Pareto extension*. Notice that $P(R^{\hat{P}}) = P(R^P)$, and $R^{\hat{P}}$ is complete and quasi-transitive. As we noted, R^F, R^H, and R^E are all complete and transitive. However, none of $P(R^{\hat{P}}, R^F)$, $P(R^{\hat{P}}, R^H)$ and $P(R^{\hat{P}}, R^E)$ is acyclic (Tadenuma, 2002, 2005). Notice that, since $P(R^{\hat{P}}) = P(R^P)$, none of $P(R^P, R^F)$, $P(R^P, R^H)$ and $P(R^P, R^E)$ is acyclic either.

The Pareto principle plays a central role in economics, but it says nothing about distributional equity. On the other hand, many binary relations based on some concepts of equity are complete and transitive. Suppose that we would like to socially rank allocations firstly by the Pareto principle, and secondly by an equity principle. When does such lexicographic applications of the Pareto and an equity principles generate an acyclic social preference relation? To answer the question, it is of special interest to investigate under what conditions $P(R^1, R^2)$ shows no cycle when $R^1 \in \mathcal{R}$ is only quasi-transitive and $R^2 \in \mathcal{R}$ is complete and transitive. Our next result gives an answer to this question.

To present the result, we define the following binary relation: for all $x, y \in X$,

$$(x, y) \in \Gamma \Leftrightarrow [(x, y) \notin P(R^1), (y, x) \notin P(R^1) \text{ and } (x, y) \in P(R^2)].$$

That is, $(x, y) \in \Gamma$ if and only if x and y are non-comparable or indifferent by the first criterion, and x is superior to y by the second criterion. Note that $P(R^1)$ and Γ decompose $P(R^1, R^2)$, that is, $P(R^1, R^2) = P(R^1) \cup \Gamma$ and $P(R^1) \cap \Gamma = \emptyset$. The relationships among any three alternatives in terms of these two components of $P(R^1, R^2)$ are the key to acyclicity of $P(R^1, R^2)$.

Proposition 6 *Let $R^1, R^2 \in \mathcal{R}$. Suppose that R^1 is quasi-transitive, and that R^2 is complete and transitive. Suppose further that the following two conditions hold:*
(A) *for all $x, y, z \in X$, if $(x, y) \in \Gamma$, $(y, z) \in P(R^1)$ and $(z, y) \in P(R^2)$, then $(x, z) \in P(R^1)$.*
(B) *for all $x, y, z \in X$, if $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$, then $(z, x) \notin P(R^1)$.*
Then, the lexicographic composition $P(R^1, R^2)$ is acyclic.

Proof. Assume that R^1 is quasi-transitive, that R^2 is complete and transitive, and that conditions (A) and (B) are satisfied. Suppose, on the contrary, that that

$P(R^1, R^2)$ has a cycle. Let $(x^1, ..., x^k)$ be a cycle of the smallest cardinality for $P(R^1, R^2)$. Since $P(R^1, R^2)$ is asymmetric, $k \geq 3$.

Assume that $(x^1, x^2), (x^2, x^3) \in P(R^1)$. Then, by quasi-transitivity of R^1, $(x^1, x^3) \in P(R^1)$ and then $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$ which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$.

Assume that $(x^1, x^2), (x^2, x^3) \in \Gamma$. Then, by definition, $(x^1, x^2), (x^2, x^3) \in P(R^2)$, and by transitivity of R^2, $(x^1, x^3) \in P(R^2)$. Moreover, by condition (B), $(x^3, x^1) \notin P(R^1)$ which implies that $(x^1, x^3) \in P(R^1, R^2)$. Then $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$ which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$.

Let $(x^1, ..., x^k)$ be one of the smallest cycles for $P(R^1, R^2)$ with $k \geq 3$. From what we have shown above, with no loss of generality, we can set $(x^1, x^2) \in \Gamma$ and $(x^2, x^3) \in P(R^1)$. We distinguish two cases.
(1) If $(x^3, x^2) \in P(R^2)$, then by condition (A), $(x^1, x^3) \in P(R^1)$. Hence, $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$, which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$.
(2) If $(x^3, x^2) \notin P(R^2)$, then by completeness of R^2, $(x^2, x^3) \in R^2$. Together with $(x^1, x^2) \in \Gamma$ and transitivity of R^2, we have $(x^1, x^3) \in P(R^2)$. If $(x^3, x^1) \in P(R^1)$, then by quasi-transitivity of R^1, $(x^2, x^1) \in P(R^1)$, which contradicts $(x^1, x^2) \in \Gamma$. Hence, $(x^3, x^1) \notin P(R^1)$. But then, $(x^1, x^3) \in P(R^1, R^2)$, and $(x^1, x^3, ..., x^k)$ is a cycle for $P(R^1, R^2)$, which contradicts the fact that $(x^1, ..., x^k)$ is a cycle of the smallest cardinality for $P(R^1, R^2)$. ■

The usefulness of the above result may be illustrated by the following example.

Example 5 For each $i \in N$, let $a_i \in \mathbb{R}^m_+$ be the *reference bundle for agent* i. (Examples of reference bundles are (i) the equal division bundle for all agents under a social resource constraint, (ii) initial endowment bundles in a private ownership economy, (iii) minimum bundles to meet some basic functionings.) Define $R^B \in \mathcal{R}$ as follows: for all $x, y \in \mathbb{R}^{mn}_+$, $(x, y) \in R^B$ if and only if $\#\{i \in N \mid (x_i, a_i) \in \succsim_i\} \geq \#\{i \in N \mid (y_i, a_i) \in \succsim_i\}$. Clearly, R^B is complete and transitive. Notice that if $(x, y) \in P(R^P)$ where R^P is the weak Pareto domination defined above, then it never occurs that $(y, x) \in P(R^B)$. Hence, condition (A) in Proposition 6 is vacuously satisfied. Furthermore, if $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$, then $(x, z) \in P(R^B)$ by transitivity of R^B, and hence $(z, x) \notin P(R^P)$. Therefore, condition (B) in Proposition 6 is also met. We can conclude that the the lexicographic composition $P(R^P, R^B)$ is acyclic. The same result holds for $P(R^{\hat{P}}, R^B)$.

Often an equity criterion dichotomizes allocations into equitable and non-equitable ones. In such a case, we can define a complete and transitive binary relation R^2 as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^2$ if and only if x is equitable or y is not equitable. Note that from this definition, $(x, y) \in P(R^2)$ if and only if x is equitable *and* y is not equitable. Moreover, in this case, R^2 has at most two indifference classes. Hence, the condition (B) in Proposition 6 is irrelevant because for all $x, y, z \in \mathbb{R}_+^{mn}$, $(x, y) \in \Gamma$ and $(y, z) \in \Gamma$ cannot occur together. Therefore, we have the following corollary.

Corollary 3 *Let $R^1, R^2 \in \mathcal{R}$. Suppose that R^1 is quasi-transitive, and that R^2 is complete and transitive, and has at most two indifference classes. Suppose further that for all $x, y, z \in X$, if $(x, y) \in \Gamma$, $(y, z) \in P(R^1)$ and $(z, y) \in P(R_2)$, then $(x, z) \in P(R^1)$. Then, the lexicographic composition $P(R^1, R^2)$ is acyclic.*

As an example of application of the above corollary, we present the lexicographic composition of the Pareto domination and the binary relation based on egalitarian-equivalence that was studied in Tadenuma (2005).

Let $A \subset \mathbb{R}_+^{mn}$ be such that for all $a, b \in A$, $a \geq b$ or $b \geq a$. An allocation $x \in \mathbb{R}_+^{mn}$ is *A-egalitarian-equivalent* if there exists $a \in A$ such that $(x_i, a) \in \sim_i$ for all $i \in N$. Let $E_A \subset \mathbb{R}_+^{mn}$ be the set of A-egalitarian-equivalent allocations. Define R^{E_A} as follows: for all $x, y \in \mathbb{R}_+^{mn}$, $(x, y) \in R^{E_A}$ if and only if $x \in E_A$ or $y \notin E_A$. Notice that $(x, y) \in P(R^{E_A})$ if and only if $x \in E_A$ and $y \notin E_A$. One can check that R^{E_A} is complete and transitive.

Corollary 4 *Let R^{E_A} be defined as above. Let R^P be the weak Pareto domination. Then, the lexicographic composition $P(R^P, R^{E_A})$ is acyclic.*

Proof. As noted above, R^{E_A} is complete and transitive, and R^P is quasi-transitive. In view of Corollary 3, it is enough to show that for all $x, y, z \in \mathbb{R}_+^{mn}$, if $(x, y) \in \Gamma$, $(y, z) \in P(R^P)$ and $(z, y) \in P(R^{E_A})$, then $(x, z) \in P(R^P)$.

Suppose that $(x, y) \in \Gamma$, $(y, z) \in P(R^P)$ and $(z, y) \in P(R^{E_A})$. Because $(x, y) \in P(R^{E_A})$, we have $x \in E_A$ and $y \notin E_A$. Thus, there exists $a \in A$ such that $(x_i, a) \in \sim_i$ for all $i \in N$. Since $(z, y) \in P(R^{E_A})$, we have $z \in E_A$ and $y \notin E_A$. Hence, there exists $b \in A$ such that $(z_i, b) \in \sim_i$ for all $i \in N$. If $(y_i, x_i) \in \sim_i$ for all $i \in N$, then $(y_i, a) \in \sim_i$ for all $i \in N$, which contradicts $y \notin E_A$. Therefore, $(y, x) \notin P(R^P)$ holds only if there exists $i^* \in N$ such that $(x_{i^*}, y_{i^*}) \in \succ_{i^*}$. Since $(y, z) \in P(R^P)$, we have $(y_i, z_i) \in \succsim_i$ for all $i \in N$, and in particular, for agent i^*. Hence, $(x_{i^*}, z_{i^*}) \in \succ_{i^*}$. We also have $(a, x_{i^*}) \in \sim_{i^*}$ and $(z_{i^*}, b) \in \sim_{i^*}$. By transitivity of $\succsim_{i^*}$, $(a, b) \in \succ_{i^*}$. Since

$a, b \in A$, either $a > b$ or $b > a$. By strict monotonicity of $\succsim_{i^*}$, we have $a > b$. Then, for all $i \in N$, $(x_i, a) \in \sim_i$, $(a, b) \in \succ_i$ and $(b, z_i) \in \sim_i$. It follows from transitivity of $\succsim_i$ that $(x_i, z_i) \in \succ_i$. Thus, we have $(x, z) \in P(R^P)$. ■

5 Lexicographic Composition of Multiple Choice Functions

In this section, we study the procedure β to compose multiple criteria, namely, we first choose the set of maximal elements for the first binary relation R^1, and then from this set we select its subset of maximal elements for the second binary relation R^2, and iterate this procedure until the last binary relation R^M. Formally, the procedure is represented by the choice function $C_{P(R^M)} \cdots C_{P(R^1)}$ defined recursively as, for every $S \in \mathcal{X}$,

$$\begin{aligned} C^0(S) &= S \\ C^m(S) &= C_{P(R^m)}(C^{m-1}(S)) \text{ for each } m = 1, \ldots, M \end{aligned}$$

and $C_{P(R^M)} \cdots C_{P(R^1)}(S) = C^M(S)$.

In contrast to procedure α, procedure β provides non-empty outcomes under very mild conditions. Indeed, if each of the original criteria, $P(R^1), \ldots, P(R^M)$, does not have a cycle, then $C_{P(R^M)} \cdots C_{P(R^1)}(S) \neq \emptyset$ for every $S \in \mathcal{X}$. However, even if there exists a cycle S for $P(R^2)$, $C_{P(R^2)} C_{P(R^1)}(S) \neq \emptyset$ holds as long as $P(R^1)$ is acyclic and $P(R^1)$ ranks at least one pair in S, and eliminate at least one alternative from $C_{P(R^1)}(S)$. A similar observation holds for every R^m, $m = 3, \ldots, M$. The following result, which was shown in Houy (2007), provides a necessary and sufficient condition for $C_{P(R^M)} \cdots C_{P(R^1)}$ to satisfy non-emptiness.

Proposition 7 *Let $R^1, \ldots, R^M \in \mathcal{R}$ be given. The choice function $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness if and only if for every $m \in \{1, \ldots, M\}$, and for every cycle $(x^1, \ldots, x^K) \subseteq X$ for $P(R^m)$, there exist $m' < m$ and $k, \ell \in \{1, \ldots, K\}$ such that $(x^k, x^\ell) \in P(R^{m'})$.*

Proof. See Houy (2007, Theorem 2). ■

Comparing Propositions 3 and 7, we can see that if $C_{P(R^1, \ldots, R^M)}$ satisfies non-emptiness (or equivalently, $P(R^1, \ldots, R^M)$ is acyclic), then $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies non-emptiness as well. In other words, when we compose lexicographically two criteria for decision making, it is more difficult to guarantee non-empty choices under procedure α than under procedure β.

Corollary 5 *Let $R^1, \ldots, R^M \in \mathcal{R}$ be given. If the choice function $C_{P(R^1,\ldots,R^M)}$ satisfies non-emptiness, then $C_{P(R^M)} \cdots C_{P(R^1)}$ also satisfies non-emptiness.*

The following example shows that the converse of Corollary 5 does not hold true.

Example 6 Let R^P be the weak Pareto domination, and let R^F be defined as in Example 1. As noted above, R^P is quasi-transitive and R^F is transitive. Hence, for every finite set $S \in \mathbb{R}^m_+$, $C_{P(R^P)}(S) \neq \emptyset$, and $C_{P(R^F)}(C_{P(R^P)}(S)) \neq \emptyset$. However, there exists a cycle for $P(R^P, R^F)$ (Tadenuma, 2002), and hence $C_{P(R^P,R^F)}$ does not satisfy non-emptiness. The same result holds for the other criteria given in Example 4, namely $C_{P(R^H)}C_{P(R^P)}$ and $C_{P(R^E)}C_{P(R^P)}$ satisfy non-emptiness whereas $C_{P(R^H,R^P)}$ and $C_{P(R^E,R^P)}$ do not.

We now examine the choice consistency properties of the lexicographic composition of multiple choice functions. First, we show a basic relationship between the choice functions derived from procedures α and β. It also implies Corollary 5 above.

Lemma 1 *For every $S \in \mathcal{X}$, $C_{P(R^1)}(S) \cap \cdots \cap C_{P(R^M)}(S) \subseteq C_{P(R^1,\ldots,R^M)}(S) \subseteq C_{P(R^M)} \cdots C_{P(R^1)}(S)$.*

Proof. To show that $C_{P(R^1)}(S)\cap\cdots\cap C_{P(R^M)}(S) \subseteq C_{P(R^1,\ldots,R^M)}(S)$, let $S \in \mathcal{X}$, $x \in S$ and $x \notin C_{P(R^1,\ldots,R^M)}(S)$. Then, there exists $y \in S$ with $(y, x) \in P(R^1, \ldots, R^M)$. By definition, $(y, x) \in P(R^m)$ for some $m \in \{1, \ldots M\}$. Hence, $x \notin C_{P(R^m)}(S)$, and $x \notin C_{P(R^1)}(S) \cap \cdots \cap C_{P(R^M)}(S)$.

To prove that $C_{P(R^1,\ldots,R^M)}(S) \subseteq C_{P(R^M)} \cdots C_{P(R^1)}(S)$, let $S \in \mathcal{X}$, $x \in S$ and $x \notin C_{P(R^M)} \cdots C_{P(R^1)}(S)$. Then, either (i) $x \notin C_{P(R^1)}(S)$ or (ii) there exist $m \in \{1, ..., M-1\}$ and $y \in S$ such that $x, y \in C_{P(R^m)} \cdots C_{P(R^1)}(S)$ and $(y, x) \in P(R^{m+1})$. In case (i), there exists $z \in X$ such that $(z, x) \in P(R^1)$, and hence $(z, x) \in P(R^1, \ldots, R^M)$. Therefore, $x \notin C_{P(R^1,\ldots,R^M)}(S)$. In case (ii), it follows that $(x, y) \notin P(R^1)\cup\cdots\cup P(R^m)$ and $(y, x) \in P(R^{n+1})$. By definition, $(y, x) \in P(R^1, \ldots, R^M)$, which implies $x \notin C_{P(R^1,\ldots,R^M)}(S)$. ■

Our next proposition shows a necessary and sufficient condition for $C_{P(R^M)} \cdots C_{P(R^1)}$ to satisfy Contraction Consistency. It is interesting to see that the condition requires a certain relationship among any three alternatives in terms of the decompositions of $P(R^1, \ldots, R^M)$ defined below.

Let $R^1, \ldots, R^M$ be given. Define $\Gamma^1, \ldots, \Gamma^M \in \mathcal{R}$ as follows: $\Gamma^1 = P(R^1)$, and for each $m \in \{2, \ldots, M\}$, and for all $x, y \in X$, $(x, y) \in \Gamma^m$ if and only if $(x, y), (y, x) \notin P(R^{m'})$ for all $m' < m$ and $(x, y) \in P(R^m)$. Note that $P(R^1, \ldots, R^M) = \Gamma^1 \cup \Gamma^2 \cup \cdots \cup \Gamma^M$ and for all $m, m' \in \{1, \ldots, M\}$ with $m \neq m'$, $\Gamma^m \cap \Gamma^{m'} = \emptyset$.

Proposition 8 *Assume that $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness. Then, $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Contraction Consistency if and only if for all $x, y, z \in X$, and for all $m, m' \in \{1, \ldots, M\}$ with $m < m'$, $[(x, y) \in \Gamma^m$ and $(y, z) \in \Gamma^{m'}]$ implies $(x, z) \in \Gamma^{m''}$ for some $m'' \in \{1, \ldots, M\}$. Moreover, if $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency, then $C_{P(R^M)} \cdots C_{P(R^1)} = C_{P(R^1,\ldots,R^M)}$.*

Proof. Sufficiency. Assume that $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness, and that for all $x, y, z \in X$, and for all $m, m' \in \{1, \ldots, M\}$ with $m < m'$, $[(x, y) \in \Gamma^m$ and $(y, z) \in \Gamma^{m'}]$ implies $(x, z) \in \Gamma^{m''}$ for some $m'' \in \{1, \ldots, M\}$.

First, we show that $P(R^1, \ldots, R^M)$ is acyclic. Suppose, on the contrary, that $P(R^1, \ldots, R^M)$ has a cycle. Let $Y = (x^1, ..., x^K) \subseteq X$ be one of the cycles with the smallest cardinality. Since $P(R^1, \ldots, R^M)$ is asymmetric, $K \geq 3$. By Proposition 7 and since $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness, it follows that for no $m \in \{1, \ldots, M\}$, Y is a cycle for Γ^m. Hence, there exist $m, m' \in \{1, \ldots, M\}$ with $m < m'$ and $k \in \{1, \ldots, K\}$ such that $(x^k, x^{k+1}) \in \Gamma^m$ and $(x^{k+1}, x^{k+2}) \in \Gamma^{m'}$, where we abuse notation by letting $(K + 1) := 1$ and $(K + 2) := 2$. By the assumption, we have $(x^k, x^{k+2}) \in \Gamma^{m''}$ for some $m'' \in \{1, \ldots, M\}$. Then, $(x^1, \ldots, x^k, x^{k+2}, \ldots, x^K)$ is a cycle for $P(R^1, \ldots, R^M)$, which contradicts the fact that Y is one of the cycles with the smallest cardinality. Thus, $P(R^1, \ldots, R^M)$ is acyclic.

Now we show that $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Contraction Consistency. Suppose, on the contrary, that $C_{P(R^M)} \cdots C_{P(R^1)}$ violates Contraction Consistency. Then, there exist $S, T \in \mathcal{X}$ with $S \subseteq T$ and $x \in S$ such that $x \in C_{P(R^M)} \cdots C_{P(R^1)}(T)$ but $x \notin C_{P(R^M)} \cdots C_{P(R^1)}(S)$.

Since $x \in C_{P(R^1)}(T)$ and $S \subseteq T$, we have $x \in C_{P(R^1)}(S)$. However, because $x \notin C_{P(R^M)} \cdots C_{P(R^1)}(S)$, there exist $n_1 \in \{2, \ldots, M\}$ and $y^1 \in C_{P(R^{n_1-1})} \cdots C_{P(R^1)}(S)$ such that $(y^1, x) \in \Gamma^{n_1}$. Then, since $x \in C_{P(R^M)} \cdots C_{P(R^1)}(T)$, it must be true that $y^1 \notin C_{P(R^{n_1-1})} \cdots C_{P(R^1)}(T)$.

Then there exist $m_1 < n_1$ and $y^2 \in C_{P(R^{m_1-1})} \cdots C_{P(R^1)}(T)$ with $(y^2, y^1) \in \Gamma^{m_1}$. By the assumption, we have $(y^2, x) \in \Gamma^{n_2}$ for some $n_2 \in \{2, \ldots, M\}$. (Note that $(y^2, x) \notin \Gamma^1 = P(R^1)$ since $x \in C_{P(R^1)}(T)$.)

Because $x \in C_{P(R^M)} \cdots C_{P(R^1)}(T)$, it must be true that $y^2 \notin C_{P(R^{n_2-1})} \cdots C_{P(R^1)}(T)$. Then, a similar argument shows that there exist $y^3 \in T$ and $m_2 \in \{1, \ldots, M\}$ such that $(y^3, y^2) \in \Gamma^{m_2}$ and $(y^3, x) \in \Gamma^{n_3}$ for some $n_3 \in \{2, \ldots, M\}$.

Repeating the above argument, we obtain a sequence $(y^1, y^2, y^3, \ldots) \in T \times T ...$ such that for every k, $(y^{k+1}, y^k) \in \Gamma^{m_k} \subset P(R^1, \ldots, R^M)$. Since T is finite, there exists ℓ such that $y^\ell = y^k$ for some $k < \ell$. This contradicts the acyclicity of $P(R^1, \ldots, R^M)$.

Necessity. Suppose that there exist $x, y, z \in X$ and $m, m' \in \{1, \dots, M\}$ with $m < m'$ such that $(x, y) \in \Gamma^m$, $(y, z) \in \Gamma^{m'}$, and for all $m'' \in \{1, \dots, M\}$, $(x, z) \notin \Gamma^{m''}$. Let $T = \{x, y, z\}$ and $S = \{y, z\}$. Then, $z \in C_{P(R^M)} \cdots C_{P(R^1)}(T)$ but $z \notin C_{P(R^M)} \cdots C_{P(R^1)}(S)$. Thus, $C_{P(R^M)} \cdots C_{P(R^1)}$ does not satisfy Contraction Consistency.

Now let us prove that if $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency, then, $C_{P(R^M)} \cdots C_{P(R^1)} = C_{P(R^1,\dots,R^M)}$. By Lemma 1, $C_{P(R^1,\dots,R^M)}(S) \subseteq C_{P(R^M)} \cdots C_{P(R^1)}(S)$ for every $S \in \mathcal{X}$. To show the converse inclusion relation, let $x \in S$ and $x \notin C_{P(R^1,\dots,R^M)}(S)$. Then, there exists $y \in S$ such that $(y, x) \in P(R^1, \dots, R^M)$. Hence, for some $m \in \{1, \dots, M\}$, $(y, x) \in \Gamma^m$. This implies that $x \notin C_{P(R^m)} \cdots C_{P(R^1)}(\{x, y\})$ and thus $x \notin C_{P(R^M)} \cdots C_{P(R^1)}(\{x, y\})$. It follows from Contraction Consistency of $C_{P(R^M)} \cdots C_{P(R^1)}$ that $x \notin C_{P(R^M)} \cdots C_{P(R^1)}(S)$. ■

The following example shows that even if $C_{P(R^1,\dots,R^M)}$ satisfies Non-Emptiness and Contraction Consistency, or equivalently, $P(R^1, \dots, R^M)$ is acyclic, then it is possible that $C_{P(R^M)} \cdots C_{P(R^1)} \neq C_{P(R^1,\dots,R^M)}$ and $C_{P(R^M)} \cdots C_{P(R^1)}$ violates Contraction Consistency.

Example 7 Let $S = \{x, y, z\}$. Assume that $P(R^1) = \{(z, y)\}$ and $P(R^2) = \{(y, x)\}$. Then, $P(R^1, R^2)$ is acyclic and $C_{P(R^1,R^2)}$ satisfies Non-Emptiness and Contraction Consistency. However, $C_{P(R^2)}C_{P(R^1)}(S) = C_{P(R^2)}(\{x, z\}) = \{x, z\}$ and $C_{P(R^1,R^2)}(S) = \{z\}$. Hence, $C_{P(R^2)}C_{P(R^1)} \neq C_{P(R^1,R^2)}$. Let $T = \{x, y\} \subset S$. Then, although $x \in T \cap C_{P(R^2)}C_{P(R^1)}(S)$, $C_{P(R^2)}C_{P(R^1)}(T) = \{y\}$. This is a violation of Contraction Consistency.

Corollary 6 *Let $R^1, \dots, R^M \in \mathcal{R}$ be given. If $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness and Contraction Consistency, then $C_{P(R^1,\dots,R^M)}$ satisfies Non-Emptiness and Contraction Consistency, or equivalently, $P(R^1, \dots, R^M)$ is acyclic. However, the converse does not hold true.*

In contrast to the results on Contraction Consistency, requiring Non-Emptiness and Path Independence for $C_{P(R^M)} \cdots C_{P(R^1)}$ is equivalent to requiring the same conditions for $C_{P(R^1,\dots,R^M)}$ as the following proposition shows.

Proposition 9 *The following three statements are equivalent.*

1. *$P(R^1, \dots, R^M)$ is quasi-transitive.*

2. *$C_{P(R^1,\dots,R^M)}$ satisfies Non-Emptiness and Path Independence.*

3. $C_{P(R^M)} \cdots C_{P(R^1)}$ *satisfies Non-Emptiness and Path Independence.*

Proof.
$1 \Leftrightarrow 2$:
This has already been shown in Corollary 2.

$1 \Rightarrow 3$:
Assume that $P(R^1, \ldots, R^M)$ is quasi transitive. Then, it is acyclic, and by Corollary 1, $C_{P(R^1,\ldots,R^M)}$ satisfies Non-Emptiness. It follows from Lemma 1 that $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness as well. Notice that if $P(R^1, \ldots, R^M)$ is quasi transitive, then the necessary and sufficient condition for Contraction Consistency of $C_{P(R^M)} \cdots C_{P(R^1)}$ in Proposition 8 is met. Hence, by Proposition 8, $C_{P(R^M)} \cdots C_{P(R^1)} = C_{P(R^1,\ldots,R^M)}$. It follows from the equivalence of 1 with 2 that $C_{P(R^1,\ldots,R^M)}$ satisfies Path Independence. Thus, $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Path Independence.

$3 \Rightarrow 2$:
Assume that $C_{P(R^M)} \cdots C_{P(R^1)}$ satisfies Non-Emptiness and Path Independence. Since Path Independence implies Contraction Consistency, it follows from Proposition 8 that $C_{P(R^M)} \cdots C_{P(R^1)} = C_{P(R^1,\ldots,R^M)}$. Hence, $C_{P(R^1,\ldots,R^M)}$ satisfies Non-Emptiness and Path Independence. ■

To illustrate this result, let us go back to Examples 1, 2 and 3. We have shown that $C_{P(R^F,R^P)}$, $C_{P(R^H,R^P)}$, and $C_{P(R^E,R^P)}$ satisfy Non-Emptiness and Path Independent. Then, by Proposition 9, we can conclude that $C_{P(R^P)}C_{P(R^F)}$, $C_{P(R^P)}C_{P(R^H)}$, and $C_{P(R^P)}C_{P(R^E)}$ also satisfy Non-Emptiness and Path Independent.

6 Order Independence of Lexicographic Compositions

This section investigates under what conditions the outcomes of each choice procedure are independent of the order of lexicographic applications of multiple criteria.

The next result, which is based on Lemma 1, shows that if procedure α satisfies order independence, then it always chooses the set of alternatives that are maximal for *every* criterion.

Define the choice function $C_{P(R^1)} \cap \cdots \cap C_{P(R^M)} : \mathcal{X} \to \mathcal{X}$ as $[C_{P(R^1)} \cap \cdots \cap C_{P(R^M)}](S) = C_{P(R^1)}(S) \cap \cdots \cap C_{P(R^M)}(S)$ for every $S \in \mathcal{X}$.

Proposition 10 *If* $C_{P(R^1,\ldots,R^M)} = C_{P(R^{\pi(1)},\ldots,R^{\pi(M)})}$ *for every permutation* π *on* $\{1, \ldots, M\}$*, then* $C_{P(R^1,\ldots,R^M)} = C_{P(R^1)} \cap \cdots \cap C_{P(R^M)}$.

Proof. Assume that $C_{P(R^1,\dots,R^M)}(S) = C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}(S)$ for every permutation π on $\{1,\dots,M\}$. Let $S \in \mathcal{X}$. By Lemma 1, it is enough to show that $C_{P(R^1,\dots,R^M)}(S) \subseteq C_{P(R^1)}(S) \cap \cdots \cap C_{P(R^M)}(S)$. Let $x \in S$ and $x \notin C_{P(R^1)}(S) \cap \cdots \cap C_{P(R^M)}(S)$. Then, there exists $i \in \{1,...,M\}$ and $y \in S$ such that $(y,x) \in P(R^i)$. Consider a permutation π on $\{1,\dots,M\}$ such that $\pi(1) = i$. Then, $(y,x) \in P(R^{\pi(1)})$. Hence, by the definition of $P(R^{\pi(1)},\dots,R^{\pi(M)})$, $(y,x) \in P(R^{\pi(1)},\dots,R^{\pi(M)})$. Thus, $x \notin C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}(S) = C_{P(R^1,\dots,R^M)}(S)$. ■

As in the case of the efficiency-equity trade-off, if x is superior to y according to one criterion where as y is better than x by another criterion, then the best choice from $\{x,y\}$ must depend on which criterion we should take first. Hence, in order for the final choice to be independent of the order of application of the two criteria, such conflict cannot arise. The following result formalizes this observation.

Proposition 11 *$C_{P(R^1,\dots,R^M)} = C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}$ for every permutation π on $\{1,\dots,M\}$ if and only if $P(R^1) \cup \cdots \cup P(R^M)$ is asymmetric. Moreover, if $C_{P(R^1,\dots,R^M)} = C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}$ for every permutation π on $\{1,\dots,M\}$, then $C_{P(R^1,\dots,R^M)} = C_{P(R^1)\cup\cdots\cup P(R^M)} = C_{P(R^1)} \cap \cdots \cap C_{P(R^M)}$.*

Proof. Necessity. Suppose that $P(R^1) \cup \cdots \cup P(R^M)$ is symmetric. Since each of the M binary relations, $P(R^1),\dots,P(R^M)$, is asymmetric by definition, there exist $x,y \in X$ such that $x \neq y$, $(x,y) \in P(R^i)$ and $(y,x) \in P(R^j)$ for some $i,j \in \{1,...M\}$ with $i \neq j$. Let π and σ be two permutations on $\{1,\dots,M\}$ such that $\pi(1) = i$ and $\sigma(1) = j$. Then, by the definition of $P(R^{\pi(1)},\dots,R^{\pi(M)})$, $(x,y) \in P(R^{\pi(1)},\dots,R^{\pi(M)})$ whereas by the definition of $P(R^{\sigma(1)},\dots,R^{\sigma(M)})$, $(y,x) \in P(R^{\sigma(1)},\dots,R^{\sigma(M)})$. Hence, $C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}(\{x,y\}) = \{x\} \neq \{y\} = C_{P(R^{\sigma(1)},\dots,R^{\sigma(M)})}(\{x,y\})$.

Sufficiency. Assume that $P(R^1) \cup \cdots \cup P(R^M)$ is asymmetric. It is clear that generally $P(R^1,\dots,R^M) \subseteq P(R^1) \cup \cdots \cup P(R^M)$. Let $x,y \in X$ and $(x,y) \in P(R^1) \cup \cdots \cup P(R^M)$. Then, $(x,y) \in P(R^i)$ for some $i \in \{1,\dots,M\}$. By the assumption, $(y,x) \notin P(R^j)$ for all $j \in \{1,\dots,M\}$. Thus, from the definition of $P(R^1,\dots,R^M)$, we have $(x,y) \in P(R^1,\dots,R^M)$. Then, we have shown that $P(R^1,\dots,R^M) = P(R^1) \cup \cdots \cup P(R^M)$. By the same argument, it can be shown that for every permutation π on $\{1,\dots,M\}$, $C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}(S) = C_{P(R^1)\cup\cdots\cup P(R^M)}(S)$. Thus, $C_{P(R^1,\dots,R^M)}(S) = C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}(S) = C_{P(R^1)\cup\cdots\cup P(R^M)}(S)$. Together with Proposition 10, this completes the proof. ■

If we also require non-emptiness as well as order independence, a further stronger condition must be called for.

Corollary 7 $C_{P(R^1,\dots,R^M)} = C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}$ *for every permutation* π *on* $\{1,\dots,M\}$ *and* $C_{P(R^1,\dots,R^M)}$ *satisfies Non Emptiness if and only if* $P(R^1)\cup\cdots\cup P(R^M)$ *is acyclic.*

Proof. Necessity. Assume that $C_{P(R^1,\dots,R^M)} = C_{P(R^{\pi(1)},\dots,R^{\pi(M)})}$ for every permutation π on $\{1,\dots,M\}$ and $C_{P(R^1,\dots,R^M)}$ satisfies Non-Emptiness. By Proposition 11, $P(R^1)\cup\cdots\cup P(R^M)$ is asymmetric and $C_{P(R^1,\dots,R^M)} = C_{P(R^1)\cup\cdots\cup P(R^M)}$. Then, $C_{P(R^1)\cup\cdots\cup P(R^M)}$ satisfies Non-Emptiness. By Corollary 1, $P(R^1)\cup\cdots\cup P(R^M)$ is acyclic.

Sufficiency. Assume that $P(R^1)\cup\cdots\cup P(R^M)$ is acyclic. Then, $P(R^1)\cup\cdots\cup P(R^M)$ is asymmetric by definition, and it follows from Proposition 11 that $C_{P(R^{\pi(1)},\dots,R^{\pi(M)})} = C_{P(R^1)\cup\cdots\cup P(R^M)} = C_{P(R^1)}(S)\cap\cdots\cap C_{P(R^M)}$ for every permutation π on $\{1,\dots,M\}$. Moreover, by Corollary 1, $C_{P(R^1)\cup\cdots\cup P(R^M)}$ satisfies Non-Emptiness. ∎

Notice that acyclicity of $P(R^1)\cup\cdots\cup P(R^M)$ is sufficient but not necessary for non-emptiness of $C_{P(R^1,\dots,R^M)}$. It is necessary for $C_{P(R^1,\dots,R^M)}$ to satisfy order independence as well as non-emptiness.

Asymmetry of $P(R^1)\cup\cdots\cup P(R^M)$ is very demanding in economic allocation problems. It implies that any two criteria are never in contradiction. This requirement is rarely met when we are concerned with the efficiency and equity criteria.

The acyclicity of $P(R^1)\cup\cdots\cup P(R^M)$ is also necessary for procedure β to be order independent. In fact, requiring order independence of procedure β is even more demanding than procedure α.

Proposition 12 *If* $C_{P(R^M)}\cdots C_{P(R^1)} = C_{P(R^{\pi(M)})}\cdots C_{P(R^{\pi(1)})}$ *for every permutation* π *on* $\{1,\dots,M\}$*, then* $C_{P(R^M)}\cdots C_{P(R^1)} = C_{P(R^1)}(S)\cap\cdots\cap C_{P(R^M)}$.

Proof. By Lemma 1, $C_{P(R^1)}\cap\cdots\cap C_{P(R^M)} \subseteq C_{P(R^M)}\cdots C_{P(R^1)}$. Let $S\in\mathcal{X}$. Let $x\in S$ and $x\notin C_{P(R^1)}(S)\cap\cdots\cap C_{P(R^M)}(S)$. Then, there exists $i\in\{1,\dots,M\}$ and $y\in S$ such that $(y,x)\in P(R^i)$. Consider a permutation π on $\{1,\dots,M\}$ such that $\pi(1)=i$. By assumption, $(y,x)\in P(R^{\pi(1)})$, and hence $x\notin C_{P(R^{\pi(1)})}(S)$. Then, $x\notin C_{P(R^{\pi(M)})}\cdots C_{P(R^{\pi(1)})}(S) = C_{P(R^M)}\cdots C_{P(R^1)}(S)$. Therefore, $C_{P(R^M)}\cdots C_{P(R^1)} \subseteq C_{P(R^1)}\cap\cdots\cap C_{P(R^M)}$. ∎

A necessary and sufficient condition for procedure β to satisfy order independence as well as Non-Emptiness was given in Houy (2007).

Proposition 13 $C_{P(R^M)}\cdots C_{P(R^1)} = C_{P(R^{\pi(M)})}\cdots C_{P(R^{\pi(1)})}$ *for every permutation* π *on* $\{1,\dots,M\}$ *and* $C_{P(R^M)}\cdots C_{P(R^1)}$ *satisfies Non-Emptiness if and only if* (i) $P(R^1)\cup$

$\cdots \cup P(R^M)$ *is acyclic, and* (ii) *for all* $x, y, z \in X$*, if* $(x, y), (y, z) \in P(R^1) \cup \cdots \cup P(R^M)$*,* $(x, y) \in P(R^i)$ *and* $(y, z) \notin P(R^i)$ *for some* $i \in \{1, \cdots, M\}$*, then* $(x, z) \in P(R^1) \cup \cdots \cup P(R^M)$*.*

Proof. See Houy (2007, Theorem 5) ■

From Propositions 11, 12, and 13, we have the following corollary.

Corollary 8 *If* $C_{P(R^M)} \cdots C_{P(R^1)} = C_{P(R^{\pi(M)})} \cdots C_{P(R^{\pi(1)})}$ *for every permutation* π *on* $\{1, \ldots, M\}$ *and* $C_{P(R^M)} \cdots C_{P(R^1)}$ *satisfies Non-Emptiness, then* $C_{P(R^1, \ldots, R^M)} = C_{P(R^{\pi(1)}, \ldots, R^{\pi(M)})} = C_{P(R^M)} \cdots C_{P(R^1)} = C_{P(R^{\pi(M)})} \cdots C_{P(R^{\pi(1)})} = C_{P(R^1) \cup \cdots \cup P(R^M)} = C_{P(R^1)} \cap \cdots \cap C_{P(R^M)}$ *for every permutation* π *on* $\{1, \ldots, M\}$

The following example shows that order independence and non-emptiness of procedure β is strictly more demanding than those of procedure α. Let $\mathcal{X} = \{x, y, z\}$, $R^1 = \{(x, y)\}$, and $R^2 = \{(y, z)\}$. Then, $P(R^1, R^2) = P(R^2, R^1) = \{(x, y), (y, z)\}$, and hence, $C_{P(R^1, R^2)} = C_{P(R^2, R^1)}$. However, $C_{P(R^2)} C_{P(R^1)}(\{x, y, z\}) = \{x, z\}$ whereas $C_{P(R^1)} C_{P(R^2)}(\{x, y, z\}) = \{x\}$.

7 Conclusion

Social or individual decision making often involves multiple criteria. Lexicographic applications of the multiple criteria seem natural and reasonable ways to make decisions in such contexts. However, there are at least two distinct procedures to lexicographically apply two (social or individual) preference relations, as studied in this paper. Procedure α constructs the lexicographic composition of multiple binary relations, and then selects its maximal elements while procedure β first selects the set of maximal elements for the first binary relation, and then chooses from that set its maximal elements for the second binary relation, and iterates the procedure until the Mth binary relation.

There are indeed essential differences between these two procedures. First, procedure α, being a more deliberate way, often ends up with empty choices, whereas procedure β, being simpler and more intuitive, provides final choices as long as each of the two original criteria itself does not have inconsistency. For instance, acyclicity of the original binary relations is sufficient for procedure β to be non-empty, but it is not so for procedure α. Precise necessary and sufficient conditions for non-emptiness have been given in Sections 4 and 5 of this paper.

Second, although procedure β scarcely becomes empty, it may fail a minimum requirement of choice-consistency, namely contraction consistency, even in the case where procedure α satisfies non-emptiness and this consistency property. In fact, procedure β satisfies contraction consistency only when it coincides with procedure α. Exactly when this happens has also been shown in Section 5. However, turning to path independence, which is stronger than contraction consistency, procedure β satisfies non-emptiness and this condition when and only when procedure α satisfies the same conditions. This requires that the lexicographic composition of multiple binary relations is quasi-transitive.

Third, the outcomes of procedure α are non-empty and independent of the order of applications of the multiple criteria if and only if the union of the original binary relations is acyclic. This is already a very strong requirement because it implies that there is no conflict between any two criteria. Still, it is not sufficient for procedure β to satisfy the same condition. We need an additional condition given in Section 6.

In reality, there are many observations of inconsistent social or individual choices. Such observations may be explained by the "gap" between procedures β and α. People may actually use the simpler approach, namely procedure β, which always gives some answers as long as the original criteria themselves do not contain contradiction, but which quite easily fails a very basic condition of choice-consistency. In order to avoid inconsistent choices, they need to take the more deliberate approach, namely procedure α. But then, it often fails to provide optimal choices. This is a fundamental dilemma between non-emptiness and choice-consistency.

We also observe cases of disagreement among individuals who respect the same list of criteria. They may be explained by differences in order of application of the multiple criteria by the individuals.

We hope that the present paper contributes in clarifying the "gap" between the two procedures of decision making by showing several distinct sets of conditions for non-emptiness, contraction consistency, path independence, or order independence of each procedure. It would be interesting to use these conditions to examine how the choice procedures with multiple criteria can explain social or individual choices in concrete problems.

References

Aizerman, M. A. (1985), "New Problems in the General Choice Theory," *Social Choice and Welfare* 2, 235-282.

Aizerman, M. A. and F. T. Aleskrov (1995), *Theory of Choice.* Amsterdam: North Holland.

Arrow, K. J. (1963), *Social Choice and Individual Values*, 2nd ed., New York: Wiley. First ed., 1951.

Arrow, K. J. (1977), “Extended Sympathy and the Possibility of Social Choice,” *American Economic Review, Papers and Proceedings* 67, 219-225.

Blair, D. H., G. Bordes, J. S. Kelly and K. Suzumura, “Impossibility Theorems without Collective Rationality,” *Journal of Economic Theory* 13, 361-379.

Chernoff, H. (1954), “Rational Selection of Decision Functions,” *Econometrica* 22, 422-443.

Feldman, A. M. and A. Kirman (1974), “Fairness and Envy,” *American Economic Review* 64, 996-1005.

Foley, D. (1967), “Resource Allocation and the Public Sector,” *Yale Economic Essays* 7, 45-98.

Hammond, P. J. (1976), “Equity, Arrow’s Conditions and Rawls’ Difference Principle,” *Econometrica* 44, 793-804.

Harsanyi, J. C. (1955), “Cardinal Welfare, Individual Ethics, and Interpersonal Comparisons of Utility,” *Journal of Political Economy* 63, 309-321.

Houy, N. (2007), “Rationality and Order-Dependent Sequential Rationality,” *Theory and Decision* 62, 119-134.

Kolm, S.-C. (1972), *Justice et Equité*, Paris: Centre National de la Recherche Scientifique.

Manzini, P. and M. Mariotti (2005) “Sequentially Rationalizable Choice,” *American Economic Review*, forthcoming.

Pattanaik, P. K. (1968), “Risk, Impersonality, and the Social Welfare Function,” *Journal of Political Economy* 76, 1152-1169.

Pazner, E. A. and D. Schmeidler (1978), “Egalitarian Equivalent Allocations: A New Concept of Economic Equity,” *Quarterly Journal of Economics* 92, 671-687.

Plott, C. R. (1973), "Path Independence, Rationality, and Social Choice," *Econometrica* 41, 1075-1091.

Sen, A. K. (1970), *Collective Choice and Social Welfare*, San Francisco: Holden-Day.

Sen A.K. (1985), *Commodities and Capabilities*, Amsterdam: North-Holland.

Suppes, P. (1966), "Some Formal Models of Grading Principles," *Synthese* 6, 284-306.

Suzumura, K. (1981a), "On Pareto-Efficiency and the No-Envy Concept of Equity," *Journal of Economic Theory* 25, 367-379.

Suzumura, K. (1981b), "On the Possibility of "Fair" Collective Choice Rule," *International Economic Review* 22, 351-364.

Suzumura, K. (1983a), *Rational Choice, Collective Decisions, and Social Welfare*, Cambridge, U. K.: Cambridge University Press.

Suzumura, K. (1983b), "Resolving Conflicting Views of Justice in Social Choice," in P. K. Pattanaik and M. Salles (eds.), *Social Choice and Welfare*, pp. 125-149, Amsterdam: North-Holland.

Tadenuma, K. (2002), "Efficiency First or Equity First? Two Principles and Rationality of Social Choice," *Journal of Economic Theory* 104, 462-472.

Tadenuma, K. (2005), "Egalitarian-Equivalence and the Pareto Principle for Social Preferences," *Social Choice and Welfare* 24, 455-473.