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A Note on the Core of a Profit-Center Game
with Incomplete Information
and Increasing Returns to Scale

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Abstract

An existence theorem of a *full-information revealing core plan* of a profit-center game with incomplete information and increasing returns to scale is given. It does not exclude *nonmarketed intermediate commodities*.

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Key Words: profit-center game with incomplete information, *ex ante* Bayesian incentive-compatible core, increasing returns to scale, distributive set, *non-marketed intermediate commodity*

1 Introduction

To deal with the *transfer payment* problem Radner [4] introduced a profit-center game, and Ichiishi and Radner [2] extended it toward the incomplete information situation. Ichiishi and Radner [2] proved that the nonemptiness of the *ex ante Bayesian incentive compatible core* in three interesting cases. But their proof for the case of increasing-returns-to-scale technology is much involved and excludes

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nonmarketed intermediate commodity. In this paper, the simple proof of the existence theorem for a *full-information revealing core plan* in a profit-center game with incomplete information and increasing returns to scale is given. It does not exclude *nonmarketed intermediate commodities*.

A profit-center game is a specific case of a strategic cooperative game with incomplete information. Several types of information revelation process are considered in this framework. Here, we use information revelation process called by *executing contract* developed by Ichiishi, Idzik, and Zhao [1]. Vohra [7] deals with another information revelation process called mediator based approach in the framework of a Bayesian pure exchange economy. For further discussion, see Ichiishi and Yamazaki [3].

2 A Profit-Center Game with Incomplete Information

Let K be the set of commodities. A generic element is $a \in K$. We denote the cardinality of K by k , so there are k commodities in the world. Assume that the commodity space is \mathbb{R}_+^k . The set of commodities K is divided into two categories K_m and K_n . A commodity $a \in K_m$ is called *marketed commodity*. A *marketed commodity* can be bought or sold on markets. So a *marketed commodity* $a \in K_m$ has a market price p_a . The market price vector for *marketed commodities* is denoted by $p \gg 0$. On the other hand, a commodity in K_n is called *nonmarketed commodity*. A *nonmarketed commodity* is owned or produced in the firm and used only internally; thus it has no market price. We denote the number of *marketed commodities* (resp. *nonmarketed commodity*) by k_m (resp. k_n). Of course, $k = k_m + k_n$.

A firm consists of finitely many *profit-centers*, that is *divisions*. Let N be the finite set of divisions. Each division is considered as an independent decision maker. A division j is characterized by exogenously given data $\{T^j, (Y^j, r^j(\cdot))\}$, where T^j is a finite set of *types*, whose generic element is t^j , Y^j is a *production possibility set*, and $r^j : T^j \rightarrow \mathbb{R}_+^k$ is a *resource function*. In this model, a type $t^j \in T^j$ is interpreted as *asset specificity* of division j .

For any $S \in \mathcal{N} := 2^N \setminus \{\emptyset\}$, we denote the set of type profiles $T^S := \prod_{j \in S} T^j$. In particular, for grand coalition N , $T := T^N$. Notice that $Y^j \subset \mathbb{R}^{k|T^j|}$. We also define $Y^S := \prod_{j \in S} Y^j$. By abuse of notation, $r^j := r^j(T^j)$.

Let π be the *ex ante* probability distribution on T . We assume that π is a

product probability of π^j , $j \in N$, where π^j is a probability on T^j ; for simplicity, $\pi^j \gg 0$. We also assume that $(\pi^j)_{j \in N}$ is common knowledge.

Definition 1. A profit-center game with incomplete information is specified by exogenously given data

$$\mathcal{D} := \left(\{T^j, (Y^j, r^j(\cdot))\}_{j \in N}, \pi, p \right).$$

A profit-center game with incomplete information \mathcal{D} is played as follows. In *ex ante* stage, which no division knows the true type of any division, several divisions form a coalition and agree on their *profit imputation plan* and *net output plan*. A *profit imputation plan* of a coalition S is a type dependent *profit imputation* $x^S : T \rightarrow \mathbb{R}^S, t \mapsto x^S(t)$, where $x^S := (x^j)_{j \in S}$. An intended interpretation is $x^j(t)$ is a *profit imputation* of division j , given a type profile $t \in T$. A *net output plan* $y^S : T \rightarrow \mathbb{R}^{k \times S}$ is similarly defined. We call a pair (x^S, y^S) a *plan*.

Once the grand coalition N decide on a plan¹ (x^N, y^N) , the game proceeds *interim stage*, i.e., the nature reveals to division j that true type is \bar{t}^j . Notice that in this stage \bar{t}^j is j 's private information. In this stage, the plan is executed. However, since the type is private information, division j may have incentive to misrepresent its true type as \tilde{t}^j (instead of \bar{t}^j). If the plan had left such incentives, the member of N would not have agree on (x^N, y^N) from the beginning. They must have agreed on an *incentive-compatible* plan in the *ex ante* stage. As a result, the plan is truthfully executed.

To define the *incentive compatibility* precisely, we take the approach developed by Ichiishi, Idzik and Zhao [1]. We postulate that the *interim stage* is divided into two period and the set of commodities K is partitioned into $\{K_1, K_2\}$. The first *interim* period is called the *setup period* and the second *interim* period the *manufacturing period*.

For any *net output plan* y^S , we define $(y_1^S, y_2^S) := y^S$, where y_i^S correspond to K_i . In the first *interim* period, y_1^N is executed and in the second *interim* period, y_2^N and x^N are executed.

Now we are ready for formal analysis. To begin with, we define the *technological attainability*. For any coalition S , $(x^S, y^S) : T \rightarrow \mathbb{R}^{1+k}$ is *technologically attainable* if (x^S, y^S) satisfies

$$y^S \in Y^S$$

and

$$\forall t \in T : \sum_{j \in S} \begin{pmatrix} x^j(t) \\ 0 \end{pmatrix} \leq \sum_{j \in S} \begin{pmatrix} p \cdot y_m^j(t) \\ y_n^j(t) + r^j(t^j) \end{pmatrix}.$$

¹It is easy to extend the equilibrium concept to allow for realization of a coalition structure.

The set of *technologically attainable plan* for coalition S is denoted by F^S .

Next, we define the measurability condition of a strategy, which is called the *allowability*. In the setup period, any types are private information, so y_1^j must be measurable with respect to his private information. In the manufacturing period, however, more information can be used because information is revealed by executing y_1^S . Suppose the true types are \bar{F}^S and the coordinated strategy y_1^S is executed truthfully. Then, it is natural postulate that in the manufacturing period all divisions know that true types are in $(y_1^S)^{-1}(y_1^S(\bar{F}^S))$.

Let \mathcal{T}^j be the algebra generated by the partition $\{\{t^j\} \times T^{N \setminus \{j\}}\}_{t^j \in T^j}$ on T , and $\mathcal{A}(y_1^j)$ be the algebra generated by a function y_1^j on T . Define $\mathcal{T}^j(y_1^S) := \mathcal{T}^j \vee (\bigvee_{j \in S} \mathcal{A}(y_1^j))$. For any coalition S , $(x^S, y^S) \in F^S$ is *allowable* if (x^S, y^S) satisfies the following conditions:

- (i) y_1^j is \mathcal{T}^j -measurable;
- (ii) (x^j, y_2^j) is $\mathcal{T}^j(y_1^S)$ -measurable.

The set of *allowable plan* for coalition S is denoted by F'^S .

Postulate 1 (Information-Pooling Rule). *The member of coalition S can design only allowable plans.*

Even if (x^S, y^S) is *allowable*, there may be a division which has an incentive to represent a false type. Hence, we require the *Bayesian incentive compatibility* as a feasibility condition of a coordinated strategy. Suppose y^S be agreed upon in a coalition S . In the setup period, since each division's type is a private information, division $j \in S$ can choose $\hat{y}_1^j \in y_1^j(T^j)$ arbitrarily; however this choice restricts the action in the manufacturing period. By the information-pooling rule, it becomes common knowledge that division j 's type is in $A^j := (y_1^j)^{-1}(\hat{y}_1^j)$ in the manufacturing period; thus division j can only choose $\hat{y}_2^j \in y_2^j(A^j, A^{S \setminus \{j\}})$, where $A^{S \setminus \{j\}} := (y_1^{S \setminus \{j\}})^{-1}(\hat{y}_2^{S \setminus \{j\}})$ and $\hat{y}_2^{S \setminus \{j\}}$ is $S \setminus \{j\}$'s choice in the setup period. The *Bayesian incentive compatibility* is the condition that no division in S has an incentive to misrepresent its type in the above restriction.

Postulate 2 (Bayesian Incentive Compatibility). *The member of a coalition S can design only Bayesian incentive-compatible plans.*

The formal definition of the *Bayesian incentive compatibility* is a bit involved; see Ichiishi and Radner [2] for the detail. We denote by \hat{F}^S the *Bayesian incentive-compatible plan* for coalition S .

The *Bayesian incentive compatibility* is too stringent to guarantee the existence of equilibrium plan. Hence, we elaborate on a role of headquarters. The grand

coalition has the headquarters as its member, so can count on the latter's ability to insure monetary gain or loss. Therefore, it can adopt plans outside F^N , as long as they can be insured. A plan $(x^S, y^S) \in F^S$ is called *weakly Bayesian incentive-compatible* if for all $j \in S$, and all $\tilde{t}^j, \tilde{t}^j \in T^j$, it follows that

$$E(x^j | \tilde{t}^j) \geq E(x^j \circ (\tilde{t}^j, \text{id}) | \tilde{t}^j).$$

It is easy to show that if (x^S, y^S) is *weakly Bayesian incentive-compatible*, then $E(x^j | \mathcal{T}^j)$ is a constant function. So, if the headquarters is risk-neutral, then the following postulate is justified.

Postulate 3 (Headquarter's Insurability). *Let (x^N, y^N) be the allowable plan and $E(x^j | \mathcal{T}^j)$ is a constant function for each $j \in N$. Then the plan $((E(x^j | \mathcal{T}^j))_{j \in N}, y^N)$ is available to the grand coalition N .*

Let H^N be the set of plans satisfies the above condition. It is known that if (x^N, y^N) is technologically attainable and y^N satisfies the information-pooling rule, and $E(x^j | \mathcal{T}^j)$ is a constant function for each $j \in N$, then the plan $((E(x^j | \mathcal{T}^j))_{j \in N}, y^N)$ is Bayesian incentive-compatible. Thus, headquarter's insurability is consistent with the other postulates. In the light of this postulate, we can define the set of feasible plans as follows:

$$\hat{F}^{*N} := \begin{cases} \hat{F}^S, & \text{if } S \neq N \\ \hat{F}^N \cup H^N, & \text{if } S = N. \end{cases}$$

We are going to define a solution of the profit-center game (a specific strategic cooperative game): it is a *core plan*.

Definition 2. $(x^N, y^N) \in \hat{F}^{*N}$ is an *ex ante core plan* of a profit-center game with incomplete information \mathcal{D} if it is not true that

$$\exists S \in \mathcal{N} : \exists (x^S, y^S) \in \hat{F}^{*S} : \forall j \in S : Ex^j > Ex^{*j}.$$

If (x^{*N}, y_2^{*N}) is \mathcal{T}^N -measurable, then we call (x^{*N}, y^{*N}) an *ex ante full-information revealing core plan*.

There are two basic assumption that guarantees the existence of a core plan.

Assumption 1 (Basic Assumptions on the Production Sets). *For each coalition S , its total production set $Y(S)$ is given as*

(i) $Y(S) := \sum_{j \in S} Y^j$.

- (ii) The production set Y^j is closed in $\mathbb{R}^{k|T|}$ for each $j \in N$;
- (iii) $0 \in Y^j$ for each $j \in N$;
- (iv) $Y^j - \mathbb{R}_+^{k|T|} \subset Y^j$ for each $j \in N$;
- (v) for each $y_n^j \in \mathbb{R}^{k_n|T|}$, the production possibility set

$$\{(y_m^j, y_n^j) \in \mathbb{R}^{k_m|T|+k_n|T|} \mid (y_m^j, y_n^j) \in Y^j\}$$

is bounded from above.

Assumption 1 (ii)-(v) are standard. Assumption 1 (i) implies that there are no external economies. It is not difficult to extend our results to the case of existence of external economies.

Assumption 2 (Basic Assumptions on the Resource Functions).

- (i) $K_{1n} \neq \emptyset$;
- (ii) the resource function r_1^j is 1-1 on T^j ;
- (iii) $r^j(t^j) \geq 0$, for all $t^j \in T^j$.

Assumption 2 (ii) plays a crucial role to ensure the existence of full-information revealing core plans.

Ichiishi and Radner [2] established three types of core existence theorem. To prove our theorem, we need their first theorem (existence theorem under a convex production possibility set).

Theorem 1 (Ichiishi and Radner). *Let \mathcal{D} be a profit-center game with incomplete information which satisfies Postulate 1-3 and Assumption 1, 2. Assume moreover that Y^j is convex for any $j \in N$. Then there exists a full-information revealing core plan of the game.*

3 Distributive Production Sets

The main result of this paper is that an *ex ante* core plan exists even if the production possibility set exhibits increasing returns to scale. Ichiishi and Radner [2]’s second theorem addressed the nonemptiness of the core given an increasing-returns-to-scale technology, but had to exclude an intermediate commodity. Our main theorem, on the other hand, overcomes this shortcoming and indeed allows for presence of intermediate commodities. To state the condition which guarantees the existence of core plan, we introduce the ideas of *nonmarketed principal commodity* and *distributive technology*. A *nonmarketed commodity* which

is only used as input is called *nonmarketed principal commodity*. The set of *nonmarketed principal commodity* is denoted by K_{np} . Let $k_{np} := \#K_{np}$. Define $\Lambda := \mathbb{R}^{(k-k_{np})|T|} \times (-\mathbb{R}_+^{k_{np}|T|})$. By the definition of *nonmarketed principal commodity*, a production possibility set satisfies the following condition:

$$Y^j \subset -\Lambda$$

for any $j \in N$. A commodity $a \in K_n \setminus K_{np}$ can be a *nonmarketed intermediate commodity*.

Next, we define the *distributiveness* of a production possibility set. This idea is introduced by Scarf [5].

Definition 3. Let Y^j be a production possibility set which satisfies basic assumption on production set. The set Y^j is called *distributive* if for any finite number of points $y^i \in Y^j$, and any non-negative α_i , the point $y = \sum \alpha_i y^i$ is also in Y^j , if y satisfies the conditions $y^i - y \in \Lambda$.

Notice that if Y^j is distributive, then Y^j exhibits nondecreasing returns to scale. See Sharkey [6] for a clear presentation of the distributiveness concept.

Assumption 3 (Distributiveness of the Total Production Possibility Set).

- (i) $K_{np} \neq \emptyset$.
- (ii) $Y(N)$ is distributive.

Assumption 3 (i) is a mild one. Indeed, as an example of a *nonmarketed principal commodity*, consider a human capital.

Theorem 2 (Scarf). Let Y be a distributive set and let $\xi \notin Y$. Then, there is a nonnegative vector ρ such that

$$\begin{aligned} \rho \cdot \xi &> 0 \text{ and;} \\ \rho \cdot y &\leq 0 \text{ for any } y \in Y \cap [\Lambda + \xi]. \end{aligned}$$

Now, we can establish our main theorem. The proof uses quite similar logic as Scarf [5, Theorem 6], the existence theorem for the social equilibrium of a production economy with the distributive production possibility set.

Theorem 3. Let \mathcal{D} be a profit-center game with incomplete information which satisfies Postulate 1-3 and Assumption 1-3. Assume moreover that $\sum_{j \in N} r^j(t^j) \gg 0$ for any $t \in T$. Then there exists a full-information revealing core plan of the game.

Proof. Define by \hat{Y} the minimum closed convex cone which includes the set

$$\sum_{j \in N} Y^j \cap \left[\Lambda - \sum_{j \in N} r^j \right].$$

Consider the profit-center game \mathcal{D}' which is the same as \mathcal{D} except that each division has an identical production possibility set \hat{Y} . Since \hat{Y} is convex, there is a full-information revealing core plan (x^{*N}, y^{*N}) of \mathcal{D}' . By the construction of \hat{Y} , if $y^{*N} \in Y^N$, then (x^{*N}, y^{*N}) is a full-information revealing core plan of \mathcal{D} .

Let $y^* := \sum_{j \in N} y^{*j}$. Then $y^{*N} \in Y^N$ is equivalent to $y^* \in Y(N)$. (Remember $Y(N) = \sum_{j \in N} Y^j$.) Suppose $y^* \notin Y(N)$. By the distributiveness of $Y(N)$ and theorem 2, there is a nonnegative vector ρ such that

$$\rho \cdot y^* > 0 \text{ and;}$$

$$\rho \cdot y \leq 0 \text{ for any } y \in \sum_{j \in N} Y^j \cap \left[\Lambda - \sum_{j \in N} r^j \right]$$

By the second inequality and the definition of \hat{Y} , for any $y \in \hat{Y}$, $\rho \cdot y \leq 0$; in particular, $\rho \cdot y^* \leq 0$ —a contradiction. □

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