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Recovery Process Model

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Abstract

Recently because of Basel Ⅱ and the subprime mortgage crisis, the quantification of recovery size and recovery rate for the debt of a defaulted company is a serious problem for financial institutions and their supervision, but there has been no study of structure of recovery process. Existent recovery models do not regard recovery progress before the time of achievement of recovery.

We directly model recovery process for the debt of a single defaulted company. We model the recovery process by a homogeneous compound Poisson process and extend our model to an inhomogeneous compound Poisson process. Interest rate is explicitly used in our model. By our model, the relationship between cumulative recovery, the increment of recovery, the initial debt amount, the last recovery possible time and interest rate can be analyzed.

We derive the expectation and the variance of the survival value of the debt and recovery rate, and also derive the probability distribution function and the expectation of the recovery completion time. For this paper we present the numerical methods of calculating the expectation and the variance based on Panjer recursion formula and the fast Fourier transformation, and give numerical result. The methods of calculating the transition density of an inhomogeneous compound Poisson process is necessary for calculating the expectation and the variance of those in the inhomogeneous compound Poisson model, however little attention has been given to such methods. Therefore we propose the new procedure for calculating it by a piecewise homogeneous compound Poisson process.

Keywords Recovery rate, Credit risk, Basel, inhomogeneous compound Poisson process, Loan
JEL classification number C61, C63, G10, G21

1 Introduction

Recently, because of Basel Ⅱ and the subprime mortgage crisis, financial institutions must develop the model of recovery for the debt. But compared to the amount of research on other kinds of credit risk, few studies indeed have been done to analyze recovery for the debt.

There are a few empirical studies of recovery rate for the debt on the bank loan for small companies. Asarnow and Edwards (1995) study recovery rate of the loans at Citibank in United States and Hurt and Felsovalyi (1998) also study them at Citibank in Latin American countries. Araten, Jr. and Varshney (2004) analyze recovery rate of the loans at J.P. Morgan Chase. Franks, de Servigny and Davydenko (2004) research the correlation between the features of company and recovery rate from the data of ten banks in United kingdom, France and Germany. Dermine and de Carvalho (2006) study the factors of changing recovery rate by generalized linear model, using the data of Banco Comercial Português. Itoh and Yamashita (2008)

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study recovery rate of the guarantee loans of Credit Guarantee Corporations in Japan. They analyze the factors which affect recovery rate of the debt for small companies by binary logit model and ordered logit model. Also the point that all of above studies indicate the cumulative recovery rate has bimodal distribution deserves careful attention.

There are very few theoretical studies of recovery rate. Jokivuolle and Peura (2003) suppose if asset value at maturity is less than the default boundary, the company is assumed to default. It is a Merton (1974) type default framework in structural models. They also suppose the collateral value process of a company follows geometric Brownian motion, and the collateral value process and the asset value process are correlated. Recovery rate is mainly determined by collateral value at default.

Guo, Jarrow and Zeng (2008) set random recovery rate in reduced form model. Reduced form model gives default exogenously and focuses on modeling the stochastic process for the default intensity.

Thus there are a few empirical and theoretical studies for the recovery of the debt of an individual company. Because the recovery data is much important for lender and borrower on business, the empirical data is seldom opened for academic study. Also because the feature of the recovery action may be different by the lender, there are few general arguments for recovery progress.

For public study as far as we know, Itoh and Yamashita (2008) study empirically for the recovery progress (the relationship between time and recovery) for each company. Figure 6, Figure 7, and Figure 8 of Itoh and Yamashita (2008) show the empirical recovery process. From these figures, we know that there are usually several times of recovery for the same debt, that the recovery points are at random and that the increments of the recoveries are at random.

On the basis of study of Itoh and Yamashita (2008), in this paper we attempt to model the recovery process for the debt of single defaulted company by a homogeneous compound Poisson process and extend to an inhomogeneous compound Poisson process. Moreover we suppose there is interest on the debt after default and thus we can analyze the change of the survival value of the debt as interest rate changes.

Our model is similar to the framework of the aggregate loss model and the ruin process model in actuarial mathematics. For the aggregate loss model, see Mikosch (2004), Klugman, Panjer and Willmot (2004), Rolski, Schmidli, Schmidt and Teugels (1999). For further information of the ruin process model, see Gerber and Shiu (1998). Wu, Wang and Zhang (2005) add to the interest rate in the model of Gerber and Shiu (1998). Lindskog and Mcneil (2003) model an insurance loss and credit risk by a Poisson process.

In the real world, there are usually several times of recovery for the same debt. But other recovery models do not consider more than one time of recovery at all. Thus they can not distinguish the increment of recovery (each recovery) from the cumulative recovery. Our model can analyze the relationship between the cumulative recovery, the increment of recovery, the initial debt amount (debt amount at default), the last recovery possible time and interest rate.

This paper is organized as follows. We introduce the model in the second section and our main results of this paper are given the third section. The fourth section sets the distribution of increment of recovery and the fifth section extends to the inhomogeneous compound Poisson model and suggests the new procedure for calculating the transition of an inhomogeneous compound Poisson process. The sixth section demonstrates numerical results. The seventh section concludes. In appendix we explain a homogeneous compound Poisson process and present the numerical methods of calculating the probability distribution of compound Poisson distribution.

2 Model

We consider a single defaulted company whose default occurs at time 0. In our model, the prescription for the debt is assumed simply as below.

Assumption 1. *Lender can recover the loan up to time T. It means T is the last recovery possible time for* *lender.*

Assumption 2. *At time 0, the initial debt amount is D. It means D is the survival value of the debt at default time.*

We suppose there is interest on debt after default.

Assumption 3. *The interest rate per a period r is invariable (constant) for all periods and continuously compounded.*

Definition 1. *Let D_t be the survival value of the debt at time t, let U_n be the n-th recovery time, let X_n be the n-th increment of recovery which means each recovery size, where $0 < U_1 < U_2 < \cdots < U_{N_t} \le t$, and let N_t be the number of recoveries up to time t.*

In the case that at each time $U_1, U_2, \cdots, U_{N_t}$, borrower repays $X_1, X_2, \cdots, X_{N_t}$ respectively. The survival value of the debt at time t is

$$\begin{aligned} D_t &= e^{rt}D - e^{r(t-U_1)}X_1 - e^{r(t-U_2)}X_2 - \cdots - e^{r(t-U_{N_t})}X_{N_t} \\ &= e^{rt}D - \sum_{n=1}^{N_t} e^{r(t-U_n)}X_n. \end{aligned}$$

Assumption 4. *The increments of recoveries $\{X_n : n \in \mathbb{N}\}$ are non negative i.i.d. random variables and have the common distribution function F_X.*

Assumption 5. *N_t, the number of recoveries up to time t, follows the Poisson process with the intensity λ where λ is positive.*

Assumption 6. *The increments of recoveries $\{X_n : n \in \mathbb{N}\}$ and the number of recoveries N_t are independent.*

Definition 2. *Let S_t be the cumulative recovery with no interest effect up to time t and $\tilde{S}_t$ be the cumulative recovery with interest effect up to time t as follows.*

$$S_t \triangleq \sum_{n=1}^{N_t} X_n, \tag{1}$$

$$\tilde{S}_t \triangleq \sum_{n=1}^{N_t} e^{r(t-U_n)}X_n. \tag{2}$$

From above assumptions and the consequence of Section A.1, the cumulative recovery with interest rate follows a compound Poisson process and the survival value of the debt D_T may be written as,

$$D_T = e^{rT}D - \tilde{S}_T. \tag{3}$$

Figure 1 shows the relationship between the survival value of the debt D_t and time t.

3 Main Result

3.1 The Modified Survival Value of Debt and Recovery Rate

In this section, we derive the expectation and the variance of the modified survival value of the debt and recovery rate.

We have modeled the survival value of the debt D_T by (3). However we do not consider accomplishment of recovery (borrower clears all the debt) up to time T in the previous section and now we make the following assumption.

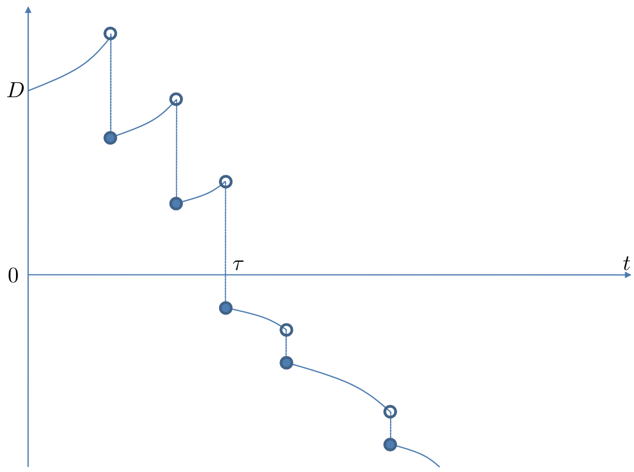


Figure 1: The relationship between the survival value of the debt D_t and time t.

Assumption 7. *Lender can not recover more than the amount of the initial value of the debt D plus interest on D.*

In view of Assumption 7, we introduce the following definition.

Definition 3. *Let τ be the first time that D_t is not positive. τ is called the recovery completion time, and we can write it as follows,*

$$\tau = \inf\left\{t \geq 0 : D_t \leq 0\right\}.$$

We employ standard convention that the infimum of the empty set is infinity.

Let M_t be the modified survival value of the debt,

$$M_t \triangleq \begin{cases} D_t, & t < \tau, \\ 0, & t \geq \tau. \end{cases}$$

By Assumption 7, once D_t crosses zero, all the debt are cleared and then the cumulative recovery amount is $De^{r\tau}$. The increment of recovery in the modified survival value of the debt is no longer independent and identically distributed. But using the probability function of $\tilde{S}_T$ (in which the increment of recovery is independent and identically distributed), we derive the expectation of the modified survival value of the debt at $T \in [0, \infty)$.

Theorem 3.1. *For $D, r > 0,\ T \geq 0$, we have*

$$E\left[M_T\right] = \int_0^{e^{rT}D} x f_{\tilde{S}_T}\left(e^{rT}D - x\right) dx, \tag{4}$$

where $f_{\tilde{S}_t}(x)$ is the probability density function of $\tilde{S}_t$ defined by (2)

$$f_{\tilde{S}_t}\left(x\right) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\left(\lambda t\right)^n}{n!} f^{*n}_{\tilde{X}_t}(x), \tag{5}$$

$f^{*n}_{\tilde{X}_t}(x)$ is the probability density function of the n-fold convolution of $\tilde{X}_t = e^{r(T-V_t)}X$, and V_t is uniform random variable on $(0, t]$.

We defer the proof of (5) until Section A.1.

Proof. At time T, we get

$$\begin{aligned} P\left\{D_T < x\right\} &= P\left\{e^{rT}D - \sum_{n=1}^{N_T} e^{r(T-U_n)}X_n < x\right\} \\ &= 1 - F_{\tilde{S}_T}(e^{rT}D - x). \end{aligned}$$

Because X_n is non negative, once $D_t = e^{rt}(D - \sum_{n=1}^{N_t} e^{-rU_n}X_n)$ becomes non positive at τ, D_t is monotone decreasing as t. Therefore only in the case $T < \tau$, D_T is almost surely positive for all sample paths. We remark $D_T \leq e^{rT}D$. Therefore, we get

$$\begin{aligned} E\left[M_T\right] &= E\left[\mathbf{1}_{\{D_T>0\}}D_T\right] \\ &= \int_0^{e^{rT}D} x dP\left\{D_T < x\right\} \\ &= \int_0^{e^{rT}D} x f_{\tilde{S}_T}\left(e^{rT}D - x\right)dx. \end{aligned}$$

□

Next we consider the variance of M_T.

Proposition 3.2.

$$Var\left(M_T\right) = \int_0^{e^{rT}D} x^2 f_{\tilde{S}_T}\left(e^{rT}D - x\right)dx - \left(\int_0^{e^{rT}D} x f_{\tilde{S}_T}\left(e^{rTt}D - x\right)dx\right)^2 \quad (6)$$

Proof. Note $\left(\mathbf{1}_{\{D_T>0\}}\right)^2 = \mathbf{1}_{\{D_T>0\}}$ and as in the proof of Theorem 3.1, we have

$$\begin{aligned} E\left[M_T^2\right] &= E\left[\mathbf{1}_{\{D_T>0\}}D_T^2\right] \\ &= \int_0^{e^{rT}D} x^2 f_{\tilde{S}_T}\left(e^{rT}D - x\right)dx. \end{aligned}$$

We obtain (6) easily. □

Let R_T be the cumulative recovery rate at T and it is defined as follows,

$$R_T = \frac{e^{rT}D - M_T}{e^{rT}D}.$$

We will derive the expectation and the variance of R_T from Theorem 3.1 and Proposition 3.2.

Corollary 3.3. *We have that*

$$\begin{aligned} E\left[R_T\right] &= \frac{e^{rT}D - E\left[M_T\right]}{e^{rT}D}, \\ Var\left(R_T\right) &= \frac{Var\left(M_T\right)}{e^{2rT}D^2}. \end{aligned}$$

3.2 Recovery Completion Time

In this section we derive the probability distribution function and the expectation of the recovery completion time τ.

First of all, we prove the following lemma. We set $\bar{X}_t = e^{-rV_t}X_1$ and let $F_{\bar{X}_t}$ be the probability distribution function of $\bar{X}_t$ where V_t is uniform random variable on $(0,t]$.

Lemma 3.4. *If F_X is continuous distribution, then*

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}F_{\bar{X}_t}(x) = \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left(\frac{(\lambda t)^n}{n!}F_{\bar{X}_t}(x)\right). \tag{7}$$

Proof. We set $g_{t,n}(x) = \frac{(\lambda t)^n}{n!}F_{\bar{X}_t}(x)$. For the proof, It is necessary to show that $\frac{\partial}{\partial t}g_{t,n}(x)$ is continuous as t, $\sum_{n=1}^{\infty} g_{t,n}(x)$ convergences and $\frac{\partial}{\partial t}g_{t,n}(x)$ convergences uniformly. We will illustrate that $\frac{\partial}{\partial t}g_{t,n}(x)$ convergences uniformly.

We use Weierstrass M-test for the check of uniform convergence.

$$\frac{\partial}{\partial t}g_{t,n}(x) = \sum_{n=1}^{\infty}\lambda\frac{(\lambda t)^{n-1}}{(n-1)!}F^{*n}_{\bar{X}_t}(x) + \sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\frac{\partial}{\partial t}F^{*n}_{\bar{X}_t}(x) \tag{8}$$

We consider each term of (8).

$$g_{t,n,1}(x) \triangleq \lambda\frac{(\lambda t)^{n-1}}{(n-1)!}F^{*n}_{\bar{X}_t}(x) \tag{9}$$

$$g_{t,n,2}(x) \triangleq \frac{(\lambda t)^n}{n!}\frac{\partial}{\partial t}F^{*n}_{\bar{X}_t}(x) \tag{10}$$

First, we show (9) converges uniformly. We set $\mathcal{M}_{n,1} = \lambda\frac{(\lambda t)^{n-1}}{(n-1)!}$, and we have $\sum_{n=1}^{\infty}\mathcal{M}_{n,1} = \lambda e^{\lambda t} < \infty$. Now, because of $0 \le F^{*n}_{\bar{X}_t}(x) \le 1$, we obtain

$$\left|g_{t,n,1}(x)\right| = \left|\lambda\frac{(\lambda t)^{n-1}}{(n-1)!}F^{*n}_{\bar{X}_t}(x)\right| \le \lambda\frac{(\lambda t)^{n-1}}{(n-1)!} = \mathcal{M}_{n,1}, \quad {}^{\forall}x\in[0,\infty), \quad {}^{\forall}t\in[0,T], \quad n=1,2,\cdots.$$

Then from Weierstrass M-test, $\sum_{n=1}^{\infty}g_{t,n,1}(x)$ converges uniformly.

Next, we show that (10) converges uniformly.

$$\frac{\partial}{\partial t}F^{*n}_{\bar{X}_t}(x) = -\frac{1}{t^2}\int_0^t F^{*n}_X\left(e^{ru}x\right)du + \frac{1}{t}F^{*n}_X(e^{rt}x) \tag{11}$$

We set $\mathcal{M}_{n,2} = \frac{(\lambda t)^n}{n!}\frac{1}{t}$, and we have $\sum_{n=1}^{\infty}\mathcal{M}_{n,2} = \left(e^{\lambda t}-1\right)\frac{1}{t} < \infty$. Now, because of $0 \le F^{*n}_{\bar{X}_t}(x) \le 1$, we obtain

$$\begin{aligned}\left|g_{t,n,2}(x)\right| &= \left|\frac{(\lambda t)^n}{n!}\frac{\partial}{\partial t}F^{*n}_{\bar{X}_t}(x)\right| = \left|\frac{(\lambda t)^n}{n!}\left(-\frac{1}{t^2}\int_0^t F^{*n}_X\left(e^{ru}x\right)du + \frac{1}{t}re^{rt}xF^{*n}_X(e^{rt}x)\right)\right| \\ &\le \left|\frac{(\lambda t)^n}{n!}\frac{1}{t}F^{*n}_X(e^{rt}x)\right| \\ &\le \frac{(\lambda t)^n}{n!}\frac{1}{t} = \mathcal{M}_{n,2}, \quad {}^{\forall}x\in[0,\infty), \quad {}^{\forall}t\in[0,T], \quad n=1,2,\cdots.\end{aligned}$$

Then from Weierstrass M-test, $\sum_{n=1}^{\infty}g_{t,n,2}(x)$ converges uniformly. Thus, because each term of (8) converges uniformly , $\frac{\partial}{\partial t}g_{t,n}(x)$ converges uniformly.

Similarly because $|g_{t,n}(x)| = |\frac{(\lambda t)^n}{n!} F_{\bar{X}_t}(x)| \leq \frac{(\lambda t)^n}{n!} \triangleq \mathcal{M}_n$ and $\sum_{n=1}^{\infty} \mathcal{M}_n = e^{\lambda t} - 1 < \infty$, then $\sum_{n=1}^{\infty} g_{t,n}(x)$ converges uniformly from Weierstrass M-test. Thus condition that $\sum_{n=1}^{\infty} g_{t,n}(x)$ converges is shown. $\frac{\partial}{\partial t} g_{t,n}$ is continuous as t in the case that $F_{\bar{X}_t}(x)$ is continuous distribution.

Therefore (7) holds. □

Next, we derive the probability distribution function and the expectation of the recovery completion time τ.

Proposition 3.5.

$$P\{\tau \leq T\} = 1 - F_{\tilde{S}_T}\left(e^{rT} D\right),$$

If F_X is continuous , then

$$E\left[\tau \mathbf{1}_{\{\tau \leq T\}}\right] = \int_0^T t \left\{ \lambda e^{-\lambda t} + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \left[\left(\lambda - \frac{n}{t}\right) F_{\bar{X}_t}^{*n}(D) + \int_0^t \frac{F_X^{*n}\left(e^{ru} D\right)}{t^2} du - \frac{F_X^{*n}(e^{rt} D)}{t} \right] \right\} dt. \quad (12)$$

Proof.

$$\begin{aligned} P\{\tau \leq T\} &= P\{D_T \leq 0\} \\ &= P\left\{ e^{rT} D - \sum_{n=1}^{N_T} e^{r(T-U_n)} X_n \leq 0 \right\} \\ &= 1 - F_{\tilde{S}_T}\left(e^{rT} D\right) \end{aligned}$$

Next we derive expectation.

$$\begin{aligned} E\left[\tau \mathbf{1}_{\{\tau \leq T\}}\right] &= \int_0^T t dP\{\tau \leq t\} \\ &= \int_0^T t \frac{\partial}{\partial t} \left(1 - F_{\tilde{S}_t}\left(e^{rT} D\right)\right) dt \end{aligned} \quad (13)$$

Similarly as in Section A.1 (cf. (29)), we have $F_{\tilde{S}_t}(e^{rt} x) = e^{-\lambda t} + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} F_{\bar{X}_t}^{*n}(x)$, and

$$\begin{aligned} F_{\bar{X}_t}(x) &= P\left\{\bar{X}_t \leq x\right\} \\ &= \frac{1}{t} \int_0^t F_X\left(e^{ru} x\right) du. \end{aligned}$$

Using Lemma 3.4, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left(1 - F_{\tilde{S}_t}\left(e^{rt} x\right)\right) \\ &= \frac{\partial}{\partial t}\left(1 - e^{-\lambda t} - e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} F_{\bar{X}_t}^{*n}(x)\right) \\ &= \lambda e^{-\lambda t} + \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} F_{\bar{X}_t}^{*n}(x) - e^{-\lambda t} \sum_{n=1}^{\infty} n \frac{\lambda^n t^{n-1}}{n!} F_{\bar{X}_t}^{*n}(x) - e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \frac{\partial}{\partial t} F_{\bar{X}_t}^{*n}(x) \\ &= \lambda e^{-\lambda t} + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \left[\left(\lambda - \frac{n}{t}\right) F_{\bar{X}_t}^{*n}(x) - \frac{\partial}{\partial t} F_{\bar{X}_t}^{*n}(x)\right]. \end{aligned} \quad (14)$$

Substituting (11) and (14) into (13), we obtain (12). □

4 Numerical Study

In this section we set notation and the distribution of the increment of recovery. In Section 4.2 we specify the distribution of the increment of recovery. Also we present the two methods of calculating the probability distribution of compound Poisson distribution in Section B: Panjer recursion formula and the fast Fourier transformation.

4.1 Notation

In order to calculate (4), we must calculate $f_{\tilde{S}_T}$. From the consequence of Section A.1, given fixed T, $\tilde{S}_T$ is considered as $\tilde{S}$ which is compound Poisson distribution with the intensity λT and increment of recovery $\tilde{X}_T$. Also from Section A.1, the method of calculating probability function or probability density function of $\tilde{S}$ is the same to that of S without interest effect for X where S is compound Poisson distribution with the intensity λT and increment of recovery X. We will explain the method of calculating probability function or probability density function of S when increment of recovery X is continuous and discrete respectively.

Suppose $\{X_n : n \in \mathbb{N}\}$ are discrete (continuous) random variables, let p_X be the probability function (f_X be the probability density function) of X.

Therefore in the case $\{X_n : n \in \mathbb{N}\}$ have a discrete distribution, p_S is the probability function of S, and in the case $\{X_n : n \in \mathbb{N}\}$ have a continuous distribution, f_S is the probability density function of S. Also let F_S be the probability distribution function of S. Let P_N is the probability function of Poisson distribution with the intensity λT. Then

$$p_S(x) = \sum_{n=0}^{\infty} p_N(n) p_X^{*n}(x), \quad x \geq 0,$$
$$f_S(x) = \sum_{n=0}^{\infty} p_N(n) f_X^{*n}(x), \quad x \geq 0.$$

Since it is difficult to calculate p_S or f_S directly by above expression, we will present the two methods of calculating it in Section B: Panjer recursion formula and the fast Fourier transformation.

4.2 Increment of Recovery

If the increment of recovery has continuous distribution, it is difficult to calculate F_S by the methods based on either Panjer recursion formula or the fast Fourier transformation, but if the increment of recovery has discrete distribution, it is easy to calculate p_S. Thus if the increment of recovery has continuous distribution, we discretize its distribution. In Section 4.2.1, we propose some methods of discretization.

Up to here, we have not supposed the increment of recovery has particular distribution, but in the remaining section let us assume the increment of recovery has exponential or Pareto distribution.

4.2.1 Discretization

If $\{X_n : n \in \mathbb{N}\}$ have a continuous distribution function, we approximate it by the following method which Panjer (2006) calls rounding method. This method spilts the probability between $(l-1)h$ and lh and assigns it to $l-1$ and l for $l = 1, 2, \cdots$.

Let $p_{X_{app}}(l)$ denote the probability placed at $lh,\ l = 0, 1, 2, \cdots$. Then we set

$$\begin{aligned}
p_{X_{app}}(0) &\triangleq P\left\{X < \frac{h}{2}\right\} = F_X\left(\frac{h}{2} - 0\right) \\
p_{X_{app}}(l) &\triangleq P\left\{lh - \frac{h}{2} \le X < lh + \frac{h}{2}\right\} \\
&= F_X\left(lh + \frac{h}{2} - 0\right) - F_X\left(lh - \frac{h}{2} - 0\right), \quad l = 1, 2, \cdots,
\end{aligned}$$

where notation $F_X(x-0)$ indicates that probability mass at x is excluded.

Panjer and Willmot (1992) suggests rounding method at not mid-point in span but left endpoint or right endpoint in span as follows.

$$\begin{aligned}
p_{X_{right}}(0) &\triangleq F_X(h-0) \\
p_{X_{right}}(l) &\triangleq F_X(lh + h - 0) - F_X(lh - 0), \quad l = 1, 2, \cdots.
\end{aligned}$$

$$\begin{aligned}
p_{X_{left}}(0) &\triangleq 0 \\
p_{X_{left}}(l) &\triangleq F_X(lh) - F_X(lh - h), \quad l = 1, 2, \cdots.
\end{aligned}$$

4.2.2 Increment of Recovery with Exponential Distribution

In this section we suppose the increment of recovery has exponential distribution. Exponential density is monotonically decreasing toward the right and the difference one minus exponential distribution is exponentially decaying its right tail. Therefore large increment of recovery is given a roughly zero probability.

Assumption 8. $\{X_n : n \in \mathbb{N}\}$ *are independent and have common exponential distribution function with mean* θ.

Under Assumption 8, we get

$$\begin{aligned}
F_X(x) &= 1 - e^{-\frac{x}{\theta}}, \quad x \ge 0, \\
f_X(x) &= \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x \ge 0.
\end{aligned}$$

Let $\tilde{X}_t$ denote the increment of recovery with interest effect and from Section A.1, it represents

$$\tilde{X}_t \triangleq e^{r(t-V_t)} X_1,$$

where V_t is uniform random variable on $(0, t]$. Let $F_{\tilde{X}_t}$ denote the probability distribution function of $\tilde{X}_t$, and then it follows from (26) that

$$\begin{aligned}
F_{\tilde{X}_t}(x) &= P\left\{\tilde{X}_t \le x\right\} \\
&= \frac{1}{t} \int_0^t 1 - e^{-\frac{xe^{-r(t-u)}}{\theta}} du.
\end{aligned}$$

Also let $f_{\tilde{X}_t}$ denote the probability density function of $\tilde{X}_t$, and we obtain

$$\begin{aligned}
f_{\tilde{X}_t}(x) &= \frac{1}{t} \int_0^t e^{-r(t-u)} f_X\left(xe^{-r(t-u)}\right) du \\
&= \frac{1}{xtr}\left(-e^{-\frac{x}{\theta}} + e^{-\frac{xe^{-rt}}{\theta}}\right).
\end{aligned}$$

4.2.3 Increment of Recovery with Pareto Distribution

In this section we suppose the increment of recovery has Pareto distribution. Pareto distribution has heavy tail and is used in actuarial mathematics when there is high probability of very large losses.

Assumption 9. $\{X_n : n \in \mathbb{N}\}$ *are independent and identically Pareto distributed with parameter* α, β, *where* $\alpha, \beta > 0$.

From Assumption 9, we obtain

$$F_X(x) = 1 - \left(\frac{\beta}{\beta + x}\right)^\alpha, \quad x > 0,$$
$$f_X(x) = \frac{\alpha\beta^\alpha}{(\beta + x)^{\alpha+1}}, \quad x > 0.$$

We have

$$E[X] = \frac{\beta}{\alpha - 1}, \quad \alpha > 1,$$
$$Var(X) = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}, \quad \alpha > 2.$$

Similar to Section 4.2.2, let $F_{\tilde{X}_t}$ denote the probability distribution function of $\tilde{X}_t$ and we get

$$\begin{aligned} F_{\tilde{X}_t}(x) &= P\left\{\tilde{X}_t \le x\right\} \\ &= \frac{1}{t}\int_0^t 1 - \left(\frac{\beta}{\beta + xe^{-r(t-u)}}\right)^\alpha du. \end{aligned}$$

5 Inhomogeneous Compound Poisson model

In the real world the expected number of recoveries per unit time may differ in time. In this section we extend our model into the case that the number of recoveries follows an inhomogeneous Poisson process. We present the procedure for approximating the probability function of an inhomogeneous compound poisson process by the probability function of a piecewise homogeneous compound Poisson process. The probability function of a piecewise compound Poisson can be calculated easily by using the Panjer recursion formula or fast Fourier transformation, and by using its Markov property.

Assumption 10. *Assume the number of recoveries follows an inhomogeneous Poisson process* N_t^I *with the intensity function* $\lambda(t)$ *which is positive deterministic function in the interval* $[0, T]$. *Let* $\Lambda(t) = \int_0^t \lambda(u)du$ *be the cumulative intensity function.*

For example, we may suppose the intensity function is an exponential function as time as follows. For $\gamma_1, \gamma_2 > 0$,

$$\lambda(t) = \gamma_1 \exp\{-\gamma_2 t\}.$$

5.1 Inhomogeneous Poisson and Piecewise Homogeneous Poisson Process

We will consider a piecewise homogeneous Poisson process, and it is a compound Poisson process with a constant intensity in each sub-interval. In this section we will also show a piecewise homogeneous Poisson process converges weakly to an inhomogeneous Poisson process.

Definition 4. *We split the interval $[0, t]$ into the sub-interval $[t_0, t_1], [t_1, t_2], \cdots, [t_{J-1}, t_J]$ where $0 = t_0 < t_1 < t_2 < \cdots < t_J = t$. Let Δ_J be the partition of $[0, t]$ and let T_j be the j-th sub-interval as follows,*

$$\Delta_J : t_0, t_1, \cdots, t_J; \quad T_j = [t_{j-1}, t_j], \quad j = 1, 2, \cdots J.$$

Let $|T_j|$ be the length of interval T_j and the norm of the partition be

$$|\Delta_J| = \max_{1 \le j \le J} |T_j|,$$

and let a Riemann sum of λ, $\Lambda_{\Delta_J}(t)$ be

$$\Lambda_{\Delta_J}(t) = \sum_{j=1}^{J} |t_j - t_{j-1}| \lambda(\xi_j),$$

for any but fixed $\xi_j \in T_j, j = 1, 2, \cdots, J$.

For any but fixed $\xi_j \in T_j, j = 1, 2, \cdots, J$, Λ_{Δ_J} is considered as the piecewise constant cumulative intensity function.

Definition 5. *Let $N^{\lambda_{\Delta_J}}$ be a Poisson process on $[0, T]$ with the piecewise constant intensity function λ_{Δ_J} defined by,*

$$\lambda_{\Delta_J}(t) = \begin{cases} \lambda(\xi_1) & t \in T_1 \\ \lambda(\xi_2) & t \in T_2 \\ \vdots & \vdots \\ \lambda(\xi_J) & t \in T_J. \end{cases}$$

$N^{\lambda_{\Delta_J}}$ is seen as a piecewise homogeneous Poisson process. We will show the piecewise homogeneous Poisson process $N^{\lambda_{\Delta_J}}$ converges weakly to the inhomogeneous Poisson process N^I.

Theorem 5.1. *If $\lambda(t)$ is Riemann integrable on $[0, T]$, then*

$$N^{\lambda_{\Delta_J}} \xrightarrow{w} N^I \text{ on } [0, T] \text{ as } |\Delta_J| \to 0 \ (J \to \infty), \tag{15}$$

where $\xrightarrow{w}$ means "converges weakly".

Proof. For $^\forall n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and fixed $t \in [0, T]$, the following convergence holds

$$\begin{aligned} P\left\{N_t^{\lambda_{\Delta_J}} = n\right\} &= e^{-\Lambda_{\Delta_J}(t)} \frac{(\Lambda_{\Delta_J}(t))^n}{n!} \\ &\to e^{-\Lambda(t)} \frac{(\Lambda(t))^n}{n!} \quad \text{as} \quad |\Delta_J| \to 0 \ (J \to \infty) \\ &= P\left\{N_t^I = n\right\}. \end{aligned}$$

Because the interarrival times are independent, the finite dimensional distribution converges. From p.137 in Daley and Vere-Jones (2008), for a point process, tightness follows from the convergence of its finite dimensional distribution. Thus (15) holds. □

An inhomogeneous Poisson process has the following property, see Proposition 12.2.1 of Rolski et al. (1999) for the detail.

Proposition 5.2. *Given the number of recoveries $N_t^I = n$, n arrival times are distributed independently in the interval $(0, t]$ with probability density function*

$$\frac{\lambda(u)}{\Lambda(t)}, \quad u \in (0, t].$$

5.2 Inhomogeneous Compound Poisson and Piecewise Homogeneous Compound Poisson Process

In this section we derive the probability distribution function of an inhomogeneous compound Poisson process $\tilde{S}_t^I$. Let $F_{\tilde{S}_t^I}$ be the probability distribution function of $\tilde{S}_t^I$ and $f_{\tilde{S}_t^I}$ be the probability density function of $\tilde{S}_t^I$. For $t, x > 0$, we obtain

$$\begin{aligned} F_{\tilde{S}_t^I}(x) &= P\left\{\tilde{S}_t^I \le x\right\} \\ &= e^{-\Lambda(t)} + \sum_{n=1}^{\infty} P\left\{\sum_{k=1}^{n} e^{r\left(t-U_k^I\right)} X_k \le x \middle| N_t^I = n\right\} P\left\{N_t^I = n\right\} \\ &= e^{-\Lambda(t)} + \sum_{n=1}^{\infty} P\left\{\sum_{k=1}^{n} e^{r\left(t-U_k^I\right)} X_k \le x \middle| N_t^I = n\right\} \frac{(\Lambda(t))^n}{n!} e^{-\Lambda(t)}. \end{aligned} \tag{16}$$

where $\left\{U_k^I : k \ge 0\right\}$ are the sequence of jump points of an inhomogeneous Poisson process N_t^I. Given $N_t^I = n$, from Proposition 5.2, it follows that

$$\begin{aligned} P\left\{\tilde{S}_t^I \le x | N_t^I = n\right\} &= P\left\{\sum_{k=1}^{n} e^{r\left(t-U_k^I\right)} X_k \le x \middle| N_t^I = n\right\} \\ &= P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{t,k}^I\right)} X_k \le x\right\}, \end{aligned} \tag{17}$$

where $V_{1,t}^I, V_{2,t}^I, \cdots, V_{k,t}^I$ are independent random variables with the probability density function $\frac{\lambda(u)}{\Lambda(t)}$ where $u \in (0, t]$, and they are independent of $\{X_n : n \in \mathbb{N}\}$. Substituting (17) into (16), we obtain

$$\begin{aligned} F_{\tilde{S}_t^I}(x) &= e^{-\Lambda(t)} + \sum_{n=1}^{\infty} P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{k,t}^I\right)} X_k \le x\right\} \frac{(\Lambda(t))^n}{n!} e^{-\Lambda(t)} \\ &= \sum_{n=0}^{\infty} P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{k,t}^I\right)} X_k \le x\right\} \frac{(\Lambda(t))^n}{n!} e^{-\Lambda(t)}. \end{aligned} \tag{18}$$

Now we define

$$\tilde{X}_{k,t}^I \triangleq e^{r(t-V_{k,t}^I)} X_k.$$

Let $F_{\tilde{X}_t^I}$ denote the distribution function of $\tilde{X}_t^I = \tilde{X}_{k,t}^I$, then

$$\begin{aligned} F_{\tilde{X}_t^I}(x) &\triangleq P\left\{e^{r(t-V_{k,t}^I)} X_k \le x\right\} \\ &= \frac{1}{\Lambda(t)} \int_0^t F_X\left(x e^{-r(t-u)}\right) \lambda(u) du. \end{aligned}$$

Let $F_{\tilde{X}_t^I}^{*n}$ denote the n-fold convolution of $F_{\tilde{X}_t^I}$, and we obtain

$$F_{\tilde{X}_t^I}^{*n}(x) = P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{k,t}^I\right)} X_k \le x\right\}. \tag{19}$$

Substituting (19) into (18), we have

$$F_{\tilde{S}_t^I}(x) = \sum_{n=0}^{\infty} e^{-\Lambda(t)} \frac{(\Lambda(t))^n}{n!} F_{\tilde{X}_t^I}^{*n}(x).$$

Definition 6. *Let* $\tilde{X}_{k,t}^{\Delta_J}$ *be the* k*-th increment of recovery compounded by the interest rate in the case the number of recoveries follows a piecewise homogeneous Poisson process* $N_t^{\lambda_{\Delta_J}}$*,*

$$\tilde{X}_{k,t}^{\Delta_J} = e^{r(t-V_{k,t}^{\Delta_J})} X_k,$$

where $V_{1,t}^{\Delta_J}, V_{2,t}^{\Delta_J}, \cdots, V_{n,t}^{\Delta_J}$ *are random variables with the probability density function* $\frac{\lambda_J(u)}{\Lambda_{\Delta_J}(t)}$ *where* $u \in (0, t]$.

We will prove that the increment of recovery with interest effect with a piecewise homogeneous Poisson process $\tilde{X}_t^{\Delta_J}$ converges weakly to the increment of recovery with interest with an inhomogeneous Poisson process $\tilde{X}_t^I$ in the following lemma.

Lemma 5.3. *If* $\lambda(t)$ *is Riemann integrable on* $[0, T]$*, and if* $|\lambda_{\Delta_J}(t)| \leq g(t)$ *holds for all* J *a.s. such that* $\int_0^t g(u)du < \infty$*, then*

$$\tilde{X}_{k,t}^{\Delta_J} \xrightarrow{w} \tilde{X}_{k,t}^I \quad \textit{as} \quad |\Delta_J| \to 0 \; (J \to \infty)$$

Proof. It follows from the dominated convergence theorem that for $^\forall t \in [0, T]$,

$$\begin{aligned} F_{\tilde{X}_t^{\Delta_J}}(x) &= \frac{1}{\Lambda_{\Delta_J}(t)} \int_0^t F_X\left(xe^{-r(t-u)}\right) \lambda_{\Delta_J}(u)du \\ &\to \frac{1}{\Lambda(t)} \int_0^t F_X\left(xe^{-r(t-u)}\right) \lambda(u)du \quad \text{as} \quad |\Delta_J| \to 0 \; (J \to \infty) \\ &= F_{\tilde{X}_t^I}(x). \end{aligned}$$

□

Definition 7. *Let* $\tilde{S}_t^{\Delta_J}$ *be a piecewise homogeneous compound Poisson process with the intensity* λ_{Δ_J}*,*

$$\tilde{S}_t^{\Delta_J} = \sum_{n=1}^{N_t^{\lambda_{\Delta_J}}} e^{r(t-U_n^{\lambda_{\Delta_J}})} X_n,$$

where $\left\{U_n^{\lambda_{\Delta_J}} : n \geq 1\right\}$ *are the sequence of jump points of a piecewise homogeneous Poisson process* $N_t^{\lambda_{\Delta_J}}$*,* $\{X_n : n \geq 0\}$ *follow Assumption 4, and* $\{X_n : n \geq 0\}$ *and* $N_t^{\lambda_{\Delta_J}}$ *are independent.*

We will prove the piecewise homogeneous compound Poisson process $\tilde{S}_t^{\Delta_J}$ converges weakly to the inhomogeneous compound Poisson process $\tilde{S}_t^I$.

Theorem 5.4. *If* $\lambda(t)$ *is Riemann integrable on* $[0, T]$*, and if* $|\lambda_{\Delta_J}(t)| \leq g(t)$ *holds for all* J *a.s. such that* $\int_0^t g(u)du < \infty$*, then*

$$\tilde{S}_t^{\Delta_J} \xrightarrow{w} \tilde{S}_t^I \quad \textit{on} \quad [0, T] \quad \textit{as} \quad |\Delta_J| \to 0 \; (J \to \infty). \tag{20}$$

Proof. Using the consequence of Lemma 5.3, For fixed $t \in [0, T]$ we obtain

$$\begin{aligned} F_{S_t^{\Delta_J}}(x) &= \sum_{n=0}^{\infty} e^{-\Lambda_{\Delta_J}(t)} \frac{(\Lambda_{\Delta_J}(t))^n}{n!} F_{\tilde{X}_t^{\Delta_J}}^{*n}(x) \\ &\to \sum_{n=0}^{\infty} e^{-\Lambda(t)} \frac{(\Lambda(t))^n}{n!} F_{\tilde{X}_t^I}^{*n}(x) \quad \text{as} \quad |\Delta_J| \to 0 \; (J \to \infty) \\ &= F_{S_t^I}(x). \end{aligned}$$

Because the increment of recovery is independent, the finite dimensional distribution converges weakly. Similarly as Theorem 5.1, (20) holds. □

5.3 Calculation Method for Piecewise Homogeneous Compound Poisson Process

We present the method of calculating the probability function of a piecewise homogeneous compound Poisson process by the absorbing Markov chain. In order to use the absorbing Markov chain, we assume the following.

Assumption 11. *The increment of recovery has discrete distribution or discretized continuous distribution. We set* $0 = h_0 < h_1 < h_2 < \cdots < h_L = De^{rT}$ *and assume* $P\{X = h_l : l \in \{0, 1, 2, \cdots L\}\} = 1$ *where* $L < \infty$.

We set

$$\mathcal{X}(x) = \{x_1, x_2, \cdots, x_J : \ x_j \geq 0 \text{ for all } j, x_1 + x_2 + \cdots + x_J = x\}.$$

We note $\tilde{S}_T^{\Delta_J}$ is Markov, and if lender recoveries $e^{-r(T-t_j)}x_j$ at t_j, it will increase to x_j at T with interest effect. Thus if in the each of sub-interval T_j, lender recovers $e^{-r(T-t_j)}x_j$ (increment of recovery in each sub-interval is $e^{-r(T-t_j)}x_j$) respectively, the cumulative recovery at T is $x = x_1 + x_2 + \cdots + x_J$. Therefore for fixed $\Xi = \{\xi_1, \xi_2, \cdots, \xi_J\}$ where ξ_j is arbitrary chosen in each interval T_j,

$$\begin{aligned}
P\left\{\tilde{S}_T^{\Delta_J} = x\right\} &= P\left\{\sum_{n=1}^{N_T^{\lambda_{\Delta_J}}} e^{r(T-U_n^{\lambda_{\Delta_J}})} X_n = x\right\} \\
&= P\left\{\sum_{n=1}^{N_{(0,t_1]}^{\lambda(\xi_1)}} e^{r(T-U_n^{\lambda(\xi_1)})} X_n + \sum_{n=1}^{N_{(t_1,t_2]}^{\lambda(\xi_2)}} e^{r(T-U_n^{\lambda(\xi_2)})} X_n + \cdots \right. \\
&\qquad \left. + \sum_{n=1}^{N_{(t_{J-1},t_J]}^{\lambda(\xi_J)}} e^{r(T-U_n^{\lambda(\xi_J)})} X_n = x\right\} \\
&= \sum_{\mathcal{X}(x)} P\left\{\tilde{S}_{(t_0,t_1]}^{\lambda(\xi_1)} = e^{-r(T-t_1)}x_1\right\} P\left\{\tilde{S}_{(t_1,t_2]}^{\lambda(\xi_2)} = e^{-r(T-t_2)}x_2\right\} \cdots P\left\{\tilde{S}_{(t_{J-1},t_J]}^{\lambda(\xi_J)} = x_J\right\}
\end{aligned}$$

where $\left\{U_n^{\lambda(\xi_j)} : n \geq 1\right\}$ are the sequence of jump points of $N_{(t_{j-1},t_j]}^{\lambda(\xi_j)}$ which is the homogeneous Poisson process on the interval $(t_{j-1}, t_j]$ with the intensity $\lambda(\xi_j)$, and $S_{(t_{j-1},t_j]}^{\lambda(\xi_j)}$ is the homogeneous compound Poisson process on the interval $(t_{j-1}, t_j]$ with the intensity $\lambda(\xi_j)$.

For calculating $P\left\{S_T^{\Delta_J} = x\right\}$, we use inhomogeneous Markov chain. Let $q_{l,t_j,T}$ be the transition kernel from t_{j-1} to t_j with the last recovery possible time T.

$$q_{l,t_j,T} = \begin{cases} P\left\{\tilde{S}_{t_j} = e^{r(T-t_j)}h_{l+l'} \middle| \tilde{S}_{t_{j-1}} = e^{r(T-t_{j-1})}h_{l'}\right\}, & l \geq 0, \\ 0, & l < 0, \end{cases}$$

and the transition matrix from t_{j-1} to t_j be

$$\underset{L\times L}{\boldsymbol{Q}_{t_j,T}} = \begin{pmatrix} q_{0,t_j,T} & q_{1,t_j,T} & q_{2,t_j,T} & \cdots & 1-\sum_{l=0}^{L-1} q_{l,t_j,T} \\ 0 & q_{0,t_j,T} & q_{1,t_j,T} & \cdots & 1-\sum_{l=0}^{L-2} q_{l,t_j,T} \\ 0 & 0 & q_{0,t_j,T} & \cdots & 1-\sum_{l=0}^{L-3} q_{l,t_j,T} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Let the probability transition vector at t_0 and t_j represent as follows,

$$\underset{L\times 1}{\mathcal{P}_{\tilde{S}_{t_0}}} = (1, 0, \cdots, 0)^\top,$$

$$\mathcal{P}_{\tilde{S}_{t_j},T} = \left(P\left(\tilde{S}_{t_j} = 0\right), P\left(\tilde{S}_{t_j} = e^{-r(T-t_j)}h_1\right), \cdots, P\left(\tilde{S}_{t_j} = e^{-r(T-t_j)}h_L\right)\right)^\top.$$

Because the increment of recovery is independent, we obtain the probability function at t_J is

$$\mathcal{P}^\top_{\tilde{S}_{t_J},T} = \mathcal{P}^\top_{\tilde{S}_{t_0}} \boldsymbol{Q}_{t_1,T}\boldsymbol{Q}_{t_2,T}\cdots\boldsymbol{Q}_{t_J,T}.$$

5.4 Numerical Procedure

We present numerical procedure for calculating the probability function of the cumulative recovery which follows the piecewise inhomogeneous compound Poisson process model $\tilde{S}_T^{\Delta_J}$.

(a) Split the interval $(0, T]$ into sub-intervals $(t_0, t_1], (t_1, t_2], \cdots, (t_{J-1}, t_J]$ where $0 = t_0 < t_1 < t_2 < \cdots < t_J = T$.

(b) Calculate $P\left\{\tilde{S}^{\lambda(\xi_1)}_{(t_0,t_1]} = e^{r(T-t_1)}h_l\right\} = q_{l,t_1,T},\ l \in \{0, 1, 2, \cdots, L\}$ by Panjer recursion formula or the fast Fourier transformation.

(c) Make the transition matrix $\boldsymbol{Q}_{t_1}$ by the consequence of (b).

(d) Similarly, calculate $\boldsymbol{Q}_{t_2}, \boldsymbol{Q}_{t_3}, \cdots, \boldsymbol{Q}_{t_k}$.

(e) Make vector $\mathcal{P}_{\tilde{S}_{t_0}} = (1, 0, \cdots, 0)^\top$ and calculate $\mathcal{P}^\top_{\tilde{S}_{t_J},T} = \mathcal{P}^\top_{\tilde{S}_{t_0}} \boldsymbol{Q}_{t_1}\boldsymbol{Q}_{t_2}\cdots\boldsymbol{Q}_{t_J}$.

5.5 Monte Calro Simulation

In this section, we present the procedure for calculating the expectation and the variance of recovery rate by using the Monte Carlo simulation.

In the cace intensity function $\lambda(t)$ is bounded by a constant $\overline{\lambda}$, Glasserman (2004) and Čížek, Härdle and Weron (2005) mention the procedure for calculating the jump points of an inhomogeneous Poisson process by using the Monte Carlo simulation. They call it thinning method. By applying thinning method, we propose the procedure for calculating expectation and variance of an inhomogeneous compound Poisson process on $[0, T]$. This procedure is as follows.

(a) Generate jump times $\overline{U}_n$ of $\overline{N}_T$, where $\overline{N}_T$ is the homogeneous Poisson process with the intensity $\overline{\lambda}$ on $[0, T]$.

(b) For each n, generate Υ_n which is uniformly distributed on $[0, 1]$.

(c) For each n, if $\Upsilon_n\overline{\lambda}_n < \lambda(\overline{U}_n)$ then accept $\overline{U}_n$ as a jump point of N^I, obtaining $U_1^I, U_2^I, \cdots, U_m^I$ which is the sequence of the jump points of N^I where $m \le n$.

(d) Generate the increments of recoveries $X_1, X_2, \cdots X_m$ from i.i.d. distribution F_X.

(e) Calculate

$$R_T = \begin{cases} \frac{\sum_{k=1}^m e^{r(T-U_k)}X_k}{e^{rT}D} & \text{if } \sum_{k=1}^m e^{r(T-U_k)}X_k < e^{rT}D \\ 1 & \text{if } \sum_{k=1}^m e^{r(T-U_k)}X_k \ge e^{rT}D. \end{cases} \tag{21}$$

(f) Repeat (a)-(e) $\mathcal{N}$ times and obtain $R_{1,T}, R_{2,T}, \cdots, R_{\mathcal{N},T}$. Calculate $E[R_t] = \frac{\sum_{i=1}^{\mathcal{N}} R_{i,T}}{\mathcal{N}}$ and $Var(R_t) = \frac{\sum_{i=1}^{\mathcal{N}} R_{i,T}^2}{\mathcal{N}} - \left(\frac{\sum_{i=1}^{\mathcal{N}} R_{i,T}}{\mathcal{N}}\right)^2$.

6 Numerical Result

In this section, we calculate the expectation and the standard deviation of the modified survival value of the debt and recovery rate which are derived in Theorem 3.1, Proposition 3.2 and Corollary 3.3, by two kinds of the method based on Panjer recursion formula and the fast Fourier transformation respectively. Parameters without the increment of recovery are as follows.

$$
\begin{aligned}
\lambda &: \text{the intensity of the number of recoveries} \\
T &: \text{the last recovery possible time} \\
r &: \text{interest rate} \\
D &: \text{the initial value of the debt}
\end{aligned}
$$

We set $\lambda = 5, T = 1, r = 0.05, D = 10$.

Also we illustrate discretization. We split the interval $[0, De^{rT}]$ into 100 sub-intervals. Let h be the length of sub-intervals and $h = \frac{De^{rT}}{100}$ by the rounding method at mid-point in Section 4.2.1. This h and h in Section 4.2.1 are the same.

6.1 Exponential Distribution

In this section, we suppose the increment of recovery has exponential distribution with mean θ. We set $\theta = 2$. We calculate the expectation and the standard deviation of the modified survival value of the debt and recovery rate by Panjer recursion formula and the fast Fourier transformation. Regardless of numerical methods, numerical results are almost the same. But the numerical speed by Panjer recursion formula is much faster than that by the fast Fourier transformation. Each Figure 4, Figure 5, Figure 6, Figure 7 and Figure 8 shows the trajectory of the expectation and the standard deviation of the survival value of the debt by changing each of the parameters. Similarly each Figure 9, Figure 10, Figure 11, Figure 12 and Figure 13 shows those of recovery rate.

Table 1 shows the expectation and the standard deviation of the modified survival value of the debt and recovery rate in the case parameters are $\theta = 2, \lambda = 5, T = 1, r = 0.05, D = 10$.

Table 2 demonstrates average time for spent on calculating the expectation and the standard deviation of the modified survival value of the debt by Panjer recursion formula algorithm and by the fast Fourier transformation algorithm. In order to calculate them, we use a personal computer with CPU Intel core 2 quad 2.4GHz, memory 4GB, OS Windows Vista, and software R.

Figure 2 and Figure 3 illustrate histogram of cumulative recovery rate as a function of T. According to Figure 2 and Figure 3, cumulative recovery rate is increasing in progress of time and has a bimodal distribution in early time of recovery.

	$E[M_T]$	$SD(M_T)$	$E[R_T]$	$SD(R_T)$
Panjer recursion formula	2.702981	3.139094	0.742884	0.298600
fast Fourier transformation	2.702981	3.139094	0.742884	0.298600

Table 1: Numerical results for parameters $\theta = 2, \lambda = 5, T = 1, r = 0.05, D = 10$.

	time
Panjer recursion formula	0.0653
fast Fourier transformation	20.9139

Table 2: Average time (second) spent on calculating the expectation and the standard deviation of the modified survival value of the debt by Panjer recursion formula algorithm and by the fast Fourier transformation algorithm in 100 trials.

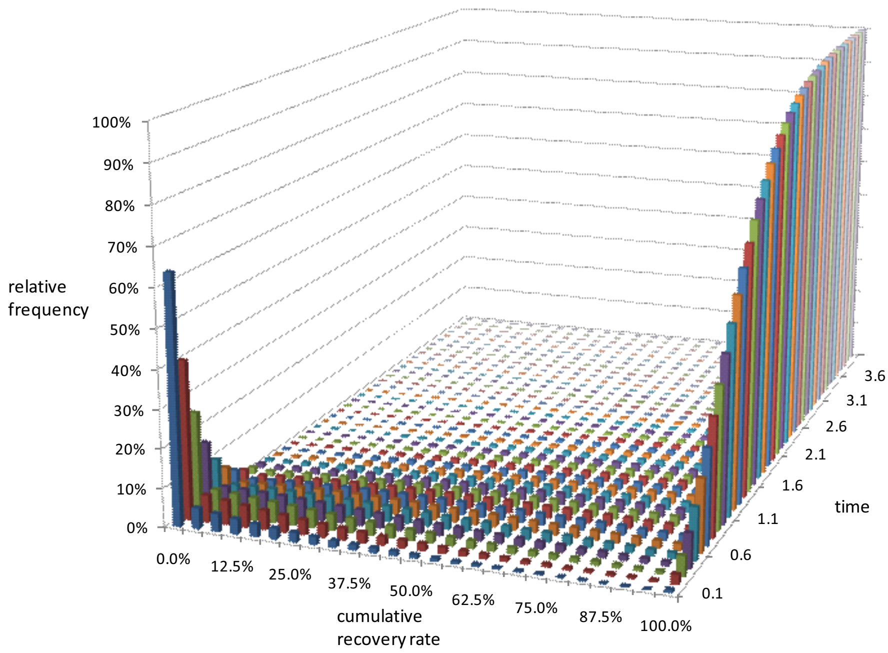


Figure 2: The histogram of cumulative recovery rate as a function of T for parameters $\theta = 2$, $\lambda = 5$, $r = 0.05$, $D = 10$.

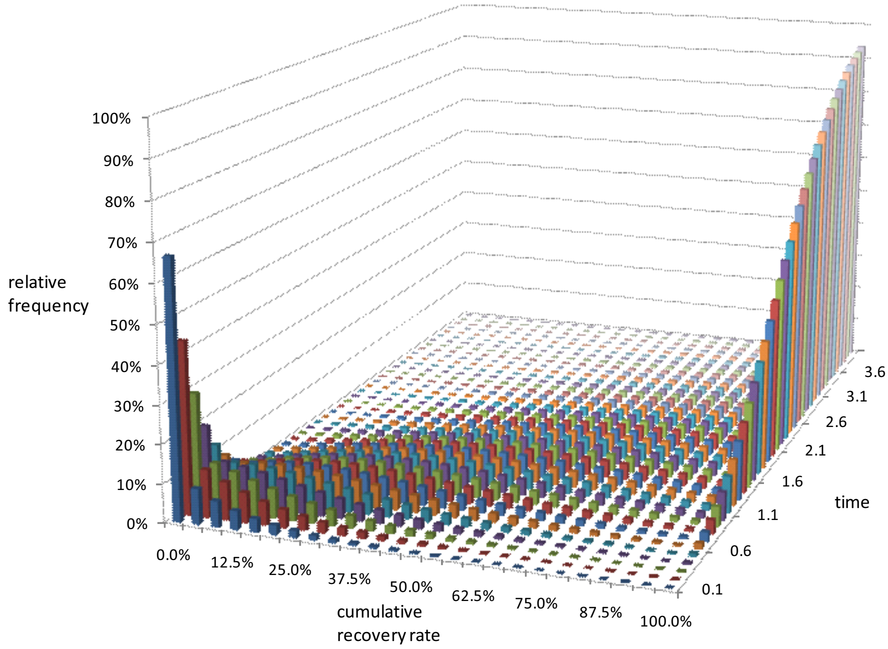


Figure 3: The histogram of cumulative recovery rate as a function of T for parameters $\theta = 1$, $\lambda = 5$, $r = 0.05$, $D = 10$.

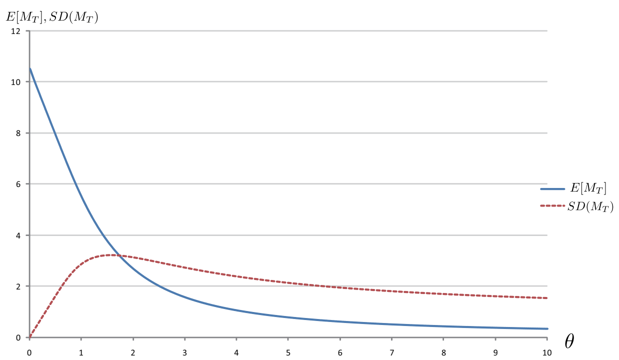


Figure 4: The expectation and the standard deviation of the modified survival value of the debt as a function of θ for parameters $\lambda = 5$, $T = 1$, $r = 0.05$, $D = 10$.

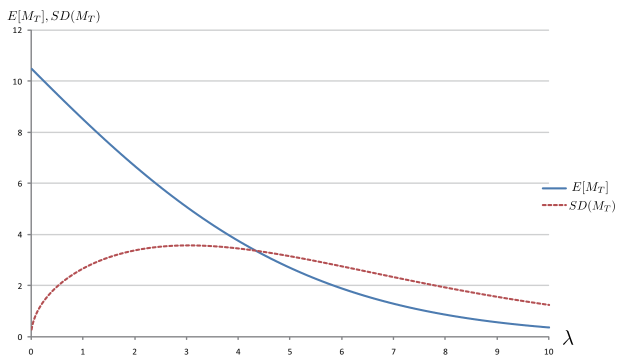


Figure 5: The expectation and the standard deviation of the modified survival value of the debt as a function of λ for parameters $\theta = 2, T = 1, r = 0.05, D = 10$.

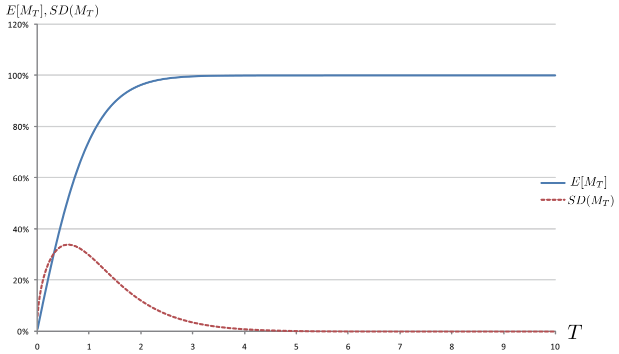


Figure 6: The expectation and the standard deviation of the modified survival value of the debt as a function of T for parameters $\theta = 2, \lambda = 5, r = 0.05, D = 10$.

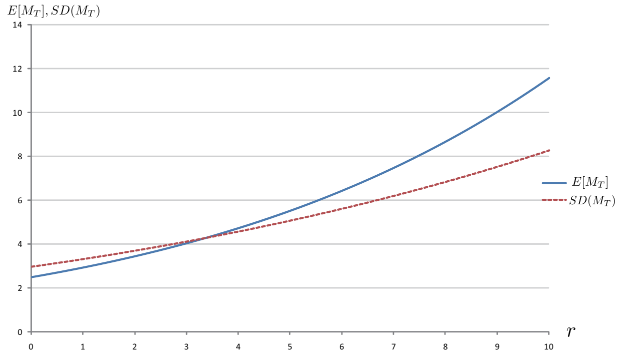


Figure 7: The expectation and the standard deviation of the modified survival value of the debt as a function of r for parameters $\theta = 2, \lambda = 5, T = 1, D = 10$.

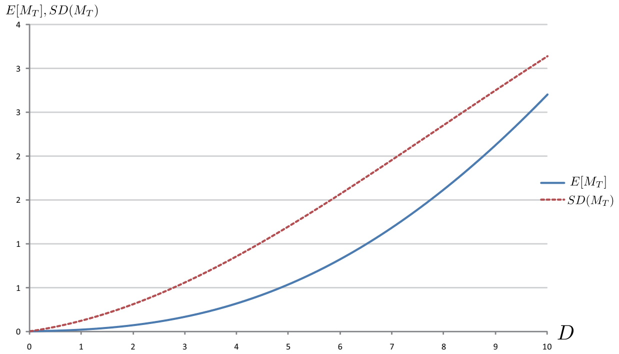


Figure 8: The expectation and the standard deviation of the modified survival value of the debt as a function of D for parameters $\theta = 2, \lambda = 5, T = 1, r = 0.05$.

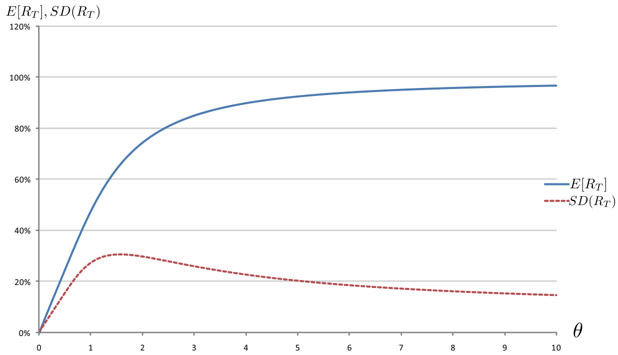


Figure 9: The expectation and the standard deviation of recovery rate as a function of θ for parameters $\lambda = 5, T = 1, r = 0.05, D = 10$.

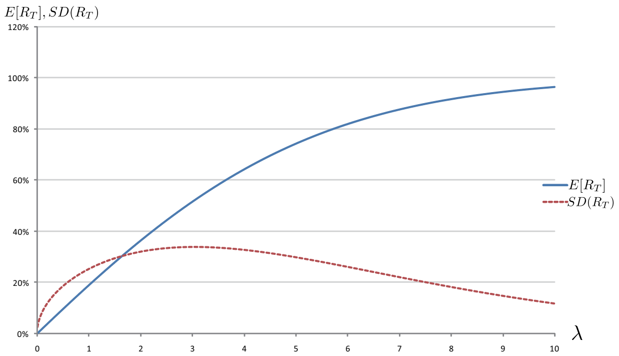


Figure 10: The expectation and the standard deviation of recovery rate as a function of λ for parameters $\theta = 2, T = 1, r = 0.05, D = 10$.

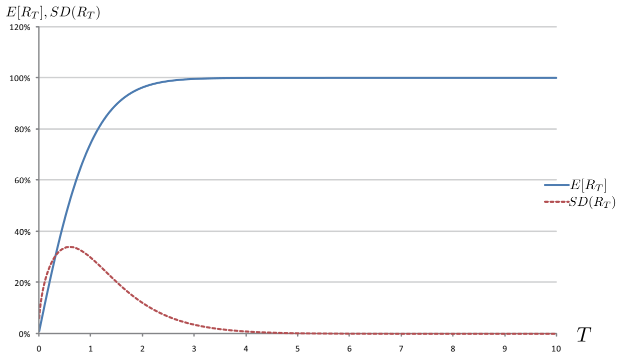


Figure 11: The expectation and the standard deviation of recovery rate as a function of T for parameters $\theta = 2, \lambda = 5, r = 0.05, D = 10$.

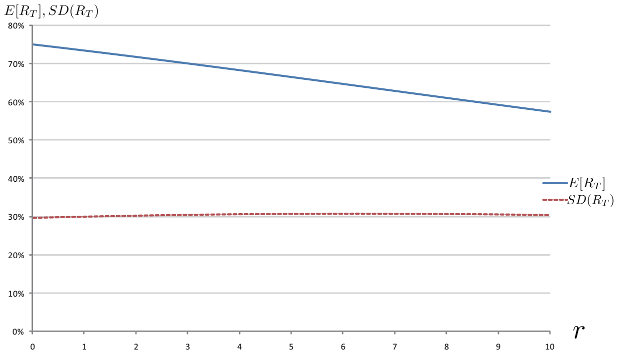


Figure 12: The expectation and the standard deviation of recovery rate as a function of r for parameters $\theta = 2, \lambda = 5, T = 1, D = 10$.

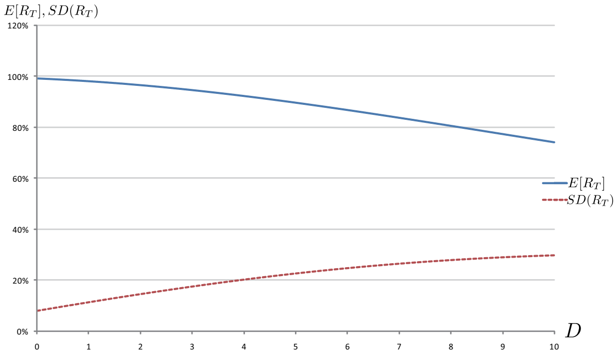


Figure 13: The expectation and the standard deviation of recovery rate as a function of D for parameters $\theta = 2, \lambda = 5, T = 1, r = 0.05$.

6.2 Pareto Distribution

In this section we compare the case the increment of recovery has exponential distribution with θ, with the case it has Pareto distribution with α, β. We set parameters as follows.

$$\theta = 2$$
$$\alpha = 4.475$$
$$\beta = 7.234$$

Figure 14 illustrates the difference one minus probability distribution function of exponential distribution with $\theta = 2$ and that of Pareto distribution with $\alpha = 4.475,\ \beta = 7.234$. Figure 14 shows that the right tail of the difference one minus Pareto distribution function is fatter than that of exponential. It means that there is higher possibility of very large increment of recovery in the case increment of recovery has Pareto one.

Each Figure 15, Figure 16 and Figure 17 shows the expectation and the standard deviation of the survival value of the debt as a function of each parameter in two cases: the increment of recovery has exponential distribution and it has Pareto distribution. Similarly each Figure 18, Figure 19 and Figure 20 show those of recovery rate.

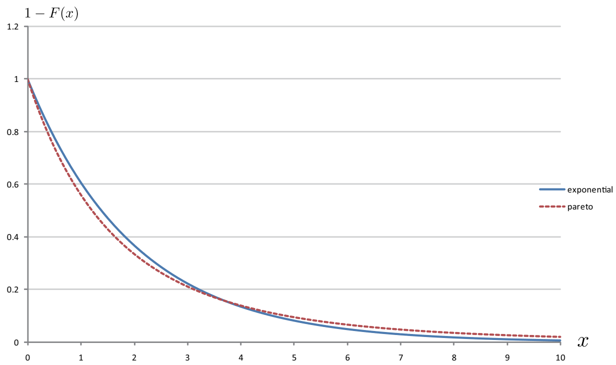


Figure 14: Exponential distribution with $\theta = 2$ and Pareto distribution with $\alpha = 4.475,\ \beta = 7.234$.

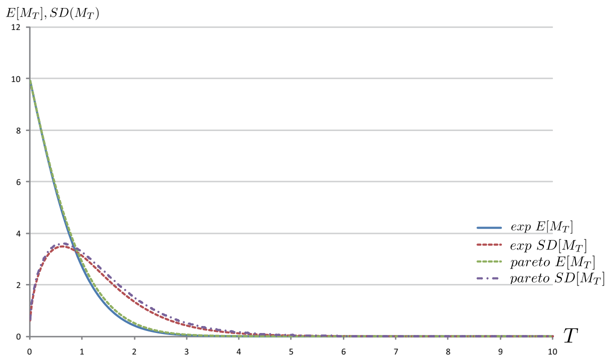


Figure 15: Expectation and standard deviation of modified survival value of debt in the case increment of recovery has an exponential distribution and in the case it has a Pareto distribution, as a function of T for parameters $\lambda = 5, r = 0.05, D = 10, \theta = 2, \alpha = 4.475$ and $\beta = 7.234$.

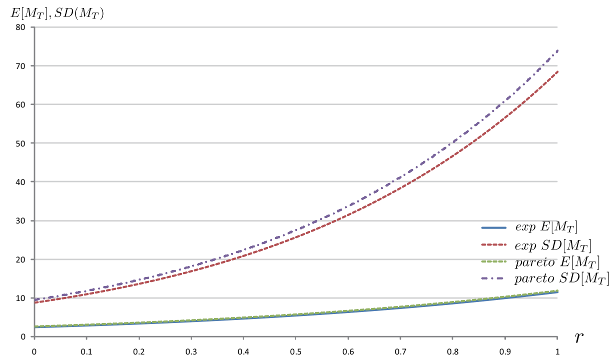


Figure 16: Expectation and standard deviation of modified survival value of debt in the case increment of recovery has an exponential distribution and in the case it has a Pareto distribution, as a function of r for parameters $\lambda = 5, T = 1, D = 10, \theta = 2, \alpha = 4.475$ and $\beta = 7.234$.

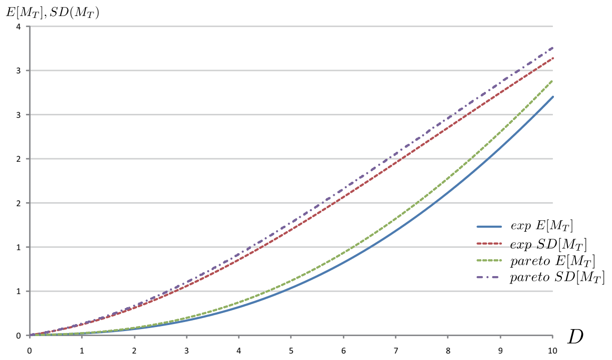


Figure 17: Expectation and standard deviation of modified survival value of debt in the case increment of recovery has an exponential distribution and in the case it has a Pareto distribution, as a function of D for parameters $\lambda = 5, T = 1, r = 0.05, \theta = 2, \alpha = 4.475$ and $\beta = 7.234$.

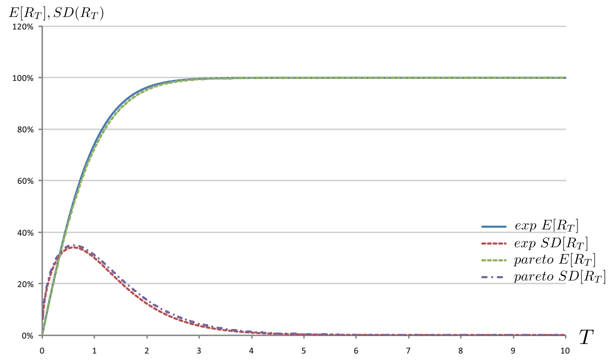
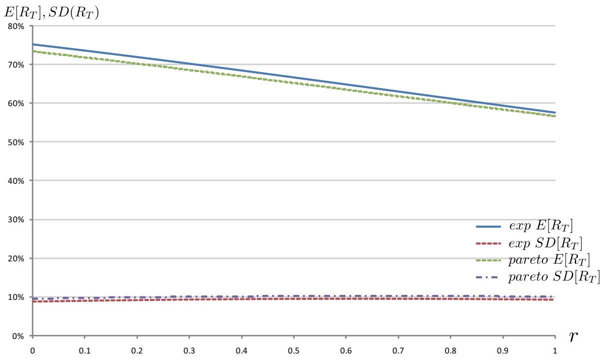


Figure 18: Expectation and standard deviation of recovery rate in the case increment of recovery has an exponential distribution and in the case it has a Pareto distribution, as a function of T for parameters $\lambda = 5, r = 0.05, D = 10, \theta = 2, \alpha = 4.475$ and $\beta = 7.234$.



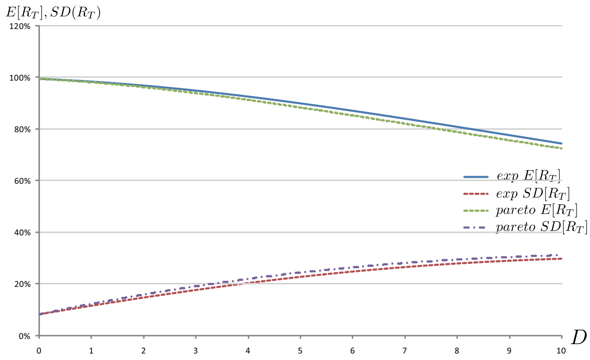


Figure 19: Expectation and standard deviation of recovery rate in the case increment of recovery has an exponential distribution and in the case it has a Pareto distribution, as a function of r for parameters $\lambda = 5, T = 1, D = 10, \theta = 2, \alpha = 4.475$ and $\beta = 7.234$.

Figure 20: Expectation and standard deviation of recovery rate in the case increment of recovery has an exponential distribution and in the case it has a Pareto distribution, as a function of D for parameters $\lambda = 5, T = 1, r = 0.05, \theta = 2, \alpha = 4.475$ and $\beta = 7.234$.

6.3 Inhomogeneous Case

We calculate the probability function, the expectation, and the standard deviation of cumulative recovery rate in the case the number of recovery has the inhomogeneous intensity. We set the intensity function as follows

$$\lambda(t) = 5e^{-t}, \tag{22}$$

and other parameters is $T = 1, r = 0.05, D = 10$, and the increment of recovery has exponential distribution with $\theta = 2$. For calculation, in the procedure (b) in Section 5.3, we set $\lambda(\xi_j) = \lambda(t_{j-1})$ which is left endpoint (and also upper bound in this case) of each sub-interval. Figure 21 shows (22). Figure 22 shows the expectation and the standard deviation of cumulative recovery rate in two case: homogeneous compound Poisson model and inhomogeneous compound Poisson model. Figure 23 and Figure 24 illustrate the histogram of cumulative recovery rate as a function of T.

We also calculate recovery rate by using Monte calro simulation and compare the result of the piecewise homogeneous compound Poisson approximation on computational speed and accuracy. We use the same personal computer in Section 6.1 to calculate them. Table 3 and Table 4 show the computational results and speed for calculating recovery rate by the two methods.

		$E[R_t^I]$	$SD(R_t^I)$
PHCPA	J =100	0.541484	0.338744
	J =400	0.539878	0.338791
	J =1000	0.539557	0.338800
Monte Calro simulation	$\mathcal{N}$ =100000	0.539367	0.338809
	$\mathcal{N}$ =300000	0.539238	0.338801
	$\mathcal{N}$ =500000	0.539371	0.338807

Table 3: Numerical results for parameter $\theta = 2, T = 1, r = 0.05, D = 10$ and the intensity function $\lambda(t) = 5e^{-t}$ by the piecewise homogeneous compound Poisson approximation (PHCPA) and the Monte Calro simulation in 100 trials. J is the number of sub-intervals and $\mathcal{N}$ is the number of simulations.

		time
PHCPA	J =100	6.5952
	J =400	25.8279
	J =1000	64.7437
Monte Carlo simulation	$\mathcal{N}$ =100000	25.3095
	$\mathcal{N}$ =300000	76.9649
	$\mathcal{N}$ =500000	127.0603

Table 4: Average time (second) spent on calculating the expectation and the standard deviation of recovery rate for parameter $\theta = 2, T = 1, r = 0.05, D = 10$ and the intensity function $\lambda(t) = 5e^{-t}$ by the piecewise homogeneous compound Poisson approximation (PHCPA) and the Monte Calro simulation in 100 trials. J is the number of sub-intervals and $\mathcal{N}$ is the number of simulations in one trial.

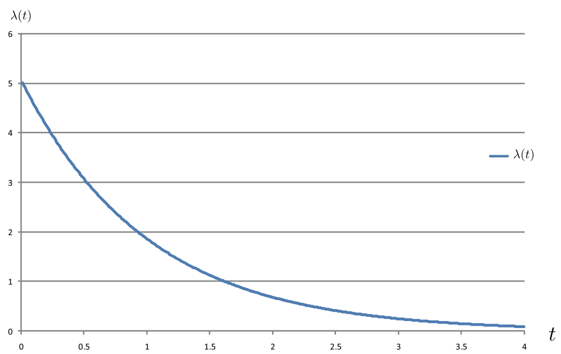


Figure 21: Intensity Function $\lambda(t) = 5e^{-t}$.

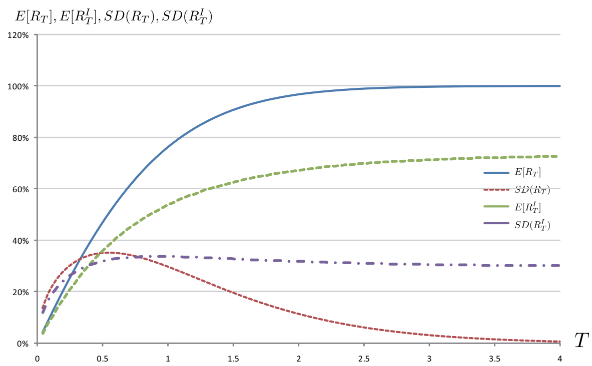


Figure 22: Expectation and standard deviation of recovery rate in two case: homogeneous case R_T and inhomogeneous case R_T'. $\lambda = 5, \lambda(T) = 5e^{-T}, \theta = 2, r = 0.05, D = 10$

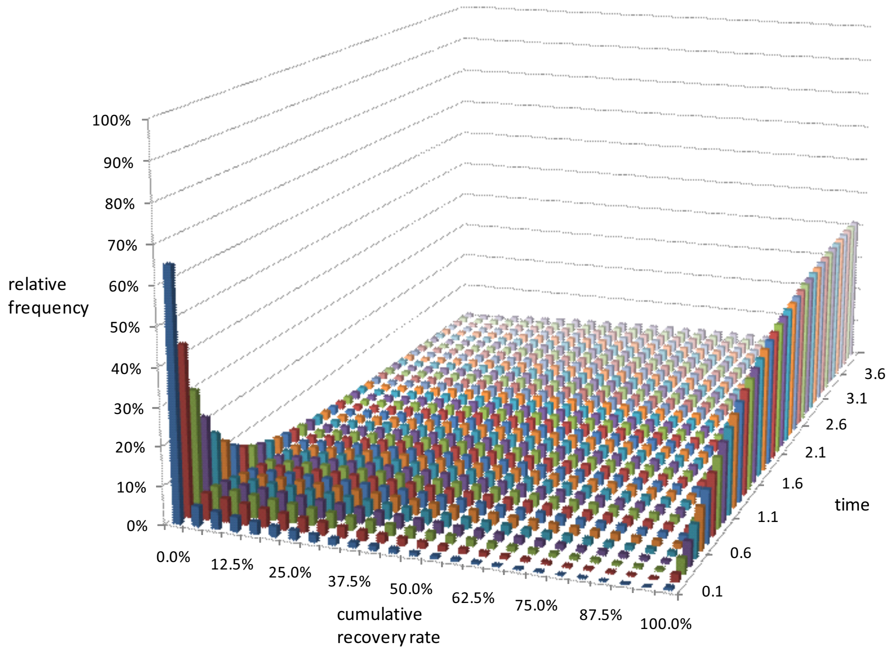


Figure 23: Histogram of cumulative recovery rate as a function of T for parameters $\theta = 2$, $r = 0.05$, $D = 10$, $\lambda(T) = 5e^{-T}$.

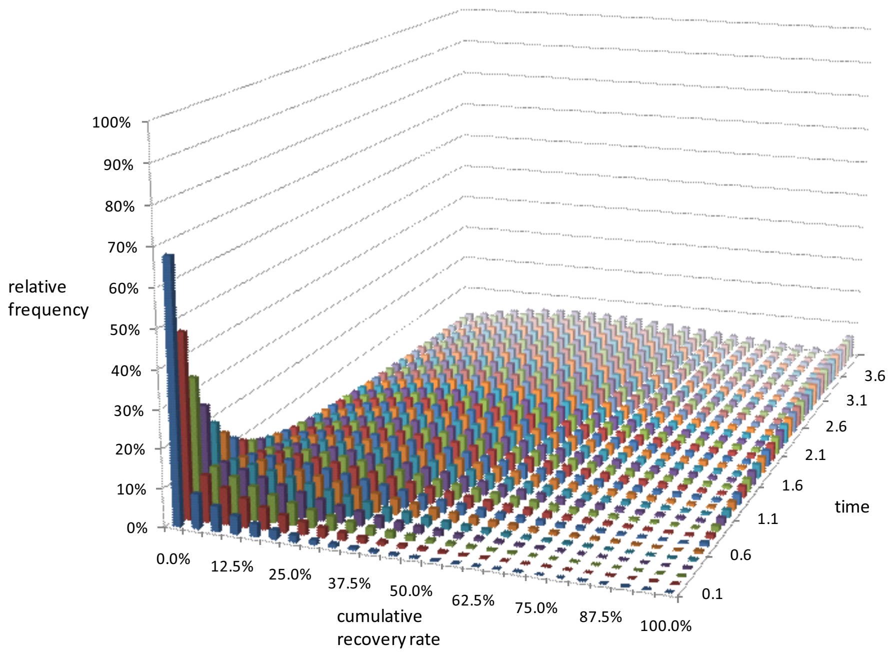


Figure 24: Histogram of cumulative recovery rate as a function of T for parameters $\theta = 1$, $r = 0.05$, $D = 10$, $\lambda(T) = 5e^{-T}$.

7 Conclusion

This paper addresses the issue of modeling and calculating the recovery process. The expectation of recovery rate is increasing function as the increment of recovery, the number of recoveries, the last recovery possible time and is decreasing function as interest rate and the initial debt amount.

On the calculation of the probability function of the modified survival value of the debt, numerical results show that Panjer recursion formula algorithm yields as much accurate as the fast Fourier transformation does. But the calculation by the method based on Panjer recursion formula is much faster than that by the method based on the fast Fourier transformation.

In the case the intensity is constant as time, the cumulative recovery rate is increasing as time. However by using the inhomogeneous compound Poisson model, we can demonstrate recovery rate is diminishing as time. This feature matches Figure 15 of Itoh and Yamashita (2008) which describes the relationship between empirical cumulative recovery rate and time. In this paper, we suggest the new procedure for calculating the transition density of an inhomogeneous compound Poisson process by the transition density of a piecewise homogeneous compound Poisson process. From the numerical experiments, the computational speed and accuracy of our method is similar to the Monte Calro simulation.

A Homogeneous Compound Poisson Process

A.1 distribution function

In this section we will derive the distribution function of a compound Poisson process. We refer the reader to Taylor and Karlin (1998) and Wu et al. (2005) for the references of the following content.

Let $\{U_n : n \in \mathbb{N}\}$ be the sequence of jump points of the Poisson process N_t with the intensity λ. Let $\{X_n : n \in \mathbb{N}\}$ be independent random variables and have distribution function F_X (density function f_X). N_t and $\{X_n : n \in \mathbb{N}\}$ are independent. Let $\tilde{S}_t$ denote

$$\tilde{S}_t = \sum_{n=1}^{N_t} e^{r(t-U_n)} X_n.$$

Let $F_{\tilde{S}_t}$ be the distribution function of $\tilde{S}_t$ and $f_{\tilde{S}_t}$ be the density function of $\tilde{S}_t$. For $t, x > 0$, we get

$$\begin{aligned} F_{\tilde{S}_t}(x) &= P\left\{\tilde{S}_t \le x\right\} \\ &= \sum_{n=0}^{\infty} P\left\{\sum_{k=1}^{n} e^{r(t-U_k)} X_k \le x \middle| N_t = n\right\} P\{N_t = n\} \\ &= e^{-\lambda t} + \sum_{n=1}^{\infty} P\left\{\sum_{k=1}^{n} e^{r(t-U_k)} X_k \le x \middle| N_t = n\right\} \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \end{aligned} \tag{23}$$

Given $N_t = n$, the joint distribution of $\{U_n : n \in \mathbb{N}\}$ is the same as that of order statistics of n uniformly distributed random variables on $(0, t]$, it follows that

$$P\left\{\sum_{k=1}^{n} e^{r(t-U_k)} X_k \le x \middle| N_t = n\right\} = P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{k,t}\right)} X_k \le x\right\}, \tag{24}$$

where $V_{1,t}, V_{2,t}, \cdots, V_{k,t}$ are independently distributed uniform random variables on $(0, t]$, and they are independent of $\{X_n : n \in \mathbb{N}\}$.

Substituting (24) into (23), we obtain

$$F_{\tilde{S}_t}(x) = e^{-\lambda t} + \sum_{n=1}^{\infty} P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{k,t}\right)} X_k \le x\right\} \frac{(\lambda t)^n}{n!} e^{-\lambda t}. \tag{25}$$

Now we define

$$\tilde{X}_{k,t} \triangleq e^{r\left(t-V_{k,t}\right)} X_k,$$

and the distribution function of $\tilde{X}_t = \tilde{X}_{k,t}$ is

$$\begin{aligned} F_{\tilde{X}_t}(x) &\triangleq P\left\{e^{r\left(t-V_{k,t}\right)} X_k \le x\right\} \\ &= \frac{1}{t}\int_0^t F_X\left(xe^{-r(t-u)}\right) du. \end{aligned} \tag{26}$$

Let $F^{*n}_{\tilde{X}_t}$ denote the n-fold convolution of $F_{\tilde{X}_t}$,

$$F^{*n}_{\tilde{X}_t}(x) = P\left\{\sum_{k=1}^{n} e^{r\left(t-V_{k,t}\right)} X_k \le x\right\}, \quad t, x > 0. \tag{27}$$

For $t, x > 0$, let $f_{\tilde{X}_t}$ denote the probability density function of $\tilde{X}_t$. From (26), we get

$$f_{\tilde{X}_t}(x) = \frac{1}{t}\int_0^t e^{-r(t-u)} f_X\left(xe^{-r(t-u)}\right) du, \quad t, x > 0. \tag{28}$$

From (25) and (27), for $T, x > 0$ we have

$$F_{\tilde{S}_t}(x) = e^{-\lambda t} + e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!} F^{*n}_{\tilde{X}_t}(x), \tag{29}$$

and

$$f_{\tilde{S}_t}(x) = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!} f^{*n}_{\tilde{X}_t}(x), \tag{30}$$

where $f^{*n}_{\tilde{X}_t}$ is the n-fold convolution of $f_{\tilde{X}_t}$.

From (29) and (30), we consider $\tilde{S}_t$ as the compound Poisson process with Poisson process N_t with the intensity λ and the increment of recovery $\tilde{X}_t$.

B Panjer Recursion Formula and Fast Fourier Transformation

In this section, we present the two methods of calculating the probability distribution of compound Poisson distribution. One of methods is based on Panjer recursion formula and the other is based on the fast Fourier transformation. Embrechts and Frei (2008) numerically evaluate two methods. We explain Panjer recursion formula in Section B.1 and the fast Fourier transformation in Section B.2.

B.1 Panjer Recursion Formula

In this section we explain Panjer recursion formula. Only in the case the number of recoveries is a member of $(a, b, 0)$-class and the increment of recovery has non negative distribution, Panjer recursion formula can be used.

Definition 8. *We suppose $p(n)$ is the probability function of discrete random variable. It is a member of $(a, b, 0)$-class, if there exist constant a, b such that*

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1), \quad n = 1, 2, \cdots . \tag{31}$$

$(a, b, 0)$-class is also called Panjer class.

It has been known that only four distributions i.e. Poisson, binomial, negative binomial and geometric belong to $(a, b, 0)$-class. It can be seen that Poisson distribution with the intensity λ satisfies the recursion and that each value of a and b is 0 and λ respectively.

If the number of recoveries is a member of $(a, b, 0)$-class and the increment of recovery has non negative discrete distribution, the following discrete type Panjer recusion formula can be used. We present the result of Panjer (1981), for the references also see Klugman et al. (2004) and Panjer (2006).

Theorem B.1. *If N is a member of $(a, b, 0)$-class and each increment of recovery $X_n \in \mathbb{N}_0$, then the recursion formula is*

$$\begin{aligned}
p_S(0) &= \psi_N\left(p_X(0)\right), \\
p_S(x) &= \frac{1}{1 - ap_X(0)} \sum_{y=1}^{x} \left(a + \frac{by}{x}\right) p_X(y) p_S(x-y), \quad x = 1, 2, \cdots
\end{aligned}$$

where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\psi_N(u) = E\left[u^N\right]$ is the probability generating function of N.

If the number of recoveries is a member of $(a, b, 0)$-class and the increment of recovery has non negative continuous distribution, the following continuous type Panjer recusion formula can be used. Panjer (1981) proves the following Theorem.

Theorem B.2. *If N is a member of $(a, b, 0)$-class and the increment of recovery X_n has non negative continuous distribution function, then recursion formula is*

$$\begin{aligned}
f_S(0) &= \psi_N\left(f_X(0)\right), \\
f_S(x) &= p_N(1) f_X(x) + \int_0^x \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y), \quad x > 0,
\end{aligned} \tag{32}$$

where $\psi_N(u) = E\left[u^N\right]$ is the probability generating function of N.

B.2 Fast Fourier Transformation

In this section, we explain the calculation method of the probability distribution function of compound Poisson distribution by the fast Fourier transformation. A number of studies have been made by the fast Fourier transformation in actuarial mathematics and finance. We refer the reader to Rolski et al. (1999), Klugman et al. (2004) and Panjer (2006) for the references.

For any integrable function $f(x)$, the Fourier transformation is defined by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(x) e^{\boldsymbol{i} zx} dx.$$

where $\boldsymbol{i}$ is imaginary unit. If $\hat{f}(z)$ is integrable, the original function can be recovered from its inverse Fourier transform as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(z)e^{-\boldsymbol{i}zx}dz.$$

When $f(x)$ is a probability density function, $\hat{f}(z)$ is its characteristic function.

Definition 9. *For some fixed $n \in \mathbb{N}$, consider a sequence $f_0, f_1, \cdots, f_{n-1}$ of arbitrary real numbers. The discrete Fourier transformation $\hat{f}_k^D$ is defined by*

$$\hat{f}_k^D = \sum_{j=0}^{n-1} f_j \exp\left\{\frac{2\pi\boldsymbol{i}}{n}kj\right\}, \quad k = 0, 1, \cdots n-1.$$

The discrete inverse Fourier transformation is defined by

$$f_j = \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}_k^D \exp\left\{-\frac{2\pi\boldsymbol{i}}{n}kj\right\}, \quad k = 0, 1, \cdots n-1.$$

The fast Fourier transformation is an algorithm that reduces the number of computations for the Fourier transformation. The procedure for calculating distribution function of the cumulative recovery in the case the increment of recovery has continuous distribution by the fast Fourier transformation is as follows. We refer to Panjer (2006) for the following procedure.

(a) Discretize the distribution of the increment of recovery X using some methods (see Section 4.2.1), obtaining the discretized increment of recovery probability function

$$p_{X_{app}}(0), p_{X_{app}}(1), \cdots, p_{X_{app}}(n-1),$$

where X_{app} is discretized random variable of X.

(b) Apply the fast Fourier transformation to this vector of values, obtaining $\phi_{X_{app}}(z) \triangleq E\left[e^{\boldsymbol{i}zX_{app}}\right]$, the characteristic function of discretized distribution X_{app}.

(c) Transform this vector using the probability generating function transformation of the number of recoveries distribution, obtaining $\phi_{S_{app}}(z) \triangleq E\left[e^{\boldsymbol{i}zS_{app}}\right] = \psi_N\left(\phi_{X_{app}}(z)\right)$, which is the characteristic function, that is, the discrete Fourier transform of the cumulative recovery distribution where S_{app} is discretized approximation of S.

(d) Apply the inverse fast Fourier transformation, which is identical to the fast Fourier transformation except for a sign change and division by n. This gives a vector of length n values representing the exact distribution of cumulative recovery for the discrete increment of recovery model.

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