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Multiple Stochastically Stable Equilibria in Coordination Games

by
Toshimasa Maruta and Akira Okada

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Toshimasa Maruta[1] and Akira Okada[2]

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Abstract: In an (n, m)-*coordination game*, each of the n players has two alternative strategies. A strategy generates positive payoff only if there are at least $m - 1$ others who choose the same, where $m > n/2$. The payoff is nondecreasing in the number of such others so that there are exactly two strict equilibria. Applying the adaptive play with mistakes (Young 1993) to (n, m)-coordination games, we point out potential complications inherent in many-person games. Focusing on games that admit simple analysis, we show that there is a nonempty open set of (n, m)-coordination games that possess multiple stochastically stable equilibria, which may be Pareto ranked, if and only if $m > (n + 3)/2$, which in turn is equivalent to the condition that there is a strategy profile against which every player has alternative best responses.
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[1]Corresponding author: Advanced Research Institute for the Sciences and Humanities, Nihon University, 12-5 Goban-cho, Chiyoda, Tokyo 102–8251, Japan. E-mail: *tosh@arish.nihon-u.ac.jp* Phone: +81 3 5275 9607 Fax: +81 3 5275 9204.

[2]Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601 Japan. E-mail: *aokada@econ.hit-u.ac.jp* Phone: +81 42 580 8599 Fax: +81 42 580 8748

1 Introduction

Many social and economic models have multiple equilibria. Whenever players face a "coordination" problem, they typically find themselves in a game with multiple strict Nash equilibria. Such a problem may arise in the context of collective decision making. One of the simplest examples is the unanimity game, in which a particular policy can be implemented only if all the participants unanimously agree. Imagine that there are two alternative policies, either of them is preferred by everyone to the inaction, the status quo. If the collective decision is governed by the unanimity rule, it gives rise to a game with two strict equilibria. One may consider this situation as a prototypical example of the equilibrium selection problem, in that the game is so simple but the problem it poses is genuine. In developing their theory, thus, Harsanyi and Selten (1988) regarded the equilibrium selection in the unanimity game as a benchmark. For unanimity games, their theory selects an equilibrium that maximizes the Nash product, *i.e.,* the risk dominant equilibrium.

Another class of selection theories that receives attention is the stochastic evolution,[1] developed by Kandori, Mailath, and Rob (1993), Young (1993) and others. It is well known that in a two-player two-strategy coordination game, theories agree in their selection outcome. A majority of stochastic evolution models pick the same outcome, and it coincides with the risk dominant equilibrium. In broader classes of games, however, the agreement collapses in a drastic way. Not only that stochastic evolution need not select the risk dominant equilibrium, but also that outcomes differ within the class of stochastic evolution models. In fact, a stark disagreement arises even in the class of many-person binary unanimity games.[2]

In one of the rare studies that focus on the stochastic equilibrium selection in many-person stage games, Kim (1996), working in a single population random matching environment, obtains a unique equilibrium selection for a symmetric n-person binary coordination game. Thus, in a unanimity game, Kim (1996) selects the Pareto dominant equilibrium, which in this case is equivalent to the risk dominant equilibrium. By contrast, Young (1998), working in a multi-population random matching environment, points out by an example that there are binary unanimity games with four or more players and non-degenerated payoffs, in which both equilibria are stochastically stable. In particular, such an indeterminacy may occur even when two equilibria are Pareto ranked.

In this paper, we characterize a class of n-person coordination games that have multiple stochastically stable equilibria. As the equilibrium selection model, we adopt the adaptive play

[1] By a "stochastic evolution" model, we mean an equilibrium selection model formulated as a Markov chain and analyzed by the notion of stochastic stability à la Freidlin and Wentzell (1984) or its adaptation by Young (1993), or a closely related method.

[2] A strategic game is *binary* if every player has exactly two strategies.

with mistakes, a perturbation of the fictitious play with histories of a fixed length, introduced by Young (1993). Not only the result is of interest, but also an examination of the argument leading to it suggests answers to the following questions, which arise naturally when one compares Kim (1996) on one hand, and Young (1998) and the present paper on the other: What is the source of the different selection outcome?[3] Is the indeterminacy specific to the unanimity rule *per se*? How the number of players in the stage game affects the stochastic stability analysis?

We apply the adaptive play with mistakes to the class of (n, m)-*coordination games*.[4] Imagine a situation in which a group of n players has to make a collective decision over two alternative policies. The decision rule stipulates that a policy can be implemented only if at least $m > n/2$ members favor it. Once a policy has been implemented, the environment in question allows those and only those who chose it to receive the positive payoffs. The payoff from the policy is nondecreasing in the size of its proponents so that there are exactly two strict equilibria. A unanimity game is an (n, n)-coordination game. The intended interpretation of an (n, m)-coordination games is a collective decision making on the provision of a club good, as opposed to a public good. An alternative interpretation is that of a generalized n-person bargaining, in the spirit of Harsanyi and Selten (1988) and Young (1998).

The best response structure of an (n, m)-coordination game depends on the relative magnitude of n and m. For our purposes, it is worth pointing out that the game has a strategy profile against which all players have alternative best responses if and only if $m > (n + 3)/2$. At such a profile, best response behavior cannot point to any particular direction the adaptive dynamics should move toward.

We show that if $m > (n + 3)/2$, then there is an open set of (n, m)-coordination games on which stochastic stability fails to discriminate the two equilibria. Conversely, if $(n + 3)/2 \geq m$, then any multiplicity is non-generic, in that a slight perturbation of payoffs in the space of (n, m)-coordination games would restore equilibrium selection one way or another. Therefore an (n, m)-coordination game may have multiple stochastically stable equilibria if and only if it has a strategy profile at which none of the players is advised to choose any particular strategy by the best response principle.

In general, each stochastic evolution model has a particular adjustment dynamics through which the stage game is played. Authors have identified modeling details that may affect the nature of their selection outcome. Ellison (1993) shows that a local interaction shorten the average waiting time to observe the stochastically stable outcome. Binmore, Samuelson, and Vaughan (1995) pointed out Kandori et al. (1993) and Young (1993) may differ in their waiting

[3]While it is easy to point out the difference in the number of populations in Kim (1996) and Young (1998), it is not clear how it relates to the difference in the selection outcome.

[4]Kim (1996, p.217) considers a similar class of games.

time distributions. Bergin and Lipman (1996), Blume (2003), and Maruta (2002) examined how the selection outcome depends on the way the vanishing rates of mistake vary on different states. Robson and Vega-Redondo (1996) and Canals and Vega-Redondo (1998) show that it depends on the way the payoff information, based on which the strategies of the agents are revised, is given. Our analysis shows that the selection outcome also depends on the variety of probability distributions on the stage game strategy profiles that an agent might face in the underlying adaptive dynamics. It turns out that the range of the distributions in the multi-population random matching (Young 1998) or in the adaptive play (Young 1993) is much wider than that in the single population model (Kim 1996). This accounts for the different selection results mentioned above.

The plan of the paper is as follows. In Section 2, we define a *binary population game*. In such a game, each player has two strategies, her payoff is increasing in the number of the others who do the same, and the two unanimous strategy configurations are strict equilibria. We apply the adaptive play with mistakes to the class of binary population games. The class is much broader than that of (n, m)-coordination games. We start with this class in order to point out potential complications one may face in identifying resistances in general many-person games. In Section 3, we evaluate the resistance. We do this by introducing a linear program, the optimal solution of which constitutes a lower bound of the resistance. Examining the possible types of its optimal solution and its relationship to the resistance, we point out complications specific in many-person games. We focus on a *doubly simple* binary population game, in which the relevant linear program admits a simple solution and its value and the resistance coincide. For such a game, the resistances are explicitly solved in terms of the payoff parameters. In Section 4, the results in Section 3 are applied to investigate the equilibrium selection problem. First, we explain why and how the multiple stochastically stable equilibria may arise in a binary population game. Subsequently, we show that there is an open set of doubly simple (n, m)-coordination games that have multiple stochastically stable equilibria if and only if $m > (n + 3)/2$. In Section 5, we discuss the relationship between the differences in the specifics of the dynamics and those of the selection outcomes among representative stochastic equilibrium selection models. The proofs are given in the Appendix.

2 Preliminaries

There are n players, denoted by $i \in I = \{1, \ldots, n\}$. Each player chooses her strategy σ^i from $\{A, B\}$. $\sigma \in \Sigma = \{A, B\}^n$ and $\sigma^{-i} \in \{A, B\}^{n-1}$ denote a generic strategy profile and a strategy profile of players other than i, respectively. Let $|\sigma|$ be the number of A-players in σ and σ^{-i}, respectively. Let the payoff of player i be given as follows:

$$u^i(\sigma) = \begin{cases} a^i\left(|\sigma|\right), & \text{if } \sigma^i = A, \\ b^i\left(n - |\sigma|\right), & \text{if } \sigma^i = B, \end{cases}$$

where $a^i(k)$ and $b^i(k)$ are functions defined on $\{1, \ldots, n\}$ such that

- $a^i(k)$ and $b^i(k)$ are nondecreasing in k,
- $a^i(n) > b^i(1)$ and $b^i(n) > a^i(1)$.

The game thus defined is called a *binary population game*. It is called *symmetric* if $a^i(k) = a(k)$ and $b^i(k) = b(k)$ for every $i \in I$. It is called *asymmetric* if it is not symmetric.

An example of a binary population game is an (n, m)-*coordination game*, in which

$$a^i(k) = b^i(k) = 0 \quad \text{for every } k < m,$$

where n and m are natural numbers such that $n \geq m > n/2$. A *unanimity game* is an (n, n)-coordination game.

By the second assumption on payoffs, both $(A, \ldots, A)$ and $(B, \ldots, B)$ are strict equilibria. If the game is either symmetric or (n, m)-coordination, one can show that it has exactly two strict equilibria. In Section 3, we discuss general binary population games. We simply assume there that the game has exactly two strict equilibria.

As an equilibrium selection model, we adopt the adaptive play with mistakes, introduced by Young (1993). We assume that the reader is familiar to the stochastic stability analysis in general, and the adaptive play with mistakes in particular. For details, the reader is referred to Young (1993). In what follows, the sizes of a history and of a sample are denoted by T and s, respectively. Let $\mathbf{A}$ and $\mathbf{B}$ denote the T-fold repetition of $(A, \ldots, A)$ and $(B, \ldots, B)$, respectively. We assume that $s \leq T/2$. Under this assumption, one can show that in the adaptive play without mistakes for an (n, m)-coordination game, starting from any state, either $\mathbf{A}$ or $\mathbf{B}$ is reached with probability one. Thus the method of Young (1993) to identify stochastically stable equilibria is applicable. In Section 3, we implicitly assume this for the general binary population games. The resistance from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $r(A, B)$. $r(B, A)$ is the resistance for the other direction. $(A, \ldots, A)$ is uniquely stochastically stable if and only if $r(A, B) > r(B, A)$.

3 The resistance and the relevant linear program

3.1 Programs

Consider the adaptive play with mistakes for a binary population game. The current state is $\mathbf{A}$. In any path from $\mathbf{A}$ to $\mathbf{B}$, there is a player who optimally chooses strategy B for the first time. Let us call that player a *first exitor*. The first exitor $i \in I$ must have a sample against which playing B is optimal. Such a sample must contain considerable number of Bs played by others. Since player i is a first exitor, all such Bs are mistakes. We are going to set up a linear program that gives us the minimum number of mistaken Bs that i must face. Its optimal solution not only gives us the number, but also reveals the way the mistaken Bs occur, a factor that becomes significant only in games played by three or more players.

Fix a player and let her stick with A. In the adaptive play, what matters are samples. A sample is simply a set of strategy profiles that she observes. Specifically, we search for a sample against which she can best respond by B, and we count the number of Bs (*i.e.*, mistakes) in it. The number of mistakes in the sample is the sum of the numbers of mistakes in constituent profiles. In each profile, in turn, the number of mistakes is at most $n-1$. In a two-person game, it follows that the number of mistakes in the sample is equal to the number of profiles that deviate from the original equilibrium. In many-person games, they differ. A consequence is that two samples may work quite differently even if they contain the same number of mistakes. Not only the *number*, but also the *distribution* of mistakes matters. The linear program introduced below takes care of the case in point. The relationship between its optimal value and the resistance $r(A, B)$ is clarified in the next subsection.

Fix a player $i \in I$. Denote $a_k = a^i(k)$ and $b_k = b^i(k)$ for $k = 1, \dots, n$ and set

$$z_k = a_n - a_{n-k} + b_{k+1} - b_1$$

for $k = 1, \dots, n-1$. Note that z_k is nonnegative and nondecreasing in k. Although z_k depends on $i \in I$, we omit the superscript most of the time. The relevant program to solve is given as follows:

$$\begin{aligned} &\min x_1 + 2x_2 + \cdots + (n-1)x_{n-1} \\ &s.t. \quad x_1 + \cdots + x_{n-1} \le s, \quad \textstyle\sum_{k=1}^{n-1} z_k x_k \ge s(a_n - b_1), \end{aligned} \tag{P^i_A}$$

together with the nonnegativity condition $x_1 \ge 0, \dots, x_{n-1} \ge 0$.[5] In this program, x_k is the number of profiles in the sample that contain exactly k mistakes. $x_1 + \cdots + x_{n-1}$ is the number

[5]To be precise, we have to consider the integer constraint for x_k, since the stochastic stability hinges on the number of mistakes. Implicitly assuming that the sample size s is sufficiently large, we ignore the rounding problem throughout.

of profiles that contain at least one mistake. The first constraint comes from the fact that this number cannot exceed the sample size. The second constraint expands into

$$b_n x_{n-1} + \cdots + b_2 x_1 + (s - \textstyle\sum_{k=1}^{n-1} x_k) b_1 \geq a_1 x_{n-1} + \cdots + a_{n-1} x_1 + (s - \sum_{k=1}^{n-1} x_k) a_n.$$

Thus it ensures that strategy B is a best response against the sample. The objective function gives the total number of mistakes (*i.e.*, Bs) in the sample. It is clear that (P^i_A) has an optimal solution.[6]

Define $k_A = \min\{\, k \mid b_{k+1} \geq a_{n-k} \,\}$. Here again, we omit the superscript most of the time although k_A depends on $i \in I$. In a single strategy profile, strategy B is a best response for the player in question if and only if there are other $k \geq k_A$ players who choose B. If there is $j < k_A$ then $z_j < a_n - b_1$. Thus if $j < k_A \leq k$ then $0 \leq z_j < z_k$. Let $(x_1, \ldots, x_{n-1})$ be a feasible solution of (P^i_A). Its *support* is the set $\{\, k \mid x_k > 0 \,\}$ of indices. When it is singleton, we identify the set and its unique element. The next result characterizes conditions that ensure an optimal solution with single support.[7]

Proposition 1. *Consider the program (P^i_A) of a player $i \in I$ in a binary population game. Denote by λ_1 and λ_2 the Lagrange multipliers for the best response constraint and the sample size constraint, respectively. The following conditions are equivalent:*

(1) *There is an optimal solution in which $\lambda_2 = 0$.*

(2) *There is $k^* \geq k_A$ such that* $\dfrac{k^*}{z_{k^*}} = \min\limits_{\substack{k \\ z_k \neq 0}} \dfrac{k}{z_k}$.

(3) *The solution*

$$(x_1^*, \ldots, x_{n-1}^* : \lambda_1^*, \lambda_2^*) = \Big(\overbrace{0, \ldots, 0, \frac{s(a_n - b_1)}{z_{k^*}}}^{k^*}, 0, \ldots, 0 : \frac{k^*}{z_{k^*}}, 0 \Big)$$

is optimal.

If these conditions are satisfied, then the optimal value is given by $\dfrac{s k^* (a_n - b_1)}{z_{k^*}}$.

In (P^i_A), let a_k and b_k be interchanged and replace z_k by w_k, where

$$w_k = b_n - b_{n-k} + a_{k+1} - a_1,$$

[6] In a binary population game, what matters to each player is the number of other A-players and B-players. Who takes which does not matter. This feature justifies the formulation of (P^i_A).

[7] Note that the result allows the relevant program to have a single support optimal solution with $\lambda_2 > 0$, in which case it must be that $x_{k^*} = s$. It also permits $x_{k^*} = s$ and $\lambda_2 = 0$ to be optimal. One can construct examples that possess these types of optimal solutions.

and name the resulting program (P^i_B). It is the program that is relevant to evaluate the resistance $r(B, A)$ from **B** to **A**. For (P^i_B), the counterpart result to Proposition 1 follows.

For each $i \in I$, there are associated programs (P^i_A) and (P^i_B). They correspond to the minimum number of mistakes when i is a first exitor. Let i_A be the player whose optimal value of (P^i_A) is the smallest among $i \in I$. Intuitively, i_A is the player who prefer equilibrium $(B, \dots, B)$ most. In what follows, set $(\mathrm{P}_A) = (\mathrm{P}^{i_A}_A)$ and define (P_B) analogously.

In asymmetric games, (P_A) and (P_B) belong to different players in general. We always assume that the optimal value of i_A is strictly less than those of the others so that slight perturbation of i_A's payoff parameters would not alter the status of i_A. The same assumption is made for i_B.

Definition. A binary population game is *simple* if both (P_A) and (P_B) have optimal solutions with $\lambda_2 = 0$.

Simple binary population games inherit a tractable property from two-person games. Imagine the adaptive play over a two-person strategic game and find a sample with the least number of mistakes that allows an optimal switch from an equilibrium to another. Unless the destination equilibrium involves a weakly dominated strategy, we know that the least mistake sample contains one or more original equilibrium profiles. Is this also true in a game with many players? Since there is no dominance relationship between two strategies, we know for sure that there are samples containing one or more original profiles that allow the switch. We do not know, however, whether the least mistake sample can be found within such type of samples. Every profile in the least mistake sample may well contain a mistake. This occurs if (P_A) has no optimal solution with $\lambda_2 = 0$.[8] Viewed thus, a simple binary population game is "simple" in that the least mistake sample for the game looks "similar" to that for a two-person game. Examples below should make the issue clear. In Figure 1 and similar ones, a strategy with an asterisk denotes one by mistake.

Example 1. Consider the following symmetric $(4, 3)$-coordination game:

$$(a_3, a_4 : b_3, b_4) = (3, 4 : 2, 6).$$

In this game, $k_A = 2$ and

$$\min_{\substack{k \\ z_k \neq 0}} \frac{k}{z_k} = \frac{3}{z_3} = \frac{3}{10}.$$

[8]This is not to say that if $\lambda_2 = 0$ then the least mistake sample contains a profile without mistake. In fact, our results in the next section hinge on the fact that in some games the relevant program has an optimal solution in which $\lambda_2 = 0$ but the corresponding sample size constraint binds.

	Phase 1 $\overbrace{\qquad}^{3s/5}$			Phase 2 $\overbrace{\qquad}^{2s/5}$			Phase 3 $\overbrace{\qquad}^{3s/5}$			Phase 4 $\overbrace{\qquad}^{s}$		
σ^1	A	$\cdots$	A	B^*	$\cdots$	B^*	A	$\cdots$	A	X	$\cdots$	X
σ^2	A	$\cdots$	A	B^*	$\cdots$	B^*	A	$\cdots$	A	X	$\cdots$	X
σ^3	A	$\cdots$	A	B^*	$\cdots$	B^*	A	$\cdots$	A	X	$\cdots$	X
σ^4	A	$\cdots$	A	A	$\cdots$	A	B	$\cdots$	B	B	$\cdots$	B

Figure 1: A simple optimal solution of (P_A).

	$\overbrace{\qquad}^{s}$			$\overbrace{\qquad}^{s/4}$			$\overbrace{\qquad}^{3s/4}$			$\overbrace{\qquad}^{s}$		
σ^1	B	$\cdots$	B	A^*	$\cdots$	A^*	A^*	$\cdots$	A^*	X	$\cdots$	X
σ^2	B	$\cdots$	B	A^*	$\cdots$	A^*	B	$\cdots$	B	X	$\cdots$	X
σ^3	B	$\cdots$	B	B	$\cdots$	B	B	$\cdots$	B	A	$\cdots$	A
σ^4	B	$\cdots$	B	B	$\cdots$	B	B	$\cdots$	B	A	$\cdots$	A

Figure 2: A non-simple optimal solution of (P_B).

Therefore there is an optimal solution of (P_A) with single support $x_3 = 2s/5$. A sample corresponding to it is depicted in Figure 1, in which $i = 4$ is the first exitor. One can verify that Program (P_B) also satisfies the conditions in Proposition 1. Thus the game is simple.

Example 2. For the following symmetric $(4,3)$-coordination game

$$(a_3, a_4 : b_3, b_4) = (3, 4 : 1, 6),$$

consider Program (P_B). Since

$$\min_{\substack{k \\ w_k \neq 0}} \frac{k}{w_k} = \frac{1}{w_1} = \frac{1}{5},$$

the game is not simple. One can verify that

$$(x_1, x_2, x_3) = \left(\frac{3s}{4}, \frac{s}{4}, 0\right)$$

is an optimal solution. A sample corresponding to it is depicted in Figure 2, in which $i = 3, 4$ are the first exitors.

3.2 Evaluating resistances

By construction, the optimal value of (P_A) is a lower bound of the resistance $r(A, B)$. The optimal value is exactly equal to the resistance in some games, but it is strictly less in others.

	Phase 1 $\overbrace{\qquad}^{2s/9}$	Phase 2 $\overbrace{\qquad}^{7s/9}$	Phase 3 $\overbrace{\qquad}^{2s/9}$	Phase 4 $\overbrace{\qquad}^{7s/9}$	Phase 5 $\overbrace{\qquad}^{2s/9}$	Phase 6 $\overbrace{\qquad}^{7s/9}$
σ^1	$A \cdots A$	$B^* \cdots B^*$	$A \cdots A$	$X \cdots X$	$B \cdots B$	$B \cdots B$
σ^2	$A \cdots A$	$B^* \cdots B^*$	$A \cdots A$	$X \cdots X$	$B \cdots B$	$B \cdots B$
σ^3	$A \cdots A$	$A \cdots A$	$B \cdots B$	$B \cdots B$	$B \cdots B$	$B \cdots B$
σ^4	$A \cdots A$	$A \cdots A$	$B \cdots B$	$B \cdots B$	$B \cdots B$	$B \cdots B$

Figure 3: A path from **A** to **B**

This is another salient feature of games with three or more players. To see this, let us go back to Example 1 and Figure 1. During Phase 4, let $i = 1, 2, 3$ sample Phases 2 and 3. Since each of them is observing strictly less number of mistakes than program (P_A) indicates, the unspecified action X in Phase 4 is actually A. It is clear that without further mistakes there will be no sample that rationalizes them playing B. In order to reach state **B**, more mistakes are needed. That is, the resistance $r(A, B)$ is greater than the optimal value of (P_A).[9]

In other games, the resistance and the optimal value coincide.

Example 3. Consider a symmetric four-person binary population game in which

$$(a_1, a_2, a_3, a_4 : b_2, b_2, b_3, b_4) = (2, 3, 6, 7 : 0, 1, 5, 8).$$

One can verify that single support solutions

$$\frac{sa_4}{z_2} = \frac{7s}{9} \quad \text{and} \quad \frac{sb_4}{w_2} = \frac{8s}{11}$$

solve (P_A) and (P_B), respectively. Figure 3 depicts a transition from **A** to **B**. The sample assignments are as follows. In Phase 4, sample for player 1 and 2 are not specified. Players 3 and 4 sample Phases 2 and 3. In Phase 5, players 1 and 2 sample Phases 3 and 4. Players 3 and 4 sample Phases 2 and 3. In Phase 6, players 1 and 2 sample Phases 4 and 5. Players 3 and 4 sample the final available segment of Phase 2, Phase 5, and the initial segment of Phase 6. These assignments are possible since $s \leq T/2$. Each player best responds against the sample. The key is that in Phase 5, players 1 and 2 are observing a block of Bs in the right shape that is dictated by the optimal solution of (P_A).

This example should convince us that in a symmetric simple n-person binary population game, where n is even, if the single support indices of the two programs are $k = n/2$,

[9] It is clear from Figure 1 that in a simple binary population game that is *symmetric*, $s(k^* + 1)(a_n - b_1)/z_{k^*}$ is an upper bound of the resistance. In the Appendix, we derive a tighter upper bound.

$b_{(n+2)/2} \geq a_{n/2}$, and $a_{(n+2)/2} \geq b_{n/2}$, then the resistances are equal to the optimal values of the relevant programs. This observation, in turn, explains why the property holds in any two-person coordination game, with or without symmetric payoff. In this sense, the class of games defined by the property is a natural generalization of two-person coordination games.

Definition. A simple binary population game, with or without symmetric payoff, is *doubly simple* if $r(A, B)$ is equal to the optimal value of (P_A) and $r(B, A)$ is equal to the optimal value of (P_B).

In a doubly simple game, one can circumscribe potential complications inherent in many-person games.

4 Equilibrium selection in (n, m)-coordination Games

4.1 Insensitivity of the resistance to payoff parameters

Young (1998) found that in a unanimity game the resistances can be insensitive to equilibrium payoffs and that this leads to multiple stochastically stable equilibria. It turns out that insensitivity and multiplicity can arise in a broader class of games. Having established the relationship between the resistance and the relevant linear program, we are ready to explain why and when the resistances may become insensitive to the payoff parameters.

Consider a simple binary population game. Let ξ be the optimal solution of (P_A) described in Proposition 1, in which the unique support is k^*. Assume that the envelope theorem is applicable at ξ.[10] Then as a function of payoff parameters, the derivatives of the optimal value function $L(\cdot)$, the Lagrangian of (P_A), are given as follows:

- $\dfrac{\partial L(\xi)}{\partial a_n} = \dfrac{sk^*}{{z_{k^*}}^2}(b_{k^*+1} - a_{n-k^*}) = -\dfrac{\partial L(\xi)}{\partial b_1}$,
- $\dfrac{\partial L(\xi)}{\partial a_{n-k^*}} = \dfrac{sk^*}{{z_{k^*}}^2}(a_n - b_1) = -\dfrac{\partial L(\xi)}{\partial b_{k^*+1}}$,

and all the other derivatives are zero. The mirror image equations for the program (P_B) also follow.[11] By Proposition 1, $k^* \geq k_A$. Thus by definition of k_A, $b_{k^*+1} \geq a_{n-k^*}$.

It is important to notice that there are only four payoff parameters, when perturbed, that possibly affect the optimal value. Specifically, while b_{k^*+1} and a_{n-k^*} directly affect it, a_n or b_1

[10] If k^* is a unique maximizer in condition (2) of Proposition 1, then ξ is a unique solution of (P_A), in which case we can apply the envelope theorem at ξ.

[11] If $i_A \neq i_B$ (see paragraphs that follow Proposition 1), then perturbation of i_A's payoff has no effect on the optimal value of (P_B), and vice versa. If $i_A = i_B$ or the game is symmetric, then the perturbation may affect both.

does only if the first two differ. Note that $b_{k^*+1} = a_{n-k^*}$ means that at the strategy profile in which exactly k^* others play B, both A and B are best responses of i_A. Therefore, insensitivity of the resistance necessarily involves multiple best responses in the stage game.

These observations imply that in a doubly simple binary population game, if all the payoff parameters are allowed to vary, then generically there is a unique stochastically stable equilibrium. As typical in games that embody collective decision rules of the majority-rule variety, however, there are games in which some of the payoff parameters are defined to be constant. What if the index k^* is such that both a_{n-k^*} and b_{k^*+1} have to be, say, zero by the underlying decision rule? Then they are not allowed to be perturbed and no permissible payoff perturbation would change the value of the resistance. This leads to a failure of equilibrium selection by means of stochastic stability. This possibility is actual in the class of (n, m)-coordination games. For expository purpose, we start with unanimity games.

4.2 Unanimity games

Let an n-person unanimity game with $n \geq 3$ be given. Consider (P^i_A). We have

$$z^i_k = a^i_n - a^i_{n-k} + b^i_{k+1} - b^i_1 = \begin{cases} a^i_n, & \text{for } k = 1, \ldots, n-2, \\ a^i_n + b^i_n, & \text{for } k = n-1. \end{cases}$$

It follows that $k_A = 1$. Therefore (P^i_A) has a single support optimal solution described in Proposition 1. That is, unanimity games are simple. Proposition 1 implies that if $1/a^i_n$ is no more than $(n-1)/(a^i_n + b^i_n)$ for every $i \in I$, then the optimal value of (P_A) is s. Otherwise, it is $s(n-1)a^i_n/(a^i_n + b^i_n)$, which is strictly less than s. In either case, the optimal value and the resistance coincide, since by the unanimity rule an s-consecutive play of single deviation is enough for everyone to switch optimally. That is, the unanimity game is doubly simple.[12] Thus we have the following result.

Proposition 2. *Consider an n-person unanimity game, where $n \geq 3$.*

(A1) If

$$\frac{1}{a^i_n} \leq \frac{n-1}{a^i_n + b^i_n}$$

for every $i \in I$, then $r(A, B) = s$.

(A2) If there is $i \in I$ such that

$$\frac{1}{a^i_n} > \frac{n-1}{a^i_n + b^i_n}$$

[12]This would be clear from modifying Figure 6, which will appear in the next subsection to demonstrate a similar claim for (n, m)-coordination games. Details are left to the reader.

	$(B1)$	$(B2)$
$(A1)$	Both	A
$(A2)$	B	A if $\alpha > \beta$ B if $\beta > \alpha$

Figure 4: Stochastically stable equilibria in a unanimity game.

then $r(A, B) = \min_{i \in I} \frac{s(n-1)a_n^i}{a_n^i + b_n^i} < s.$

The analogous results $(B1)$ *and* $(B2)$ *follow for the resistance* $r(B, A)$*.*

By Proposition 2, the stochastically stable equilibria are determined as in Figure 4, where

$$\alpha = \min_{i \in I} \frac{a_n^i}{a_n^i + b_n^i}, \qquad \beta = \min_{i \in I} \frac{b_n^i}{a_n^i + b_n^i}.$$

The diagonal cases in Figure 4 are of interest. First, if both $(A1)$ and $(B1)$ apply, then the stochastic stability fails to discriminate the two equilibria. When do they apply? It is easy to see that when $n = 3$, they are simultaneously satisfied only if $a_n^i = b_n^i$ for every $i \in I$. Hence for three-person unanimity games, the stochastic stability selects one or the other in every game of interest. In contrast, if $n \geq 4$ then they place only very loose restriction on the equilibrium payoffs. Specifically, $(A1)$ and $(B1)$ apply if and only if $b_n^i \leq (n-2)a_n^i$ and $a_n^i \leq (n-2)b_n^i$, a pair of very generous conditions, and they become more so as n increases. In particular, both equilibria can be stochastically stable even if one of them Pareto-dominates the other.

Second, if $(A2)$ and $(B2)$ apply, then there is a unique stochastically stable equilibrium, which is determined by the risk dominance in a two-strategy two-player unanimity game played by the distinguished two players, i_A and i_B. It should be noted, however, that the unique outcome may not be risk dominant in the original game.

Example 4. Consider a three-person unanimity game in which

$$(a_3^1, a_3^2, a_3^3) = (7, 4, 4) \quad \text{and} \quad (b_3^1, b_3^2, b_3^3) = (3, 9, 9).$$

Player 1 prefers the former equilibrium, but the others prefer the latter. Comparing Nash products, we know that (B, B, B) is risk dominant. On the other hand, it follows from Proposition 2 that (A, A, A) is uniquely stochastically stable since (A, A) is risk dominant in the *two-player* game, depicted in Figure 5, that is played by $i_A = 2$ and $i_B = 1$.

It is instructive to consider a four-person game in which

$$(a_4^1, a_4^2, a_4^3, a_4^4) = (7, 4, 4, 4) \quad \text{and} \quad (b_4^1, b_4^2, b_4^3, b_4^4) = (3, 9, 9, 9).$$

	A	B
A	$7, 4$	$0, 0$
B	$0, 0$	$3, 9$

Figure 5: A two-person unanimity game.

	Phase 1 $\overbrace{\hphantom{xx}}^{T}$			Phase 2 $\overbrace{\hphantom{xx}}^{s}$			Phase 3 $\overbrace{\hphantom{xx}}^{s}$			Phase 4 $\overbrace{\hphantom{xx}}^{s}$		
σ_1	A	$\cdots$	A	B^*	$\cdots$	B^*	A	$\cdots$	A	B	$\cdots$	B
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_{n-m+1}	A	$\cdots$	A	B^*	$\cdots$	B^*	A	$\cdots$	A	B	$\cdots$	B
σ_{n-m+2}	A	$\cdots$	A	A	$\cdots$	A	B	$\cdots$	B	B	$\cdots$	B
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_n	A	$\cdots$	A	A	$\cdots$	A	B	$\cdots$	B	B	$\cdots$	B

Figure 6: A path from $\mathbf{A}$ to $\mathbf{B}$.

One can verify that $(A2)$ and $(B2)$ still apply, so $(A, \ldots, A)$ is uniquely stochastically stable. In the analogous five-person game, however, they no longer apply but $(A1)$ and $(B1)$ do. Thus the indeterminacy occurs.

In general, even if a unique selection outcome may be obtained in a "small"-person game, "replicating" it with respect to players eventually leads to multiplicity.

4.3 Multiple stochastically stable equilibria in (n, m)-coordination games

By definition of an (n, m)-coordination game, $m > n/2$. In this subsection, we consider games in which $m \geq (n+3)/2$. In such a game, we have

$$z_k^i = a_n^i - a_{n-k}^i + b_{k+1}^i - b_1^i = \begin{cases} a_n^i - a_{n-k}^i, & \text{for } k = 1, \ldots, n-m, \\ a_n^i, & \text{for } k = n-m+1, \ldots, m-2, \\ a_n^i + b_{k+1}^i, & \text{for } k = m-1, \ldots, n-1, \end{cases} \tag{$\star$}$$

and w_k^i is given by flipping a_k^i and b_k^i. Note that $m \geq (n+3)/2$ implies $n - m + 1 < m - 1$. Thus there is at least one k such that $z_k^i = a_n^i$. It follows that $k_A \leq n - m + 1$.

Inspecting $(\star)$, if $x_{n-m+1} = s$ solves (P_A^i), then by Proposition 1 the optimal value of (P_A^i) is $(n - m + 1)s$, a value that depends on neither the players nor the payoff parameters. If this is true for some $i \in I$, then $(n - m + 1)s$ is an upper bound of the resistance $r(A, B)$. In Figure

6, let $i \in \{1, \ldots, n-m+1\}$ and $j \in \{n-m+2, \ldots, n\}$. In Phase 3, let j sample Phase 2. Then both A and B are best responses since $n-m+2 < m$. In Phase 4, let i sample Phase 3. Then i can choose B since respective strategies yields $b^i_m \geq 0$ and $a^i_{n-m+1} = 0$. Letting j sample the final available segment of Phase 2 and the initial segment of Phase 4, we make her choose B in Phase 6 as well. In this way, an s-consecutive $(B, \ldots, B)$ arises. It follows that if $k = n-m+1$ is the single support solution index of (P^i_A) and (P^i_B) for every $i \in I$, then the game is doubly simple and the two resistances are equal to $(n-m+1)s$. Formally, the crucial condition is given by

$$\frac{n-m+1}{a^i_n} = \min_{\substack{k \\ z^i_k \neq 0}} \frac{k}{z^i_k} \quad \text{and} \quad \frac{n-m+1}{b^i_n} = \min_{\substack{k \\ w^i_k \neq 0}} \frac{k}{w^i_k} \qquad \text{for every } i \in I. \tag{\textbf{M}}$$

The question is, therefore, when this can be true. It turns out that if $m > (n+3)/2$ then there is an open set of (n,m)-coordination games in which (**M**) is satisfied.

A sequence of nonempty open intervals $\gamma_1, \gamma_2, \ldots, \gamma_K$ in $\mathbb{R}$ is *strictly increasing* if every element of γ_k is strictly less than every element of γ_{k+1}.

Proposition 3. *If $m > (n+3)/2$, then there are strictly increasing nonempty open intervals $\alpha_m, \alpha_{m+1}, \ldots, \alpha_n$ and $\beta_m, \beta_{m+1}, \ldots, \beta_n$ of positive numbers such that for every $a^i_k \in \alpha_k$ and $b^i_k \in \beta_k$, where $k = m, m+1, \ldots, n$ and $i \in I$, the (n,m)-coordination game $(a^i_m, a^i_{m+1}, \ldots, a^i_n : b^i_m, b^i_{m+1}, \ldots, b^i_n)_{i \in I}$ is doubly simple in which both equilibria are stochastically stable.*

4.4 Characterizing multiplicity

An (n,m)-coordination game has *generic payoff* if both a^i_m and b^i_m are positive, a^i_n and b^i_n differ, and both a^i_k and b^i_k are strictly increasing in $k \geq m$. With generic payoff, we can prove a sort of converse to Proposition 3.

Proposition 4. *Let there be a simple (n,m)-coordination game with generic payoff such that the optimal values of (P_A) and (P_B) coincide. If $m \leq (n+3)/2$, then a slight perturbation of payoff results in another (n,m)-coordination game in which the optimal values differ.*

Proposition 3 and 4 characterize multiplicity in the class of doubly simple[13] (n,m)-coordination games: there is an open set of games with multiple stochastically stable equilibria if and only if $(n+3)/2 > m$. The latter condition, in turn, is a necessary and sufficient condition for an (n,m)-coordination game to have a strategy profile against which both strategies are best responses for every player. Thus the characterization indicates a close relationship between multiplicity of equilibria and of best responses.

[13] We are simply assuming that after the slight payoff perturbation the game remains to be doubly simple.

5 Concluding Remarks

Working in the adaptive play with mistakes, we have characterized the indeterminacy in doubly simple (n, m)-coordination games. Our analysis have shown that it is not specific to unanimity games but may present itself in games with alternative best responses. The result should be compared with Kim (1996), in which a unique selection result for binary coordination games has been obtained, and Young (1998), where the possibility of indeterminacy in unanimity games has been pointed out. In what follows, we reconcile the different selection results that may appear to be contradicting each other.

Stochastic evolution models vary in their own specific adjustment dynamics through which the stage game is played. Nonetheless, it is possible to describe a general structure. Typically, the dynamics run as follows.[14] There is a mapping f from its state space Z into the set $\Delta(\Sigma)$ of all probability distributions on the set Σ of stage game pure strategy profiles. At state $z \in Z$, each player may or may not best respond against the probability distribution $f(z) \in \Delta(\Sigma)$. Their strategies determine the next state. Naturally, different models generate different mappings. For equilibrium selection, the range of the mapping f matters.

Take an n-person symmetric binary unanimity game and consider its static best response. By the unanimity rule, both strategies are best responses to a strategy profile in which everyone but exactly one chooses the same. In a primitive sense, therefore, the minimum number of deviations to upset either equilibrium is one. This is definitely one of the forces working behind the indeterminacy result.[15] In order for this feature to translate into the selection outcome, it seems necessary and sufficient that there is a state z such that the corresponding distribution $f(z)$ places probability one to the one-deviation profile.

There is such a state in the adaptive play with mistakes. Setting $n = m$ in Figure 6, agents $i = 2, \ldots, n$ are observing the one-deviation profile with certainty. This is also true if the game is played, as in Young (1998), through a multi-population random matching. In such a model, there are n different populations, each of which is the set of agents that act as one of the players in the stage game. Consider a state in which every agent in $n-1$ populations chooses the same strategy but every agent in the remaining population chooses the other. Then every agent in the $n-1$ populations is observing the one-deviation profile with probability one. In fact, it is clear in both models that for each pure strategy profile $\sigma \in \Sigma$ there is a state z such that $f(z)$ places the unit mass on σ. This observation leads to the conjecture that results analogous to

[14] The description covers at least representative models such as the single population random matching, the multi-population random matching, and the adaptive play with mistakes. For the latter, a state should be refined to include a profile of sample assignments to the agents.

[15] We say "one of the forces" here since if it were the only one then the indeterminacy would occur in every unanimity game.

ours should be true in a multi-population random matching environment.

By contrast, there is no such state if the game is played, as in Kim (1996), through a single population random matching mechanism. In such a model, the state space is $\{0, 1, \ldots, N\}$, where $N > n$ is the population size and each state represents the number of agents who choose a particular action. Consider state $N - 1$, in which exactly one agent plays differently. In this state, everyone except the deviator faces the probability distribution $f(N-1)$ in which the unanimous profile has probability $\binom{N-2}{n-2}/\binom{N-1}{n-1}$ and the one-deviation profile has probability $\binom{N-2}{n-1}/\binom{N-1}{n-1}$. It is clear that there is no state z such that $f(z)$ places probability one to the one-deviation profile.[16] This is the source of the difference between the adaptive play and the multi-population matching on one hand, and the single population random matching on the other.[17] In this way, the range of the mapping f affects the selection outcome of the stochastic equilibrium selection.

One might be tempted to think that binary coordination games are particularly simple class of games. As far as equilibrium selection is concerned, such a view may be ill-founded. The selection outcome of a binary coordination game is by no means obvious.[18]

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[16]If the stage game is an n-person binary unanimity game, we can take as $\Delta(\Sigma)$ the n-dimensional unit cube, vertices of which correspond to pure strategy profiles. The relevant mixed strategy profiles in the single population model are those on the diagonal of $\Delta(\Sigma)$, which links $(0, \ldots, 0)$ to $(1, \ldots, 1)$. Thus, except for the two unanimities, no pure strategy profile in Σ can be the value $f(z)$ of a state $z \in Z$.

[17]Concerning the stability properties of versions of the replicator dynamics, Weibull (1995) discusses the difference between the single population model and the multi-population model.

[18]Recent studies in other strands of equilibrium selection appear to support this view. Morris and Ui (2005) and Oyama, Takahashi, and Hofbauer (2005) generalize previous results in equilibrium selection, respectively, by robustness with respect to incomplete information (Kajii and Morris 1997) and by perfect foresight dynamics (Matsui and Matsuyama 1995). For an n-person unanimity game, however, no clear-cut result has been obtained. Meanwhile, Hofbauer (1999) presents a dynamic selection model in which the risk dominant equilibrium is selected in n-person unanimity games.

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Appendix

A.1 Proof of Proposition 1

Proof. Consider the dual program of (P_A):

$$\max s(a_n - b_1)\lambda_1 - s\lambda_2$$

$$s.t. \quad z_1\lambda_1 - \lambda_2 \le 1,\ \ldots,\ z_k\lambda_1 - \lambda_2 \le k,\ \ldots,\ z_{n-1}\lambda_1 - \lambda_2 \le n-1, \tag{D$_A$}$$

together with the nonnegativity condition $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$. By the duality theorem, nonnegative vectors $(x_1, \ldots, x_{n-1})$ and (λ_1, λ_2) are optimal solutions of (P$_A$) and (D$_A$), respectively, if and only if

- Primal Feasibility: $(x_1, \ldots, x_{n-1})$ is a feasible solution of (P$_A$),
- Dual Feasibility: (λ_1, λ_2) is a feasible solution of (D$_A$),
- Complementary Slackness:
  - For each x_k, if $x_k > 0$ then $z_k\lambda_1 - k = \lambda_2$,
  - If $\lambda_1 > 0$, then $\sum_{k=1}^{n-1} z_k x_k = s(a_n - b_1)$,
  - If $\lambda_2 > 0$, then $\sum_{k=1}^{n-1} x_k = s$.

Since (3) implies (1), it suffices to show the following two implications.

(1) $\Rightarrow$ (2). Let $\lambda_2 = 0$. Thus $z_k\lambda_1 \le k$. Thus $\lambda_1 \le k/z_k$ for every $z_k \ne 0$. By complementary slackness,

$$\lambda_1 = \frac{k}{z_k} \quad \text{for every } k \text{ such that } x_k \ne 0.$$

Therefore

$$\lambda_1 = \min_{\substack{k \\ z_k \ne 0}} \frac{k}{z_k} > 0.$$

Let

$$\arg\min_{\substack{k \\ z_k \ne 0}} \frac{k}{z_k} = \{k_1, \ldots k_l\}.$$

Assume that $k_j < k_A$ for every $j = 1, \ldots, l$. Then $z_{k_j} < a_n - b_1$ by the definition of k_A. Thus

$$\begin{aligned} z_{k_1}x_{k_1} + \cdots + z_{k_l}x_{k_l} &< (a_n - b_1)x_{k_1} + \cdots + (a_n - b_1)x_{k_l} \\ &= (a_n - b_1)(x_1 + \cdots + x_n) \le s(a_n - b_1). \end{aligned}$$

Therefore $\sum_k z_k x_k < s(a_n - b_1)$ since $x_k > 0$ implies $k = k_j$ for some $j = 1, \ldots, l$. But this contradicts the complementary slackness since $\lambda_1 > 0$. Therefore there is $k^* \ge k_A$ such that $k^*/z_{k^*} = \min_{k, z_k \ne 0} k/z_k$.

(2) $\Rightarrow$ (3). Consider the solution given in (3). Nonnegativity constraints are all satisfied. Since $a_{n-k^*} \le b_{k^*+1}$, $z_{k^*} \ge a_n - b_1$, which implies $x_{k^*} \le s$. Thus $(x_1, \ldots, x_{n-1})$ is primal feasible. Since $\lambda_2^* = 0$, the dual constraint is given by $z_k\lambda_1^* \le k$, which is satisfied by the definition of λ_1^*. Thus $(\lambda_1^*, \lambda_2^*)$ is dual feasible. It is straightforward to verify complementary slackness. Thus the solution is optimal by the duality theorem. □

A.2 Proof of Proposition 3

To prove Proposition 3, we use the following lemma.

Lemma 1. *Consider a parameter set* $(a_m, a_{m+1}, \ldots, a_n : b_m, b_{m+1}, \ldots, b_n)$ *for an* (n, m)*-coordination game such that* $m \geq (n+3)/2$*.* (**M**) *in Section 4.3 holds true if and only if*

$$\eta_{n-k} a_n \leq a_k \leq \eta_{k-1} b_n \quad \text{and} \quad \eta_{n-k} b_n \leq b_k \leq \eta_{k-1} a_n \tag{$\dagger$}$$

for $k = m, m+1, \ldots, n$*, where*

$$\eta_j = \frac{|n-m+1-j|}{n-m+1}, \quad j \in \{0, 1, \ldots, n-m\} \cup \{m-1, \ldots, n-1\}.^{19}$$

Proof. It suffices to consider (P_A) only. We are looking for a necessary and sufficient condition for

$$\frac{n-m+1}{a_n} \leq \min_{\substack{k \\ z_k \neq 0}} \frac{k}{z_k}. \tag{A.1}$$

We have discussed unanimity games in Section 4.2. In what follows, thus, we assume $n-m \geq 1$. In an (n, m)-coordination game with $m \geq (n+3)/2$,

$$z_k = \begin{cases} a_n - a_{n-k}, & \text{for } k = 1, \ldots, n-m, \\ a_n, & \text{for } k = n-m+1, \ldots, m-2, \\ a_n + b_{k+1}, & \text{for } k = m-1, \ldots, n-1. \end{cases}$$

Assume (A.1). Then for every $k \leq n-m$ such that $a_n > a_{n-k}$,

$$\frac{n-m+1}{a_n} \leq \frac{k}{a_n - a_{n-k}},$$

which is equivalent to

$$(n-m+1-k) a_n \leq (n-m+1) a_{n-k}.$$

Hence,

$$\eta_k a_n \leq a_{n-k}, \quad \text{where } \eta_k = \frac{n-m+1-k}{n-m+1}. \tag{A.2}$$

Since $\eta_k \leq 1$, (A.2) is true, a posteriori, every $k \leq n-m$ such that $a_n = a_{n-k}$. Thus we have (A.2) for every $k = 1, \ldots, n-m$. By changing the variable and noting that $\eta_0 = 1$, we have $\eta_{n-k} a_n \leq a_k$ for every $k = m, \ldots, n$.

Similarly, (A.1) implies

$$\frac{n-m+1}{a_n} \leq \frac{k}{a_n + b_{k+1}}$$

[19] Since $m \geq (n+3)/2$, $n-m+1 < m-1$. Thus the numerator of η_j never be zero.

for every $k = m-1, \dots, n-1$,[20] which is equivalent to

$$(n-m+1)b_{k+1} \leq (k-n+m-1)a_n.$$

Thus

$$b_{k+1} \leq \eta_k a_n, \quad \text{where } \eta_k = \frac{k-n+m-1}{n-m+1}$$

for every $k = m-1, \dots, n-1$. By changing the variable, we have $b_k \leq \eta_{k-1} a_n$ for every $k = m, \dots, n$. Thus we have $\eta_{n-k} a_n \leq a_k$ and $b_k \leq \eta_{k-1} a_n$ for every $k = m, \dots, n$, the half of the inequalities in (†). Conversely, it is clear that these two imply (A.1). □

We are now ready to prove Proposition 3.

Proof. Let us say that a sequence of nonempty open intervals $\delta_1, \delta_2, \dots, \delta_K$ in $\mathbb{R}$ is *weakly increasing* if $\inf \delta_k < \inf \delta_{k+1}$. It is clear that if $\delta_1, \delta_2, \dots, \delta_K$ is weakly increasing then there are nonempty open intervals $\gamma_k \subset \delta_k$ such that $\gamma_1, \gamma_2, \dots, \gamma_K$ is strictly increasing.

By the preceding observation and Lemma 1, it suffices to construct weakly increasing sequences of nonempty open intervals $\alpha_m, \dots, \alpha_n$ and $\beta_m, \dots, \beta_n$ of positive numbers that satisfy the following.

(B.1) $a_n \leq \eta_{n-1} b_n$ and $b_n \leq \eta_{n-1} a_n$ for every $(a_n, b_n) \in \alpha_n \times \beta_n$.

(B.2) For every $(a_n, b_n) \in \alpha_n \times \beta_n$,

$$\alpha_k \subset [\eta_{n-k} a_n,\, \eta_{k-1} b_n] \quad \text{and} \quad \beta_k \subset [\eta_{n-k} b_n,\, \eta_{k-1} a_n]$$

for every $k = m, m+1, \dots, n-1$.

Let $m > (n+3)/2$. Then

$$\eta_1 = \frac{n-m}{n-m+1} < 1 < \frac{m-2}{n-m+1} = \eta_{n-1}.$$

One can verify that under this condition we can choose $\overline{e} > \underline{e} > 0$ such that

(B.3) $\eta_1 \overline{e} < \underline{e}$,

(B.4) $\max\left\{\dfrac{a_n}{b_n}, \dfrac{b_n}{a_n}\right\} < \eta_{n-1}$ for every $a_n, b_n \in (\underline{e}, \overline{e})$.

Letting $\alpha_n = \beta_n = (\underline{e}, \overline{e})$, we obtain (B.1). One can verify that

$$\frac{\eta_{k-1}}{\eta_{n-k}} = \frac{k-n+m-2}{k-m+1}$$

[20]Note that $m-1 > k_A$.

is strictly decreasing in $k \geq m$. Note that $\eta_{n-1}/\eta_0 = \eta_{n-1}$. Then by (B.4),

$$\frac{\overline{e}}{\underline{e}} \leq \eta_{n-1} < \frac{\eta_{k-1}}{\eta_{n-k}}$$

for every $k = m, \ldots, n-1$. Therefore

$$\eta_{n-k} a_n < \eta_{n-k} \overline{e} < \eta_{k-1} \underline{e} < \eta_{k-1} b_n$$

for every $k = m, \ldots, n-1$ and every $(a_n, b_n) \in \alpha_n \times \beta_n$. Setting

$$\alpha_k = \beta_k = (\eta_{n-k} \overline{e}, \eta_{k-1} \underline{e}),$$

we have (B.2).

It remains to show that $\alpha_m, \ldots, \alpha_n$ is weakly increasing. Since η_{n-k} is strictly increasing in k, $\inf \alpha_k = \eta_{n-k} \overline{e} < \eta_{n-k-1} \overline{e} = \inf \alpha_{k+1}$ for $k = m, \ldots, n-2$. Together with (B.3), it follows that $\alpha_m, \ldots, \alpha_n$ is weakly increasing. □

A.3 Proof of Proposition 4

Proof. Let a simple (n, m)-coordination game with generic payoff be given. Assume that the optimal values v_A and v_B of (P_A) and (P_B) to be equal. In Section 4.1 we saw that v_A is a decreasing function of b_{k^*+1}, where k^* is the single support optimal solution of (P_A). If $k^* + 1 \geq m$, then by generic payoff $b_{k^*+1} < b_{k^*+2}$. Thus we can increase $b^i_{k^*+1}$ to decrease v_A but to weakly increase v_B.[21] Hence it suffices to show that $k^* \geq m - 1$.

If $m \leq (n+2)/2$, then $n - (m-2) \geq m$. Thus $b^i_{(m-2)+1} < a^i_{n-(m-2)}$ by generic payoff. Therefore the definition of k_A implies that $m - 1 \leq k_A \leq k^*$. The case that $m = (n+3)/2$ only remains. In this case, $k_A = n - m + 1 = m - 2 = (n-1)/2$ (see ($\star$) in the opening paragraph of Section 4.3). Assume that $k^* = m - 2$. Then it follows from the first equation in (**M**) (Section 4.3) that for every $i \in I$,

$$\frac{n-1}{2a^i_n} = \frac{n-m+1}{a^i_n} \leq \frac{(n-1)}{a^i_n + b^i_n},$$

which implies $b^i_n \leq a^i_n$. Thus if, in addition, k^{**}, the support of (P_B), is also $m - 2$, then $a^i_n = b^i_n$ for every $i \in I$. But such a case is excluded by generic payoff assumption. Thus $\max\{k^*, k^{**}\} \geq m - 1$. By renaming the payoff parameters if necessary, we can conclude that $k^* \geq m - 1$. □

A.4 An upper bound of the resistance

Even if the resistance $r(A, B)$ and the optimal value of (P_A) differ the latter is still useful in deriving an upper bound of the former if the game is symmetric.

[21] If (P_A) and (P_B) belong to the same player $i \in I$, as they would in a symmetric game, then an increase of b_{k^*+1} would change v_B as well. But we know that $\partial L_B / \partial b_k$, the partial of the Lagrangian of (P_B), is nonnegative.

	Phase 1	Phase 2_t	Phase 2_{1-t}	Phase 3	Phase 4
	$s-x$	tx	$(1-t)x$	$s-x$	x
σ_1	$A \cdots A$	$B^* \cdots B^*$	$B^* \cdots B^*$	$A \cdots A$	$B \cdots B$
$\vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$
σ_{k-1}	$A \cdots A$	$B^* \cdots B^*$	$B^* \cdots B^*$	$A \cdots A$	$B \cdots B$
σ_k	$A \cdots A$	$B^* \cdots B^*$	$B^* \cdots B^*$	$A \cdots A$	$B \cdots B$
σ_{k+1}	$A \cdots A$	$B^* \cdots B^*$	$A \cdots A$	$B \cdots B$	$B \cdots B$
σ_{k+2}	$A \cdots A$	$A \cdots A$	$A \cdots A$	$B \cdots B$	$B \cdots B$
$\vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$	$\vdots \cdots \vdots$
σ_n	$A \cdots A$	$A \cdots A$	$A \cdots A$	$B \cdots B$	$B \cdots B$

Figure 7: A path from **A** to **B**.

To derive the upper bound, take a simple and symmetric binary population game and let the unique support solution be given by $x = s(a_n - b_1)/z_k$. Consider the sequence of events depicted in Figure 7. At the beginning of Phase 1, the current state of the adaptive play is **A**, which we call Phase zero. In Phase 1, everyone best responds to Phase zero, plays A. In Phase 2, players $i = 1, \dots, k$ make mistakes as exactly as the solution of (P$_A$) indicates. Players $i = k+2, \dots, n$ best respond to Phase zero. Pay a special attention to player $k+1$, who makes mistakes in Phase 2_t but best responds in Phase 2_{1-t}, where $t \in [0,1]$. In Phase 3, each player samples Phases 1 and 2 and best responds. The number of Bs that players $i = 1, \dots, k$ face is weakly less than the optimal value of (P$_A$).[22] Thus they can choose A. Player $k+1$ faces the solution of (P$_A$). Thus she can chooses B. Players $i = k+2, \dots, n$ face the solution of (P$_A$) with some extra Bs. Therefore they choose B. In Phase 4, each player samples Phases 2 and 3 and best responds. Since players $i = k+1, \dots, n$ see more Bs than in Phase 3, they continue playing B. In contrast, what players $i = 1, \dots, k$ choose depends on t. They choose B if $g(t) \geq 0$, where

$$g(t) = \{(s-x)b_{n-k+1} + txb_{k+1} + (1-t)xb_k\} - \{(s-x)a_k + txa_{n-k} + (1-t)xa_{n-k+1}\}.$$

It may well be the case that $g(0) \geq 0$. In this case, the resistance $r(A,B)$ is equal to the optimal value of (P$_A$). Even if $g(0) < 0$, there is $t \in (0,1]$ such that $g(t) \geq 0$. This is because by plugging $x = s(a_n - b_1)/z_k$ in , we have

$$(s-x)b_1 + xb_{k+1} = (s-x)a_n + xa_{n-k}.$$

By monotonicity of payoff parameters, it follows that

$$(s-x)b_{n-k+1} + xb_{k+1} \geq (s-x)a_k + xa_{n-k},$$

[22] In other words, if $t < 1$ they face a strictly less number of Bs so that they must choose A. If $t = 1$, they face the optimal solution so that A and B are alternative best responses. Recall that we ignore rounding issues.

which means that $g(1) \geq 0$. Therefore if $t \geq t^* = \min \{ t \in [0,1] \mid g(t) \geq 0 \}$, everyone can play B in Phase 4. In Phase 5, each player samples Phases 3 and 4. By monotonicity again, everyone continues playing B. Note that this sequence of events can happen with positive probability under the assumption that $s \leq T/2$. Now it is clear that we can bound the resistance as

$$\frac{sk(a_n - b_1)}{z_k} \leq r(A, B) \leq \frac{s(k + t^*)(a_n - b_1)}{z_k}.$$

It does not appear to be straightforward to bound the resistance any tighter. Note that the argument that leads to this inequality depends on the symmetric payoff assumption in several steps.