Do Irrelevant Commodities Matter?*

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October 2005

Abstract

We study the possibility of making social evaluations of allocations independently of individuals' preferences over unavailable commodities. This is related to the well-known problem of performing international comparisons of standard of living across countries with different consumption goods. We prove impossibility results which suggest that such evaluations encounter difficulties when the objects of evaluation are allocations of ordinary commodities. We show how possibility results can be retrieved when the objects of evaluation are allocations of composite commodities, characteristics or human functionings.

Keywords: consumer preferences, social choice, irrelevant commodities, functionings.

Journal of Economic Literature Classification Numbers: D63, D71.

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*We thank J. Bone, R. Bradley, P.A. Chiappori, R. Deb, M. Mariotti, H. Moulin, K. Roberts, K. Suzumura, W. Thomson, and P. Young for comments, as well as participants in the seminars at the LSE, Oxford, Columbia, Rice, SMU and Waseda, and in the Asian Decentralization Conference 2005. A first draft of this paper was prepared while the second author was visiting the University of Pau. He is grateful for the hospitality and financial support of the University. Financial support from the Ministry of Education, Culture, Sports, Science and Technology of Japan for the 21st Century Center of Excellence Project on the Normative Evaluation and Social Choice of Contemporary Economic Systems is also gratefully acknowledged.

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1 Introduction

In economic models, the analyst often focuses on interior allocations and seldom realizes how irrealistic it is. In real life, ordinary individuals consume only a tiny fraction of the available commodities. Moreover, the available commodities in an integrated economic space represent only a fraction of the whole set of conceivable or simply producible commodities. As a consequence, very little is known about preferences over unavailable commodities. Japanese preferences over the various kinds of sheep cheese from different valleys in the French Pyrenees are hardly known, just like French preferences over varieties of *sake* are difficult to guess. Modern preferences over the use of pigeon post are unknown, just like 17th century preferences over cellular phones.

This is the source of well-known problems in international comparisons of living standards. The fact that different countries have different consumption goods in their national markets makes it difficult to compare prices, and the computation of purchasing power parities has to rely on gross approximations. For instance, the few items that are common in two countries are used to compute indexes of relative prices for broad categories of goods. A similar problem occurs in time series for the evaluation of growth over many periods of time. Moreover, in order to evaluate the true living standards, one should presumably take account of individuals' preferences. Even if the usual computations of real incomes in purchasing power parity units are useful, the relation between such evaluations and true population preferences is rather loose.

Given the constraints of available commodities, it seems inevitable to focus only on individuals' preferences over available commodities in social welfare evaluations of resource allocations, and to disregard those over unavailable ones. It is simply impossible to take account of commodities we do not know, for instance those which are yet to be invented. Furthermore, one may argue that even in the case of known but unavailable commodities, it makes little sense to take account of individuals' preferences over them in order to allocate available commodities. The welfare evaluation of Japanese consumption may ignore apple fritters, just like the evaluation of French consumption may ignore *fugu*.

In this paper, we examine the possibility of making social welfare evaluations on the sole basis of individuals' preferences over available commodities. More precisely, we introduce a condition of *Independence of Irrelevant*

Commodities (IIC), stating that when two allocations have zero quantities of some commodities, the social ranking of these two allocations should be independent of individuals' preferences over these commodities. Our framework follows the theory of social choice in economic environments, surveyed, for instance, by Le Breton (1997) and Le Breton and Weymark (2000).

IIC is similar to the well-known condition due to Arrow (1951), *Independence of Irrelevant Alternatives (IIA)*, but it turns out to be much weaker and, we believe, much less controversial. Let us briefly explain why. IIA states that the ranking of two allocations should depend only on individuals' preferences *over these two allocations*. This requirement is often implausible, especially in economic environments. For instance, suppose that half of the population prefers allocation x and half of the population prefers allocation y. How can we rank x and y with this limited information? Under IIA, we may not explore whether any one of these allocations is Pareto-efficient (with respect to the total quantity distributed in this allocation), envy-free, or a Walrasian equilibrium with equal budgets because such exploration requires checking how bundles in x or y are ranked in individual preferences with respect to other bundles. IIA forbids the use of information about individual preferences that is considered directly relevant to the welfare or fairness evaluation of allocations in standard economic analyses.[1] In contrast, IIC allows us to use all the information about individuals' preferences over available commodities. This information encompasses that retained in IIA, namely, how individuals rank x and y, and extends much more. In particular, the information retained in IIC is sufficient to assess whether any of these allocations is Pareto-efficient, envy-free, or a Walrasian equilibrium with equal budgets.

We may even claim that our condition is actually more faithful to Arrow's initial vision. As a defense of his condition for applications to the evaluation of resource allocations, Arrow wrote (1950, p. 346; see also 1951, p. 73):

> 'Suppose that there are just two commodities, bread and wine. A distribution, deemed equitable by all, is arranged, with the wine-lovers getting more wine and less bread than the abstainers. Suppose now that all the wine is destroyed. Are the wine-lovers entitled, because of that fact, to more than an equal share of

[1] For a more detailed criticism of Arrow's condition in economic contexts, see e.g. Mayston (1974), Pazner (1979), Fleurbaey and Maniquet (1996, 2005), Fleurbaey, Suzumura and Tadenuma (2005a,b).

> bread? The answer is, of course, a value judgment. My own feeling is that tastes for unattainable alternatives should have nothing to do with the decision among the attainable ones; desires in conflict with reality are not entitled to consideration.'

This example suggests that when wine is not available, preferences over wine should be disregarded. This is exactly what our IIC states. Arrow's IIA says much more than this, requiring that the ranking of any two allocations should depend only on how people rank these two allocations. Besides, Arrow's mention of "equal shares" is in fact an introduction of fairness considerations which cannot be accommodated within the informational straitjacket of Arrow's condition.

We do not consider Arrow's impossibility theorem to be a serious obstacle to social choice in economic environments because IIA imposes a too severe restriction on information about individuals' preferences. Unfortunately, as we show, our condition still entails a similar impossibility result even though it is much weaker than IIA. Combined with a Pareto condition embodying the respect for unanimous individuals' preferences, it implies that the social preference rule must be dictatorial: social preferences must always obey one particular agent's strict preferences. We consider this result to be much more disturbing than Arrow's theorem.

More precisely, however, we show that the whole domain of possible population preferences is partitioned into two subdomains, and the dictatorship result holds on one of these subdomains, whereas a non-dictatorial rule can be constructed on the complementary subdomain. The latter subdomain of possibility, in fact, contains population preferences which are quite unlikely for preferences about ordinary commodities. Indeed, we obtain nice social preferences only for particular allocations and preference profiles such that every commodity is "indispensable" for at least one individual, in the sense that if this commodity were unavailable, then this individual would be worse off than at these allocations. For ordinary commodities, with their innumerable varieties, this seems far-fetched. We argue that this configuration is however reasonable if the objects of individual preferences are not ordinary commodities but composite commodities, "characteristics" (Lancaster, 1971) or human "functionings" (Sen, 1985). In our view, thus, our results advocate that social evaluations should be made with respect to characteristics or functionings rather than ordinary commodities.

Our work is related to the small part of the social choice literature which has been critical of Arrow's independence condition and has examined how to construct fair social preferences when this condition is relaxed. Mayston (1974, 1982), Pazner (1979), Fleurbaey and Maniquet (1986), Fleurbaey, Suzumura and Tadenuma (2005a,b) studied how to rely on individual indifference surfaces in order to construct "fair" social preferences. In this literature, the main independence condition, among the less restrictive, allows us to use information about the whole indifference surface of every individual at contemplated allocations in order to rank the allocations. As shown by these authors, nice social preference rules can be constructed under this independence condition. Our IIC is neither weaker nor stronger than this. It allows us to retain information about the whole preference maps in the subspace of consumed commodities while discarding the rest of the indifference surfaces. More closely related to our condition of independence is Donaldson and Roemer's (1987) consistency condition, which essentially requires social evaluation to disregard commodities with identical quantities in the allocations to be compared. They also derive an impossibility theorem, but their framework is different since it involves utility functions, whereas we consider only ordinal non-comparable preferences. Moreover, their consistency condition is somehow stronger than ours. We only require social evaluation to disregard individual preferences over commodities with zero quantities, but require nothing concerning commodities with identical positive quantities.

The rest of the paper is organized as follows. Section 2 presents a simple parable conveying the main intuition for our results. Sections 3 and 4 introduce the formal framework and the main notions. Section 5 states and proves the main result. Section 6 introduces a stronger variant of the independence condition. Section 7 examines the prospects for possibility results in light of the results.

2 A tale of two commodities

With a simple example, this section shows why it may entail a difficulty to require social evaluations to disregard individual preferences about unavailable commodities. On Robinson and Friday's island, two commodities may be available, bread and wine. Assume that both Robinson and Friday have strictly monotonic and self-centered preferences, so that when only one commodity is available, there is no question about their preferences over various allocations of this commodity. If we want to evaluate allocations made of one commodity only, and disregard individuals' preferences for the other commodity, then such social evaluation may be made irrespectively of individuals' preferences altogether.

Suppose that we consider allocation x better than allocation y, when these allocations are as described in the following table, along two other allocations z and w.

	Robinson		Friday	
	bread	wine	bread	wine
x	8	0	5	0
y	10	0	1	0
z	0	8	0	5
w	0	10	0	1

As explained above, the evaluation of x and y should be valid independently of individuals' preferences for wine. Then it can be argued that allocation z should also be considered better than allocation w, also independently of individual preferences. Indeed, suppose that Robinson's and Friday's preferences rank these allocations in this way (in decreasing order):

Robinson	Friday
y	z
w	x
z	y
x	w

One sees that both individuals prefer z to x and y to w. This strongly suggests that z is better than x and y is better than w. But since x is better than y, by transitivity one concludes that z is better than w. Of course, this reasoning was made with particular individual preferences and by comparisons of allocations containing bread (x and y) to allocations containing wine (z and w), but since z and w contain only wine, this particular judgment about these two allocations should be upheld independently of individual preferences over bread.

By a similar construction, one can show that whenever an allocation containing only one commodity is better for Friday, it should be declared better if we want social evaluations to be consistent with the initial judgment that x is better than y. In other words, Friday is what social choice theory calls a dictator over such one-commodity allocations: whenever he prefers an allocation to another, the social evaluation obeys.[2]

Now, consider any pair of two-commodity allocations a and b. Suppose, for instance, that Friday prefers a to b. If Robinson prefers a to b as well, one concludes from the Pareto principle that a is better than b. What if Robinson prefers b? Suppose that there is a pair of bread-only allocations, x' and y', such that individual preferences are as follows:

Robinson	Friday
y'	a
b	x'
a	y'
x'	b

Then, by the same reasoning as above, one can conclude that a is better. It seems that Friday is dictator for all allocations, including two-commodity allocations.

This conclusion, however, would be hasty. Suppose that Robinson prefers the bundle made of 1 unit of bread and 1 unit of wine to any one-commodity bundle. If b were better than this (1,1) bundle for him, it would be impossible to find a bread-only allocation y' better than b for him, and the above reasoning would fail. It is then indeed possible to rank b above a, against Friday's preference. Friday's dictatorship need not extend to all allocations.

This simple example has provided some intuition for the main elements of our analysis: 1) the condition of Independence of Irrelevant Commodities may still entail a dictatorship result; 2) impartial social evaluation is however possible in some cases; 3) a key fact is whether individuals, according to their own preferences, can find better bundles containing less commodities.

3 Sufficient and dispensable commodities

Let $L := \{1, \dots, \ell\}$ be the set of *commodities*, and $N := \{1, \dots, n\}$ the set of *agents*, where $2 \leq \ell < \infty$ and $2 \leq n < \infty$. Denoting $\mathbb{R}_+$ the set of all

[2]Interestingly, with an iteration of this kind of argument, one can show that dictatorship over one-commodity allocations can even be obtained when one restricts individual preferences to be quasi-linear with respect to one given commodity. This line of analysis therefore appears robust to severe domain restrictions, but we will not explore this issue further in this paper.

non-negative real numbers, $\mathbb{R}^L_+$ is the set of all consumption bundles. Agent i's *consumption bundle* is a vector $x_i := (x_{i1}, ..., x_{i\ell}) \in \mathbb{R}^L_+$. An *allocation* is a vector $x := (x_1, ..., x_n) \in (\mathbb{R}^L_+)^n$. The *set of all allocations* is $(\mathbb{R}^L_+)^n$. The set of allocations such that no individual bundle x_i is equal to the zero vector is denoted X, i.e., $X := \left(\mathbb{R}^L_+ \setminus \{0\}\right)^n$.

In order to study allocations in which some of the ℓ commodities are absent, we introduce the following notion of subspace. For each $K \subseteq L$, define $\mathbb{R}^K_+ \subseteq \mathbb{R}^L_+$ by

$$\mathbb{R}^K_+ := \{x_i \in \mathbb{R}^L_+ \mid \forall k \in L\backslash K,\ x_{ik} = 0\}.$$

Notice that $\mathbb{R}^K_+$ is a subset of $\mathbb{R}^L_+$, so that $x_i \in \mathbb{R}^K_+$ is a full vector with ℓ components, some of which are simply null.

An *ordering* is a reflexive and transitive binary relation. For each $i \in N$, agent i's *preference relation* is a complete ordering R_i on $\mathbb{R}^L_+$, that is, on i's personal bundles. This means that, as is standard in microeconomics, we restrict attention to self-centered preferences. The strict preference relation and the indifference relation associated to R_i are denoted P_i and I_i, respectively. Let $\mathcal{R}$ be the set of continuous, convex, and strictly[3] monotonic preference relations.

We now introduce some notions which will play a key role in our analysis. Consider an arbitrary subset K of commodities, and some agent's preference relation R_i. For this preference relation, we may ask whether satisfaction is bounded or unbounded with bundles containing only those commodities, that is, with bundles $x_i \in \mathbb{R}^K_+$. The expression "bounded satisfaction" here does not refer to a utility representation of preferences. In our terminology, satisfaction is bounded in $\mathbb{R}^K_+$ when there are some indifference surfaces which cannot be reached with bundles in $\mathbb{R}^K_+$. Satisfaction is unbounded when any arbitrary indifference surface can be reached with bundles in $\mathbb{R}^K_+$. Now, if satisfaction is unbounded in $\mathbb{R}^K_+$, we will call K a *sufficient* set. Otherwise, it may be called an *insufficient* set. For instance, with only water, bread and lodging in log cabins, satisfaction may be bounded, so that this set of three commodities is insufficient. On the contrary, with all the commodities typically available in a given country, one can reach any indifference surface, so that this forms a sufficient set.

[3]For a discussion of the role of strict monotonicity in our analysis, see a remark in the appendix.

When K is a sufficient set, we will say that the complement set $M = L \backslash K$ is a *dispensable set*. Indeed, this means that satisfaction is unbounded in absence of the commodities in M. Conversely, when satisfaction is bounded in absence of commodities in M, M will be called an *indispensable set*. For instance, apple fritters and fugu may be dispensable for some preferences, whereas water and newspapers may be indispensable for these same preferences. The following table summarizes these notions.

	Satisfaction is bounded	Satisfaction is unbounded
With K	insufficient	sufficient
Without K	indispensable	dispensable

Formally, for any set $K \subseteq L$, and for any $R_i \in \mathcal{R}$, K is a *sufficient set* for R_i if

$$\forall x_i \in \mathbb{R}^L_+, \exists y_i \in \mathbb{R}^K_+,\ y_i\ R_i\ x_i.$$

and K is an *insufficient set* for R_i if it is not a sufficient set, that is, if

$$\exists x_i \in \mathbb{R}^L_+, \forall y_i \in \mathbb{R}^K_+,\ x_i\ P_i\ y_i.$$

K is a *dispensable set* for R_i if $L \setminus K$ is a sufficient set, and K is an *indispensable set* for R_i if $L \setminus K$ is an insufficient set.

If K is sufficient (resp., indispensable) for R_i, then any $K' \supseteq K$ is also sufficient (resp., indispensable) for R_i. If K is insufficient (resp., dispensable) for R_i, then any $K' \subseteq K$ is also insufficient (resp., dispensable) for R_i. The set L is always sufficient and indispensable. Notice that the complement of a sufficient (resp., dispensable) set need not be insufficient (resp., indispensable). For instance, with all the typical Japanese commodities, one can reach any indifference surface, while at the same time, one can do so with all the typical French commodities.

4 Social ordering functions

Any evaluation of public policy, or any comparison of social welfare in various times or regions requires some comparisons between different social states. Here the social state is described in terms of allocations of commodities. Hence, we need a well-defined rule to socially rank allocations based on individuals' preferences. In other words, we look for a social ordering function which associates to each profile of individuals' preferences a consistent ranking of allocations.

A *profile of preference relations* is a list $R_N := (R_1, ..., R_n) \in \mathcal{R}^n$. A *social ordering function* (SOF) is a mapping Ψ defined on some set $\mathcal{D} \subseteq \mathcal{R}^n$, such that for all $R_N \in \mathcal{D}$, $\Psi(R_N)$ is a complete ordering on the set of all allocations $(\mathbb{R}_+^L)^n$. $\Psi(R_N)$ is interpreted as the social ordering of all allocations when agents' preferences are R_N. We simply denote by R (with no subscript) the social ordering $\Psi(R_N)$, by R' the social ordering $\Psi(R'_N)$, and so on, when no confusion may arise.

We will repeatedly require the SOF to obey the Weak Pareto condition saying that unanimous strict preference must be respected. This is a very basic condition of respect of individual preferences. It is especially compelling when dealing, as here, with self-centered preferences. It then means that individuals are sovereign over their personal consumption.

Weak Pareto: $\forall R_N \in \mathcal{D}$, $\forall x, y \in (\mathbb{R}_+^L)^n$, if $\forall i \in N, x_i \; P_i \; y_i$, then $x \; P \; y$.

We also need to define the notion of dictatorship. Let Y be a subset of $(\mathbb{R}_+^L)^n$. We say that an agent $i_0 \in N$ is a *dictator for the SOF* Ψ *over* Y if for all $R_N \in \mathcal{D}$, all $x, y \in Y$, $x_{i_0} \; P_{i_0} \; y_{i_0}$ implies $x \; P \; y$. The SOF Ψ is *dictatorial over* Y if there is a dictator for Ψ over Y.

Let us recall Arrow's theorem for this model. It involves Arrow's independence condition, requiring that for any given pair of allocations, a change of individuals' preferences about a third allocation should not alter the social ranking between the given two allocations.

Independence of Irrelevant Alternatives: $\forall R_N, R'_N \in \mathcal{D}$, $\forall x, y \in (\mathbb{R}_+^L)^n$, if $\forall i \in N$, R_i and R'_i agree on $\{x, y\}$, then R and R' agree on $\{x, y\}$.

On the economic domain studied here, the following version of Arrow's theorem was established by Bordes and Le Breton (1989).

Theorem 1 *On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and Independence of Irrelevant Alternatives, then it is dictatorial over X.*

The whole domain $\mathcal{R}^n$ may be partitioned in two subdomains $\mathcal{D}^+$ and $\mathcal{D}^-$. The former is the subset of profiles such that every proper subset $K \subseteq L$ is insufficient (and therefore every non-empty subset is indispensable) for at least one agent $i \in N$, and the latter is the complement, that is, it is the subset of profiles such that there is a proper subset $K \subseteq L$ that is sufficient (and its non-empty complement is dispensable) for all $i \in N$. Intuitively,

$\mathcal{D}^+$ is the set of profiles such that the absence of any commodity in the economy makes some agent's satisfaction bounded. In contrast, for profiles in $\mathcal{D}^-$, there is always some subset of commodities which can be absent in the allocations without limiting any agent's satisfaction as long as enough quantities of commodities in its complement are available.

Formally:

$$\begin{aligned}
\mathcal{D}^+ &= \{R_N \in \mathcal{R}^n \mid \forall K \subseteq L, K \neq L, \exists i \in N,\ K \text{ is insufficient for } R_i\} \\
&= \{R_N \in \mathcal{R}^n \mid \forall K \subseteq L, K \neq \emptyset, \exists i \in N,\ K \text{ is indispensable for } R_i\}, \\
\mathcal{D}^- &= \mathcal{R}^n \setminus \mathcal{D}^+ \\
&= \{R_N \in \mathcal{R}^n \mid \exists K \subseteq L, K \neq L, \forall i \in N,\ K \text{ is sufficient for } R_i\} \\
&= \{R_N \in \mathcal{R}^n \mid \exists K \subseteq L, K \neq \emptyset, \forall i \in N,\ K \text{ is dispensable for } R_i\}.
\end{aligned}$$

How relevant are the two cases epitomized by these two subdomains? If one thinks of applications in intertemporal or international studies of standards of living, certainly the $\mathcal{D}^-$ case is more relevant. Among the millions of different kinds of consumption goods and services available in the world economy, many of them could be dispensed with, and this would not significantly reduce the prospects for human development and satisfaction. On the other hand, in simple models where a limited number of commodities represent important dimensions of individual achievements, then $\mathcal{D}^+$ is the relevant domain to consider, because thriving is presumably impossible in absence of any of the considered dimensions.

5 Independence of Irrelevant Commodities

As explained in the introduction, we require the social ranking of two allocations to depend only on individual preferences for commodities that are available in these allocations. The motivation is that individual preferences for unavailable commodities are not only hard to observe, but also irrelevant, as illustrated in Arrow's bread-and-wine example. Formally, our condition states that a change of individual preferences for unavailable commodities should not alter the social ranking.

Independence of Irrelevant Commodities (IIC): $\forall R_N, R'_N \in \mathcal{D}, \forall x, y \in (\mathbb{R}^L_+)^n$, if $\exists K \subseteq L$ such that $x, y \in (\mathbb{R}^K_+)^n$ and $\forall i \in N$, R_i and R'_i agree on $\mathbb{R}^K_+$, then R and R' agree on $\{x, y\}$.

Our condition is logically weaker than Arrow's IIA. If a SOF satisfies IIA, it must also satisfy IIC since when individual preferences remain the same on $\mathbb{R}_+^K$, they must remain the same on any pair $x, y \in (\mathbb{R}_+^K)^n$. On the other hand, many SOFs satisfying IIC do not satisfy IIA whenever the ranking of any two allocations $x, y \in (\mathbb{R}_+^K)^n$ depends on preferences for a third allocation $z \in (\mathbb{R}_+^K)^n$. Contrary to Arrow's condition, our independence condition makes it possible for the SOF to rely on all the information about preferences that is considered relevant in welfare economics, especially in the theory of fair allocation. Indeed, when R_i and R_i' agree on $\mathbb{R}_+^K$ for all $i \in N$, the status of any allocation $x \in (\mathbb{R}_+^K)^n$ does not change with respect to such criteria as Pareto-efficiency,[4] envy-freeness,[5] minimal equality,[6] egalitarian-equivalence,[7] Walrasian equality.[8] The criticism that Arrow's condition excludes ethically relevant and important information would not apply to our condition.

Our main result is that, with our independence condition, Arrow's impossibility carries over to one of the two subdomains distinguished above, but not to the other. This implies in particular that the impossibility does not hold any more on the whole domain $\mathcal{R}^n$.

Theorem 2 *On the domain $\mathcal{D}^-$, if a SOF satisfies Weak Pareto and Independence of Irrelevant Commodities, then it is dictatorial over X. On the domain $\mathcal{D}^+$, there exists a non-dictatorial SOF satisfying Weak Pareto and Independence of Irrelevant Commodities.*

The proof is in the appendix. It is worth noting that the proof of the possibility result involves an example of a SOF that is not only non-dictatorial, but actually has nice fairness properties on some important subset of allocations. On this subset, it corresponds to a variant of the SOF introduced in Pazner and Schmeidler (1978) and Pazner (1979), based on the notion

[4] An allocation x is *Pareto-efficient* when there is no other allocation y such that $\sum_{i \in N} y_i \leq \sum_{i \in N} x_i$, $y_i \; R_i \; x_i$ for all $i \in N$, and $y_i \; P_i \; x_i$ for some $i \in N$.

[5] An allocation x is envy-free (Foley 1967, Kolm 1972) when $x_i \; R_i \; x_j$ for all $i, j \in N$.

[6] An allocation satisfies *minimal equality* (Steinhaus 1948) if $x_i \; R_i \; \left(\sum_{j \in N} x_j\right)/n$ for all $i \in N$.

[7] An allocation x is *egalitarian-equivalent* (Pazner and Schmeidler 1978) if there exists a bundle x_0, proportional to $\sum_{i \in N} x_i$, such that $x_i \; I_i \; x_0$ for all $i \in N$.

[8] An allocation is *egalitarian Walrasian* when there exist $p \in \mathbb{R}_{++}^L$ and $\alpha \in \mathbb{R}_{++}$ such that for all $i \in N$, $p.x_i = \alpha$ and $x_i \; R_i \; y_i$ for all y_i such that $p.y_i \leq \alpha$.

of egalitarian-equivalence. Given a reference bundle $x_0 \in \mathbb{R}^L_{++}$, the SOF is defined on this subset by: $x \; R \; y$ if and only

$$\min_i \min \{\lambda \in \mathbb{R}_+ \mid \lambda x_0 \; R_i \; x_i\} \geq \min_i \min \{\lambda \in \mathbb{R}_+ \mid \lambda x_0 \; R_i \; y_i\} .$$

Intuitively, this amounts to evaluating individual situations by the minimal fraction of x_0 which individuals would be willing to substitute for x_i, and applying the maximin criterion to the vector of such individual measures.

The domain $\mathcal{D}^-$ is rather wide and contains preference profiles which may appear unrealistic (the same criticism could a fortiori be raised against $\mathcal{R}^n$). In particular it contains preferences profiles for which one commodity alone is sufficient. But from the argument in the proof of the impossibility part of the above theorem (see Lemma 1 in the appendix), one can see that the dictatorship result still holds if one restricts attention to the subdomain $\mathcal{D}_{KK'}$ related to some fixed subsets of commodities K and K' with $K \cap K' = \emptyset$, and containing all profiles R_N such that for all $i \in N$, K and K' are each sufficient for R_i and no proper subset of K or K' is itself sufficient. Think of K and K' as two rich sets of commodities all providing all the usual resources for a flourishing life. Therefore the above result does not depend upon the consideration of unrealistic cases.

6 An individualistic condition

In this section, we examine the idea of applying the independence principle to each individual separately. When two allocations give a zero quantity of some commodity to an individual, one could argue that there is no reason to take account of his preferences for this commodity in the social ranking of these two allocations. This may seem particularly suitable for international studies when different populations consume different subsets of commodities. For instance, the evaluation of global allocations of resources might ignore Japanese preferences about apple fritters when the Japanese do not consume any, and French preferences about fugu for the same reason.

The "individualistic" version of Independence of Irrelevant Commodities requires the SOF to disregard agent i's preferences over non-consumed commodities.

Individualistic Independence of Irrelevant Commodities (IIIC): $\forall R_N, R'_N \in \mathcal{D}$, $\forall x, y \in \left(\mathbb{R}^L_+\right)^n$, if $\forall i \in N$, $\exists K_i \subseteq L$ such that $x_i, y_i \in \mathbb{R}^{K_i}_+$ and R_i and R'_i agree on $\mathbb{R}^{K_i}_+$, then R and R' agree on $\{x, y\}$.

This independence condition is logically stronger than the original version introduced in Section 5 and with it, dictatorship extends to a larger domain. Formally, we define the new domain of preference profiles as follows.

$$\mathcal{D}^{-*} = \{R_N \in \mathcal{R}^n \mid \forall i \in N, \exists K \subseteq L, K \neq L,\ K \text{ is sufficient for } R_i\}.$$

Notice that $\mathcal{D}^- \subseteq \mathcal{D}^{-*}$. By allowing heterogeneity of sufficient subsets across individuals, we do not bar the former homogeneous configuration. The complement of this domain is defined as

$$\mathcal{D}^{+*} = \{R_N \in \mathcal{R}^n \mid \exists i \in N, \forall K \subseteq L, K \neq L,\ K \text{ is insufficient for } R_i\},$$

and obviously, $\mathcal{D}^{+*} \subseteq \mathcal{D}^+$.

Theorem 3 *On the domain $\mathcal{D}^{-*}$, if a SOF satisfies Weak Pareto and Individualistic Independence of Irrelevant Commodities, then it is dictatorial over X. On the domain $\mathcal{D}^{+*}$, there exists a non-dictatorial SOF satisfying Weak Pareto and Individualistic Independence of Irrelevant Commodities.*

Individualistic Independence of Irrelevant Commodities might be criticized for being too individualistic and too demanding. In the reference to international studies above we referred to different communities (countries), not to different individuals. This is not a trivial difference. One may argue that Japanese preferences over goods which are not available in Japan do not matter, but if a particular individual in Japan does not consume sodas, which are available in the local market, this tells something about her preferences about sodas, and it may be relevant to take account of such preferences in general. It would, however, be a little cumbersome to formulate the "community" version of the independence condition in our model, and the results would be essentially similar (with impossibility obtained on a domain in between $\mathcal{D}^-$ and $\mathcal{D}^{-*}$).

Another remark must be made, however. Independence of Irrelevant Commodities has been contrasted with Independence of Irrelevant Alternatives, in Section 5, by noticing that the former, contrary to the latter, does make it possible to refer to efficiency and fairness properties of allocations. On this issue, however, *Individualistic* Independence of Irrelevant Commodities is similar to Arrow's condition. Consider two allocations in which Robinson consumes bread only and Friday consumes wine only. In virtue of Individualistic Independence, their preferences over the commodity consumed by the other must be disregarded, making it impossible to assess efficiency and fairness of either allocation. This would also be true for the community version of the axiom. After all, the world allocation might be very inefficient simply because the Japanese life-style would suit the French better, and conversely. In order to check this fact, preferences over the other community's consumptions must be examined, even if they are quite hard to elicit. This observation leads us to conclude that the impossibility results obtained with the individualistic or community version of Independence of Irrelevant Commodities have less importance from truly normative considerations, but rather they reveal practical difficulties in making social rankings of global allocations under informational constraints in real life.

7 Absent commodities

The possibility parts of Theorems 2 and 3 give a special role to allocations in which every commodity is consumed by at least one agent (Th. 2) or even in which all commodities are consumed by at least one agent (Th. 3). But the examples remain dictatorial for the other allocations. In this section, we show that the scope of dictatorship extends to the whole domain of preference profiles with IIC when one focuses on allocations in which some commodities are absent (and, as above, no agent has the null bundle). Similarly, with Individualistic IIC, dictatorship prevails in allocations such that no agent consumes all commodities.

Let $\overline{X}$ be the subset of X such that at least one commodity is absent from the allocation, i.e.

$$\overline{X} = \left\{ x \in X \mid \sum_{i \in N} x_i \notin \mathbb{R}^L_{++} \right\}.$$

Theorem 4 *On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and Independence of Irrelevant Commodities, then it is dictatorial on $\overline{X}$.*

Let $\overline{\overline{X}}$ be the subset of X such that every individual bundle x_i has some zero component, i.e.

$$\overline{\overline{X}} = \left\{ x \in X \mid \forall i \in N,\ x_i \notin \mathbb{R}^L_{++} \right\}.$$

Theorem 5 *On the domain $\mathcal{R}^{\mathsf{n}}$, if a SOF satisfies Weak Pareto and Individualistic Independence of Irrelevant Commodities, then it is dictatorial on* $\overline{\overline{X}}$.

It may be worth emphasizing that even if $\overline{X} \subsetneq \overline{\overline{X}} \subsetneq X$, $\overline{X}$ and $\overline{\overline{X}}$ are large subsets of X that are quite relevant to real life. For instance, let us consider allocations in one particular country. Commodities produced in some other regions in the world are often unavailable in this country. In this context, $\overline{X}$ contains all relevant allocations. If we consider global allocations of resources, for the same reason $\overline{\overline{X}}$ is a suitable set to consider. Similarly, when we examine growth paths, the consumption bundles available in the 21th century do not include the commodities that are to be invented in the future, nor many commodities that were in substantial use in the 17th century, making $\overline{X}$ or $\overline{\overline{X}}$ the relevant set of allocations depending on whether we consider allocations at given times or the whole growth paths.

8 What is left for possibility?

We consider that our results seriously challenge the possibility of making reasonable social evaluations of allocations on the basis of ordinary commodities, in contexts where individuals do not consume all commodities, and/or there are some disjoint subsets of commodities that may each be sufficient for individuals to attain any level of satisfaction (i.e., any indifference surface).

One may think of the following easy escape. If we consider a small number of composite commodities such as food, lodging, transportation, etc., then every individual usually consumes some positive amount of every commodity. In such a case, our conditions of independence have no bite and the construction of appealing SOFs is then possible. This solution, however, has two weaknesses. First, it is rare that composite commodities can be defined in a way that makes it possible to define individuals' preferences consistently. One must assume either that prices are fixed over all contemplated allocations, or that preferences are separable. Second, restricting attention to interior allocations (i.e. $x \gg 0$) is artificial when non-interior allocations are also obviously feasible. An important fact observed about composite commodities is not that typical individuals consume positive quantities of each of them, but that individuals' reasonable preferences regard each of them as indispensable. What we should seek is a natural restriction on the domain of preference profiles which leads to a possibility of reasonable social evaluations, rather than an artificial reduction of the set of allocations.

Such favorable configurations of preferences are hard to obtain with ordinary commodities but may be more likely with different objects of preferences, such as characteristics (Lancaster, 1971) or functionings (Sen, 1985). If that is true, the main conclusion emerging from this analysis is that welfare economics should migrate from the space of commodities to the space of characteristics or to the space of functionings.

The space of functionings may actually, in all generality, be as diverse as the space of commodities. For instance, Sen (1992) defines a functioning as any kind of achievement attained by the individual. Our negative results can then be reproduced in the space of functionings if many disjoint subsets of functionings can be sufficient for individuals' preferences. One can live well without ever performing tightrope walking above the Thames, or without ever extending Arrow's theorem. But if we consider broadly categorized, important functionings, then it is more likely that each of them becomes indispensable. Moreover, there is enough homogeneity in human beings to provide means for easy measurement and interpersonal comparison of achievement levels in these functionings. For instance, measures of comparisons are available for literacy and education level, nutritional intake, health, etc. whereas it is much harder to think of measuring such composite commodities as educational resources or food.

Similar observations may be made about characteristics. The list of characteristics is virtually endless, and different commodities may provide similar characteristics in some dimensions but quite different characteristics in other dimensions. A particular Japanese meal may provide the same calorie, fat and protein quantities as a particular French meal, but who would deny that they provide quite different characteristics in some other dimensions? However, one may hope that there are indispensable, essential characteristics so that the scope of impossibility is substantially reduced if one considers characteristics rather than ordinary commodities.

As an example of the above observations, consider again Robinson and Friday, confronted with three functionings: nutrition, singing and hiking. Suppose that both Robinson and Friday need nutrition for survival, and hence {nutrition} is indispensable. However, each of them cannot lead an enjoyable life without some kind of recreation. Thus, {nutrition} is insufficient, but {nutrition,singing} as well as {nutrition,hiking} is sufficient (and hence {hiking} and {singing} are each dispensable for everyone). We restrict attention to allocations in which the total quantity of nutrition is always positive, and the total quantity of either singing or hiking is also positive. Consider the following allocations, similar to those of Section 2.

	Robinson			Friday		
	nutrition	singing	hiking	nutrition	singing	hiking
x	8	8	0	5	5	0
y	10	10	0	1	1	0
z	8	0	8	5	0	5
w	10	0	10	1	0	1

Suppose that x is better than y. Then, by a similar argument as in Section 2, one can show that Weak Pareto and Independence of Irrelevant Commodities imply that z is better than w. Friday is then shown to be a dictator over two-functioning allocations (one of the functionings being nutrition). And, since {nutrition,singing} and {nutrition,hiking} are sufficient for everyone, the argument can be extended to show that Friday is a dictator over three-functioning allocations as well.

Dictatorship is avoided only if there is no functioning that is dispensable for everyone. Let us now recategorize functionings so that {recreation} includes both {singing} and {hiking}. Then, not only {nutrition} but also {recreation} is indispensable for Robinson and Friday because without any recreation, they cannot reach high levels of indifference surfaces. We can then find a non-dictatorial SOF, and the reader is referred to the proof of Theorem 2 (the possibility part) for an example. A favorable configuration of this sort is more likely to be obtained as functionings are more broadly categorized. A similar observation can be made regarding Individualistic Independence of Irrelevant Commodities, for which the possibility domain requires at least one individual for whom every dimension is indispensable.

To summarize, a restricted class of preference profiles which leads to a possibility of reasonable social evaluations can be more naturally obtained if the objects of preferences are characteristics or functionings that are defined (and measured) in sufficiently broad terms. A similar conclusion may be derived from the consideration of composite commodities, but it is indeed hoped that one can make suitable categorization in a more satisfactory way with functionings (or characteristics) than with commodities.

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Appendix: Proofs

In the proofs, some additional notations are needed. For any $R_i \in \mathcal{R}$ and any $x_i \in \mathbb{R}^L_+$, the (*closed*) *upper contour set* for R_i at x_i is defined as

$$uc(x_i; R_i) := \left\{ y_i \in \mathbb{R}^L_+ \mid y_i \; R_i \; x_i \right\}.$$

For any $x_i \in \mathbb{R}^L_+$, the *cone* generated by x_i is defined as

$$C(x_i) := \left\{ y_i \in \mathbb{R}^L_+ \mid \exists \alpha \in \mathbb{R}_+, \; y_i = \alpha x_i \right\}.$$

Proof of Theorem 2: The proof of the first part relies on Arrow's theorem, and we need to define a variant of Arrow's independence condition.

Weak Independence of Irrelevant Alternatives: $\forall R_N, R'_N \in \mathcal{D}$, $\forall x, y \in X$, if $\forall i \in N$, R_i and R'_i agree on $\{x, y\}$, and for no i, $x_i \; I_i \; y_i$, then R and R' agree on $\{x, y\}$.

Lemma 1 *On the domain $\mathcal{D}^-$, if a SOF satisfies Weak Pareto and Independence of Irrelevant Commodities, then it satisfies Weak Independence of Irrelevant Alternatives.*

Proof of Lemma 1. For any R_N, let $\mathcal{M}(R_N)$ be the set of non-empty subsets of commodities which are dispensable for all agents:

$$\mathcal{M}(R_N) = \{K \subseteq L \mid K \neq \emptyset \text{ and } \forall i \in N, \; K \text{ is dispensable for } R_i\}.$$

By definition of $\mathcal{D}^-$, for all $R_N \in \mathcal{D}^-$, $\mathcal{M}(R_N)$ is not empty. Let $R_N, R'_N \in \mathcal{D}^-$ and $x, y \in X$ be such that for all $i \in N$, R_i and R'_i agree on $\{x, y\}$, and for no $i \in N$, $x_i \; I_i \; y_i$. Assume that $x \; P \; y$.

First case: $\mathcal{M}(R_N) \cap \mathcal{M}(R'_N) \neq \emptyset$. Choose any $M \in \mathcal{M}(R_N) \cap \mathcal{M}(R'_N)$ and let $K := L \backslash M$. Since $x, y > 0$,[9] by monotonicity of preferences we know

[9] Vector inequalities are denoted $\gg, >, \geq$.

that $y_i \; P_i \; 0$ and $x_i \; P'_i \; 0$ for all i. Since M is dispensable for all i in R_N and R'_N, we can choose $z, w, z', w' \in (\mathbb{R}^K_+ \setminus \{0\})^n$ such that for all $i \in N$,
(i) if $x_i \; P_i \; y_i$ (and hence $x_i \; P'_i \; y_i$ as well), then $z_i \; P_i \; x_i \; P_i \; y_i \; P_i \; w_i$ and $x_i \; P'_i \; z'_i \; P'_i \; w'_i \; P'_i \; y_i$,
(ii) if $y_i \; P_i \; x_i$ (and hence $y_i \; P'_i \; x_i$ as well), then $y_i \; P_i \; w_i \; P_i \; z_i \; P_i \; x_i$ and $w'_i \; P'_i \; y_i \; P'_i \; x_i \; P'_i \; z'_i$, and
(iii) there is $\lambda_i, \lambda'_i \in \mathbb{R}_{++}$ such that $w_i = \lambda_i z_i$, $w'_i = \lambda'_i z'_i$.

By Weak Pareto, we have $z \; P \; x$ and $y \; P \; w$. By transitivity of P, $z \; P \; w$. It also follows from Weak Pareto that $x \; P' \; z'$ and $w' \; P' \; y$.

Next, choose $a, b, a', b', a'', b'' \in (\mathbb{R}^M_+ \setminus \{0\})^n$ such that for all $i \in N$,
(i) if $x_i \; P_i \; y_i$, then $a'_i > a''_i > a_i > b_i > b''_i > b'_i$, and
(ii) if $y_i \; P_i \; x_i$, then $b_i > b''_i > b'_i > a'_i > a''_i > a_i$. and
(iii) there is $\mu_i, \mu'_i \in \mathbb{R}_{++}$ such that $b_i = \mu_i a_i$, $b'_i = \mu'_i a'_i$.

For every $i \in N$, choose two increasing functions $\gamma_i, \gamma'_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma_i(0) = \gamma'_i(0) = 0$, $\gamma_i(1) = \gamma'_i(1) = 1$ and $\gamma_i(\lambda_i) = \mu_i$, $\gamma'_i(\lambda'_i) = \mu'_i$. Such functions always exist because $\lambda_i > 1 \Leftrightarrow \mu_i > 1$ and $\lambda'_i > 1 \Leftrightarrow \mu'_i > 1$.

Let $R_0 \in \mathcal{R}$ be an arbitrary preference relation on $\mathbb{R}^L_+$. We now define a new preference relation for i, R^*_i, as follows. The upper contour set for R^*_i at any $q \in C(z_i)$ such that $q = \alpha z_i$ for some α is constructed as

$$uc(q; R^*_i) := co\left\{\left(uc(q; R_i) \cap \mathbb{R}^K_+\right) \cup \left(uc(\gamma_i(\alpha) a_i; R_0) \cap \mathbb{R}^M_+\right)\right\},$$

where co denotes the convex hull. More generally, for any $c \in \mathbb{R}^L_+$, we define

$$uc(c; R^*_i) := \bigcap_{q \in C(z_i), c \in uc(q; R^*_i)} uc(q; R^*_i).$$

As a convex hull, $uc(q; R^*_i)$ is convex for all $q \in C(z_i)$, and as an intersection of convex sets, $uc(c; R^*_i)$ is convex for all $c \in \mathbb{R}^L_+$. This means that R^*_i is convex. Clearly, it is also continuous and strictly monotonic, so that $R^*_i \in \mathcal{R}$. Moreover, R^*_i and R_i agree on $\mathbb{R}^K_+$. Indeed, if $c \in \mathbb{R}^K_+$,

$$\begin{aligned} uc(c; R^*_i) \cap \mathbb{R}^K_+ &= \bigcap_{q \in C(z_i), c \in uc(q; R^*_i)} uc(q; R^*_i) \cap \mathbb{R}^K_+ \\ &= \bigcap_{q \in C(z_i), c \in uc(q; R_i) \cap \mathbb{R}^K_+} uc(q; R_i) \cap \mathbb{R}^K_+ \\ &= uc(c; R_i) \cap \mathbb{R}^K_+. \end{aligned}$$

Similarly, R_i^* and R_0 agree on $\mathbb{R}_+^M$. Finally, $z_i\ I_i^*\ a_i$, because

$$\begin{aligned} uc(z_i; R_i^*) \cap \mathbb{R}_+^M &= co\left(\left[uc(z_i; R_i) \cap \mathbb{R}_+^K\right] \cup \left[uc(\gamma_i(1)a_i; R_0) \cap \mathbb{R}_+^M\right]\right) \cap \mathbb{R}_+^M \\ &= co\left(\left[uc(z_i; R_i) \cap \mathbb{R}_+^K\right] \cup \left[uc(a_i; R_0) \cap \mathbb{R}_+^M\right]\right) \cap \mathbb{R}_+^M \\ &= uc(a_i; R_0) \cap \mathbb{R}_+^M, \end{aligned}$$

and $w_i\ I_i^*\ b_i$, because

$$\begin{aligned} uc(w_i; R_i^*) \cap \mathbb{R}_+^M &= co\left(\left[uc(w_i; R_i) \cap \mathbb{R}_+^K\right] \cup \left[uc(\gamma_i(\lambda_i)a_i; R_0) \cap \mathbb{R}_+^M\right]\right) \cap \mathbb{R}_+^M \\ &= co\left(\left[uc(w_i; R_i) \cap \mathbb{R}_+^K\right] \cup \left[uc(b_i; R_0) \cap \mathbb{R}_+^M\right]\right) \cap \mathbb{R}_+^M \\ &= uc(b_i; R_0) \cap \mathbb{R}_+^M, \end{aligned}$$

Similarly, we construct $R_i'^* \in \mathcal{R}$ by

$$uc(q; R_i'^*) := co\left(\left[uc(q; R_i') \cap \mathbb{R}_+^K\right] \cup \left[uc(\gamma_i'(\alpha)a_i'; R_0) \cap \mathbb{R}_+^M\right]\right)$$

for $q \in C(z_i')$, $q = \alpha z_i'$, and

$$uc(c; R_i'^*) := \bigcap_{q \in C(z_i'), c \in uc(q; R_i'^*)} uc(q; R_i'^*).$$

for any $c \in \mathbb{R}_+^L$. The orderings $R_i'^*$ and R_i' agree on $\mathbb{R}_+^K$, $R_i'^*$ and R_0 agree on $\mathbb{R}_+^M$, $z_i'\ I_i'^*\ a_i'$ and $w_i'\ I_i'^*\ b_i'$. In particular, R_i^* and $R_i'^*$ agree on $\mathbb{R}_+^M$. Let $R_N^* := (R_1^*, \ldots, R_n^*)$ and $R_N'^* := (R_1'^*, \ldots, R_n'^*)$. By construction, $R_N^*, R_N'^* \in \mathcal{D}^-$.

Recall that $a_i'' > a_i$, and $b_i > b_i''$. By transitivity and strict monotonicity of preferences, $a_i''\ P_i^*\ z_i$ and $w_i\ P_i^*\ b_i''$ for all $i \in N$. By Weak Pareto, $a''\ P^*\ z$ and $w\ P^*\ b''$. Since $z\ P\ w$, it follows from Independence of Irrelevant Commodities (IIC) that $z\ P^*\ w$. By transitivity of R^*, we have $a''\ P^*\ b''$.

Next recall that $a_i' > a_i''$, and $b_i'' > b_i'$. By transitivity and strict monotonicity of preferences and Weak Pareto, $z'\ P_i'^*\ a''$ and $b''\ P_i'^*\ w'$. On the other hand, it follows from IIC and $a''\ P^*\ b''$ that $a''\ P'^*\ b''$. By transitivity of P'^*, $z'\ P'^*\ w'$. From IIC (applied to $R_N'^*$ and R_N'), we have $z'\ P'\ w'$. Recall that $x\ P'\ z'$ and $w'\ P'\ y$. By transitivity, $x\ P'\ y$.

We have shown that $x\ P\ y \Rightarrow x\ P'\ y$. By symmetry, $x\ P'\ y \Rightarrow x\ P\ y$, and $y\ P\ x \Leftrightarrow y\ P'\ x$. Hence, it also holds that $x\ I\ y \Leftrightarrow x\ I'\ y$.

Second case: $\mathcal{M}(R_N) \cap \mathcal{M}(R_N') = \emptyset$. Let $R_N'' \in \mathcal{D}^-$ be such that $\mathcal{M}(R_N'') \cap \mathcal{M}(R_N) \neq \emptyset$ and $\mathcal{M}(R_N'') \cap \mathcal{M}(R_N') \neq \emptyset$, and such that R_N'' and R_N (and R_N' as well) agree on $\{x, y\}$. It is easy to find such R_N''. For instance, if

R''_i is linear for all i (that is, there is $p_i \in \mathbb{R}^L_{++}$ such that $x_i\ R_i\ y_i$ if and only if $p_i.x_i \geq p_i.y_i$), then $\mathcal{M}(R''_N) = \mathcal{L} \setminus \{L\}$ so that $\mathcal{M}(R''_N) \cap \mathcal{M}(R_N) = \mathcal{M}(R_N)$ and $\mathcal{M}(R''_N) \cap \mathcal{M}(R'_N) = \mathcal{M}(R'_N)$. By the first case argument, R and R'' agree on $\{x, y\}$, and so do R' and R''. Therefore R and R' agree on $\{x, y\}$. ■

Let $Y \subseteq X$ be given. An agent $i_0 \in N$ is called a *quasi-dictator over Y* if for all $R_N \in \mathcal{D}$, all $x, y \in Y$, $x\ P\ y$ whenever $x_{i_0}\ P_{i_0}\ y_{i_0}$ and for no $i \in N$, $x_i\ I_i\ y_i$. A pair of allocations $\{x, y\} \subseteq X$ is called a *trivial pair on $\mathcal{D}$* if there is $i \in N$ such that for all $R_N, R'_N \in \mathcal{D}$, R_i and R'_i agree on $\{x, y\}$. By strict monotonicity of preferences, this happens when either $x > y$ or $x < y$. A set of three allocations $\{x, y, z\} \subseteq X$ is called a *free triple on $\mathcal{D}$* if for every n-tuple of orderings O_N on $\{x, y, z\}$, there exists $R_N \in \mathcal{D}$ such that R_N and O_N agree on $\{x, y, z\}$. Let $\mathcal{R}_K \subset \mathcal{R}$ be the subset of preferences R_i such that K is sufficient for R_i.

Lemma 2 *Let $K \subseteq L$. Let $\{x, y\}, \{z, w\} \subset X$ be non-trivial pairs on $(\mathcal{R}_K)^n$. There exist $v^1, ..., v^m \in X$ such that*

$$v^1 = x,\ v^2 = y,\ v^{m-1} = z,\ v^m = w$$

and for all $q = 1, ..., m-2$, $\{v^q, v^{q+1}, v^{q+2}\}$ is a free triple on $(\mathcal{R}_K)^n$.

Proof of Lemma 2. Since $\{x, y\}$ is a non-trivial pair, there is $p \in \mathbb{R}^L_{++}$ be such that $px = py$. Let

$$\begin{aligned} x' &= \frac{2}{3}x + \frac{1}{3}y, \\ y' &= \frac{1}{3}x + \frac{2}{3}y. \end{aligned}$$

For every $\varepsilon \in \mathbb{R}_{++}$ there is $x'', y'' \in \left(\mathbb{R}^L_{++}\right)^n$ such that $\|x'' - x'\| < \varepsilon$, $\|y'' - y'\| < \varepsilon$ (where $\|.\|$ denotes the Euclidean distance), and $\{x, y, u\}$, $\{v, x'', y''\}$ are free triples for every $u \in \{x'', y''\}$ and $v \in \{x, y\}$. Similarly, one constructs $z'', w'' \in \left(\mathbb{R}^L_{++}\right)^n$ such that $\{z, w, u\}$, $\{v, z'', w''\}$ are free triples for every $u \in \{z'', w''\}$ and $v \in \{z, w\}$.

The pairs $\{x'', y''\}, \{z'', w''\}$ are non-trivial, with $x'', y'', z'', w'' \in \left(\mathbb{R}^L_{++}\right)^n$. Pick $i \in N$. Let $\bar{p}, \bar{p}' \in \mathbb{R}^L_{++}$ be such that $\bar{p}x''_i = \bar{p}y''_i$ and $\bar{p}'z''_i = \bar{p}'w''_i$. Consider the set

$$B_i = \left\{ q \in \mathbb{R}^L_{++} \mid \bar{p}q > \bar{p}x''_i,\ \bar{p}'q > \bar{p}'z''_i,\ q \not> x''_i, y''_i, z''_i, w''_i \right\}.$$

Because $x_i'', y_i'', z_i'', w_i'' \gg 0$, there exists $\bar{p}'' \in \mathbb{R}^L_{++}$ and $q_i', q_i'' \in B_i$ such that

$$\bar{p}''q' = \bar{p}''q'' < \bar{p}''x_i'', \bar{p}''y_i'', \bar{p}''z_i'', \bar{p}''w_i''.$$

This construction can be made for every $i \in N$. One checks that $\{x'', y'', u\}$, $\{z'', w'', u\}$ and $\{v, q', q''\}$ are free triples for every $u \in \{q', q''\}$ and $v \in \{x'', y'', z'', w''\}$.

We can now connect $\{x, y\}, \{z, w\}$ by the following sequence of free triples: $\{x, y, x''\}$, $\{y, x'', y''\}$, $\{x'', y'', q'\}$, $\{y'', q', q''\}$, $\{q', q'', z''\}$, $\{q'', z'', w''\}$, $\{z'', w'', z\}$, $\{w'', z, w\}$. ■

Lemma 3 *Let $K \subseteq L$. Let $R_N \in (\mathcal{R}_K)^n$ and $x, y \in X$ be such that for no $i \in N$, $x_i \; I_i \; y_i$. Then, there exists $z \in X$ such that $\{x, z\}$ and $\{z, y\}$ are non-trivial on $(\mathcal{R}_K)^n$ and for all $i \in N$, either $x_i \; P_i \; z_i \; P_i \; y_i$ or $y_i \; P_i \; z_i \; P_i \; x_i$.*

Proof of Lemma 3. Pick $i \in N$ and assume, without loss of generality, that $x_i \; P_i \; y_i$.

First case, $x_i > y_i$. Suppose, again without loss of generality, that $x_{i1} > y_{i1}$.

First subcase, $y_{i1} > 0$. One can find $\varepsilon, \eta \in \mathbb{R}_{++}$ such that

$$z_i = (y_{i1} - \varepsilon, x_{i2} + \eta, y_{i3}, ..., y_{i\ell})$$

satisfies $x_i \; P_i \; z_i \; P_i \; y_i$.

Second subcase, $y_{i1} = 0$ and (without loss of generality) $y_{i2} > 0$. One can find $\varepsilon, \eta \in \mathbb{R}_{++}$ such that

$$z_i = (x_{i1} + \varepsilon, y_{i2} - \eta, y_{i3}, ..., y_{i\ell})$$

satisfies $x_i \; P_i \; z_i \; P_i \; y_i$.

Second case, $x_i \not> y_i$. For $\lambda \in (0, 1)$, let

$$z_i = \lambda x_i + (1 - \lambda) y_i.$$

By convexity of preferences, for λ close enough to 0, one has $x_i \; P_i \; z_i \; P_i \; y_i$. ■

Lemma 4 *Let $K \subseteq L$. Suppose that for every triple $\{x, y, z\} \subset X$ that is free on $(\mathcal{R}_K)^n$, there is a quasi-dictator over $\{x, y, z\}$. Then there is a quasi-dictator over X.*

Proof of Lemma 4. Pick any free triple $\{a, b, c\} \subset X$ and let i_0 be its quasi-dictator. Let $R_N \in (\mathcal{R}_K)^n$ and $x, y \in X$ be such that $x_{i_0}\ P_{i_0}\ y_{i_0}$ and for no $i \in N$, $x_i\ I_i\ y_i$. By Lemma 3, there is $z \in X$ such that $\{x, z\}$ and $\{z, y\}$ are non-trivial on $(\mathcal{R}_K)^n$ and for all $i \in N$, either $x_i\ P_i\ z_i\ P_i\ y_i$ or $y_i\ P_i\ z_i\ P_i\ x_i$. In particular, one has $x_{i_0}\ P_{i_0}\ z_{i_0}\ P_{i_0}\ y_{i_0}$.

By Lemma 2, there exist $v^1, ..., v^m \in X$ such that

$$v^1 = a,\ v^2 = b,\ v^{m-1} = x,\ v^m = z$$

and for all $q = 1, ..., m-2$, $\{v^q, v^{q+1}, v^{q+2}\}$ is a free triple on $(\mathcal{R}_K)^n$. Similarly, there exist $w^1, ..., w^t \in X$ such that

$$w^1 = a,\ w^2 = b,\ w^{t-1} = y,\ w^t = z$$

and for all $q = 1, ..., t-2$, $\{w^q, w^{q+1}, w^{q+2}\}$ is a free triple on $(\mathcal{R}_K)^n$.

Necessarily i_0 is a quasi-dictator for all $\{v^q, v^{q+1}, v^{q+2}\}$, for $q = 1, ..., m-2$, as well as for all $\{w^q, w^{q+1}, w^{q+2}\}$, for $q = 1, ..., t-2$. This implies that i_0 is a quasi-dictator over $\{x, z\}$ and over $\{z, y\}$. Therefore, $x\ P\ z$ and $z\ P\ y$. By transitivity, $x\ P\ y$. ■

Lemma 5 *Let $K \subseteq L$. On the domain $(\mathcal{R}_K)^n$, if a SOF satisfies Weak Pareto and Weak Independence of Irrelevant Alternatives, then for every free triple $\{x, y, z\} \subseteq X$, there exists a quasi-dictator over $\{x, y, z\}$.*

Proof of Lemma 5. This is a direct application of the variant of Arrow's theorem which considers the case of strict preferences (no indifference). ■

Lemma 6 *Let $K \subseteq L$. On the domain $(\mathcal{R}_K)^n$, if $i_0 \in N$ is a quasi-dictator over X, then i_0 is a dictator over X.*

Proof of Lemma 6. Let $x, y \in X$ and $R_N \in (\mathcal{R}_K)^n$ be such that $x_{i_0}\ P_{i_0}\ y_{i_0}$. By continuity and strict monotonicity of preferences, there exists $z \in X$ such that $x_{i_0}\ P_{i_0}\ z_{i_0}\ P_{i_0}\ y_{i_0}$ and for all $i \in N$, either $x_i\ P_i\ z_i\ P_i\ y_i$ or $y_i\ R_i\ x_i\ P_i\ z_i$. It follows that $x\ P\ z$ (by Weak Pareto)and $z\ P\ y$ (because i_0 is a quasi-dictator). By transitivity, $x\ P\ y$. ■

Lemma 7 *On the domain $\mathcal{D}^-$, if a SOF satisfies Weak Pareto and Weak Independence of Irrelevant Alternatives, then there exists a dictator over X.*

Proof of Lemma 7. $\mathcal{D}^- = \bigcup_{K \subsetneq L} (\mathcal{R}_K)^n$. By Lemmas 4, 5 and 6, we know that there is a dictator i_K over X for every domain $(\mathcal{R}_K)^n$. Let $x, y \in X$, $K, K' \subset L$ and $R_N \in \bigcap_{K \subsetneq L} (\mathcal{R}_K)^n$ be such that $x_{i_K}\ P_{i_K}\ y_{i_K}$ and $y_{i_{K'}}\ P_{i_{K'}}\ x_{i_{K'}}$. One must have $x\ P\ y$ and $y\ P\ x$, an impossibility. Therefore the same agent must be the dictator for all K. ■

The first part of the theorem is a direct consequence of Lemmas 1 and 7.

For the second part, let $R_N \in \mathcal{D}^+$ and define the following correspondences:

$$\begin{aligned}
x_i(k) &= \left\{ x_i \in \mathbb{R}_+^L \mid \forall y_i \in \mathbb{R}_+^{L \setminus \{k\}},\ x_i\ P_i\ y_i \right\} \\
i(k) &= \{ i \in N \mid x_i(k) \neq \emptyset \}, \\
K(i) &= \{ k \in L \mid i \in i(k) \}.
\end{aligned}$$

Since $R_N \in \mathcal{D}^+$, $i(k) \neq \emptyset$ for all $k \in L$ and $\bigcup_{i \in N} K(i) = L$. Let us also prove that for any $i \in N$ and $k \in L$, the set $x_i(k)$ is closed and is equal to $uc(\overline{x}_i; R_i)$ for some $\overline{x}_i \in \mathbb{R}_+^L$. Take any sequence $(z_i^t)_{t \in \mathbb{N}}$ in $x_i(k)$ which converges to some $z_i \in \mathbb{R}_+^L$. Assume that there is $y_i \in \mathbb{R}_+^{L \setminus \{k\}}$ such that $y_i\ R_i\ z_i$. Then, by strict monotonicity of preferences, there is $y_i' \in \mathbb{R}_+^{L \setminus \{k\}}$ such that $y_i'\ P_i\ z_i$. By continuity of preferences, there is $t \in \mathbb{N}$ such that $y_i'\ P_i\ z_i^t$, contradicting the fact that $z_i^t \in x_i(k)$. This proves that $x_i(k)$ is closed. Moreover, for any $z_i, z_i' \in \mathbb{R}_+^L$ such that $z_i' \in x_i(k)$ and $z_i\ R_i\ z_i'$, one has $z_i \in x_i(k)$. Take any $z_i \in x_i(k)$ and let

$$\overline{x}_i = \min \{ \alpha \in \mathbb{R}_+ \mid \alpha z_i \in x_i(k) \}\, z_i.$$

By construction, $\overline{x}_i \in x_i(k)$ and for any $x_i' \in \mathbb{R}_+^L$ such that $x_i'\ R_i\ \overline{x}_i$, $x_i' \in x_i(k)$; conversely, for any $x_i' \in x_i(k)$, $x_i'\ R_i\ \overline{x}_i$. Otherwise, if $\overline{x}_i\ P_i\ x_i'$, there is exists $\alpha' < \min \{ \alpha \in \mathbb{R}_+ \mid \alpha z_i \in x_i(k) \}$ such that $\alpha' z_i\ P_i\ x_i'$, implying that $\alpha' z_i \in x_i(k)$, contradicting the fact that $\alpha' < \min \{ \alpha \in \mathbb{R}_+ \mid \alpha z_i \in x_i(k) \}$.

Let

$$uc_i^k = \begin{cases} x_i(k) & \text{if } i \in i(k) \\ \mathbb{R}_+^L & \text{if } i \notin i(k), \end{cases}$$

and

$$\overline{uc}_i = \bigcap_{k \in L} uc_i^k.$$

By construction, for every i there is $\overline{x}_i \in \mathbb{R}_+^L$ such that $\overline{uc}_i = uc(\overline{x}_i; R_i)$. As a consequence, for all allocations $x \in \prod_{i \in N} \overline{uc}_i$ and $y \notin \prod_{i \in N} \overline{uc}_i$, there is i such that $x_i\ R_i\ \overline{x}_i\ P_i\ y_i$, implying that y does not Pareto-dominate x.

When $K(i) \neq \emptyset$ and $x_i \in \overline{uc}_i$, then for all $k \in K(i)$, $x_i \in uc_i^k = x_i(k)$. Therefore for all $y_i \in \mathbb{R}_+^{L\setminus\{k\}}$, $x_i \; P_i \; y_i$, so that necessarily $x_i \notin \mathbb{R}_+^{L\setminus\{k\}}$. In summary, one has $x_{ik} > 0$ for any $x_i \in \overline{uc}_i$ and $k \in K(i)$. Since $\bigcup_{i\in N} K(i) = L$, one then has $\sum_{i\in N} x_i \gg 0$ whenever $x_i \in \overline{uc}_i$ for all $i \in N$. Therefore, for any $K \subsetneq L$ and any $x \in (\mathbb{R}_+^K)^n$, $x \notin \prod_{i\in N} \overline{uc}_i$.

We can now define the social ordering $\Psi(R_N)$ as follows. Choose a reference bundle $x_0 \in \mathbb{R}_{++}^L$. For all $x, y \in \left(\mathbb{R}_+^L\right)^n$, $x \; R \; y$ if one of the following conditions holds:

(i) $x, y \in \prod_{i\in N} \overline{uc}_i$ and

$$\min_i \min \{\lambda \in \mathbb{R}_+ \mid \lambda x_0 \; R_i \; x_i\} \geq \min_i \min \{\lambda \in \mathbb{R}_+ \mid \lambda x_0 \; R_i \; y_i\} ;$$

(ii) $x \in \prod_{i\in N} \overline{uc}_i$ and $y \notin \prod_{i\in N} \overline{uc}_i$;

(iii) $x, y \notin \prod_{i\in N} \overline{uc}_i$ and $x_1 \; R_1 \; y_1$.

For any R_N this social ordering is transitive because it partitions the set $(\mathbb{R}_+^L)^n$ in two subsets, $\prod_{i\in N} \overline{uc}_i$ and its complement, ranks all allocations in $\prod_{i\in N} \overline{uc}_i$ above the others, and espouses transitive rankings within each subset. Weak Pareto is satisfied because no allocation from the complement of $\prod_{i\in N} \overline{uc}_i$ can Pareto-dominate an allocation in $\prod_{i\in N} \overline{uc}_i$, and also because the specific rankings for $\prod_{i\in N} \overline{uc}_i$ and its complement satisfy Weak Pareto. Independence of Irrelevant Commodities is satisfied because when $x, y \in (\mathbb{R}_+^K)^n$ for $K \subsetneq L$, necessarily $x, y \notin \prod_{i\in N} \overline{uc}_i$, so that when R_N and R'_N agree on $\mathbb{R}_+^K$, obviously R_1 and R'_1 agree on $\{x, y\}$, and the social ordering as well. ■

Remark. The impossibility result no longer holds if the set $\mathcal{R}$ is extended to include non strictly monotonic preferences. Let $A(R_N) = \{x \in X \mid \forall i \in N, \; x_i \; P_i \; 0\}$ and $B(R_N) = \{x \in X \mid \exists i \in N, \; x_i \; I_i \; 0\}$. Consider the following SOF R. It is such that $x \; R \; y$ whenever one of the conditions below holds:
(i) $x \in A(R_N)$ and $y \in B(R_N)$;
(ii) $x, y \in A(R_N)$ and $x_1 \; R_1 \; y_1$;
(iii) $x, y \in B(R_N)$ and $x_2 \; R_2 \; y_2$.
This SOF satisfies Weak Pareto and IIC, but is not dictatorial. However, this example obviously displays clear dictatorial features, and the essence of our results does not really depend on strict monotonicity of preferences.

In particular on the subdomain of $\mathcal{D}^-$ (extended to non-strictly monotonic preferences) such that for every i, every $x_i > 0$, $x_i \; P_i \; 0$, the dictatorship result is preserved.

Proof of Theorem 3: For the impossibility, the only part of the proof of Th. 2 which needs to be changed is the proof of Lemma 1, which is now reformulated as follows.

Lemma 8 *If a SOF satisfies Weak Pareto and Individualistic Independence of Irrelevant Commodities, then it satisfies Weak Independence of Irrelevant Alternatives.*

Proof of Lemma 8. Let $R_N, R'_N \in \mathcal{D}^{-*}$ and $x, y \in X$ be such that for all $i \in N$, R_i and R'_i agree on $\{x, y\}$, and for no $i \in N$, $x_i \; I_i \; y_i$. Assume that $x \; P \; y$. For any R_i, let $\mathcal{M}(R_i)$ be the set of non-empty subsets of commodities which are dispensable for R_i:

$$\mathcal{M}(R_i) = \{K \subseteq L \mid K \neq \emptyset \text{ and } K \text{ is dispensable for } R_i\}.$$

Let R''_N be such that for all i, $\mathcal{M}(R''_i) = \mathcal{L} \setminus \{L\}$ (for instance, this is obtained with linear preferences) and R_i and R''_i agree on $\{x, y\}$.

Let $i \in N$ be given. Since $R_N \in \mathcal{D}^{-*}$, $\mathcal{M}(R_i) \neq \emptyset$. Choose some non-empty $M_i \in \mathcal{M}(R_i)$ and define $K_i := L \setminus M_i$. Since $x, y > 0$, by monotonicity of preferences we know that $y_i \; P_i \; 0$ and $x_i \; P''_i \; 0$. We can choose $z_i, w_i, z'_i, w'_i \in \mathbb{R}^{K_i}_{++}$ such that:
(i) if $x_i \; P_i \; y_i$ (and hence $x_i \; P'_i \; y_i$ as well), then $z_i \; P_i \; x_i \; P_i \; y_i \; P_i \; w_i$ and $x_i \; P''_i \; z'_i \; P''_i \; w'_i \; P''_i \; y_i$,
(ii) if $y_i \; P_i \; x_i$ (and hence $y_i \; P'_i \; x_i$ as well), then $y_i \; P_i \; w_i \; P_i \; z_i \; P_i \; x_i$ and $w'_i \; P''_i \; y_i \; P''_i \; x_i \; P''_i \; z'_i$, and
(iii) there is $\lambda_i, \lambda'_i \in \mathbb{R}_{++}$ such that $w_i = \lambda_i z_i$, $w'_i = \lambda'_i z'_i$.

Next, choose $a_i, b_i, a'_i, b'_i, a''_i, b''_i \in \mathbb{R}^{M_i}_{++}$ such that:
(i) if $x_i \; P_i \; y_i$, then $a'_i > a''_i > a_i > b_i > b''_i > b'_i$, and
(ii) if $y_i \; P_i \; x_i$, then $b_i > b''_i > b'_i > a'_i > a''_i > a_i$. and
(iii) there is $\mu_i, \mu'_i \in \mathbb{R}_{++}$ such that $b_i = \mu_i a_i$, $b'_i = \mu'_i a'_i$.

Let $R_0 \in \mathcal{R}$ be an arbitrary preference relation on $\mathbb{R}^L_+$. By the same way as in the proof of Lemma 1, we can construct a preference relation $R^*_i \in \mathcal{R}$ such that:
(i) R^*_i and R_i agree on $\mathbb{R}^{K_i}_+$.
(ii) R^*_i and R_0 agree on $\mathbb{R}^{M_i}_+$.

(iii) $z_i \ I_i^* \ a_i$ and $w_i \ I_i^* \ b_i$.
Similarly, we construct $R_i''^* \in \mathcal{R}$ such that:
(i) $R_i''^*$ and R_i'' agree on $\mathbb{R}_+^{K_i}$.
(ii) $R_i''^*$ and R_0 agree on $\mathbb{R}_+^{M_i}$.
(iii) $z_i' \ I_i''^* \ a_i'$ and $w_i' \ I_i''^* \ b_i'$.
Notice that R_i^* and $R_i''^*$ agree on $\mathbb{R}_+^{M_i}$ because they both agree with R_0.

Having defined $z_i, w_i, z_i', w_i' \in \mathbb{R}_{++}^{K_i}$, $a_i, b_i, a_i', b_i', a_i'', b_i'' \in \mathbb{R}_{++}^{M_i}$, R_i^* and $R_i''^*$ for every $i \in N$, we obtain allocations $z, w, z', w', a, b, a', b', a'', b'' \in \mathbb{R}_+^L$ and preference profiles $R_N^* := (R_1^*, \ldots, R_n^*)$ and $R_N''^* := (R_1''^*, \ldots, R_n''^*)$. By construction (see the proof of Lemma 1), $R_N^*, R_N''^* \in \mathcal{D}^{-*}$.

By Weak Pareto, we have $z \ P \ x$ and $y \ P \ w$. By our supposition, $x \ P \ y$. Hence, transitivity of P implies $z \ P \ w$. On the other hand, for every $i \in N$, since $a_i'' > a_i$, $b_i > b_i''$, $z_i \ I_i^* \ a_i$ and $w_i \ I_i^* \ b_i$, strict monotonicity and transitivity of preferences imply $a_i'' \ P_i^* \ z_i$ and $w_i \ P_i^* \ b_i''$. By Weak Pareto, $a'' \ P^* \ z$ and $w \ P^* \ b''$. It follows from $z \ P \ w$ and Individualistic Independence of Irrelevant Commodities (IIIC) that $z \ P^* \ w$. By transitivity of R^*, we have $a'' \ P^* \ b''$.

Similarly, by Weak Pareto, $x \ P'' \ z'$ and $w' \ P'' \ y$. By transitivity and strict monotonicity of preferences and Weak Pareto, we have $z' \ P''^* \ a''$ and $b'' \ P''^* \ w'$. On the other hand, it follows from $a'' \ P^* \ b''$ and IIIC that $a'' \ P''^* \ b''$. By transitivity of P''^*, $z' \ P''^* \ w'$. From IIIC (applied to $R_N''^*$ and R_N''), we have $z' \ P'' \ w'$. By transitivity, $x \ P'' \ y$.

We have shown that $x \ P \ y \Rightarrow x \ P'' \ y$. By symmetry, $x \ P'' \ y \Rightarrow x \ P \ y$, and $y \ P \ x \Leftrightarrow y \ P'' \ x$. Hence, it also holds that $x \ I \ y \Leftrightarrow x \ I'' \ y$. This means that R and R'' agree on $\{x, y\}$.

By a similar reasoning, we can prove that R' and R'' agree on $\{x, y\}$. Therefore R and R' agree on $\{x, y\}$. ■

For the second part of Th. 3, let $R_N \in \mathcal{D}^{+*}$. Choose any $i_0 \in N$ such that every $K \subseteq L$ is essential for R_{i_0}. We define the social ordering $\Psi(R_N)$ as follows. For all $x, y \in \left(\mathbb{R}_+^L\right)^n$, $x \ R \ y$ if one of the following conditions holds:

(i) $x_{i_0}, y_{i_0} \in \overline{uc}_{i_0}$ and $x_2 \ R_2 \ y_2$;

(ii) $x_{i_0} \in \overline{uc}_{i_0}$ and $y_{i_0} \notin \overline{uc}_{i_0}$;

(iii) $x_{i_0}, y_{i_0} \notin \overline{uc}_{i_0}$ and $x_1 \ R_1 \ y_1$. ■

Proof of Theorem 4:

The proof relies on the following lemmas.

Lemma 9 *Let $K \subsetneq L$, $K \neq \emptyset$. On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and Independence of Irrelevant Commodities, then it satisfies Weak Independence of Irrelevant Alternatives restricted to allocations in $\left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$.*

Proof of Lemma 9: Let $x, y \in \left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$ and $R_N, R'_N \in \mathcal{R}^n$ be such that for all $i \in N$, R_i and R'_i agree on $\{x, y\}$ and for no $i \in N$, $x_i \; I_i \; y_i$. Let $M = L \setminus K$. Suppose $x \; P \; y$.

By a similar method as in the proof of Lemma 1, one constructs $R^*_N, R'^*_N \in \mathcal{R}^n$ and $a, a', a'', b, b', b'' \in \mathbb{R}^M_+ \setminus \{0\}$ such that for all $i \in N$:
(i) if $x_i \; P_i \; y_i$ then $a'_i > a''_i > a_i > b_i > b''_i > b'_i$;
(ii) if $y_i \; P_i \; x_i$ then $b_i > b''_i > b'_i > a'_i > a''_i > a_i$;
(iii) R_i and R^*_i agree on $\mathbb{R}^K_+$;
(iv) R'_i and R'^*_i agree on $\mathbb{R}^K_+$;
(v) R^*_i and R'^*_i agree on $\mathbb{R}^M_+$;
(vi) $x_i \; I^*_i \; a_i$ and $y_i \; I^*_i \; b_i$;
(vii) $x_i \; I'^*_i \; a'_i$ and $y_i \; I'^*_i \; b'_i$.

By Weak Pareto, $a'' \; P^* \; x$ and $y \; P^* \; b''$. By IIC, $x \; P^* \; y$ so that $a'' \; P^* \; b''$. By IIC again, $a'' \; P'^* \; b''$. By Weak Pareto, $x \; P'^* \; a''$ and $b'' \; P'^* \; y$ so that $x \; P'^* \; y$. By IIC, $x \; P' \; y$. As in Lemma 1, one then easily deduces that R and R' agree on $\{x, y\}$. ■

Lemma 10 *Let $K \subsetneq L$, with cardinal $|K| > 1$. On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and Weak Independence of Irrelevant Alternatives restricted to allocations in $\left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$, then there is a dictator over $\left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$.*

This is proved like the sequence of Lemmas 4, 5 and 6.

Lemma 11 *Let $K \subsetneq L$, $|K| = 1$. On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and Independence of Irrelevant Commodities, then there is a dictator over $\left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$.*

Proof of Lemma 11: By IIC and monotonicity of preferences, the social ranking over $\left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$ does not depend on individual preferences. Consider $x, y, z, w \in \left(\mathbb{R}^K_+ \setminus \{0\}\right)^n$ such that for every $i \in N$, $x_i \gtrless y_i$ if and only if $z_i \gtrless w_i$ and for no $i \in N$, $x_i = y_i$.

Let $K' \subseteq L \setminus K$, $|K'| = 1$. Let $a, b \in \left(\mathbb{R}^{K'}_+ \setminus \{0\}\right)^n$ be such that for every $i \in N$, $x_i \gtrless y_i$ if and only if $a_i \gtrless b_i$. Suppose $x \; P \; y$. One can construct $R_N \in \mathcal{R}^n$ such that for every $i \in N$, either $a_i \; P_i \; x_i \; P_i \; y_i \; P_i \; b_i$ or $y_i \; P_i \; b_i \; P_i \; a_i \; P_i \; x_i$. By Weak Pareto, $a \; P \; x$ and $y \; P \; b$, so that by transitivity, $a \; P \; b$. By IIC and monotonicity of preferences, this ranking does not depend on individual preferences since $a, b \in \left(\mathbb{R}_+^{K'} \setminus \{0\}\right)^n$ and $|K'| = 1$. By a similar reasoning, therefore, one shows that $a \; P \; b$ implies $z \; P \; w$. Similarly, if $y \; P \; x$, one proves that $w \; P \; z$. In summary, $x \; P \; y$ if and only if $z \; P \; w$, and $y \; R \; x$ if and only if $w \; R \; z$. Let us call this property "neutrality".

The rest of the proof mimics part of the proof of Arrow's theorem (see, e.g., Sen 1970, ch. 3*). We present it for completeness. Take any $G \subseteq N$ such that for all $x, y \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$, $x \; P \; y$ if $x_i > y_i$ for all $i \in G$. (This property holds for $G = N$ by Weak Pareto.) Partition G into non-empty subsets G_1 and G_2. Let $x, y \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$ be such that $x_i > y_i$ for all $i \in N \setminus G_2$ and $x_i < y_i$ for all $i \in G_2$. Construct $z \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$ such that $x_i > y_i > z_i$ for all $i \in G_1$, $x_i < z_i < y_i$ for all $i \in G_2$ and $z_i > x_i > y_i$ for all $i \in N \setminus G$. One has $y \; P \; z$ because $y_i > z_i$ for all $i \in G$. Now, either $x \; P \; z$ or $z \; R \; x$. In the former case, by neutrality this implies that for all $a, b \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$, $a \; P \; b$ whenever $a_i > b_i$ for all $i \in G_1$ and $a_i < b_i$ for all $i \in N \setminus G_1$. In the latter, this implies $y \; P \; x$, so that by neutrality, for all $a, b \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$, $a \; P \; b$ whenever $a_i > b_i$ for all $i \in G_2$ and $a_i < b_i$ for all $i \in N \setminus G_2$.

Let us pursue the former case. Let $a, b \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$ be such that $a_i > b_i$ for all $i \in G_1$. Take $c \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$ such that $a_i > c_i > b_i$ for all $i \in G_1$ and $c_i > \max\{a_i, b_i\}$ for all $i \in N \setminus G_1$. Since $a_i > c_i$ for all $i \in G_1$ and $c_i > a_i$ for all $i \in N \setminus G_1$, $a \; P \; c$ and by Weak Pareto, $c \; P \; b$, implying $a \; P \; b$. Therefore, for all $a, b \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$, $a \; P \; b$ whenever $a_i > b_i$ for all $i \in G_1$.

Repeating this argument, one ultimately finds a subset containing a single individual i_0 such that for all $a, b \in \left(\mathbb{R}_+^K \setminus \{0\}\right)^n$, $a \; P \; b$ whenever $a_{i_0} > b_{i_0}$. That is the dictator. ■

Lemma 12 *On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and for all $K \subsetneq L$, $K \neq \emptyset$, there is a dictator over $\left(\mathbb{R}_+^K \setminus \{0\}\right)^n$, then there is a dictator over $\overline{X}$.*

Proof of Lemma 12: First we prove that the same dictator rules over all $\left(\mathbb{R}_+^K \setminus \{0\}\right)^n$.

Case 1: $\ell \geq 3$. Suppose not, with i_1 being a dictator over $\left(\mathbb{R}_+^K \setminus \{0\}\right)^n$ and i_2 over $\left(\mathbb{R}_+^{K'} \setminus \{0\}\right)^n$ Let $K'' \subsetneq L$ be such that $K \cap K'' \neq \emptyset$ and $K' \cap K'' \neq \emptyset$, with i_3 the dictator over $\left(\mathbb{R}_+^{K''} \setminus \{0\}\right)^n$. Since

$\left(\mathbb{R}_+^{K\cap K''}\setminus\{0\}\right)^n \subset \left(\mathbb{R}_+^{K}\setminus\{0\}\right)^n \cap \left(\mathbb{R}_+^{K''}\setminus\{0\}\right)^n$, both i_1 and i_3 must be dictators over $\left(\mathbb{R}_+^{K\cap K''}\setminus\{0\}\right)^n$, implying $i_1 = i_3$. Similarly $i_2 = i_3$, so that $i_1 = i_2$. This proves that there is only one dictator i_0.

Case 2: $\ell = 2$. By Lemma 11, there is a dictator i_1 over $(\mathbb{R}_{++} \times \{0\})^n$. Let $x, y \in (\{0\} \times \mathbb{R}_{++})^n$ be such that $x_{i_1} > y_{i_1}$. Then, there exist $z, w \in (\mathbb{R}_{++} \times \{0\})^n$ such that for all $i \in N$, (i) if $x_i > y_i$, then $z_i > w_i$, and (ii) if $x_i \leq y_i$, then $z_i < w_i$. One can construct $R'_N \in \mathcal{R}^n$ such that for all $i \in N$, either $x_i\ P_i\ z_i\ P_i\ w_i\ P_i\ y_i$ or $w_i\ P_i\ y_i\ R_i\ x_i\ P_i\ z_i$. In particular, $z_i\ P_{i_1}\ w_i$. Since i_1 is a dictator over $(\mathbb{R}_{++} \times \{0\})^n$, one has $z\ P\ w$. By Weak Pareto, $x\ P\ z$ and $w\ P\ y$. By transitivity, $x\ P\ y$. By IIC and monotonicity of preferences, this ranking does not depend on individual preferences since $x, y \in (\{0\} \times \mathbb{R}_{++})^n$. This means that i_1 is a dictator over $(\{0\} \times \mathbb{R}_{++})^n$ as well.

Next, let $x, y \in \overline{X}$ and $R_N \in \mathcal{R}^n$ be such that $x_{i_0}\ P_{i_0}\ y_{i_0}$. Let K, K' be such that $x \in \left(\mathbb{R}_+^{K}\setminus\{0\}\right)^n$, $y \in \left(\mathbb{R}_+^{K'}\setminus\{0\}\right)^n$. We have to prove that $x\ P\ y$. Let $z \in \left(\mathbb{R}_+^{K'}\setminus\{0\}\right)^n$ be such that $x_{i_0}\ P_{i_0}\ z_{i_0}\ P_{i_0}\ y_{i_0}$ and for all $i \neq i_0$, $x_i\ P_i\ z_i$ (this is possible by continuity and the fact that $x_i\ P_i\ 0$). Because i_0 is a dictator over $\left(\mathbb{R}_+^{K'}\setminus\{0\}\right)^n$, $z\ P\ y$, while $x\ P\ z$ by Weak Pareto. Therefore $x\ P\ y$ by transitivity. ∎

Proof of Theorem 5:

The proof has exactly the same structure as in the first part.

Lemma 13 *On the domain $\mathcal{R}^n$, if a SOF satisfies Weak Pareto and Individualistic Independence of Irrelevant Commodities, then it satisfies Weak Independence of Irrelevant Alternatives restricted to allocations in $\overline{\overline{X}}$.*

Proof of Lemma 13: Let $R_N, R'_N \in \mathcal{R}^n$ and $x, y \in \overline{\overline{X}}$ be such that for all $i \in N$, R_i and R'_i agree on $\{x, y\}$, and for no $i \in N$, $x_i\ I_i\ y_i$. Assume that $x\ P\ y$.

For every $i \in N$, if $x_i\ P_i\ y_i$ then let $K_i = \{k \in L \mid x_{ik} > 0\}$ and if $y_i\ P_i\ x_i$ then let $K_i = \{k \in L \mid y_{ik} > 0\}$. Let $M_i = L \setminus K_i$. Since $x, y \in \overline{\overline{X}}$, we know that $K_i, M_i \neq \emptyset$.

Since $x_i, y_i > 0$, by monotonicity of preferences we know that $y_i\ P_i\ 0$ and $x_i\ P'_i\ 0$ for all i. We can choose $z, w, z', w' \in \prod_{i\in N} \mathbb{R}_+^{K_i}$ such that for all $i \in N$,
(i) if $x_i\ P_i\ y_i$ (and hence $x_i\ P'_i\ y_i$ as well), then $z_i\ P_i\ x_i\ P_i\ y_i\ P_i\ w_i$ and $x_i\ P'_i\ z'_i\ P'_i\ w'_i\ P'_i\ y_i$,

(ii) if $y_i \; P_i \; x_i$ (and hence $y_i \; P'_i \; x_i$ as well), then $y_i \; P_i \; w_i \; P_i \; z_i \; P_i \; x_i$ and $w'_i \; P'_i \; y_i \; P'_i \; x_i \; P'_i \; z'_i$, and
(iii) there is $\lambda_i, \lambda'_i \in \mathbb{R}_{++}$ such that $w_i = \lambda_i z_i$, $w'_i = \lambda'_i z'_i$.

Next, choose $a, b, a', b', a'', b'' \in \prod_{i \in N} \mathbb{R}_+^{M_i}$ such that for all $i \in N$,
(i) if $x_i \; P_i \; y_i$, then $a'_i > a''_i > a_i > b_i > b''_i > b'_i$,
(ii) if $y_i \; P_i \; x_i$, then $b_i > b''_i > b'_i > a'_i > a''_i > a_i$, and
(iii) there is $\mu_i, \mu'_i \in \mathbb{R}_{++}$ such that $b_i = \mu_i a_i$, $b'_i = \mu'_i a'_i$.

The rest is as in the proof of Lemma 8. ■

The rest of the proof applies without change for the case $\ell \geq 3$. When there are only two commodities, a more direct proof is necessary, because there are no free triples. Indeed, in this case, in $\overline{\overline{X}}$ any individual bundle has only one commodity in positive quantity, so that for any triple of bundles, there are at least two bundles with the same commodity. Strictly monotonic preferences rank this pair of bundles according to their respective quantity for this commodity.

Here is a proof for $\ell = 2$. By Lemmas 11 and 12 and the fact that IIIC implies IIC, there is a dictator i_0 over $\overline{X} = (\mathbb{R}_{++} \times \{0\})^n \cup (\{0\} \times \mathbb{R}_{++})^n$.

Let $x, y \in \overline{\overline{X}}$ and $R_N \in \mathcal{R}^n$ be such that $x_{i_0} \; P_{i_0} \; y_{i_0}$ and for all $i \in N$, $x_i . y_i > 0$ (meaning that the two bundles have a positive quantity for the same commodity). One can construct $z, w \in \overline{X}$ and $R'_N \in \mathcal{R}^n$ such that for every $i \in N$, either $x_i \; P_i \; y_i$ and $x_i \; P'_i \; z_i \; P'_i \; w_i \; P'_i \; y_i$, or $y_i \; R_i \; x_i$ and $w_i \; P'_i \; y_i \; R'_i \; x_i \; P'_i \; z_i$. By dictatorship, $z \; P' \; w$ and by Weak Pareto, $x \; P' \; z$ and $w \; P' \; y$. By transitivity, $x \; P' \; y$ and by IIIC and monotonicity of preferences, $x \; P \; y$.

Let $x, y \in \overline{\overline{X}}$ and $R_N \in \mathcal{R}^n$ be such that $x_{i_0} \; P_{i_0} \; y_{i_0}$. One can construct $z \in \overline{\overline{X}}$ such that for every $i \in N$, $y_i . z_i > 0$ and either $x_i \; P_i \; z_i \; P_i \; y_i$ or $y_i \; R_i \; x_i \; P_i \; z_i$. By the above property, one has $z \; P \; y$. By Weak Pareto, $x \; P \; z$, implying $x \; P \; y$. This shows that i_0 is a dictator over $\overline{\overline{X}}$. ■