Point optimal test for cointegration with unknown variance-covariance matrix

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Abstract

This paper examines a point optimal invariant (POI) test for the null hypothesis of cointegration. Our test is different from Jansson's (2005) test in that we consider location invariance in wider directions and that we assume an unknown variance-covariance matrix for the error term, while it is assumed to be known in Jansson (2005). As the variance-covariance matrix is unknown in our paper, we consider the POI test among a class of tests that are invariant to scale change, as well as location shift, in the dependent variable. As a special case of the POI test, we also derive the locally best invariant and unbiased (LBIU) test. We find that our POI test has the same asymptotic distribution as Jansson's (2005) test, which is a point optimal test among a class of location invariant tests. On the other hand, our LBIU test is shown to have a different characteristic from the locally best invariant test in Shin (1994). We also propose a modification of our tests to accommodate more general assumptions on the error term. Monte Carlo simulation is conducted to investigate the finite sample properties of the tests, and it is shown that our modified tests perform better in finite samples than either the Jansson or Shin tests.

JEL classification: C12; C22

Key Words: Cointegration; point optimal test; locally best test; invariance; power envelope

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1. Introduction

This paper considers a single equation cointegrating model and discusses the optimality of tests for the null hypothesis of cointegration. Following the seminal work of Engle and Granger (1987), tests of cointegration have been intensively investigated in the econometric literature. For a single equation model, tests for the null of cointegration are proposed by Hansen (1992a), Quintos and Phillips (1993), Shin (1994), and Jansson (2005), while the null of no cointegration is considered in Engle and Granger (1987) and Phillips and Ouliaris (1990), among others. A system equations approach is also considered in a number of studies, whereas this paper deals only with a single equation model. See Hubrich, Lütkepohl, and Saikkonen (2001) for a useful review of system equations methods.

For the null hypothesis of cointegration, Shin (1994) proposes the locally best invariant (LBI) test for the i.i.d. normal errors, while Jansson (2005) develops the point optimal invariant (POI) test and derives the asymptotic local power envelope. These optimal tests are derived for a simple stylized model and modified such that the limiting distributions of the test statistics become independent of nuisance parameters under general assumptions. According to Jansson (2005), the POI test performs better than the LBI test in a wide range of alternatives, both asymptotically and in finite samples.

In Jansson (2005), the analysis of the cointegrating regression model proceeds under the assumption of the known variance-covariance matrix of the error term, and the optimal test is derived among a class of tests that are invariant to location shift in the dependent variable. As the variance-covariance matrix is assumed to be known, it does not consider scale invariance, and hence only location invariance is considered. Of some interest is that the limiting distribution of the POI test statistic does not depend on the true variance-covariance matrix. As a result, it is not too difficult to generalize the POI test to accommodate the general assumptions of the unknown variance-covariance matrix and the weakly dependent error term.

In this paper, we assume the unknown variance-covariance matrix and investigate the POI test for the null of cointegration. Although the properties of the POI test by Jansson

(2005) are asymptotically independent of the variance-covariance matrix as discussed, in finite samples they are surely not. Because, in general, we do not know the variance-covariance matrix for the error term, we proceed with our analysis assuming an unknown variance-covariance matrix. As the variance-covariance matrix is unknown, it is natural to introduce scale invariance in addition to location invariance. In fact, the testing problem considered in this paper is seen to be invariant not only in translations but also in scale transformations. We develop the point optimal test by taking account of these two kinds of transformations.

One interesting finding is that our approach leads to a different test statistic to that in Jansson because we consider a class of tests invariant to scale change as well as location shift in wider directions, but the asymptotic local power envelope of our test is the same as that of Jansson's test. This implies we can impose scale invariance in addition to location invariance in wider directions without sacrificing local asymptotic power. As a special case of the POI test, we also investigate the LBI test by considering location and scale invariance. We show that the first derivative of the log-likelihood function of the maximal invariant evaluated under the null hypothesis becomes identically equal to zero; we then derive the LBI and unbiased (LBIU) test. The asymptotic local power of the LBIU test is compared with that of the LBI test considered in Shin (1994), and we show that the LBIU test is more powerful in a wider range of local alternatives. The other main finding in this paper is that Jansson's and Shin's tests are greatly affected by the initial value condition on the stochastic regressors, while our POI and LBIU tests are shown to be free of the initial value condition. We show that our tests perform better than either Jansson's or Shin's tests in finite samples.

The remainder of the paper is organized as follows. In Section 2 we derive the POI and LBIU tests for a stylized model with the unknown variance-covariance matrix of the error term. Location and scale invariance is introduced and the limiting local power function is obtained. Section 3 generalizes the assumptions by allowing the error term to be weakly dependent; we modify the test statistics such that their limiting distributions are independent of nuisance parameters. The finite sample properties of our tests are investigated through Monte Carlo simulations in Section 4. Section 5 concludes the paper.

2. The POI and LBIU tests

Let us consider the following model:

$$y_t = \alpha' d_t + \beta' x_t + v_t, \quad (1-L)v_t = u_t^y - \theta u_{t-1}^y, \tag{1}$$

$$x_t = \alpha_x' d_t + x_t^0, \quad (1-L)x_t^0 = u_t^x, \tag{2}$$

where $d_t = [1, \cdots, t^p]'$ with $p \geq 0$, y_t and x_t are 1 and k dimensional observations, L is the lag operator, and $v_0 = u_0^y = 0$. For the error process we consider the following assumption in this section.

Assumption 1 $u_t = [u_t^y, u_t^{x\prime}]' \sim$ *i.i.d.*$N(0, \Sigma)$ *with* $\Sigma > 0$.

We partition Σ conformably with u_t as

$$\Sigma = \begin{bmatrix} \sigma_{yy} & \sigma_{yx} \\ \sigma_{xy} & \Sigma_{xx} \end{bmatrix}.$$

Since (2) includes a constant term we assume $x_0^0 = 0$ without loss of generality. We proceed with this restricted assumption in this section but we will relax the assumption of normality and consider the dependent case in the next section.

The model is expressed in the vectorized form as

$$y = D\alpha + X\beta + v, \quad L_1 v = L_\theta u^y,$$

$$X = D\alpha_x + \Psi_0^{1/2} U^x,$$

where $y = [y_1, \cdots, y_T]'$, $D = [d_1, \cdots, d_T]'$, and the other vectors and matrices are defined similarly, $\Psi_\theta = \Psi_\theta^{1/2} \Psi_\theta^{1/2\prime}$ with

$$\Psi_\theta^{1/2} = \begin{bmatrix} 1 & & & 0 \\ 1-\theta & 1 & & \\ \vdots & \ddots & \ddots & \\ 1-\theta & \cdots & 1-\theta & 1 \end{bmatrix}, \quad \text{and} \quad L_\theta = \begin{bmatrix} 1 & & & 0 \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ 0 & & -\theta & 1 \end{bmatrix}.$$

Since $L_1^{-1} L_\theta = \Psi_0^{1/2} L_\theta = \Psi_\theta^{1/2}$ because $L_1^{-1} = \Psi_0^{1/2}$, the above system can also be expressed as

$$
\begin{aligned}
y &= D\alpha + X\beta + \Psi_\theta^{1/2} u^y, \\
\Psi_0^{-1/2} X &= \Psi_0^{-1/2} D\alpha_x + U^x.
\end{aligned} \tag{3}
$$

Note that the first column of $\Psi_0^{-1/2} D$ consists of $e_1 = [1, 0, \cdots, 0]'$ while the other columns are obtained by the nonsingular transformation of the first p columns of D, which corresponds to $[1, \cdots, t^{p-1}]$.

Let us suppose that we are interested in the following testing problem:

$$H_0 : \ \theta = 1 \quad \text{v.s.} \quad H_1 : \ \theta < 1.$$

Under the null hypothesis, $v_t = u_t^y$ and then y_t and x_t are cointegrated, while they are not cointegrated under the alternative because v_t is a unit root process when $\theta \neq 1$.

Noting that x_t is weakly exogenous for θ, it is sufficient for us to consider the distribution of y conditional on X as far as the hypothesis about θ is concerned. It is easy to see that the conditional distribution $y|X$ is given by $N(D\alpha + X\beta + \Psi_\theta^{1/2} U^x \Sigma_{xx}^{-1} \sigma_{xy}, \sigma_{yy\cdot x} \Psi_\theta)$, where $\sigma_{yy\cdot x} = \sigma_{yy} - \sigma_{yx} \Sigma_{xx}^{-1} \sigma_{xy}$. Using (3) the conditional distribution is also expressed as

$$y|X \sim N\left(D\alpha^* + X\beta^* + \Psi_0^{-1/2} X\gamma^* + e_1\delta^*, \sigma_{yy\cdot x}\Psi_\theta\right), \tag{4}$$

where α^*, β^*, γ^*, and δ^* are defined appropriately, and we used the relation $\Psi_\theta^{1/2} \Psi_0^{-1/2} = L_\theta = \theta \Psi_0^{-1/2} + (1-\theta) I_T$. Then, it is seen that the testing problem is invariant under the group of transformations

$$
\begin{gathered}
y \to sy + Da + Xb + \Psi_0^{-1/2} Xc + e_1 d \\
(\theta, \alpha^*, \beta^*, \gamma^*, \delta^*, \sigma_{yy\cdot x}) \to (\theta, s\alpha^* + a, s\beta^* + b, s\gamma^* + c, s\delta^* + d, s^2 \sigma_{yy\cdot x}),
\end{gathered} \tag{$\mathcal{G}_y$}
$$

where a, b, c, d, and s are $p+1$, k, k, 1, and 1 dimensional vectors with $0 < a < \infty$. Note that in a classical regression context, location shift in y is considered only in the directions of the regressors, D and X, while we additionally consider the directions of $\Psi_0^{-1/2} X$ and e_1. Of importance is that in our model the I(I) regressors, X, are correlated with the error term,

u^y, and then the conditional mean of y depends on $\Psi_0^{-1/2}X\gamma^*$ and $e_1\delta^*$ in addition to $D\alpha^*$ and $X\beta^*$ as is seen in (4). Since it is natural to consider location shift in y in the directions of the conditional mean, $[D, X, \Psi_0^{-1/2}X, e_1]$ provides appropriate directions of shift in y in our case. We can also see that invariance in the directions of e_1 implies that tests do not depend on the initial value condition. In the following, we develop the POI test under $(\mathcal{G}_y)$.

Let us define $M = I - Z(Z'Z)^{-1}Z'$, where $Z = [D, X, \Psi_0^{-1/2}X, e_1]$, and choose a $T \times (T-q)$ matrix H such that $H'H = I_{T-q}$ and $HH' = M$, where $q = 2k+p+2$. As $H'Z = 0$ we have

$$H'y|X \sim N(0, \sigma_{yy\cdot x}H'\Psi_\theta H).$$

Then, we can see that the distribution of $H'y|X$ is free from nuisance parameters α^*, β^*, γ^*, and δ^*. In addition, it is shown that $\eta = H'y/\sqrt{y'HH'y}$ conditional on X is a maximal invariant under the group of transformations $(\mathcal{G}_y)$. In this section we assume $\sigma_{yy\cdot x} = 1$ without loss of generality because $\eta|X$ is invariant to scale change in y. As the probability density function of $\eta|X$ is given by (see Kariya, 1980, and King, 1980)

$$f(\eta|X;\theta) = \frac{1}{2}\Gamma\left(\frac{T-q}{2}\right)\pi^{-(T-q)/2}|H'\Psi_\theta H|^{-1/2}\left(\eta'(H'\Psi_\theta H)^{-1}\eta\right)^{-(T-q)/2}, \tag{5}$$

we can construct invariant tests based on $f(\eta|X;\theta)$. According to the Neyman–Pearson lemma, the POI test against $\theta = \bar{\theta}$ is given by $f(\eta|X;\bar{\theta})/f(\eta|X;1)$, which is normalized to have a limiting distribution as

$$\begin{aligned}
\mathcal{R}_T(\bar{\theta}) &= T\left\{1 - \left(\frac{f(\eta|X;\bar{\theta})}{f(\eta|X;1)}\right)^{-2/(T-q)}\right\} \\
&= T\left\{1 - \left(\frac{|H'\Psi_{\bar{\theta}}H|}{|H'\Psi_1 H|}\right)^{1/(T-q)}\frac{y'H(H'\Psi_{\bar{\theta}}H)^{-1}H'y}{y'H(H'\Psi_1 H)^{-1}H'y}\right\} \\
&= T\left\{1 - \left(\frac{|Z'\Psi_{\bar{\theta}}^{-1}Z|}{|Z'Z|}\right)^{1/(T-q)}\frac{y'(\Psi_{\bar{\theta}}^{-1} - \Psi_{\bar{\theta}}^{-1}Z(Z'\Psi_{\bar{\theta}}^{-1}Z)^{-1}Z'\Psi_{\bar{\theta}}^{-1})y}{y'My}\right\},
\end{aligned}$$

where the third equality holds using the relations $\Psi_1 = I_T$, $H'H = I_{T-q}$, $HH' = M$, $|H'\Psi_{\bar{\theta}}H| = |Z'\Psi_{\bar{\theta}}^{-1}Z||Z'Z|^{-1}$, and $H(H'\Psi_{\bar{\theta}}H)^{-1}H' = \Psi_{\bar{\theta}}^{-1} - \Psi_{\bar{\theta}}^{-1}Z(Z'\Psi_{\bar{\theta}}^{-1}Z)^{-1}Z'\Psi_{\bar{\theta}}^{-1}$ (see Rao, 1973, and Jansson, 2005). The null hypothesis is rejected when $\mathcal{R}_T(\bar{\theta})$ takes large values.

Note that $\mathcal{R}_T(\bar{\theta})$ has a different expression from the Jansson's POI test statistic, which is constructed by considering only location invariance. The latter test statistic is expressed as

$$\mathcal{P}_T(\bar{\theta}) = \log \frac{|R'R|}{|R'\Psi_{\bar{\theta}}^{-1}R|} + \sigma_{yy\cdot x}^{-1}\left[y(1)'My(1) - y(\bar{\theta})'(\Psi_{\bar{\theta}}^{-1} - \Psi_{\bar{\theta}}^{-1}R(R'\Psi_{\bar{\theta}}^{-1}R)^{-1}R'\Psi_{\bar{\theta}}^{-1})y(\bar{\theta})\right],$$

where $R = [X, D]$ and $y(\theta) = y - \theta\Psi_0^{-1/2}X\Sigma_{xx}^{-1}\sigma_{xy}$. One of the reasons for the difference between the two test statistics is the directions of location shift: Jansson (2005) considers location invariance in the directions of R, while we introduced invariance in the directions of $[\Psi_0^{-1/2}X, e_1]$ in addition to R. As our analysis is based on the conditional distribution of y given X as in (4) and the conditional mean of y depends on Z, it is natural to consider location invariance in the directions of Z. The other reason for the difference comes from the introduction of scale change, which leads to the distributional difference between the two maximal invariants: the maximal invariant η in our analysis has a nonnormal distribution as given by (5), while the maximal invariant with only location invariance has a normal density as shown in Jansson (2005).

To investigate the asymptotic properties of the POI test we localize the parameters θ and $\bar{\theta}$ such that $\theta = 1 - \lambda/T$ and $\bar{\theta} = 1 - \bar{\lambda}/T$. Then, the limiting distribution of $\mathcal{R}_T(\bar{\theta})$ is given in the following theorem, in which an integral such as $\int_0^1 X(s)dY(s)'$ is written simply as $\int XdY'$ to achieve notational economy.

Theorem 1 *Under Assumption 1, the limiting distribution of $\mathcal{R}_T(\bar{\theta})$ is given by*

$$\begin{aligned}
\mathcal{R}_T(\bar{\theta}) \;\Rightarrow\; & 2\bar{\lambda}\int_0^1 V_\lambda^{\bar{\lambda}}dV_\lambda - \bar{\lambda}^2\int_0^1 (V_\lambda^{\bar{\lambda}})^2 ds \\
& + \left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)'\left(\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds\right)^{-1}\left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right) \\
& - \left(\int_0^1 QdV_\lambda\right)'\left(\int_0^1 QQ'ds\right)^{-1}\left(\int_0^1 QdV_\lambda\right) - \log\left|\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds\right| + \log\left|\int_0^1 QQ'ds\right|,
\end{aligned}$$

where $\Rightarrow$ signifies weak convergence of the associated probability measures, $Q(s) = [1, s, \cdots, s^p, W(s)']'$ with $W(s)$ being a k-dimensional standard Brownian motion, $Q^{\bar{\lambda}}(s) =$

$\int_0^s \exp(-\bar{\lambda}(s-r))dQ(r)$, $V_\lambda(s) = V(s) + \lambda \int_0^s V(r)dr$ with $V(s)$ being a univariate standard Brownian motion independent of $W(s)$, and $V_\lambda^{\bar{\lambda}}(s) = \int_0^s \exp(-\bar{\lambda}(s-r))dV_\lambda(r)$.

Remark 1: Although our test statistic $\mathcal{R}_T(\bar{\theta})$ is different from Jansson's $\mathcal{P}_T(\bar{\theta})$, the limiting distribution of $\mathcal{R}_T(\bar{\theta})$ is the same as that of $\mathcal{P}_T(\bar{\theta})$. This is because the additional deterministic and I(0) regressors, e_1 and $\Psi_0^{-1/2}X$, do not contribute to the asymptotic local distribution, as is shown in the proof of the theorem in the Appendix. Our result implies we can impose scale invariance in addition to location invariance in wider directions without sacrificing local asymptotic power. However, we will see in Section 4 that these additional regressors, especially e_1, play an important role in finite samples.

In practice, we specify a value of $\bar{\theta}$ or $\bar{\lambda}$ to implement the feasible point optimal test of our version. We follow Elliott et al. (1996) and Jansson (2005) to choose $\bar{\lambda}$. Their approach is to select $\bar{\lambda}$ such that the asymptotic local power against the local alternative $\bar{\theta} = 1 - \bar{\lambda}/T$ is approximately 50% when we use the 5% test based on $\mathcal{R}_T(\bar{\theta})$. The recommended values of $\bar{\lambda}$ are given by Table 1 in Jansson (2005).

The other possible value of $\bar{\theta}$ is $\bar{\theta} \to 0$, in which case the test becomes the LBI test. According to Ferguson (1967), the LBI test is given by $d\log f(\eta|X;\theta)/d\theta|_{\theta=1}$, but it is shown in the Appendix that $d\log f(\eta|X;\theta)/d\theta|_{\theta=1} = 0$. Then, instead of the LBI test, we consider the LBI and unbiased (LBIU) test, which is, according to Ferguson (1967), given by

$$\left.\frac{d^2 \log f(\eta|X;\theta)}{d\theta^2}\right|_{\theta=1} + \left(\left.\frac{d\log f(\eta|X;\theta)}{d\theta}\right|_{\theta=1}\right)^2 > c_1 + c_2 \left.\frac{d\log f(\eta|X;\theta)}{d\theta}\right|_{\theta=1},$$

where c_1 and c_2 are some constants. As shown in the Appendix, the LBIU test statistic is given by

$$\mathcal{L}_T = \frac{y'M\Psi_0 My/T^2}{y'My/(T-q)} + \frac{1}{T^2}\mathrm{tr}\left\{(Z'Z)^{-1}(Z'\Psi_0 Z)\right\}. \tag{6}$$

The null hypothesis is rejected when $\mathcal{L}_T$ takes large values.

Corollary 1 *Under Assumption 1, the limiting distribution of $\mathcal{L}_T$ is given by*

$$\mathcal{L}_T \Rightarrow \int_0^1 \left\{ V_\lambda - \int_0^s Q'dr \left(\int_0^1 QQ'dr\right)^{-1} \int_0^1 QdV_\lambda \right\}^2 ds$$

$$+tr\left\{\left(\int_0^1 QQ'dr\right)^{-1}\int_0^1\left(\int_s^1 Qdr\right)\left(\int_s^1 Q'dr\right)ds\right\}.$$

Figure 1 depicts the Gaussian power envelope of the 5% test based on $\mathcal{R}_T(\theta)$ along with the local asymptotic power functions of four cointegration tests in the constant mean case with $k = 1$.[3] Two of these are the feasible tests proposed in this paper, denoted by $\mathcal{R}_T$ and $\mathcal{L}_T$, respectively. The other two are the feasible tests given by Shin (1994) and Jansson (2005), denoted by $\mathcal{S}_T$ and $\mathcal{P}_T$, respectively. Since it is found out that the asymptotic power functions of $\mathcal{P}_T$ and $\mathcal{R}_T$ are the same, only one line is indicated in Figure 1. $\mathcal{S}_T$, which is the most commonly used test in applications, is locally optimal under his assumptions. Therefore it becomes a convenient benchmark for assessing our new tests, $\mathcal{R}_T$ and $\mathcal{L}_T$.

The local asymptotic powers of $\mathcal{P}_T$ and $\mathcal{R}_T$ are close to the envelope for all values of λ. Whereas the local asymptotic powers of $\mathcal{S}_T$ and $\mathcal{L}_T$ are close to the envelope for small values of λ due to their local optimal properties, they are well below the envelope as are those of $\mathcal{P}_T$ and $\mathcal{R}_T$ for large values of λ. The local asymptotic power of $\mathcal{L}_T$ is closer to the envelope than that of $\mathcal{S}_T$ for large values of λ. Figure 2 shows the linear trend case. What we observed for the constant mean case is also true for this case, although the magnitude of the differences is dampened.

3. Extension to general cases

The POI and LBIU tests in the previous section are based on the assumption that the error process is normal and serially independent. However, this assumption is too restrictive in practice and so we consider more general assumptions where the error term is weakly dependent. The purpose of this section is to construct test statistics that have the same local asymptotic properties as given in Theorem 1 and Corollary 1 under general assumptions.

To construct the feasible test statistics we define the long-run variance of u_t and its

[3] The curves are obtained from 20,000 replications from the distribution of the discrete approximation based on 2,000 steps to the limiting distribution given in Theorem 1.

one-sided version as

$$\Omega = \Sigma + \Pi + \Pi' \quad \text{and} \quad \Gamma = \Sigma + \Pi,$$

$$\text{where} \quad \Sigma = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E[u_t u_t'] \quad \text{and} \quad \Pi = \lim_{T\to\infty} T^{-1}\sum_{j=1}^{T-1}\sum_{t=1}^{T-j} E[u_t u_{t+j}'].$$

We partition these matrices conforming with u_t, as in the previous section. We also define the last k rows of Γ as Γ_x; that is, $\Gamma_x = [0, I_k]\Gamma$.

Assumption 2 *(a) $\{u_t\}$ is mean-zero and strong mixing with mixing coefficients of size $-p\alpha/(p-\alpha)$ and $E|u_t|^p < \infty$ for some $p > \alpha > 5/2$.*
(b) The matrix Ω exists with finite elements, $\Omega > 0$, $\omega_{yy} > 0$, and $\Omega_{xx} > 0$.

Assumption 2 ensures that the functional central limit theorem can be applied to the partial sums of u_t.

Let $u_t^* = [u_t^{y\cdot x}, u_t^{x\prime}]$ where $u_t^{y\cdot x} = \kappa' u_t = u_t^y - \omega_{yx}\Omega_{xx}^{-1}u_t^x$ with $\kappa' = [1, -\omega_{yx}\Omega_{xx}^{-1}]$, and let $\hat{u}_t^* = [\hat{u}_t^{y\cdot x}, \hat{u}_t^{x\prime}]'$ where $\hat{u}_t^{y\cdot x}$ and $\hat{u}_t^x$ are the regression residuals of y_t on z_t and x_t on d_t. We define Ω^*, Σ^*, Π^*, and Γ^* from u_t^* analogously to Ω, Σ, Π, and Γ, which are defined from u_t, and partition them conformably with u_t^* such that ω_{11}^*, ω_{12}^*, and Ω_{22}^* are $(1,1)$, $(1,2)$, and $(2,2)$ blocks of Ω^* and Γ_x^* is the last k rows of Γ^*. Let $\hat{\omega}_{11}^*$, $\hat{\Sigma}^*$, $\hat{\pi}_{11}^*$, and $\hat{\Gamma}_x^*$ be consistent estimators of ω_{11}^*, Σ^*, π_{11}^*, and Γ_x^*, which can be obtained by the typical kernel estimators as investigated in Andrews (1991). The proposed test statistics are

$$\begin{aligned}
\mathcal{R}_T^+(\bar{\theta}) &= \hat{\omega}_{11}^{*-1}\left\{y'M^+y - y'(\Psi_{\bar{\theta}}^{-1} - \Psi_{\bar{\theta}}^{-1}Z^+(Z^{+\prime}\Psi_{\bar{\theta}}^{-1}Z^+)^{-1}Z^{+\prime}\Psi_{\bar{\theta}}^{-1})y - 2\bar{\lambda}\hat{\pi}_{11}^*\right\} \\
&\quad - \log\left|Z^{+\prime}\Psi_{\bar{\theta}}^{-1}Z^+\right| + \log\left|Z^{+\prime}Z^+\right|, \\
\mathcal{L}_T^+ &= \frac{1}{T^2}\hat{\omega}_{11}^{*-1}y'M^+\Psi_0M^+y + \frac{1}{T^2}\mathrm{tr}\left\{(Z^{+\prime}Z^+)^{-1}(Z^{+\prime}\Psi_0Z^+)\right\},
\end{aligned}$$

where $M^+ = I_T - Z^+(Z^{+\prime}Z^+)^{-1}Z^{+\prime}$ and $Z^+ = [D, X^+, \Psi_0^{-1/2}X, e_1]$ with the transpose of the t-th row of X^+ defined by $x_t^+ = x_t - \hat{\Gamma}_x^*\hat{\Sigma}^{*-1}\hat{u}_t^*$. The following theorem gives the limiting distributions of these test statistics.

Theorem 2 *Under Assumption 2, $\mathcal{R}_T^+(\bar{\theta})$ and $\mathcal{L}_T^+$ have the same limiting distributions as $\mathcal{R}_T(\bar{\theta})$ and $\mathcal{L}_T$.*

Although our correction of the test statistics is basically the same as that proposed by Phillips and Hansen (1990), Park (1992), and Jansson (2005), we do not have to modify y_t to obtain the test statistics that are asymptotically independent of nuisance parameters; therefore our correction of the test statistics is relatively simple. This is because, as explained in the proof of Theorem 2, we can replace y_t in the test statistics by $v_{\theta t}^*$ where $v_{\theta t}^* = \theta u_t^{y \cdot x} + (\lambda/T)\sum_{j=1}^{t} u_j^{y \cdot x}$. As $u_t^{y \cdot x}$ are (asymptotically) uncorrelated with u_t^x, Brownian motions induced by the partial sums of them are independent of each other and hence we do not need a "simultaneous bias correction" for our test statistics.

4. Finite sample evidence

In this section we investigate the finite sample properties of the tests proposed in Section 3. The data-generating process we consider is the same as in Jansson (2005). The data are generated according to the system of (1) and (2) with α, β, and α_x normalized to zero. The error term u_t is generated by

$$u_t = \psi(L)\Theta(\rho)\varepsilon_t, \tag{7}$$

where $\varepsilon_t = (\varepsilon_t^x, \varepsilon_t^y)' \sim i.i.d.N(0, I_2)$, $\psi(L) = (1-a)\sum_{i=0}^{\infty} a^i L^i$ and

$$\Theta(\rho) = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}.$$

The parameters a and ρ control the persistence of the error and the endogeneity of the regressor, respectively. We set $a = 0, 0.5, 0.8$, $\rho = 1, 0.975, 0.95, 0.925, 0.90$, and sample size $T = 200$. The initial value, u_0, is drawn from its stationary distribution, and y_0 is set equal to zero. We experiment with two kinds of initial values for x_0, which is set to 0 or 10.

We also use the same estimation method for Σ, Ω, and Γ as in Jansson (2004)[4]. We estimate Σ using $\hat{\Sigma} = T^{-1}\sum_{t=1}^{T} \hat{u}_t^* \hat{u}_t^{*\prime}$ and Ω and Γ using the VAR(1) prewhitened kernel estimator. Rejection frequencies for the 5% level tests are reported in Tables 2 and 3 for the case of the constant mean and linear trend, respectively (we suppress the superscript $^+$ and the argument $\bar{\theta}$ from the test statistics). Case 1 describes the result for the case of

[4]The Matlab code provided by Michael Jansson was very helpful in conducting our simulation experiments.

$x_0 = 0$ and Case 2 for the case o $x_0 = 10$. We also show the results for the feasible versions of $\mathcal{P}_T$ and $\mathcal{S}_T$ for the sake of comparison. $\mathcal{S}_T$ is based not on the parametric approach by Shin (1994) but on the nonparametric approach by Choi and Ahn (1995).

For Case 1, the results are consistent with the analysis of the local asymptotic powers shown in Figures 1 and 2 when the persistence and the endogeneity are moderate, i.e. $a \leq 0.5$ and $\rho \leq 0.5$. The empirical sizes of all the tests are satisfactorily close to the nominal one and $\mathcal{P}_T$ and $\mathcal{R}_T$ dominate $\mathcal{L}_T$ and $\mathcal{S}_T$ in the case of the moderate persistence $a \leq 0.5$. When the persistence is not present, i.e. $a = 0$, the robustness of $\mathcal{R}_T$ and $\mathcal{L}_T$ to the endogeneity is pronounced. For $\rho = 0.8$ the results show nontrivial power gain by $\mathcal{R}_T$ and $\mathcal{L}_T$. This is because $\mathcal{R}_T$ and $\mathcal{L}_T$ are invariant under $(\mathcal{G}_y)$, which takes the location shift in the direction of $\Psi_0^{-1/2}X$ into account, although the advantage of $\mathcal{R}_T$ and $\mathcal{L}_T$ is obscured when the persistence is present. For $a = 0.8$, all tests have an empirical size far from the nominal one and low power, except that the empirical size of $\mathcal{L}_T$ is highly stable for all cases considered.

For Case 2, the observations of $\mathcal{R}_T$ and $\mathcal{L}_T$ for Case 1 are still true although all the nice properties of $\mathcal{P}_T$ and $\mathcal{S}_T$ remarked for Case 1 are lost unless the endogeneity is absent, i.e. ρ=0. This shows the importance of considering the group of transformations that contains the term involving "e_1" such as $(\mathcal{G}_y)$. This is the term which makes our test robust to changes in initial values. Although we could ignore it asymptotically as Jansson (2005) does, it can play an important role in finite samples as illuminated in Tables 2 and 3. Since the exact meaning of these initial values in economic applications is still open to discussion, the properties of $\mathcal{R}_T$ and $\mathcal{L}_T$ are clearly more desirable than other tests whose performance is largely affected by changes in initial values.

5. Conclusions

In this paper we investigate POI tests for the null hypothesis of cointegration when the variance-covariance matrix is unknown. We derive the POI and LBIU tests among a class of tests that are invariant to scale change, as well as location shift, in the dependent variable.

We find that although our POI test is different from Jansson's (2005) test, they have the same local limiting distribution. We observe that the local asymptotic power of the POI test is relatively close to the local asymptotic power envelope as shown by Jansson (2005), while the LBIU test performs better than the LBI test proposed by Shin (1994) in a wide range of local alternatives. In finite samples, we show that our tests perform better than either Jansson's or Shin's tests in view of the empirical sizes of the tests. In particular, the size of the LBIU test proposed in our paper is very close to the nominal one, even when the data generating process is relatively persistent.

Appendix

Proof of Theorem 1

The POI test statistic can be written as

$$\begin{aligned}\mathcal{R}_T(\bar\theta) &= T\left(1-\mathcal{R}_{1T}^{1/T}(\bar\theta)\mathcal{R}_{2T}(\bar\theta)\right)\\ &= \mathcal{R}_{1T}^{1/T}(\bar\theta)\times T\left(1-\mathcal{R}_{2T}(\bar\theta)\right)+T\left(1-\mathcal{R}_{1T}^{1/T}(\bar\theta)\right),\end{aligned}$$

$$\text{where}\quad \mathcal{R}_{1T}(\bar\theta)=\frac{|Z'\Psi_{\bar\theta}^{-1}Z|}{|Z'Z|},\qquad \mathcal{R}_{2T}(\bar\theta)=\frac{y'(\Psi_{\bar\theta}^{-1}-\Psi_{\bar\theta}^{-1}Z(Z'\Psi_{\bar\theta}^{-1}Z)^{-1}Z'\Psi_{\bar\theta}^{-1})y}{y'My},$$

and we replaced $T-q$ with T for simplicity without loss of generality. We first show that

$$\mathcal{R}_{1T}(\bar\theta)\Rightarrow\frac{|\int_0^1 Q^{\bar\lambda}Q^{\bar\lambda\prime}ds|}{|\int_0^1 QQ'ds|}. \tag{A.1}$$

To show (A.1), notice from (3) that there exist a $k\times(p+1)$ matrix G_{31} and a $k\times 1$ vector g_{34} such that $u_t^x=G_{31}d_t+(1-(1-1_t)L)x_t+g_{34}1_t$ where $1_t=1$ for $t=1$ and $1_t=0$ otherwise. Then, we can transform z_t using a $q\times q$ nonsingular matrix G such that

$$z_t^*=Gz_t\quad\text{where}\quad G=\begin{bmatrix} I_p & 0 & 0 & 0\\ -\Sigma_{xx}^{-1/2}\alpha_x' & \Sigma_{xx}^{-1/2} & 0 & 0\\ \Sigma_{xx}^{-1/2}G_{31} & 0 & \Sigma_{xx}^{-1/2} & \Sigma_{xx}^{-1/2}g_{34}\\ 0 & 0 & 0 & 1\end{bmatrix},$$

and $z_t^*=[z_{1t}^{*\prime},z_{2t}^{*\prime}]'$ with $z_{1t}^*=[d_t',(\Sigma_{xx}^{-1/2}x_t^0)']'$ and $z_{2t}^*=[(\Sigma_{xx}^{-1/2}u_t^x)',1_t]'$. This is also expressed as $ZG'=Z^*=[Z_1^*,Z_2^*]$ in the matrix form. Then, we have

$$\begin{aligned}\mathcal{R}_{1T}(\bar\theta) &= \left|\frac{1}{T}\Upsilon_T^{-1}GZ'\Psi_{\bar\theta}^{-1}ZG'\Upsilon_T^{-1}\right|\Big/\left|\frac{1}{T}\Upsilon_T^{-1}GZ'ZG'\Upsilon_T^{-1}\right|\\ &= \left|\frac{1}{T}\Upsilon_T^{-1}Z^{\bar\theta\prime}Z^{\bar\theta}\Upsilon_T^{-1}\right|\Big/\left|\frac{1}{T}\Upsilon_T^{-1}Z^{*\prime}Z^*\Upsilon_T^{-1}\right|,\end{aligned}$$

where $\Upsilon_T=\text{diag}\{\Upsilon_{1T},\Upsilon_{2T}\}$ with $\Upsilon_{1T}=\text{diag}\{1,T,\cdots,T^p,T^{1/2}I_k\}$ and $\Upsilon_{2T}=\text{diag}\{I_k,T^{-1/2}\}$ and $Z^{\bar\theta}=\Psi_{\bar\theta}^{-1/2}Z^*$. Note that the transpose of the t-th row of $Z^{\bar\theta}$ is expressed as

$$z_t^{\bar\theta}=\bar\theta z_{t-1}^{\bar\theta}+(1-L)z_t^*\quad\text{with}\quad z_1^{\bar\theta}=z_1^*.$$

We partition $z_t^{\bar\theta}$ into $z_{1t}^{\bar\theta}$ and $z_{2t}^{\bar\theta}$ conformably with z_{1t}^* and z_{2t}^*.

Lemma A.1 *For $0 \leq s \leq 1$, the following convergences hold jointly.*

$$\begin{array}{rrcl} (i) & \Upsilon_{1T}^{-1} z_{1[Ts]}^{*} & \Rightarrow & Q(s), \\ (ii) & \Upsilon_{1T}^{-1} z_{1[Ts]}^{\bar{\theta}} & \Rightarrow & Q^{\bar{\lambda}}(s). \end{array}$$

Proof of Lemma A.1: (i) is obtained using the functional central limit theorem (FCLT). (ii) From the definition of $z_{1t}^{\bar{\theta}}$, we can express $z_{1t}^{\bar{\theta}}$ as

$$z_{1t}^{\bar{\theta}} = z_{1t}^{*} - \frac{\bar{\lambda}}{T} \sum_{j=1}^{t-1} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-j-1} z_{1j}^{*}. \tag{A.2}$$

See also the proof of Lemma 7 in Jansson (2004). Then, according to (i) and the continuous mapping theorem (CMT), we have

$$\begin{aligned} \Upsilon_{1T}^{-1} z_{1[Ts]}^{\bar{\theta}} &\Rightarrow Q(s) - \bar{\lambda} \int_0^s e^{-\bar{\lambda}(s-r)} Q dr \\ &= \int_0^s e^{-\bar{\lambda}(s-r)} dQ(r), \end{aligned}$$

where the last equality holds by the partial integration formula.□

From Lemma A.1 (ii) and the CMT we have

$$\frac{1}{T} \Upsilon_{1T}^{-1} Z_1^{\bar{\theta}\prime} Z_1^{\bar{\theta}} \Upsilon_{1T}^{-1} \Rightarrow \int_0^1 Q^{\bar{\lambda}} Q^{\bar{\lambda}\prime} ds. \tag{A.3}$$

In exactly the same way as (A.2) $z_{2t}^{\bar{\theta}}$ is expressed as

$$\begin{aligned} z_{2t}^{\bar{\theta}} &= z_{2t}^{*} - \frac{\bar{\lambda}}{T} \sum_{j=1}^{t-1} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-j-1} z_{2j}^{*} \\ &= \left[\begin{array}{c} \Sigma_{xx}^{-1/2} \left(u_t^x - \frac{\bar{\lambda}}{T} \sum_{j=1}^{t-1} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-j-1} u_j^x \right) \\ 1_t - (1 - 1_t) \frac{\bar{\lambda}}{T} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-2} \end{array} \right]. \end{aligned} \tag{A.4}$$

Then, according to the weak law of large numbers (WLLN) and Theorem 4.1 in Hansen (1992b) we have

$$\frac{1}{T} \Upsilon_{2T}^{-1} Z_2^{\bar{\theta}\prime} Z_2^{\bar{\theta}} \Upsilon_{2T}^{-1} \xrightarrow{p} I_{k+1}, \tag{A.5}$$

where $\xrightarrow{p}$ signifies convergence in probability and

$$\frac{1}{T}\Upsilon_{1T}^{-1}Z_1^{\bar{\theta}\prime}Z_2^{\bar{\theta}}\Upsilon_{2T}^{-1}\xrightarrow{p}0. \tag{A.6}$$

Combining (A.3), (A.5), and (A.6) we obtain

$$\frac{1}{T}\Upsilon_T^{-1}Z^{\bar{\theta}\prime}Z^{\bar{\theta}}\Upsilon_T^{-1}\Rightarrow\begin{bmatrix}\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds & 0\\ 0 & I_{k+1}\end{bmatrix}. \tag{A.7}$$

In a similar manner we have $T^{-1}\Upsilon_T^{-1}Z^{*\prime}Z^*\Upsilon_T^{-1}\Rightarrow\text{diag}\{\int_0^1 Q(s)Q(s)'ds, I_{k+1}\}$. We then obtain (A.1).

Using (A.1), we can show that

$$\mathcal{R}_{1T}^{1/T}\xrightarrow{p}1 \qquad \text{and} \qquad T\left(1-\mathcal{R}_{1T}^{1/T}(\bar{\theta})\right)\Rightarrow-\log\left|\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds\right|+\log\left|\int_0^1 QQ'ds\right| \tag{A.8}$$

because $a^{1/T}\to 1$ and $T(1-a^{1/T})\to-\log a$ for a given $a>0$ as $T\to\infty$.

Next, we investigate the asymptotic behavior of $T(1-\mathcal{R}_{2T}(\bar{\theta}))$. To do this, we decompose v_t as

$$\begin{aligned} v_t &= u_t^y+(1-\theta)\sum_{j=1}^{t-1}u_j^y\\ &= \theta u_t^y+\frac{\lambda}{T}\sum_{j=1}^{t}u_j^y\\ &= v_{\theta t}^*+r_{\theta t}, \end{aligned}$$

where $v_{\theta t}^*=\theta u_t^{y\cdot x}+(\lambda/T)\sum_{j=1}^t u_j^{y\cdot x}$ with $u_t^{y\cdot x}=u_t^y-\sigma_{yx}\Sigma_{xx}^{-1}u_t^x$ and $r_{\theta t}=\theta\sigma_{yx}\Sigma_{xx}^{-1}u_t^x+(\lambda/T)\sigma_{yx}\Sigma_{xx}^{-1}\sum_{j=1}^t u_j^x$. Let v_θ^* and r_θ be the vectorized forms of $v_{\theta t}^*$ and $r_{\theta t}$. Since

$$\begin{aligned} r_\theta &= \left\{\theta U^x+(\lambda/T)\Psi_0^{1/2}U^x\right\}\Sigma_{xx}^{-1}\sigma_{xy}\\ &= \left\{\theta\left(\Psi_0^{-1/2}X-\Psi_0^{-1/2}D\alpha_x\right)+(\lambda/T)\left(X-D\alpha_x\right)\right\}\Sigma_{xx}^{-1}\sigma_{xy}, \end{aligned}$$

the conditional likelihood is independent of change in the direction of r_θ, so that we can replace y in the test statistic by v_θ^*. Then, we can observe that

$$\begin{aligned} T\left(1-R_{2T}(\bar{\theta})\right) &= T\left(1-\frac{v_\theta^{*\prime}(\Psi_{\bar{\theta}}^{-1}-\Psi_{\bar{\theta}}^{-1}Z(Z'\Psi_{\bar{\theta}}^{-1}Z)^{-1}Z'\Psi_{\bar{\theta}}^{-1})v_\theta^*}{v_\theta^{*\prime}Mv_\theta^*}\right)\\ &= \frac{\mathcal{R}_{21T}(\bar{\theta})+\mathcal{R}_{22T}(\bar{\theta})}{v_\theta^{*\prime}Mv_\theta^*/T}, \end{aligned} \tag{A.9}$$

where

$$\begin{aligned}\mathcal{R}_{21T}(\bar{\theta}) &= 2\left(v_\theta^* - \Psi_{\bar{\theta}}^{-1/2} v_\theta^*\right)' v_\theta^* - \left(v_\theta^* - \Psi_{\bar{\theta}}^{-1/2} v_\theta^*\right)' \left(v_\theta^* - \Psi_{\bar{\theta}}^{-1/2} v_\theta^*\right) \\ &= 2\left(v_\theta^* - v_\theta^{\bar{\theta}}\right)' v_\theta^* - \left(v_\theta^* - v_\theta^{\bar{\theta}}\right)' \left(v_\theta^* - v_\theta^{\bar{\theta}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_{22T}(\bar{\theta}) &= v_\theta^{*\prime} \Psi_{\bar{\theta}}^{-1} Z (Z' \Psi_{\bar{\theta}}^{-1} Z)^{-1} Z' \Psi_{\bar{\theta}}^{-1} v_\theta^* - v_\theta^{*\prime} Z (Z'Z)^{-1} Z' v_\theta^* \\ &= \left(\frac{1}{\sqrt{T}} v_\theta^{\bar{\theta}\prime} Z^{\bar{\theta}} \Upsilon_T^{-1}\right) \left(\frac{1}{T} \Upsilon_T^{-1} Z^{\bar{\theta}\prime} Z^{\bar{\theta}} \Upsilon_T^{-1}\right)^{-1} \left(\frac{1}{\sqrt{T}} \Upsilon_T^{-1} Z^{\bar{\theta}\prime} v_\theta^{\bar{\theta}}\right) \\ &\quad - \left(\frac{1}{\sqrt{T}} v_\theta^{*\prime} Z^* \Upsilon_T^{-1}\right) \left(\frac{1}{T} \Upsilon_T^{-1} Z^{*\prime} Z^* \Upsilon_T^{-1}\right)^{-1} \left(\frac{1}{\sqrt{T}} \Upsilon_T^{-1} Z^{*\prime} v_\theta^*\right)\end{aligned}$$

with $v_\theta^{\bar{\theta}} = \Psi_{\bar{\theta}}^{-1/2} v_\theta^*$. As the denominator in (A.9) is shown to converge to $\sigma_{yy\cdot x} = 1$ in probability by the WLLN under the local alternative, we concentrate on the derivation of the limiting distributions of $\mathcal{R}_{21T}(\bar{\theta})$ and $\mathcal{R}_{22T}(\bar{\theta})$ in the following.

Lemma A.2 *For $0 \le s \le 1$, the following convergences hold jointly.*

$$\begin{aligned} (i) \quad \frac{1}{\sqrt{T}} \sum_{t=1}^{[Ts]} v_{\theta t}^* &\Rightarrow V_\lambda(s), \\ (ii) \quad \frac{1}{\sqrt{T}} \sum_{t=1}^{[Ts]} v_{\theta t}^{\bar{\theta}} &\Rightarrow V_\lambda^{\bar{\lambda}}(s), \\ (iii) \quad \sqrt{T}\left(v_{\theta[Ts]}^* - v_{\theta[Ts]}^{\bar{\theta}}\right) &\Rightarrow \bar{\lambda} V_\lambda^{\bar{\lambda}}(s). \end{aligned}$$

Proof of Lemma A.2: (i) is obtained from the definition of $v_{\theta t}^*$, the FCLT, and the CMT. (ii) From the definition of $v_{\theta t}^{\bar{\theta}}$ we have

$$\sum_{j=1}^{t} v_{\theta j}^{\bar{\theta}} - \bar{\theta} \sum_{j=1}^{t-1} v_{\theta j}^{\bar{\theta}} = v_{\theta t}^*.$$

Then, in exactly the same way as (A.2), it is seen that

$$\sum_{j=1}^{t} v_{\theta j}^{\bar{\theta}} = \sum_{j=1}^{t} v_{\theta j}^* - \frac{\bar{\lambda}}{T} \sum_{j=1}^{t-1} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-j-1} \left(\sum_{i=1}^{j} v_{\theta i}^*\right).$$

Using (i), the CMT, and the partial integration formula, we obtain (ii).

(iii) From the definition of $v_{\theta t}^{\bar{\theta}}$ we have

$$v_{\theta t}^{*} - v_{\theta t}^{\bar{\theta}} = (1-\bar{\theta}) \sum_{j=1}^{t-1} v_{\theta j}^{\bar{\theta}}.$$

Then, (iii) is obtained using (ii).□

Using Lemma A.2, the CMT, and Theorem 4.1 in Hansen (1992b) we have

$$\mathcal{R}_{21T}(\bar{\theta}) \Rightarrow 2\bar{\lambda} \int_0^1 V_{\lambda}^{\bar{\lambda}} dV_{\lambda} - \bar{\lambda}^2 \int_0^1 (V_{\lambda}^{\bar{\lambda}})^2 ds. \tag{A.10}$$

For $\mathcal{R}_{22T}(\bar{\theta})$, we can see that

$$\frac{1}{\sqrt{T}} \Upsilon_{1T}^{-1} Z_1^{\bar{\theta}\prime} v_{\theta}^{\bar{\theta}} \Rightarrow \int_0^1 Q^{\bar{\lambda}} dV_{\lambda}^{\bar{\lambda}}, \tag{A.11}$$

using Lemmas A.1, A.2, and Theorem 4.1 in Hansen (1992b), while

$$\frac{1}{\sqrt{T}} \Upsilon_{2T}^{-1} Z_2^{\bar{\theta}\prime} v_{\theta}^{\bar{\theta}} = \left[\begin{array}{c} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\Sigma_{xx}^{-1/2} u_t^{x\bar{\theta}}) v_{\theta t}^{\bar{\theta}} \\ \sum_{t=1}^{T} 1_t^{\bar{\theta}} v_{\theta t}^{\bar{\theta}} \end{array} \right],$$

where $u_t^{x\bar{\theta}}$ is the transpose of the t-th row of $\Psi_{\bar{\theta}}^{-1/2} U^x$. In exactly the same way as (A.2) we have

$$v_{\theta t}^{\bar{\theta}} = v_{\theta t}^{*} - \frac{\bar{\lambda}}{T} \sum_{j=1}^{t-1} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-j-1} v_{\theta j}^{*}. \tag{A.12}$$

Then, from (A.4) and (A.12) we can see that

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\Sigma_{xx}^{-1/2} u_t^{x\bar{\theta}}) v_{\theta t}^{\bar{\theta}} &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\Sigma_{xx}^{-1/2}) u_t^{x} u_t^{y\cdot x} + O_p(T^{-1/2}) \\ &\Rightarrow N(1), \end{aligned} \tag{A.13}$$

by the FCLT, where $N(s)$ is a k dimensional standard Brownian motion that is independent of $W(s)$ and $V(s)$.

On the other hand, using (A.4) we have

$$\begin{aligned} \sum_{t=1}^{T} 1_t^{\bar{\theta}} v_{\theta t}^{\bar{\theta}} &= v_{\theta 1}^{\bar{\theta}} - \frac{\bar{\lambda}}{T} \sum_{t=2}^{T} \left(1 - \frac{\bar{\lambda}}{T}\right)^{t-2} v_{\theta t}^{\bar{\theta}} \\ &\xrightarrow{p} v_{\theta 1}^{\bar{\theta}} = v_{\theta 1}^{*}. \end{aligned} \tag{A.14}$$

Then, combining (A.7), (A.11), (A.13), and (A.14) we have

$$\left(\frac{1}{\sqrt{T}}v_\theta^{\bar{\theta}\prime}Z^{\bar{\theta}}\Upsilon_T^{-1}\right)\left(\frac{1}{T}\Upsilon_T^{-1}Z^{\bar{\theta}\prime}Z^{\bar{\theta}}\Upsilon_T^{-1}\right)^{-1}\left(\frac{1}{\sqrt{T}}\Upsilon_T^{-1}Z^{\bar{\theta}\prime}v_\theta^{\bar{\theta}}\right)$$
$$\Rightarrow \left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)'\left(\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds\right)^{-1}\left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)+N(1)^2+v_{\theta 1}^{*2}.$$

In exactly the same way we have

$$\left(\frac{1}{\sqrt{T}}v_\theta^{*\prime}Z^*\Upsilon_T^{-1}\right)\left(\frac{1}{T}\Upsilon_T^{-1}Z^{*\prime}Z^*\Upsilon_T^{-1}\right)^{-1}\left(\frac{1}{\sqrt{T}}\Upsilon_T^{-1}Z^{*\prime}v_\theta^*\right)$$
$$\Rightarrow \left(\int_0^1 QdV_\lambda\right)'\left(\int_0^1 QQ'ds\right)^{-1}\left(\int_0^1 QdV_\lambda\right)+N(1)^2+v_{\theta 1}^{*2}.$$

Then, we can see that

$$\begin{aligned}\mathcal{R}_{22T}(\bar{\theta}) \;\Rightarrow\; & \left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)'\left(\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds\right)^{-1}\left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)\\ & -\left(\int_0^1 QdV_\lambda\right)'\left(\int_0^1 QQ'ds\right)^{-1}\left(\int_0^1 QdV_\lambda\right). \qquad \text{(A.15)}\end{aligned}$$

By combining (A.10) and (A.15), we have

$$\begin{aligned}T\left(1-\mathcal{R}_{2T}(\bar{\theta})\right) \;\Rightarrow\; & 2\bar{\lambda}\int_0^1 V_\lambda^{\bar{\lambda}}dV_\lambda-\bar{\lambda}^2\int_0^1(V_\lambda^{\bar{\lambda}})^2ds\\ & +\left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)'\left(\int_0^1 Q^{\bar{\lambda}}Q^{\bar{\lambda}\prime}ds\right)^{-1}\left(\int_0^1 Q^{\bar{\lambda}}dV_\lambda^{\bar{\lambda}}\right)\\ & -\left(\int_0^1 QdV_\lambda\right)'\left(\int_0^1 QQ'ds\right)^{-1}\left(\int_0^1 QdV_\lambda\right). \qquad \text{(A.16)}\end{aligned}$$

The required distribution is obtained from (A.8) and (A.16).□

Proof of Corollary 1

We first derive the LBIU test statistic (6). Note that

$$\begin{aligned}\frac{d\Psi_\theta}{d\theta} &= (I_T-\Psi_0^{1/2})\Psi_\theta^{1/2\prime}+\Psi_\theta^{1/2}(I_T-\Psi_0^{1/2\prime})\\ \frac{d^2\Psi_\theta}{d\theta^2} &= 2(I_T-\Psi_0^{1/2})(I_T-\Psi_0^{1/2\prime})=2(\Psi_0-i_Ti_T'),\end{aligned}$$

where $i_T=[1,\cdots,1]$ is a $T\times 1$ vector and we used the relation $I_T-\Psi_0^{1/2}-\Psi_0^{1/2\prime}=-i_Ti_T'$. Then, as $\Psi_1^{1/2}=I_T$, $H'H=I_{T-q}$, and $H'i_T=0$, we have

$$\left.\frac{d(H'\Psi_\theta H)}{d\theta}\right|_{\theta=1} = H'(I_T-i_Ti_T')H=I_{T-q} \qquad \text{(A.17)}$$

$$\left.\frac{d^2(H'\Psi_\theta H)}{d\theta^2}\right|_{\theta=1} = 2H'\Psi_0 H. \quad \text{(A.18)}$$

From (A.17) and the standard matrix differential calculus we can show that

$$\begin{aligned}\left.\frac{d\log f(\eta|X;\theta)}{d\theta}\right|_{\theta=1} &= -\frac{1}{2}\mathrm{tr}\left\{(H'\Psi_1 H)^{-1}\left.\frac{d(H'\Psi_\theta H)}{d\theta}\right|_{\theta=1}\right\} \\ &\quad +\frac{T-q}{2}\frac{\eta'(H'\Psi_1 H)^{-1}\left.\frac{d(H'\Psi_\theta H)}{d\theta}\right|_{\theta=1}(H'\Psi_1 H)^{-1}\eta}{\eta'(H'\Psi_1 H)^{-1}\eta} = 0. \quad \text{(A.19)}\end{aligned}$$

For the second derivative, note that

$$\begin{aligned}\left.\frac{d^2\log|H'\Psi_\theta H|}{d\theta^2}\right|_{\theta=1} &= \mathrm{tr}\left\{-I_{T-q}+2H'\Psi_0 H\right\} \\ &= -(T-q)+2\mathrm{tr}\left\{M\Psi_0\right\} \\ &= (T^2+q)-2\mathrm{tr}\left\{(Z'Z)^{-1}Z'\Psi_0 Z\right\}, \quad \text{(A.20)}\end{aligned}$$

which is obtained using (A.17), (A.18), and $HH' = M$, and

$$\begin{aligned}\left.\frac{d^2\log\{\eta'(H'\Psi_\theta H)^{-1}\eta\}}{d\theta^2}\right|_{\theta=1} &= 1-2\eta' H\Psi_0 H\eta \\ &= 1-2\frac{y'M\Psi_0 My}{y'My}. \quad \text{(A.21)}\end{aligned}$$

Then, from (A.20) and (A.21) we have

$$\left.\frac{d^2\log f(\eta|X;\theta)}{d\theta^2}\right|_{\theta=1} = const+\frac{y'M\Psi_0 My}{y'My/(T-q)}+\mathrm{tr}\left\{(Z'Z)^{-1}Z'\Psi_0 Z\right\},$$

so that we obtain (6).

Next, we derive the limiting distribution of the LBIU test statistic. For the same reason as in the proof of Theorem 1 we can replace y in the test statistic by v_θ^* and then we have $\Psi_0^{1/2\prime}My = \Psi_0^{1/2\prime}Mv_\theta^*$. Noting that $\Psi_0^{1/2\prime} = i_T i_T' - \bar{\Psi}_0^{1/2}$ where $\bar{\Psi}_0^{1/2}$ is a $T\times T$ lower triangular matrix with diagonal elements 0 and the other lower elements 1, we have

$$\begin{aligned}\frac{1}{\sqrt{T}}\Psi_0^{1/2\prime}My &= \frac{1}{\sqrt{T}}(i_T i_T' - \bar{\Psi}_0^{1/2})Mv_\theta^* \\ &= -\frac{1}{\sqrt{T}}\left\{\bar{\Psi}_0^{1/2}v_\theta^* - \bar{\Psi}_0^{1/2}Z\left(Z'Z\right)^{-1}Z'v_\theta^*\right\} \\ &= -\left\{\frac{1}{\sqrt{T}}\bar{\Psi}_0^{1/2}v_\theta^* - \frac{1}{T}\bar{\Psi}_0^{1/2}Z^*\Upsilon_T^{-1}\left(\frac{1}{T}\Upsilon_T^{-1}Z^{*\prime}Z^*\Upsilon_T^{-1}\right)^{-1}\frac{1}{\sqrt{T}}\Upsilon_T^{-1}Z^{*\prime}v_\theta^*\right\},\end{aligned}$$

where the second equality holds because $i_T'M = 0$. As the t-th rows of $\bar{\Psi}_0^{1/2}v_\theta^*$ and $\bar{\Psi}_0^{1/2}Z^*$ are $\sum_{j=1}^{t-1} v_{\theta j}^*$ and $\sum_{j=1}^{t-1} z_t^{*\prime}$, we have, using Lemmas A.1, A.2, the FCLT, and the CMT,

$$\frac{v_\theta^{*\prime} M\Psi_0 M v_\theta^*/T^2}{v_\theta^{*\prime} M v_\theta^*/(T-q)} \Rightarrow \int_0^1 \left\{ V_\lambda(s) - \int_0^s Q' dr \left(\int_0^1 QQ' dr \right)^{-1} \int_0^1 Q dV_\lambda \right\}^2 ds. \qquad \text{(A.22)}$$

Similarly, we can also see that

$$\frac{1}{T^2}\mathrm{tr}\left\{(Z'Z)^{-1}(Z'\Psi_0 Z)\right\} = \mathrm{tr}\left\{ \left(\frac{1}{T}\Upsilon_T^{-1}Z^{*\prime}Z^*\Upsilon_T^{-1}\right)\left(\frac{1}{T^3}\Upsilon_T^{-1}Z^{*\prime}\Psi_0 Z^*\Upsilon_T^{-1}\right)\right\}.$$

Noting that the transpose of the t-th row of $\Psi_0^{1/2\prime}Z^*$ is given by $\sum_{j=t}^{T} z_j^*$, we have, from Lemma A.1 and the CMT,

$$\frac{1}{T^2}\mathrm{tr}\left\{(Z'Z)^{-1}(Z'\Psi_0 Z)\right\} \Rightarrow \mathrm{tr}\left\{ \left(\int_0^1 QQ' dr\right)^{-1} \int_0^1 \left(\int_s^1 Q dr\right)\left(\int_s^1 Q' dr\right) ds \right\}. \qquad \text{(A.23)}$$

From (A.22) and (A.23) we obtain the result.□

Proof of Theorem 2

The proof proceeds in the same way as the proof of theorem 1 and the proof of Theorem 2 in Jansson (2005); therefore we provide only an outline. First, note that we can obtain the same results in Lemma A.1 by replacing $\Sigma_{xx}^{-1/2}$ in G with $\Omega_{xx}^{-1/2}$. We can also see that, as in the proof of Theorem 1, y_t in the test statistics can be replaced by $v_{\theta t}^*$ where under general assumptions $u_t^{y\cdot x}$ is defined as $u_t^{y\cdot x} = u_t^y - \omega_{yx}\Omega_{xx}^{-1}u_t^x$, so that the limiting distributions in Lemma A.2 should be multiplied by $\omega_{11}^{*1/2}$. Then, applying Lemma 1 in Sims, Stock, and Watson (1990) and Lemma 7 in Jansson (2004), we can see that

$$2(v_\theta^* - v_\theta^{\bar{\theta}})'v_\theta^* \Rightarrow 2\bar{\lambda}\omega_{11}^* \int_0^1 V_\lambda^{\bar{\lambda}} dV_\lambda + 2\bar{\lambda}\pi_{11}^*,$$

and then

$$\mathcal{R}_{2T}(\bar{\theta}) \Rightarrow \omega_{11}^* \left(2\bar{\lambda}\int_0^1 V_\lambda^{\bar{\lambda}} dV_\lambda - \bar{\lambda}^2 \int_0^1 (V_\lambda^{\bar{\lambda}})^2 ds \right) + 2\bar{\lambda}\pi_{11}^*.$$

Similarly, we can see that

$$\begin{aligned} \frac{1}{\sqrt{T}}\Upsilon_{1T}^{-1}Z_1^{+\bar{\theta}\prime}v_\theta^{\bar{\theta}} &\Rightarrow \omega_{11}^{*1/2}\int_0^1 Q^{\bar{\lambda}} dV_\lambda^{\bar{\lambda}} \\ \frac{1}{\sqrt{T}}\Upsilon_{1T}^{-1}Z_1^{+*\prime}v_\theta^* &\Rightarrow \omega_{11}^{*1/2}\int_0^1 Q dV_\lambda. \end{aligned}$$

By combining these results we obtain the theorem.□

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Table 1. Percentiles of the Limiting Distribution of $\mathcal{L}_T$

(a) constant mean	$k=1$	2	3	4	5	6
90%	0.6095	0.5739	0.5512	0.5376	0.5303	0.5246
95%	0.6803	0.6235	0.5823	0.5609	0.5483	0.5387
97.5%	0.7632	0.6795	0.6182	0.5874	0.5706	0.5538
99%	0.8940	0.7667	0.6825	0.6320	0.6037	0.5750
(b) linear trend	$k=1$	2	3	4	5	6
90%	0.5419	0.5348	0.5277	0.5228	0.5196	0.5165
95%	0.5651	0.5527	0.5425	0.5352	0.5297	0.5255
97.5%	0.5894	0.5716	0.5594	0.5490	0.5410	0.5352
99%	0.6223	0.5997	0.5831	0.5674	0.5570	0.5475

Table 2. Size and Power (conatant mean, $T = 200$)

Case 1		$\mathcal{R}_T$			$\mathcal{P}_T$			$\mathcal{L}_T$			$\mathcal{S}_T$		
		ρ			ρ			ρ			ρ		
a	θ	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8
	1	0.055	0.055	0.052	0.052	0.055	0.063	0.059	0.055	0.055	0.049	0.045	0.040
	0.975	0.223	0.214	0.219	0.223	0.199	0.177	0.237	0.222	0.226	0.208	0.195	0.192
0	0.95	0.527	0.530	0.524	0.528	0.510	0.434	0.475	0.467	0.470	0.435	0.423	0.410
	0.925	0.738	0.735	0.737	0.736	0.716	0.640	0.647	0.640	0.637	0.596	0.591	0.579
	0.9	0.854	0.855	0.851	0.858	0.849	0.787	0.752	0.751	0.745	0.712	0.703	0.693
	1	0.049	0.049	0.045	0.039	0.040	0.057	0.054	0.054	0.055	0.043	0.049	0.065
	0.975	0.197	0.184	0.188	0.178	0.161	0.183	0.216	0.206	0.211	0.182	0.184	0.225
0.5	0.95	0.457	0.452	0.452	0.436	0.423	0.432	0.426	0.420	0.416	0.371	0.380	0.420
	0.925	0.641	0.642	0.637	0.628	0.614	0.618	0.566	0.566	0.559	0.495	0.517	0.553
	0.9	0.739	0.734	0.718	0.735	0.729	0.729	0.643	0.643	0.634	0.566	0.587	0.637
	1	0.027	0.027	0.030	0.019	0.038	0.104	0.047	0.052	0.053	0.038	0.061	0.115
	0.975	0.085	0.085	0.089	0.063	0.094	0.199	0.174	0.161	0.172	0.128	0.158	0.244
0.8	0.95	0.130	0.137	0.143	0.106	0.164	0.322	0.287	0.277	0.275	0.201	0.246	0.364
	0.925	0.117	0.121	0.140	0.103	0.159	0.349	0.301	0.288	0.281	0.188	0.251	0.394
	0.9	0.087	0.098	0.139	0.090	0.132	0.307	0.264	0.247	0.239	0.145	0.202	0.366

Table 2 (continued)

Case 2		$\mathcal{R}_{\mathcal{T}}$			$\mathcal{P}_{\mathcal{T}}$			$\mathcal{L}_{\mathcal{T}}$			$\mathcal{S}_{\mathcal{T}}$		
		ρ			ρ			ρ			ρ		
a	θ	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8
	1	0.053	0.055	0.051	0.061	0.335	0.908	0.059	0.056	0.055	0.050	0.096	0.331
	0.975	0.224	0.215	0.222	0.231	0.478	0.914	0.236	0.222	0.227	0.210	0.241	0.407
0	0.95	0.529	0.530	0.527	0.535	0.689	0.940	0.475	0.468	0.471	0.434	0.453	0.557
	0.925	0.739	0.738	0.740	0.746	0.822	0.958	0.648	0.642	0.638	0.599	0.610	0.673
	0.9	0.856	0.858	0.854	0.863	0.904	0.970	0.753	0.753	0.746	0.712	0.719	0.754
	1	0.044	0.044	0.043	0.062	0.464	0.983	0.054	0.055	0.054	0.050	0.128	0.494
	0.975	0.193	0.181	0.190	0.209	0.549	0.980	0.217	0.206	0.213	0.188	0.245	0.528
0.5	0.95	0.460	0.453	0.458	0.475	0.694	0.977	0.425	0.422	0.418	0.376	0.418	0.612
	0.925	0.648	0.653	0.651	0.655	0.787	0.972	0.567	0.564	0.559	0.502	0.537	0.675
	0.9	0.756	0.759	0.748	0.760	0.840	0.970	0.649	0.649	0.643	0.569	0.608	0.713
	1	0.025	0.026	0.031	0.130	0.646	0.979	0.051	0.053	0.058	0.060	0.224	0.633
	0.975	0.103	0.098	0.105	0.210	0.641	0.970	0.173	0.169	0.180	0.154	0.287	0.617
0.8	0.95	0.203	0.200	0.199	0.309	0.618	0.941	0.297	0.295	0.289	0.227	0.347	0.602
	0.925	0.217	0.226	0.217	0.350	0.568	0.884	0.335	0.317	0.315	0.238	0.335	0.563
	0.9	0.199	0.205	0.198	0.373	0.529	0.811	0.302	0.285	0.279	0.220	0.294	0.510

Table 3. Size and Power (linear trend, $T = 200$)

Case 1		$\mathcal{R}_T$			$\mathcal{P}_T$			$\mathcal{L}_T$			$\mathcal{S}_T$		
		ρ			ρ			ρ			ρ		
a	θ	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8
	1	0.054	0.052	0.051	0.052	0.058	0.055	0.060	0.065	0.062	0.051	0.051	0.032
	0.975	0.110	0.111	0.107	0.113	0.108	0.080	0.135	0.131	0.136	0.119	0.104	0.084
0	0.95	0.307	0.304	0.306	0.305	0.269	0.179	0.314	0.312	0.315	0.286	0.272	0.235
	0.925	0.529	0.524	0.530	0.526	0.473	0.344	0.508	0.508	0.504	0.467	0.460	0.418
	0.9	0.705	0.709	0.713	0.709	0.665	0.518	0.662	0.654	0.651	0.621	0.611	0.575
	1	0.046	0.051	0.049	0.040	0.039	0.043	0.057	0.059	0.056	0.047	0.052	0.065
	0.975	0.097	0.097	0.095	0.086	0.076	0.073	0.119	0.114	0.122	0.099	0.102	0.124
0.5	0.95	0.254	0.250	0.243	0.234	0.209	0.195	0.274	0.268	0.270	0.234	0.236	0.277
	0.925	0.421	0.424	0.413	0.402	0.376	0.342	0.424	0.427	0.422	0.369	0.387	0.425
	0.9	0.558	0.556	0.551	0.542	0.518	0.477	0.549	0.540	0.541	0.483	0.498	0.540
	1	0.023	0.026	0.023	0.014	0.028	0.067	0.047	0.049	0.049	0.032	0.058	0.122
	0.975	0.037	0.036	0.039	0.022	0.037	0.087	0.091	0.087	0.093	0.063	0.095	0.177
0.8	0.95	0.063	0.062	0.078	0.041	0.064	0.144	0.161	0.156	0.160	0.111	0.162	0.276
	0.925	0.071	0.070	0.096	0.045	0.071	0.178	0.199	0.192	0.189	0.125	0.189	0.340
	0.9	0.056	0.065	0.104	0.041	0.060	0.172	0.189	0.180	0.174	0.110	0.177	0.356

Table 3 (continued)

Case 2		$\mathcal{R}_{\mathcal{T}}$			$\mathcal{P}_{\mathcal{T}}$			$\mathcal{L}_{\mathcal{T}}$			$\mathcal{S}_{\mathcal{T}}$		
		ρ			ρ			ρ			ρ		
a	θ	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8	0	0.5	0.8
	1	0.053	0.052	0.051	0.069	0.392	0.909	0.061	0.065	0.061	0.056	0.120	0.474
	0.975	0.112	0.110	0.108	0.130	0.428	0.909	0.134	0.132	0.136	0.124	0.186	0.492
0	0.95	0.306	0.307	0.307	0.325	0.557	0.914	0.314	0.313	0.317	0.289	0.335	0.562
	0.925	0.530	0.527	0.531	0.539	0.700	0.927	0.507	0.509	0.507	0.473	0.498	0.649
	0.9	0.707	0.711	0.717	0.720	0.813	0.942	0.664	0.657	0.653	0.626	0.644	0.727
	1	0.041	0.046	0.044	0.082	0.551	0.989	0.058	0.060	0.056	0.058	0.168	0.702
	0.975	0.091	0.093	0.095	0.131	0.570	0.987	0.119	0.118	0.124	0.114	0.219	0.685
0.5	0.95	0.244	0.241	0.242	0.285	0.628	0.985	0.272	0.269	0.274	0.248	0.335	0.700
	0.925	0.422	0.417	0.419	0.457	0.697	0.978	0.428	0.426	0.422	0.382	0.459	0.715
	0.9	0.563	0.561	0.558	0.588	0.750	0.967	0.549	0.541	0.539	0.494	0.546	0.740
	1	0.020	0.020	0.029	0.164	0.717	0.985	0.051	0.055	0.063	0.074	0.338	0.818
	0.975	0.035	0.035	0.041	0.202	0.697	0.981	0.094	0.100	0.103	0.113	0.342	0.797
0.8	0.95	0.067	0.069	0.069	0.261	0.663	0.965	0.174	0.163	0.174	0.172	0.373	0.766
	0.925	0.082	0.086	0.087	0.298	0.607	0.933	0.218	0.209	0.207	0.207	0.369	0.726
	0.9	0.081	0.084	0.088	0.330	0.546	0.876	0.216	0.206	0.196	0.206	0.343	0.670

Figure 1. 5% Level tests, m=1 constant mean

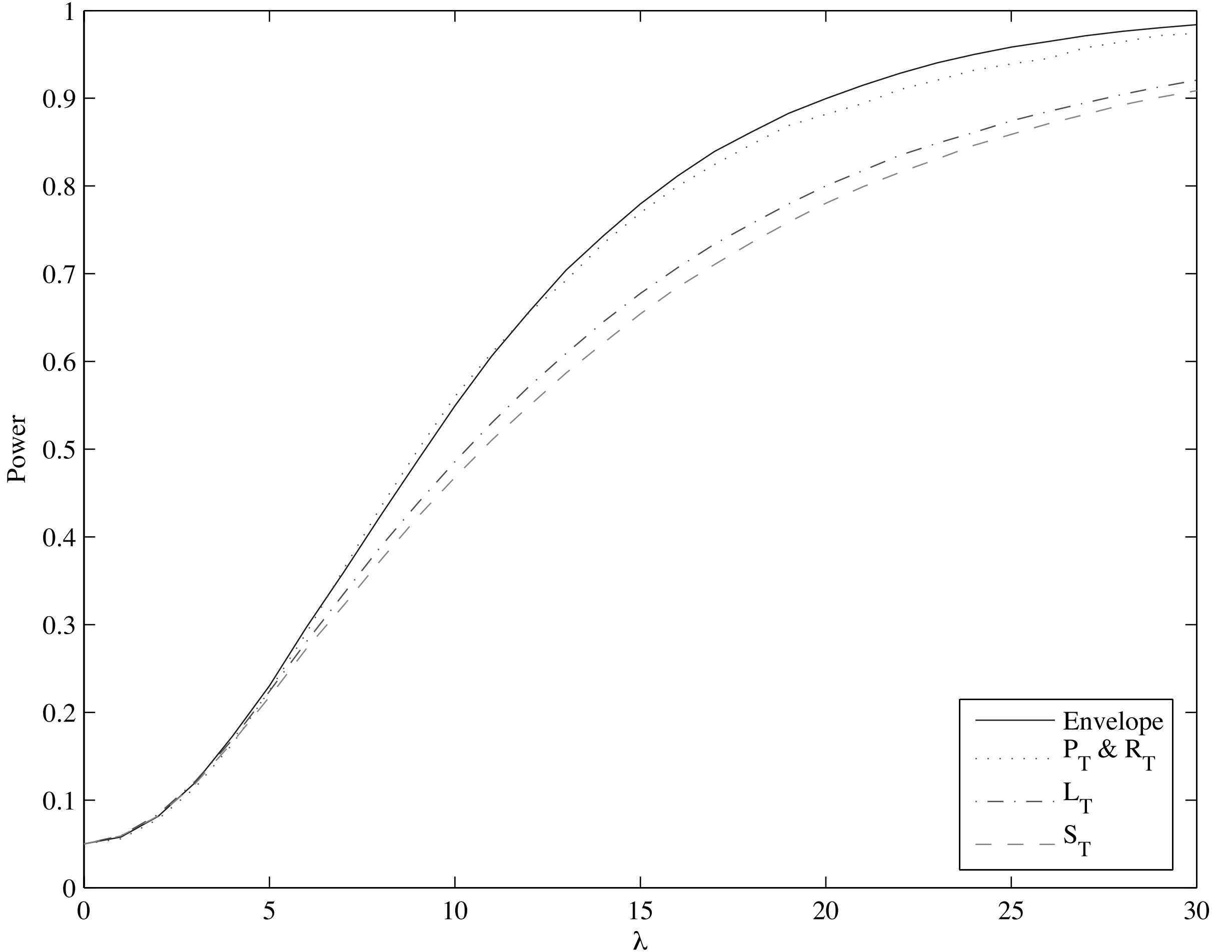

Figure 2. 5% Level tests, m=1 linear trend