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Coalition-proofness and Dominance Relations

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Abstract

This paper examines the relationship between coalition-proof Nash equilibria based on different dominance relations. Konishi, Le Breton, and Weber (1999) pointed out that the set of coalition-proof Nash equilibria under weak domination does not necessarily coincide with that under strict domination. We show that, if a game satisfies the conditions of anonymity, monotone externality, and strategic substitutability, then the set of coalition-proof Nash equilibria under strict domination contains that under weak domination. The above three conditions are met by standard Cournot oligopoly games and participation games in a mechanism producing a public good.

Key Words: Coalition-proof Nash equilibrium, Strict domination, Weak domination, Anonymity, Monotone externality, Strategic substitutability.

JEL Classification Numbers: C72.

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1 Introduction

This paper examines the relationship between coalition-proof Nash equilibria based on different dominance relations. The notion of a coalition-proof Nash equilibrium was introduced by Bernheim, Peleg, and Whinston (1987) and is known as a refinement of Nash equilibria based on the stability against credible coalitional deviations. However, there are two ways for a coalition to improve payoffs to its members. We consider the following two dominance relations:

- (i) Strategy profile x *strictly dominates* strategy profile y if there exists a coalition S such that all members of S can be better off by switching y to x , taking the strategies of the players outside S as given.
- (ii) Strategy profile x *weakly dominates* strategy profile y if there exists a coalition S such that all members are not worse off and at least one member of the coalition is better off by deviating from y to x , holding the strategies of the others fixed.

Under the notion of strict domination, all of the deviating players are better off, while, under that of weak domination, all members of a coalition are at least as well off, and at least one of them is better off. Thus, the set of equilibria under weak domination may be a subset of that under strict domination. This indeed applies to the strong Nash equilibrium introduced by Aumann (1959) and the core.

However, the set of coalition-proof Nash equilibria under strict domination does not contain that under weak domination. Konishi, Le Breton, and Weber (1999) provided an example in which the set of coalition-proof Nash equilibria under weak domination and that under strict domination are both non-empty and their intersection is empty.¹ They also showed that, in the class of common agency games, any coalition-proof Nash

¹Another solution concept that is sensitive to the notions of coalitional deviation is von Neumann and Morgenstern stable set. See Greenberg (1992).

equilibria under weak domination is that under strict domination.

In this study, we consider the class of games with n players in which the strategy space of each player is a subset of the real line. This class includes standard Cournot oligopoly games as well as voluntary participation games in a mechanism producing a public good.² We show that, if a game satisfies the conditions of *anonymity*, *monotone externality*, and *strategic substitutability*, then the set of coalition-proof Nash equilibria under weak domination is included in that under strict domination. Anonymity states that the payoffs of every player depend on his strategy and on the sum of other players' strategies. Monotone externality requires that a switch in a player's strategies change the payoffs to all the other players in the same direction. Strategic substitutability means that the incentive to every player to reduce his strategy gets higher as the sum of the other players' strategies increases. Yi (1999) introduced all of the conditions and proved that, under the three conditions, the set of coalition-proof Nash equilibria with strict domination coincides with the weakly Pareto efficient frontier of the set of Nash equilibria. However, we do not apply Yi's (1999) result to the proof of our result. Furthermore, our result cannot be derived from Yi's (1999) result. The inclusion relation between the sets of coalition-proof Nash equilibria under the two different dominance relations holds for the games that have interested economists, such as standard Cournot oligopoly games and voluntary participation games in a mechanism producing public goods.

2 The Model

We consider a strategic game $G = [N, (X_i)_{i \in N}, (u_i)_{i \in N}]$, where N is a finite set of players, X_i is the set of pure strategies of player i that is a subset of real numbers, and $u_i: \prod_{j \in N} X_j \rightarrow \mathbb{R}$ is the payoff function of player i . In this paper, we focus solely on pure strategy equilibria. Before the coalition-proof Nash equilibria are defined, some

²Note that common agency games do not belong to this class.

notations will be introduced. For any coalition $S \subseteq N$, $x_S \in \prod_{j \in S} X_j$ designates a strategy profile of S . For any $x_N \in \prod_{j \in N} X_j$, x_N is denoted by x . For any set of players S , $-S$ represents the complement of S .

The notions of coalition-proof Nash equilibria are defined under strict domination and weak domination. First, *restricted games* are introduced. For any coalition $S \subseteq N$ and any strategy profile of the complement of S , \bar{x}_{-S} , denote the game restricted by \bar{x}_{-S} by $G|\bar{x}_{-S}$ in which S is the set of players, $\prod_{j \in S} X_j$ is the set of pure strategy profiles, and $u_i(\cdot, \bar{x}_{-S}): \prod_{j \in S} X_j \rightarrow \mathbb{R}$ is player i 's payoff function. Now, the definitions of coalition-proof Nash equilibria are provided under the different dominance relations.

Definition 1 A *coalition-proof Nash equilibrium under weak domination* is defined inductively with respect to the number of players n in the game:

- (i) If $n = 1$, then $x_1^* \in X_1$ is a coalition-proof Nash equilibrium under weak domination if and only if $u_1(x_1^*) \geq u_1(\tilde{x}_1)$ for any $\tilde{x}_1 \in X_1$.
- (ii) Let $n > 1$. Assume that the coalition-proof Nash equilibria under weak domination have been defined for games with fewer than n players.
 - (a) For any game G with n players, $x^* \in \prod_{j \in N} X_j$ is a *self-enforcing strategy profile under weak domination* if, for all $S \subsetneq N$, $x_S^* \in \prod_{j \in S} X_j$ is a coalition-proof Nash equilibrium under weak domination of the reduced game $G|x_{-S}^*$.
 - (b) Profile x^* is a coalition-proof Nash equilibrium under weak domination of G if it is a self-enforcing strategy profile under weak domination and there is no other self-enforcing strategy profile under weak domination $\hat{x} \in \prod_{j \in N} X_j$ such that $u_i(\hat{x}) \geq u_i(x^*)$ for all $i \in N$ and $u_i(\hat{x}) > u_i(x^*)$ for some $i \in N$.

Definition 2 A *coalition-proof Nash equilibrium under strict domination* is defined inductively with respect to the number of players n in the game:

- (i) If $n = 1$, then $x_1^* \in X_1$ is a coalition-proof Nash equilibrium under strict domination if and only if $u_1(x_1^*) \geq u_1(\tilde{x}_1)$ for any $\tilde{x}_1 \in X_1$.
- (ii) Let $n > 1$, and assume that a coalition-proof Nash equilibrium under strict domination is defined for games with fewer than n players.
 - (a) For any game G with n players, $x^* \in \prod_{j \in N} X_j$ is a *self-enforcing strategy profile under strict domination* if, for all $S \subsetneq N$, $x_S^* \in \prod_{j \in S} X_j$ is a coalition-proof Nash equilibrium under strict domination of the reduced game $G|_{x_{-S}^*}$.
 - (b) Profile x^* is a coalition-proof Nash equilibrium under strict domination if it is a self-enforcing strategy profile under strict domination and if there does not exist another self-enforcing strategy profile under strict domination $\hat{x} \in \prod_{j \in N} X_j$ of G such that $u_i(\hat{x}) > u_i(x^*)$ for all $i \in N$.

The main difference between the two notions of coalition-proof Nash equilibria lies in the notions of coalitional deviations. The idea behind the coalition-proof Nash equilibria under weak domination is that a coalition deviates if all members in the coalition are at least as well off and at least one of them is better off. On the other hand, under strict domination, a coalition deviates only if every member of the coalition is better off. Note that, under either of the dominance relations, coalition-proof Nash equilibria and self-enforcing strategy profiles are Nash equilibria. In games with two players, the set of self-enforcing strategy profiles coincides with that of Nash equilibria, since coalitions consist of only two or fewer players. Hence, the set of coalition-proof Nash equilibria under weak domination coincides with the (strictly) Pareto efficient frontier of the set of Nash equilibria, and so does the set of coalition-proof Nash equilibria under strict domination with the weakly Pareto efficient frontier of that of Nash equilibria. Therefore, the set of coalition-proof Nash equilibria under weak domination is a subset of that under strict domination in two-player games. However, the inclusion relation between the sets

of coalition-proof Nash equilibria under the two different dominance relations does not necessarily hold in games with more than two players. Konishi, Le Breton, and Weber (1999) presented an example in which the sets of coalition-proof Nash equilibria under strict and weak dominations are disjoint.

Example 1 (Konishi, Le Breton, and Weber, 1999) Consider the game with three players depicted in Table 1, in which agent 1 chooses rows, agent 2 chooses columns, and agent 3 chooses matrices. The first entry in each box is agent 1's payoff, the second is agent 2's, and the third is agent 3's. There exist two pure strategy Nash equilibria, (a_1, b_1, c_2) and (a_2, b_2, c_1) , where the former is a coalition-proof Nash equilibrium under weak domination but not that under strict domination, and the latter is a coalition-proof Nash equilibrium under strict domination but not that under weak domination. In this example, the set of coalition-proof Nash equilibria under weak domination is not a subset of that under strict domination.

3 Result

In this section, we establish sufficient conditions under which the set of coalition-proof Nash equilibria under weak domination is a subset of that of a coalition-proof Nash equilibrium under strict domination.

The first condition is that of *anonymity*.

Anonymity. For all $i \in N$, all $x_i \in X_i$, and all $x_{-i}, \hat{x}_{-i} \in \prod_{j \neq i} X_j$, if $\sum_{j \neq i} x_j = \sum_{j \neq i} \hat{x}_j$, then $u_i(x_i, x_{-i}) = u_i(x_i, \hat{x}_{-i})$.

The anonymity condition means that the payoff function of every player depends on his strategy and on the aggregate strategy of all other players.

The next condition is that of *monotone externality*. The condition states that the

payoffs to every player are either non-increasing or non-decreasing with respect to the sum of strategies of the other players.

Monotone externality. For all $i \in N$, all $x_i \in X_i$, and all x_{-i} and $\hat{x}_{-i} \in \prod_{j \neq i} X_j$, if $\sum_{j \neq i} x_j > \sum_{j \neq i} \hat{x}_j$, then either $u_i(x_i, x_{-i}) \geq u_i(x_i, \hat{x}_{-i})$ or $u_i(x_i, x_{-i}) \leq u_i(x_i, \hat{x}_{-i})$ holds. If the former holds, the condition means *positive* externalities, and it represents *negative* externalities if the latter is satisfied.

The third condition is that of *strategic substitutability*. Under this condition, the incentive of every player to reduce his strategy gets higher as the sum of the other players' strategies increases.

Strategic substitutability. For all $i \in N$, all $x_i, \hat{x}_i \in X_i$, and all $x_{-i}, \hat{x}_{-i} \in \prod_{j \neq i} X_j$, if $x_i > \hat{x}_i$ and $\sum_{j \neq i} x_j > \sum_{j \neq i} \hat{x}_j$, then $u_i(x_i, x_{-i}) - u_i(\hat{x}_i, x_{-i}) < u_i(x_i, \hat{x}_{-i}) - u_i(\hat{x}_i, \hat{x}_{-i})$.

Proposition. Suppose that a game satisfies anonymity, monotone externality, and strategic substitutability. Then, any coalition-proof Nash equilibrium under weak domination is a coalition-proof Nash equilibrium under strict domination.

Proof. If the set of coalition-proof Nash equilibria under weak domination is empty, then the statement of proposition is vacuously true. Hence, we consider the case in which there is a coalition-proof Nash equilibria under weak domination in the game. Let us assume that a game satisfies anonymity, positive externality, and strategic substitutability.³ We show by induction that the set of coalition-proof Nash equilibria under weak domination is a subset of that under strict domination. Clearly, the statement is true for all games with a single player. In any two-player game, under either dominance relations, the set of self-enforcing strategy profiles coincides with that of Nash equilibria. Hence, the set of coalition-proof Nash equilibria under weak domination coincides with

³We can similarly show the statement in the case of negative externality.

the Pareto efficient frontier of the set of Nash equilibria, and so does the set of coalition-proof Nash equilibria under strict domination with the weakly Pareto efficient frontier of that of Nash equilibria. As a result, the set of coalition-proof Nash equilibria under strict domination contains that under weak domination in every two-player game.

Let $n \geq 3$, and suppose that any coalition-proof Nash equilibrium under weak domination is a coalition-proof Nash equilibria under strict domination for any game with fewer than n players as an induction hypothesis. Let x^* denote a coalition-proof Nash equilibrium under weak domination of a game with n players. We need to show that x^* is a self-enforcing strategy profile under strict domination and that there is not other self-enforcing strategy profile under strict domination \tilde{x} where $u_i(\tilde{x}) > u_i(x^*)$ for every $i \in N$.

Lemma 1 Any self-enforcing strategy profile under weak domination is that under strict domination.

The lemma above can be shown in the following way. Let x be a self-enforcing strategy profile under weak domination of G . Then, by definition, x_C is a coalition-proof Nash equilibrium under weak domination in the restricted game $G|x_{-C}$ for every proper subset C of N . By the induction hypothesis, x_C is also a coalition-proof Nash equilibrium under strict domination in $G|x_{-C}$. That is, for all proper subsets C of N , x_C is a coalition-proof Nash equilibrium under strict domination of $G|x_{-C}$. Hence, x is a self-enforcing strategy profile under strict domination of G .

By Lemma 1, x^* is a self-enforcing strategy profile under strict domination.

Lemma 2 There is no other self-enforcing strategy profile under strict domination \tilde{x} such that $u_i(\tilde{x}) > u_i(x^*)$ for all $i \in N$.

Proof of Lemma 2. Let us suppose, on the contrary, that there is a self-enforcing strategy profile under strict domination \tilde{x} , at which $u_i(\tilde{x}) > u_i(x^*)$ for all $i \in N$. Then,

\tilde{x} must satisfy the following condition.

Claim 1 It follows that $\sum_{j \neq i} x_j^* < \sum_{j \neq i} \tilde{x}_j$ for all $i \in N$.

Proof of Claim 1. Let us suppose, on the contrary, that there is player $i \in N$ such that $\sum_{j \neq i} x_j^* \geq \sum_{j \neq i} \tilde{x}_j$. If $\sum_{j \neq i} x_j^* = \sum_{j \neq i} \tilde{x}_j$, then we have $u_i(x^*) \geq u_i(\tilde{x}_i, x_{-i}^*) = u_i(\tilde{x})$ by the definition of Nash equilibrium, which is a contradiction. If $\sum_{j \neq i} x_j^* > \sum_{j \neq i} \tilde{x}_j$, then we have $u_i(x^*) \geq u_i(\tilde{x}_i, x_{-i}^*)$ by the definition of Nash equilibrium and $u_i(\tilde{x}_i, x_{-i}^*) \geq u_i(\tilde{x})$ by positive externality. Therefore, we obtain $u_i(x^*) \geq u_i(\tilde{x})$. This is a contradiction. \parallel

Claim 2 The strategy profile \tilde{x} is not a Nash equilibrium of G .

By Claim 1, it is satisfied that $\sum_{k \in N} \sum_{j \neq k} x_j^* < \sum_{k \in N} \sum_{j \neq k} \tilde{x}_j$. Hence, $\sum_{k \in N} x_k^* < \sum_{k \in N} \tilde{x}_k$. Therefore, $i \in N$ exists such that $\tilde{x}_i > x_i^*$. By strategic substitutability, for player i , we have $u_i(x_i^*, \tilde{x}_{-i}) - u_i(\tilde{x}) > u_i(x^*) - u_i(\tilde{x}_i, x_{-i}^*)$. Since x^* is a Nash equilibrium, $u_i(x^*) - u_i(\tilde{x}_i, x_{-i}^*) \geq 0$. Therefore, $u_i(x_i^*, \tilde{x}_{-i}) > u_i(\tilde{x})$, which implies that \tilde{x} is not a Nash equilibrium of G . This contradicts the idea that \tilde{x} is a self-enforcing strategy profile under strict domination. Thus, there is no self-enforcing strategy profile under strict domination that dominates x^* , which implies that x^* is a coalition-proof Nash equilibrium under strict domination in the n -person game. \blacksquare

Remark 1 We use none of three conditions in the proof of Lemma 1. Thus, Lemma 1 holds true in every game. Note that we use the conditions only when we prove Lemma 2.

Many interesting games in economics satisfy the conditions above. For instance, Cournot oligopoly games and the other games that have been studied as a part of industrial organization theory satisfy the conditions. For details, refer to Yi (1999). Here, we give an example in the context of the provision of pure public goods.

Example 2 (Participation games in the mechanism implementing the Lindahl allocation) Consider an economy in which there is one pure public good, one private good, n players, and a mechanism that implements the Lindahl allocation rule.⁴ Saijo and Yamato (1999) introduced a model of voluntary participation in a public good mechanism. Their model consists of two stages. In the first stage, players decide simultaneously whether or not they will participate in the public good mechanism. In the second stage, only the participants play the mechanism. Following the rule of the mechanism, they produce the public good and distribute its cost. On the other hand, non-participants can enjoy the public good produced by the participants at no cost. As a result, the participants bear the cost of the public good, but the non-participants can free-ride the public good.

In this example, we suppose that the preference relations of all players are represented by the same quasi-linear utility function $u_i(z, t_i) = \alpha\sqrt{z} + t_i$, where $\alpha > 0$, $z \geq 0$ denotes the public good and $t_i \leq 0$ represents a payment of i for producing the public good. We assume that one unit of the public good is provided from one unit of the private good. We fix an outcome of the mechanism as the Lindahl allocation only for the preferences of participants.

First, we will characterize the equilibrium outcome of the second stage. Let p be the number of participants in the mechanism. The equilibrium allocation of the second stage, when p players enter the mechanism, is denoted by $(z^p, t_1^p, \dots, t_n^p)$. Then, the public good provision z^p maximizes the total surplus of participants $p\alpha\sqrt{z} - z$. Hence, we have $z^p = (\alpha p/2)^2$. Since every participant shares the cost of z^p equally at the Lindahl allocation in the case of identical agents, every participant i pays $t_i^p = -z^p/p = -\alpha^2 p/4$. On the other hand, $t_j^p = 0$ for every non-participant j . If the payoffs of participants and non-participants are denoted by $u^1(p)$ and $u^0(p)$, respectively, then we have $u^1(p) =$

⁴Many authors have constructed the mechanisms that attain the Lindahl allocation in their equilibrium. See Tian (2000), for example.

$$\alpha\sqrt{(\alpha p/2)^2} - \alpha^2 p/4 = \alpha^2 p/4 \text{ and } u^0(p) = \alpha\sqrt{(\alpha p/2)^2} = \alpha^2 p/2.$$

Given the equilibrium outcome of the second stage, the participation decision stage can be reduced to the following simultaneous game. In the game, each player i chooses either $x_i = 1$ (participation) or $x_i = 0$ (non-participation), simultaneously. In this stage, the set of actions of player i is $X_i = \{0, 1\}$.⁵ Let p^x be the number of participants at an action profile $x = (x_1, \dots, x_n)$. Then, p^x participants obtain the utility $u^1(p^x)$, and $n - p^x$ non-participants have the payoff $u^0(p^x)$ at the action profile x . The payoff matrix of the participation stage game in the case of $n = 3$ appears in Table 2. In this example, the anonymity condition is satisfied, since payoffs to both participants and non-participants depend on the number of participants. The participation decision game satisfies the positive externality condition because the utilities of participants and non-participants get higher as the number of participants increases. The difference $u^1(p) - u^0(p - 1) = \alpha^2(-p + 2)/4$ is decreasing with respect to the number of participants p , which implies that the incentive to participate in the mechanism decreases as the number of participants increases. Therefore, the strategic substitutability condition holds in this example. From Proposition, the set of coalition-proof Nash equilibria under strict domination contains that under weak domination. In fact, two agents choose participation in every coalition-proof Nash equilibrium under weak domination, while one agent or two agents participate in the mechanism in coalition-proof Nash equilibria under strict domination in this example.⁶

Remark 2 The conditions of anonymity, monotone externality, and strategic substitutability do not necessarily guarantee equivalence between coalition-proof Nash equilibria under the two different dominance relations. In fact, the participation decision game in Example 2 satisfies all of the conditions, but there is a coalition-proof Nash

⁵Only the pure strategies are considered.

⁶Shinohara (2003) characterized the set of coalition-proof Nash equilibria under weak domination of the participation decision game in the public good mechanism.

equilibrium under strict domination which is not that under weak domination.

4 Concluding Remarks

In this paper, the relationship between coalition-proof Nash equilibria under strict and weak dominations was examined. In many equilibrium concepts, such as the core and strong Nash equilibria, the set of equilibrium concept under weak domination is a subset of that under strict domination. However, the set of coalition-proof Nash equilibria under strict domination and that under weak domination are not necessarily related by inclusion. We showed that, if a game satisfies the properties of anonymity, monotone externality, and strategic substitutability, then the set of coalition-proof Nash equilibria under weak domination is a subset of that of coalition-proof Nash equilibria under strict domination. This implies that the inclusion relation between the two sets of coalition-proof Nash equilibria holds true in such interesting games studied in economics as the participation game in a public good mechanism and the standard Cournot oligopoly game.

The coalition-proof Nash equilibrium is well known as a refinement of the Nash equilibrium. However, little is known about the structure of the equilibria. This paper has focused on the relationship between coalition-proof Nash equilibria and dominance relations. Although different strategies may be used in coalition-proof Nash equilibria under the different dominance relations, their relationship had not been studied so far. The objective of this paper was to give an answer to this problem. Clarifying other properties of this equilibrium concept may be left for future researches.

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	b_1	b_2
a_1	1, 0, -5	-5, -5, 0
a_2	-5, -5, 0	0, 0, 10

c_1

	b_1	b_2
a_1	-1, -1, 5	5, -5, 0
a_2	-5, -5, 0	-2, -2, 0

c_2

Table 1: The payoff matrix of the example presented by Konishi, Le Breton, and Weber (1999). (Example 1)

	1	0
1	$\frac{3\alpha^2}{4}, \frac{3\alpha^2}{4}, \frac{3\alpha^2}{4}$	$\frac{\alpha^2}{2}, \alpha^2, \frac{\alpha^2}{2}$
0	$\alpha^2, \frac{\alpha^2}{2}, \frac{\alpha^2}{2}$	$\frac{\alpha^2}{2}, \frac{\alpha^2}{2}, \frac{\alpha^2}{4}$

1

	1	0
1	$\frac{\alpha^2}{2}, \frac{\alpha^2}{2}, \alpha^2$	$\frac{\alpha^2}{4}, \frac{\alpha^2}{2}, \frac{\alpha^2}{2}$
0	$\frac{\alpha^2}{2}, \frac{\alpha^2}{4}, \frac{\alpha^2}{2}$	0, 0, 0

0

Table 2: Payoff matrix of the participation decision game with $n = 3$. (Example 2)