Discussion Paper #2006-22

Noncentral limit theorems for bounded functions of linear processes without finite mean

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March 2007
Revised August 2007

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Abstract

We derive noncentral limit theorems for the partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$, where $K(x)$ is a bounded function and $\{X_i\}$ is a linear process. We assume the innovations of $\{X_i\}$ are independent and identically distributed and that the distribution of the innovations is an α-stable law ($0 < \alpha < 1$) or belongs to the domain of attraction of an α-stable law ($0 < \alpha < 1$). Then we establish the finite-dimensional convergence in distribution of the partial sum process to an $\alpha\beta$-stable Lévy motion. The parameter β determines how fast the coefficients of the linear process decay and we assume that $1 < \alpha\beta < 2$. We also derive the asymptotic distribution of the kernel density estimator of the marginal density function of $\{X_i\}$ by exploiting one of the noncentral limit theorems.

Keywords: linear process; martingale; stable law; $\alpha\beta$-stable Lévy motion; kernel density estimator

*I am grateful to Professor Jan Mielniczuk for his helpful comments on the earlier version.

1 Introduction

We consider a linear process defined in (1) below and derive noncentral limit theorems for bounded functions of the linear process. The linear process is defined by

$$X_i = \sum_{j=1}^{\infty} b_j \epsilon_{i-j}, \quad i = 1, 2, \ldots, \tag{1}$$

where $b_j \sim c_0 j^{-\beta} (j \geq 1)$, $c_0 > 0$ and $\{\epsilon_i\}$ are independent and identically distributed. We assume that ϵ_1 belongs to the domain of attraction an α-stable law $(0 < \alpha < 2)$ unless we say otherwise. Let $\mathrm{E}\{\epsilon_1\} = 0$ when $\alpha > 1$. In this paper $a_j \sim a'_j$ means $a_j / a'_j \to 1$ as $j \to \infty$. Then a sufficient condition for the existence of X_i is that $\alpha\beta > 1$.

We shall derive the asymptotic distributions of finite-dimensional distributions of the partial sum process defined in (2) below when $0 < \alpha < 1$ and $1 < \alpha\beta < 2$. The partial sum process we consider is

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{[nt]} (K(X_i) - \mathrm{E}\{K(X_i)\}), \quad 0 \leq t \leq 1, \tag{2}$$

where $K(x)$ is any bounded function and $[a]$ stands for the largest integer less than or equal to a when it appears in subscripts. We can include the case of $\alpha = 1$ if we deal with slowly varying functions in Lemmas 5.1-2 below. However, we do not include the case of $\alpha = 1$ to make this paper readable and easy to understand.

In Theorem 2.1 below, we establish the convergence in distribution of finite-dimensional distributions of (2) to those of an $\alpha\beta$-stable Lévy motion under a set of assumptions on ϵ_1. See Samorodnitsky and Taqqu (1994) for details on stable laws and α-stable Lévy motions. If $K(x)$ is integrable, the assumptions can be relaxed. See Theorem 2.2 below, which is useful to the derivation of the asymptotic distribution of the kernel density estimator.

A lot of researchers have been studying the asymptotic properties of partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ and given the asymptotic distributions when X_i

is defined in (1) and ϵ_1 may have variance. Concentrating on the cases where ϵ_1 belongs to the domain of attraction of an α-stable law $(0 < \alpha < 2)$, we refer to the relevant results here.

a.$0 < \alpha < 2$ and $\alpha\beta > 2$: Hsing (1999); Pipiras and Taqqu (2003)

b.$1 < \alpha < 2$, $\beta > 1$, and $\alpha\beta < 2$: Surgailis (2002)

c.$1 < \alpha < 2$, $\beta < 1$, and $1 < \alpha\beta$: Koul and Surgailis (2001)

The limiting distributions are not Gaussian in the cases of **b** and **c**. Only in the case of **a**, the limiting distributions are Gaussian. To our knowledge, there has been no result in the case where $0 < \alpha < 1$ and $1 < \alpha\beta < 2$.

We show in Section 2 that the asymptotic distribution in the case of **b** carrys over to the case of this paper. However, we need new techniques to tackle the case of this paper, for example, Propositions 2.5 and Lemmas 5.2 below. Koul and Surgailis (2001) and Surgailis (2002) focused on the asymptotic properties of empirical processes of $\{X_i\}$ in the cases of **b** and **c**, respectively. The proofs for the partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ are not given in the two papers. Surgailis (2004) also deals with asymptotics of the partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ in some other setups where $\mathrm{E}\{|\epsilon_1|^2\} < \infty$. See also Wu (2003). As for the partial sum processes, only convergence in distribution of finite-dimensional distributions is established in Koul and Surgailis (2001), Surgailis (2002,2004), and Pipiras and Taqqu (2003).

All the above papers crucially depend on Ho and Hsing (1996, 1997). The authors of the two papers devised the martingale decomposition approach and successfully studied the asymptotic properties of the partial sum processes of $K(X_i) - \mathrm{E}\{K(X_i)\}$ and empirical processes of $\{X_i\}$ when $\{X_i\}$ is a long-range dependent linear process with $\mathrm{E}\{|\epsilon_1|^4\} < \infty$. Our results crucially depend on Ho and Hsing (1996, 1997) through Surgailis (2002), too. Koul and Surgailis (2002) is an excellent expository paper on the martingale decomposition approach. Hannan (1979) is an earlier paper related to martingale decomposition approaches. Some results in Pipiras and Taqqu (2003) are also important to the arguments in this paper. See Lemmas 5.1-2 below.

We apply the result of Theorem 2.2 to derive the asymptotic distribution of the kernel density estimator of the density function of X_i. A lot of researchers have been studying the asymptotic properties of the kernel density estimator of the marginal density function of a linear process defined in (1). All of them assumed that $\mathrm{E}\{|\epsilon_1|^2\} < \infty$ in (1). Honda (2006) is only one exception. Wu and Mielniczuk (2002) deals with both short-range dependent and long-range dependent linear processes. The asymptotic properties are fully examined in Wu and Mielniczuk (2002) when $\mathrm{E}\{|\epsilon_1|^4\} < \infty$. Some of the arguments of Wu and Mielniczuk (2002) carry over to the cases where $\mathrm{E}\{|\epsilon_1|^{2+\delta}\} < \infty$ for some positive δ. When $\mathrm{E}\{|\epsilon_1|^2\} = \infty$, Honda (2006) derived the asymptotic distribution in the cases of **a**, **b**, and **c** defined above by applying the results of Hsing (1999), Koul and Surgailis (2001), Surgailis (2002,2004), and Pipiras and Taqqu (2003). See the references in Wu and Mielniczuk (2002) and Honda (2006) for other relevant works. Peng and Yao (2004) applied the results in Hsing (1999), Koul and Surgailis (2001), and Surgailis (2002) to nonparametric regression.

The paper is organized as follows. We state the assumptions and the main theorems in Section 2. The main theorems are proved and the propositions for the proofs are also presented in the section. Those propositions are verified in Section 3. We consider the kernel density estimator in Section 4. All the technical lemmas and the proofs are confined to Section 5.

2 Noncentral limit theorems

We begin with the assumptions and the notation. Next we state the main theorems. Then we describe the propositions for the proofs of the theorems and prove the theorems at the end of this section. Hereafter we assume that $0 < \alpha < 1$ and

$1 < \alpha\beta < 2$, where $b_j \sim c_0 j^{-\beta}$.

In this paper, C, C_1, and C_2 stand for generic positive constants and their values change from place to place. The range of integration is the whole real line when it is omitted.

We denote the distribution function and the characteristic function of ϵ_1 by $G(x)$ and $\phi(\theta)$, respectively. Assumptions **A1** and **A2** are on $G(x)$ and $\phi(\theta)$, respectively. Assumption **A1** means that ϵ_1 belongs to the domain of an α-stable law. Assumption **A2** ensures some of the necessary properties of density functions.

A1: There is an $\alpha \in (0,1)$ satisfying

$$\lim_{x\to-\infty} G(x)|x|^\alpha = c_1, \quad \lim_{x\to\infty}(1-G(x))x^\alpha = c_2, \quad \text{and} \quad c_1 + c_2 > 0.$$

A2: $|\phi(\theta)| < C(1+|\theta|)^{-\delta}$ for some positive δ.

We always assume that assumptions **A1** and **A2** hold.

We introduce some notation to define another assumption. We decompose X_i into

$$X_i = X_{i,j} + \tilde{X}_{i,j}, \tag{3}$$

where

$$X_{i,j} = \sum_{l=1}^{j-1} b_l \epsilon_{i-l} \quad \text{and} \quad \tilde{X}_{i,j} = \sum_{l=j}^{\infty} b_l \epsilon_{i-l}.$$

Let $F_j(x)$ and $\tilde{F}_j(x)$ stand for the distribution functions of $X_{i,j}$ and $\tilde{X}_{i,j}$, respectively. Lemma 1 in Giraitis et al. (1996) and assumption **A2** imply that $\tilde{F}_j(x)$, $j = 1, 2, \ldots$, are three times continuously differentiable and that $F_j(x)$, $j = s_0, s_0+1, \ldots$, are three times continuously differentiable for a large positive integer s_0. Besides all the derivatives of $F_j(x)$ are bounded up to the third order uniformly in x and j. We write $f(x)$ and $F(x)$ for $f_\infty(x)$ and $F_\infty(x)$ for notational convenience. Then $f(x)$ is the density function of X_1.

We define assumption **A3**.

A3: We can choose a positive $\gamma \in (0, \alpha)$ such that

$$|F''(x)| + |F_j''(x)| \leq C(1 + |x|)^{-(1+\gamma)}$$

and

$$|F''(y) - F''(x)| + |F_j''(y) - F_j''(x)| \leq C|x - y|(1 + |x|)^{-(1+\gamma)}$$

for $|x - y| \leq 1$, uniformly in x and $j \geq s_0$.

When ϵ_1 follows an α-stable law $(0 < \alpha < 1)$, assumptions **A1-3** hold. See Remark 2.1 below about assumption **A3**.

Remark 2.1 Let $S_\alpha(\sigma, \eta, \mu)$ stand for α-stable law. Then the characteristic function of $S_\alpha(\sigma, \eta, \mu)$ has the form

$$\begin{cases} \exp\{-\sigma^\alpha|\theta|^\alpha(1 - \mathrm{i}\eta \mathrm{sign}(\theta)\tan(\pi\alpha/2)) + \mathrm{i}\mu\theta\} & , \ \alpha \neq 1 \\ \exp\{-\sigma|\theta|(1 + \frac{2}{\pi}\mathrm{i}\eta \mathrm{sign}(\theta)\log|\theta|) + \mathrm{i}\mu\theta\} & , \ \alpha = 1 \end{cases},$$

where i stands for the imaginary unit. When ϵ_1 follows an α-stable law $(\alpha \neq 1)$, the characteristic function of $X_{1,l}$, $\phi_l(\theta)$, is given by

$$\phi_l(\theta) = \exp\Big\{ - \Big(\sum_{j=1}^{l-1} b_j^\alpha\Big)|\theta|^\alpha(1 - \mathrm{i}\eta \mathrm{sign}(\theta)\tan(\pi\alpha/2)) + \mathrm{i}\Big(\sum_{j=1}^{l-1} b_j\Big)\mu\theta\Big\} \tag{4}$$

and $f_l^{(k)}(x)$, $k = 1, 2, \ldots$, is represented as

$$f_l^{(k)}(x) = \frac{(-\mathrm{i})^k}{2\pi}\int_0^\infty \theta^k\phi_l(\theta)e^{-\mathrm{i}\theta x}d\theta + \frac{(-\mathrm{i})^k}{2\pi}\int_{-\infty}^0 \theta^k\phi_l(\theta)e^{-\mathrm{i}\theta x}d\theta. \tag{5}$$

By appealing to integration by parts as in the proof in Lemma 3 in Hsing (1999), we can show that assumption **A3** holds.

Even if ϵ_1 does not follow an α-stable law, assumption **A1** implies that for any $r \in (0, \alpha)$, there is a positive constant C_r such that

$$\mathrm{E}\{|X_{1,j}|^r\} \leq C_r, \quad j = s_0, s_0 + 1, \ldots. \tag{6}$$

Hence we can say that assumption **A3** is a natural one in terms of (6) and the existence of bounded and twice continuously differentiable density functions of $X_{1,j}$, $j = s_0, s_0 + 1, \ldots$.

When $K(x)$ is bounded and integrable, we use **A4** below instead of assumption **A3**. **A4** is not an assumption and follows from assumptions **A1** and **A2**.

A4: There is a positive C such that

$$|F''(y) - F''(x)| + |F_j''(y) - F_j''(x)| \leq C|x-y|, \quad j = s_0, s_0+1, \dots$$

We define S_m by

$$S_m = \sum_{i=1}^{m} (K(X_i) - \mathrm{E}\{K(X_i)\}). \tag{7}$$

Then $n^{-1/(\alpha\beta)}S_{[nt]}$, $t \in [0,1]$, belongs to $D[0,1]$. See Embrechts et al. (1997, Appendix A2) for the definition of $D[0,1]$, the metric, and the weak convergence. We cannot deal with weak convergence in $D[0,1]$. See a comment just before Proposition 2.3.

We are ready to state the main theorems of this paper. We omit $n \to \infty$ since it is obvious from the context.

Theorem 2.1 *Suppose that assumptions* **A1-3** *hold and that $K(x)$ is a bounded function. Then finite-dimensional distributions of $n^{-1/(\alpha\beta)}S_{[nt]}$, $t \in [0,1]$, converge in distribution to those of an $\alpha\beta$-stable Lévy motion on $[0,1]$. The distribution at t of the $\alpha\beta$-stable Lévy motion is given by*

$$t^{1/(\alpha\beta)}(c_2^{1/(\alpha\beta)}c_K^+L^+ + c_1^{1/(\alpha\beta)}c_K^-L^-),$$

where

$$c_K^\pm = \sigma\int_0^\infty (K_\infty(\pm t) - K_\infty(0))t^{-(1+1/\beta)}dt,$$
$$K_\infty(x) = \mathrm{E}\{K(X_1+x)\}, \qquad \sigma = \Big\{\frac{c_0^\alpha(\alpha\beta-1)}{\Gamma(2-\alpha\beta)|\cos(\pi\alpha\beta/2)|\beta^{\alpha\beta}}\Big\}^{1/(\alpha\beta)},$$

and L^- and L^+ are mutually independent random variables whose distribution are $S_{\alpha\beta}(1,1,0)$, respectively.

When $K(x)$ is bounded and integrable, assumption **A3** is not necessary.

Theorem 2.2 *Suppose that assumptions* **A1-2** *hold and that $K(x)$ is a bounded integrable function. Then we have the same convergence in distribution as in Theorem 2.1.*

Note that we have the same asymptotic distribution as in Surgailis (2002). The proofs of the theorems are postponed to the end of this section. We give only the detailed proof of Theorem 2.1 and give just a brief remark on the proof of Theorem 2.2 since Theorem 2.2 can be proved in the same way. In fact we formulated Theorem 2.2 to deal with the kernel density estimator of $f(x)$. Then $K(x)$ is bounded and integrable. See (47), not (44), in Section 4.

We introduce decompositions of S_n before we state the propositions necessary to prove the theorems. The same kind of decomposition appears in Surgailis (2002). However, (8) is a new one.

$$
\begin{aligned}
S_n &= (S_n - T_n) + (T_n - W_n) + W_n && (8)\\
&= (S_n - T_n) + (T_n - T'_n) + (T'_n - W'_n) + W'_n && (9)
\end{aligned}
$$

where

$$
\begin{aligned}
T_n &= \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}(K_j(b_j\epsilon_{i-j}) - \mathrm{E}\{K_j(b_j\epsilon_{i-j})\}), && (10)\\
T'_n &= \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}(K_\infty(b_j\epsilon_{i-j}) - \mathrm{E}\{K_\infty(b_j\epsilon_{i-j})\}), && (11)\\
K_j(x) &= \mathrm{E}\{K(X_{1,j}+x)\}, \quad j \ge s_0, && (12)\\
W_n &= \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}(K_j(b_j\epsilon_i) - \mathrm{E}\{K_j(b_j\epsilon_i)\}), && (13)\\
W'_n &= \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}(K_\infty(b_j\epsilon_i) - \mathrm{E}\{K_\infty(b_j\epsilon_i)\}). && (14)
\end{aligned}
$$

Since

$$
K_j(x) = \int K(\xi) f_j(\xi - x)d\xi,
$$

it is easy to see from assumption **A3** that $K_j(x)$, $j \ge s_0$, and $K_\infty(x)$ are continuously differentiable and that all the derivatives are uniformly bounded in j and x.

If we have established the weak convergence of $n^{-1/(\alpha\beta)}W_{[nt]}$ in $D[0,1]$ and that $n^{-1/(\alpha\beta)}(S_{[nt]}-T_{[nt]})$ and $n^{-1/(\alpha\beta)}(T_{[nt]}-W_{[nt]})$ are asymptotically negligible for any t, Theorem 2.1 follows from (8).

In the application to the kernel density estimator, $K_j(x)$ depends on the sample size through the bandwidth. Thus the dependence on j should be avoided to make the proof simpler. The decomposition (9) is for Theorem 2.2.

We give some comments on the propositions before we state the propositions. Propositions 2.1 and 2.2 correspond to Lemma 5.1 in Surgailis (2002) and Propositions 2.3 and 2.4 correspond to Lemma 5.2 in Surgailis (2002). Proposition 2.5 corresponds to Lemma 3.1 in Surgailis (2002) and that Proposition 2.6 is essentially Lemma 3.1 in Surgailis (2002). All the proofs of the propositions are given in Section 3.

Proposition 2.1 *Suppose that assumptions* **A1-3** *hold and that $K(x)$ is a bounded function. Then for any r satisfying $\{2(\alpha\beta-1)\}\vee(\alpha\beta)<r<2\wedge(2\alpha\beta-1)$, there is a positive constant C such that*

$$\mathrm{E}\{|S_n-T_n|^r\}<C(n^{-2\alpha\beta+2+r}+n)\quad \textit{for any positive integer } n.$$

Proposition 2.2 *Suppose that assumptions* **A1-2** *hold and that $K(x)$ is bounded and integrable. Then we have the same result as in Proposition 2.1. Besides if we choose another r satisfying $(\alpha\beta+\alpha^2\beta-\alpha-1)\vee(\alpha\beta)<r<(\alpha\beta+\alpha^2\beta-\alpha)\wedge\{\alpha\beta(\alpha+1)\}\wedge 2$, there is a positive constant C such that*

$$\mathrm{E}\{|T_n-T'_n|^r\}<C(n^{-\alpha\beta-\alpha^2\beta+\alpha+1+r}+n)\quad \textit{for any positive integer } n.$$

Since $-\alpha\beta+r+1<r/(\alpha\beta)<1$ in Propositions 2.3-4 below, it is difficult to show that $n^{-1/(\alpha\beta)}(T_{[nt]}-W_{[nt]})$ and $n^{-1/(\alpha\beta)}(T'_{[nt]}-W'_{[nt]})$ are asymptotically negligible in $D[0,1]$. For example, see Theorem 12.3 in Billingsley (1968),

Proposition 2.3 *Suppose that assumptions* **A1-3** *hold and that $K(x)$ is a bounded function. Then for any r satisfying $1 \le r < \alpha\beta$, there is a positive constant C such that*

$$\mathrm{E}\{|T_n - W_n|^r\} < Cn^{-\alpha\beta+r+1} \quad \textit{for any positive integer } n.$$

Proposition 2.4 *Suppose that assumptions* **A1-2** *hold and that $K(x)$ is bounded and integrable. Then for any r satisfying $1 \le r < \alpha\beta$, there is a positive constant C such that*

$$\mathrm{E}\{|T'_n - W'_n|^r\} < Cn^{-\alpha\beta+r+1} \quad \textit{for any positive integer } n.$$

Proposition 2.5 *Suppose that assumptions* **A1-3** *hold and that $K(x)$ is a bounded function. Then*

$$\sum_{j=s_0}^{\infty} (K_j(b_j\epsilon_1) - \mathrm{E}\{K_j(b_j\epsilon_1)\})$$

belongs to the domain of attraction of an $\alpha\beta$-stable law. As a result,

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{n} \sum_{j=s_0}^{\infty} (K_j(b_j\epsilon_1) - \mathrm{E}\{K_j(b_j\epsilon_1)\})$$

converges in distribution to $c_2^{1/(\alpha\beta)} c_K^+ L^+ + c_1^{1/(\alpha\beta)} c_K^- L^-$. See Theorem 2.1 for the definitions of $c_K^\pm$ and $L^\pm$.

Proposition 2.6 *Suppose that assumptions* **A1-2** *hold and that $K(x)$ is bounded and integrable. Then*

$$\sum_{j=s_0}^{\infty} (K_\infty(b_j\epsilon_1) - \mathrm{E}\{K_\infty(b_j\epsilon_1)\})$$

belongs to the domain of attraction of an $\alpha\beta$-stable law. As a result,

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{n} \sum_{j=s_0}^{\infty} (K_\infty(b_j\epsilon_1) - \mathrm{E}\{K_\infty(b_j\epsilon_1)\})$$

converges in distribution to $c_2^{1/(\alpha\beta)} c_K^+ L^+ + c_1^{1/(\alpha\beta)} c_K^- L^-$. See Theorem 2.1 for the definitions of $c_K^\pm$ and $L^\pm$.

We are prepared to prove Theorem 2.1.

Proof of Theorem 2.1) Note that

$$\begin{aligned} & n^{-1/(\alpha\beta)} \sum_{i=1}^{[nt]} (K(X_i) - \mathrm{E}\{K(X_i)\}) \qquad (15) \\ = \;& n^{-1/(\alpha\beta)}(S_{[nt]} - T_{[nt]}) + n^{-1/(\alpha\beta)}(T_{[nt]} - W_{[nt]}) + n^{-1/(\alpha\beta)} W_{[nt]}. \end{aligned}$$

Since $\sum_{j=s_0}^{\infty}(K_j(b_j\epsilon_1) - \mathrm{E}\{K_j(b_j\epsilon_1)\})$ belongs to the domain of attraction of the $\alpha\beta$-stable law in Proposition 2.5, the weak convergence of $n^{-1/(\alpha\beta)}W_{[nt]}$, $0 \le t \le 1$, in $D[0,1]$ follows from Theorem 2.4.10 in Embrechts et al.(1997).

Next we deal with the first and second terms on the right hand side of (15).

First we choose r_1 satisfying the condition of Proposition 2.1. Then we have

$$\mathrm{E}\{|n^{-1/(\alpha\beta)}(S_{[nt]} - T_{[nt]})|^{r_1}\} \le C n^{-r_1/(\alpha\beta)}([nt]^{-2\alpha\beta+2+r_1} + [nt]) \qquad (16)$$

for any t larger than $1/n$. Note that $0 < -2\alpha\beta+2+r_1 < r_1/(\alpha\beta)$ and $1 < r_1/(\alpha\beta)$.

Next we choose r_2 satisfying the condition of Proposition 2.3. Then we have

$$\mathrm{E}\{|n^{-1/(\alpha\beta)}(T_{[nt]} - W_{[nt]})|^{r_2}\} \le C n^{-r_2/(\alpha\beta)}[nt]^{-\alpha\beta+1+r_2} \qquad (17)$$

for any t larger than $1/n$. Note that $0 < -\alpha\beta+1+r_2 < r_2/(\alpha\beta)$.

The theorem follows from the weak convergence of $n^{-1/(\alpha\beta)}W_{[nt]}$, $0 \le t \le 1$, (16), and (17). Hence the proof of the theorem is complete.

In the proof of Theorem 2.2, we use **A4**, Propositions 2.2, 2.4, and 2.6 instead of assumption **A3**, Propositions 2.1, 2.3, and 2.5. When we examine the tail conditions, $K_\infty(b_j\epsilon_i)$ is more tractable than $K_j(b_j\epsilon_i)$ in some applications. See Section 4.

3 Proofs of propositions

In this section we prove Propositions 2.1, 2.2, 2.3, and 2.5.

When $K(x)$ is a bounded integrable function, the arguments in the proofs of Propositions 2.1, 2.3, and 2.5 are still true without assumption **A3** since $\int |K(\xi)|d\xi <$

∞ and **A4** holds. Therefore we omit the proofs of the first half of Proposition 2.2 and Proposition 2.4.

An argument similar to the proof of Proposition 2.3 is found in Surgailis (2004, p.337). Proposition 2.5 is a modified version of Lemmas 3.1 in Surgailis (2002). Proposition 2.6 is essentially Lemma 3.1 in Surgailis (2002) and we omit the proof.

We write $\mathcal{F}_i$ for the σ-field generated by $\{\epsilon_j \mid j \le i\}$.

Proof of Proposition 2.1) We represent S_n and T_n as

$$S_n = \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int K(\xi)(f_j(\xi - b_j\epsilon_{i-j} - \tilde{X}_{i,j+1}) - f_{j+1}(\xi - \tilde{X}_{i,j+1}))d\xi \qquad (18)$$

$$+\sum_{i=1}^{n}\sum_{j=0}^{s_0-1}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\} - \mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}],$$

$$T_n = \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int K(\xi)(f_j(\xi - b_j\epsilon_{i-j}) - \mathrm{E}\{f_j(\xi - b_j\epsilon_{i-j})\})d\xi. \qquad (19)$$

The right hand side of (18) is typical of the martingale decomposition approach. By using the von Bahr and Esseen inequality (von Bahr and Esseen 1965) and the boundedness of $K(x)$, we obtain

$$\mathrm{E}\{|\sum_{i=1}^{n}\sum_{j=1}^{s_0-1}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\} - \mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}]|^r\} < Cn. \qquad (20)$$

We evaluate

$$S_n - T_n - \sum_{i=1}^{n}\sum_{j=0}^{s_0-1}[\mathrm{E}\{K(X_i)|\mathcal{F}_{i-j}\} - \mathrm{E}\{K(X_i)|\mathcal{F}_{i-j-1}\}] = \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int K(\xi)U_{i,j}(\xi)d\xi \qquad (21)$$

where

$$U_{i,j}(\xi) = f_j(\xi - b_j\epsilon_{i-j} - \tilde{X}_{i,j+1})$$

$$-f_{j+1}(\xi - \tilde{X}_{i,j+1}) - f_j(\xi - b_j\epsilon_{i-j}) + \mathrm{E}\{f_j(\xi - b_j\epsilon_{i-j})\}.$$

As in Surgailis (2002), the expression in (22) below is useful in evaluating (21).

$$\int K(\xi)U_{i,j}(\xi)d\xi \qquad (22)$$

$$= \int\Big[\int\Big\{\int_{-b_j u}^{-b_j\epsilon_{i-j}}(f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z))dz\Big\}K(\xi)d\xi\Big]G(du).$$

We consider 8 cases to treat (22) and give upper bounds of (22) to each case.

1. $|b_j\epsilon_{i-j}| \geq 1,\ |\tilde{X}_{i,j+1}| \geq 1,\ |b_j u| \geq 1$

2. $|b_j\epsilon_{i-j}| \geq 1,\ |\tilde{X}_{i,j+1}| \geq 1,\ |b_j u| < 1$

3. $|b_j\epsilon_{i-j}| \geq 1,\ |\tilde{X}_{i,j+1}| < 1,\ |b_j u| \geq 1$

4. $|b_j\epsilon_{i-j}| \geq 1,\ |\tilde{X}_{i,j+1}| < 1,\ |b_j u| < 1$

5. $|b_j\epsilon_{i-j}| < 1,\ |\tilde{X}_{i,j+1}| \geq 1,\ |b_j u| \geq 1$

6. $|b_j\epsilon_{i-j}| < 1,\ |\tilde{X}_{i,j+1}| \geq 1,\ |b_j u| < 1$

7. $|b_j\epsilon_{i-j}| < 1,\ |\tilde{X}_{i,j+1}| < 1,\ |b_j u| \geq 1$

8. $|b_j\epsilon_{i-j}| < 1,\ |\tilde{X}_{i,j+1}| < 1,\ |b_j u| < 1$

We present the upper bounds and the proofs.

1. $|(22)| \leq CI(|b_j\epsilon_{i-j}| \geq 1)I(|\tilde{X}_{i,j+1}| \geq 1)|b_j|^\alpha$.

Proof) We should deal with

$$\begin{aligned}&\int_{|b_j u|\geq 1}\Big[\int\{f_j(\xi - b_j\epsilon_{i-j} - \tilde{X}_{i,j+1})\\&\qquad -f_j(\xi - b_j\epsilon_{i-j}) - f_j(\xi - b_j u - \tilde{X}_{i,j+1}) + f_j(\xi - b_j u)\}K(\xi)d\xi\Big]dG(u)\\&\qquad \times I(|b_j\epsilon_{i-j}| \geq 1)I(|\tilde{X}_{i,j+1}| \geq 1).\end{aligned}$$

The expression inside [ ] is bounded since $\int f_j(\xi)d\xi = 1$. Lemma 5.1 and $I(|b_j u| \geq 1)$ yield $|b_j|^\alpha$.

2. $|(22)| \leq CI(|b_j\epsilon_{i-j}| \geq 1)I(|\tilde{X}_{i,j+1}| \geq 1)$.

Proof) The proof is similar to that of **1**. We have no $|b_j|^\alpha$ in **2** since $I(|b_j u| \geq 1)$ is replaced with $I(|b_j u| < 1)$.

3. $|(22)| \leq CI(|b_j\epsilon_{i-j}| \geq 1)|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)|b_j|^\alpha$.

Proof) We should deal with

$$\begin{aligned}&\int_{|b_j u|\geq 1}\Big\{\int(|f_j(\xi - b_j\epsilon_{i-j} - \tilde{X}_{i,j+1})\\&\qquad -f_j(\xi - b_j\epsilon_{i-j})| + |f_j(\xi - b_j u - \tilde{X}_{i,j+1}) - f_j(\xi - b_j u)|)d\xi\Big\}dG(u)\\&\qquad \times I(|b_j\epsilon_{i-j}| \geq 1)I(|\tilde{X}_{i,j+1}| < 1).\end{aligned} \tag{23}$$

The bound follows from (23), assumption **A3**, and Lemma 5.1.

4. $|(22)| \le CI(|b_j\epsilon_{i-j}| \ge 1)|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)$.

Proof) The proof is similar to that of **3**. We have no $|b_j|^\alpha$ in **4** since $I(|b_j u| \ge 1)$ is replaced with $I(|b_j u| < 1)$.

5. $|(22)| \le CI(|\tilde{X}_{i,j+1}| \ge 1)|b_j|^\alpha$.

Proof) The proof is similar to that of **1**. In **5**, $I(|b_j\epsilon_{i-j}| \ge 1)$ is replaced with $I(|b_j\epsilon_{i-j}| < 1)$.

6. $|(22)| \le CI(|\tilde{X}_{i,j+1}| \ge 1)(|b_j\epsilon_{i-j}|I(|b_j\epsilon_{i-j}| < 1) + |b_j|^\alpha\ )$.

Proof) We should deal with

$$\begin{aligned}&\int_{|b_j u|<1}\Big[\int_{-b_j u}^{-b_j\epsilon_{i-j}}\Big\{\int |f_j'(\xi + z - \tilde{X}_{i,j+1}) \\ &\quad -f_j'(\xi+z)|d\xi\Big\}dz\Big]dG(u)I(|\tilde{X}_{i,j+1}| \ge 1)I(|b_j\epsilon_{i-j}| < 1).\end{aligned} \tag{24}$$

The expression inside { } is bounded due to assumption **A3**. Hence (24) is bounded by

$$CI(|\tilde{X}_{i,j+1}| \ge 1)\Big\{|b_j\epsilon_{i-j}|I(|b_j\epsilon_{i-j}| < 1) + \int_{|b_j u|<1}|b_j u|dG(u)\Big\}. \tag{25}$$

The bound follows from (25) and Lemma 5.2.

7. $|(22)| \le C|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)|b_j|^\alpha$.

Proof) We should deal with

$$\begin{aligned}&\Big[\int_{|b_j u|\ge 1}\Big\{\int_0^{-b_j\epsilon_{i-j}}\Big(\int |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi+z)|d\xi\Big)dz\Big\}dG(u) \\ &\quad + \int_{|b_j u|\ge 1}\Big(\int |f_j(\xi - \tilde{X}_{i,j+1}) - f_j(\xi) - f_j(\xi - b_j u - \tilde{X}_{i,j+1}) \\ &\quad + f_j(\xi - b_j u)|d\xi\Big)dG(u)\Big]I(|\tilde{X}_{i,j+1}| < 1)I(|b_j\epsilon_{i-j}| < 1).\end{aligned} \tag{26}$$

Assumption **A3** implies that (26) is bounded by

$$C|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)\int_{|b_j u|\ge 1}dG(u).$$

Hence the bound follows from Lemma 5.2.

8. $|(22)| \le C|\tilde{X}_{i,j+1}|I(|\tilde{X}_{i,j+1}| < 1)(|b_j\epsilon_{i-j}|I(|b_j\epsilon_{i-j}| < 1) + |b_j|^\alpha\ )$.

Proof) This bound follows from assumption **A3**, (22), (4.11) in Surgailis (2002), and Lemma 5.2. Actually, in this case, we have

$$\int \Big[\int \Big\{ \int_0^{|b_j \epsilon_{i-j}|} |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z)| dz \Big\} d\xi \Big] G(du) \le C|\tilde{X}_{i,j+1}||b_j \epsilon_{i-j}|$$

and

$$\int \Big[\int \Big\{ \int_0^{|b_j u|} |f_j'(\xi + z - \tilde{X}_{i,j+1}) - f_j'(\xi + z)| dz \Big\} d\xi \Big] I(|b_j u| < 1) G(du) \\ \le C|\tilde{X}_{i,j+1}||b_j|^\alpha.$$

The above bounds for **1**–**8** and Lemmas 5.1-2 yield

$$\mathrm{E}\Big\{\Big|\int K(\xi) U_{i,j}(\xi) d\xi\Big|^r\Big\} \le C j^{1-2\alpha\beta}. \tag{27}$$

We evaluate (21) by using (27). We divide (21) into A_n and B_n, where

$$A_n = \sum_{i=1}^{n} \sum_{l=s_0}^{n-i+1} \int K(\xi) U_{i+l-1,l}(\xi) d\xi, \tag{28}$$

$$B_n = \sum_{l=1}^{\infty} \sum_{j=1}^{n} \int K(\xi) U_{j,j+l}(\xi) |(j + l \ge s_0) d\xi. \tag{29}$$

We can apply the von Bahr and Esseen inequality to A_n and B_n because $U_{i,j}$ is $\mathcal{F}_{i-j}$-measurable and $\mathrm{E}\{U_{i,j}|\mathcal{F}_{i-j-1}\} = 0$ almost everywhere.

By (27), von Bahr and Esseen inequality, and Minkowski's inequality, we can derive the bounds for $\mathrm{E}\{|A_n|^r\}$ and $\mathrm{E}\{|B_n|^r\}$.

Noticing that $-2\alpha\beta + 1 < -r$, we have

$$\mathrm{E}\{|A_n|^r\} \le 2 \sum_{i=1}^{n} \Big(\sum_{l=1}^{n-i+1} (l^{-2\alpha\beta+1})^{1/r} \Big)^r \le Cn. \tag{30}$$

As for B_n, we have

$$\mathrm{E}\{|B_n|^r\} \le 2 \sum_{l=1}^{\infty} \Big(\sum_{j=1+l}^{n+l} (j^{-2\alpha\beta+1})^{1/r} \Big)^r \tag{31}$$

$$\begin{aligned} &\le Cn^{-2\alpha\beta+2+r} \sum_{l=1}^{\infty} \frac{1}{n} \Big\{ \Big(\frac{l}{n}\Big)^{(-2\alpha\beta+1)/r+1} - \Big(1 + \frac{l}{n}\Big)^{(-2\alpha\beta+1)/r+1} \Big\}^r \\ &\le Cn^{-2\alpha\beta+2+r} \int_0^\infty \Big\{ u^{(-2\alpha\beta+1)/r+1} - \Big(1 + \frac{1}{n} + u\Big)^{(-2\alpha\beta+1)/r+1} \Big\}^r du \\ &\le Cn^{-2\alpha\beta+2+r}. \end{aligned}$$

The proposition follows from (30) and (31). Hence the proof of the proposition is complete.

Proof of Proposition 2.2) The first half of the proposition can be verified in the same way as Proposition 2.1 since $\int |K(\xi)|d\xi < \infty$. We only have to replace $d\xi$ with $|K(\xi)|d\xi$ and exploit the integrability of $K(\xi)$ in the proof of Proposition 2.1.

We prove the second half of the proposition. We represent $T_n - T_n'$ as

$$\begin{aligned} & T_n - T_n' \qquad (32) \\ = & \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int (f_j(\xi - b_j\epsilon_{i-j}) - f(\xi - b_j\epsilon_{i-j}) \\ & \qquad -\mathrm{E}\{f_j(\xi - b_j\epsilon_{i-j})\} + \mathrm{E}\{f(\xi - b_j\epsilon_{i-j})\})K(\xi)d\xi \\ = & \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}\int \Big[\int \Big\{\int_{-b_j u}^{-b_j\epsilon_{i-j}} (f_j'(\xi+z) - f'(\xi+z))dz\Big\}K(\xi)d\xi\Big]dG(u) \end{aligned}$$

By using Lemma 5.3, we obtain

$$\Big|\int \Big\{\int_0^{-b_j\epsilon_{i-j}} (f_j'(\xi+z) - f'(\xi+z))dz\Big\}K(\xi)d\xi\Big| \le C(|b_j\epsilon_{i-j}|j^{-\alpha\beta+1}) \wedge 1. \qquad (33)$$

Lemmas 5.1-2 and (33) imply

$$\mathrm{E}\Big\{\Big|\int \Big\{\int_0^{-b_j\epsilon_{i-j}} (f_j'(\xi+z) - f'(\xi+z))dz\Big\}K(\xi)d\xi\Big|^r\Big\} \le Cj^{-\alpha\beta-\alpha^2\beta+\alpha}. \qquad (34)$$

When ϵ_{i-j} is replaced with u in (33), we can treat it by applying Jensen's inequality.

With (34), we can proceed as in the proof of Proposition 2.1 with $-2\alpha\beta+1$ replaced with $-\alpha\beta-\alpha^2\beta+\alpha$. Then we have

$$\mathrm{E}\{|T_n - T_n'|^r\} \le C(n^{-\alpha\beta-\alpha^2\beta+\alpha+1+r} + n)$$

Hence the proof of the proposition is complete.

Proof of Proposition 2.3) First we consider the properties of $K_j(x)$. Let $K_j(0) = 0$ by redefining $K_j(x)$ by $K_j(x) - K_j(0)$. Then we have

$$|K_j(x)| \le C(1 \wedge |x|) \qquad (35)$$

by the Taylor expansion at 0 and the uniform boundedness of the derivatives. By using (35) and Lemmas 5.1-2, we get

$$\begin{aligned}
&\mathrm{E}\{|K_j(b_j\epsilon_1)|^r\} && (36)\\
\le\ & C(\mathrm{P}(|b_j\epsilon_1|\ge 1)+\mathrm{E}\{|b_j\epsilon_1|^r I(|b_j\epsilon_1|<1)\})\\
\le\ & C|b_j|^\alpha \le Cj^{-\alpha\beta}
\end{aligned}$$

We represent $T_n - W_n$ as (37) below to apply the von Bahr and Esseen inequality.

$$T_n - W_n = -\sum_{k=1}^{n} A_n(k) + \sum_{k=1}^{\infty} B_n(k), \tag{37}$$

where

$$\begin{aligned}
A_n(k) &= \sum_{j=k\vee s_0}^{\infty}(K_j(b_j\epsilon_{n+1-k})-\mathrm{E}\{K_j(b_j\epsilon_{n+1-k})\}),\\
B_n(k) &= \sum_{j=k\vee s_0}^{k+n-1}(K_j(b_j\epsilon_{1-k})-\mathrm{E}\{K_j(b_j\epsilon_{1-k})\}).
\end{aligned}$$

We evaluate the two terms on the right hand side of (37) by using von Bahr and Esseen inequality, Minkowski's inequality, and (36). Then we have

$$\mathrm{E}\{|\sum_{k=1}^{n} A_n(k)|^r\} \le 2\sum_{k=1}^{n}\Big(\sum_{j=k}^{\infty} j^{-\alpha\beta/r}\Big)^r \le Cn^{-\alpha\beta+r+1}, \tag{38}$$

$$\begin{aligned}
&\mathrm{E}\{|\sum_{k=1}^{\infty} B_n(k)|^r\} \le 2\sum_{k=1}^{\infty}\Big(\sum_{j=k}^{k+n-1} j^{-\alpha\beta/r}\Big)^r && (39)\\
\le\ & Cn^{-\alpha\beta+r+1}\sum_{k=1}^{\infty}\frac{1}{n}\Big\{\Big(\frac{k}{n}\Big)^{-\alpha\beta/r+1}-\Big(1+\frac{k}{n}\Big)^{-\alpha\beta/r+1}\Big\}^r\\
\le\ & Cn^{-\alpha\beta+r+1}\int_0^\infty \Big\{u^{-\alpha\beta/r+1}-\Big(1+\frac{1}{n}+u\Big)^{-\alpha\beta/r+1}\Big\}^r du \le Cn^{-\alpha\beta+r+1}.
\end{aligned}$$

(37)-(39) yield the result of the proposition. Hence the proof of the proposition is complete.

Proof of Proposition 2.5) Put

$$\eta_K(z) = \sum_{j=s_0}^{\infty}[K_j(b_jz)-\mathrm{E}\{K_j(b_j\epsilon_1)\}].$$

We deal with only the case where $c_K^- < 0 < c_K^+$. The other cases can be treated in the same way. If we establish

$$\lim_{z \pm \infty} |z|^{-1/\beta} \eta_K(z) = \frac{c_0^{1/\beta}}{\beta} \int_0^\infty (K_\infty(\pm s) - K_\infty(0)) s^{-(1+1/\beta)} ds, \tag{40}$$

the proposition follows from the argument in Lemma 3.1 in Surgailis (2002). Note that

$$\lim_{x \to -\infty} |x|^{\alpha\beta} \mathrm{P}(\eta_K(\epsilon_1) < x) = c_1 \frac{c_0^\alpha}{\beta^{\alpha\beta}} \Big(- \int_0^\infty (K_\infty(-s) - K_\infty(0)) s^{-(1+1/\beta)} ds \Big)^{\alpha\beta}$$

and

$$\lim_{x \to \infty} x^{\alpha\beta} \mathrm{P}(\eta_K(\epsilon_1) > x) = c_2 \frac{c_0^\alpha}{\beta^{\alpha\beta}} \Big(\int_0^\infty (K_\infty(s) - K_\infty(0)) s^{-(1+1/\beta)} ds \Big)^{\alpha\beta}$$

when we have (40).

We will establish only (40) when $z \to \infty$. We can proceed in the same way when $z \to -\infty$. Consider $K_j(x) - K_j(0)$ as in the proof of Proposition 2.3 and remember (35). Then it is easy to see that $\eta_K(z)$ is well defined.

As in Surgailis (2002), we represent $\eta_K(z)$ as

$$z^{-1/\beta} \eta_K(z) = z^{-1/\beta} \int_{s_0}^\infty (K_{[t]}(b_{[t]} z) - K_{[t]}(0)) dt + O(z^{-1/\beta}). \tag{41}$$

By making a change of variables of $zc_0 t^{-\beta} = s$, we obtain

$$\begin{aligned} & z^{-1/\beta} \eta_K(z) \\ = \; & \frac{c_0^{1/\beta}}{\beta} \int_0^{zc_0 s_0^{-\beta}} (K_{[(zc_0/s)^{1/\beta}]}(b_{[(zc_0/s)^{1/\beta}]} z) - K_{[(zc_0/s)^{1/\beta}]}(0)) s^{-(1+1/\beta)} ds + O(z^{-1/\beta}). \end{aligned} \tag{42}$$

In (41)-(43), $[a]$ in subscripts stands for the largest integer which is smaller than or equal to a.

(35) implies that

$$|K_{[(zc_0/s)^{1/\beta}]}(b_{[(zc_0/s)^{1/\beta}]} z) - K_{[(zc_0/s)^{1/\beta}]}(0)| < C_1(\{(zc_0/s)^{-1} z\} \wedge C_2). \tag{43}$$

(40) follows from (42), (43), Lemma 5.4, and the dominated convergence theorem. Hence the proof of the proposition is complete.

4 Density estimation

In this section, we investigate the asymptotic properties of the kernel density estimator of $f(x_0)$, where x_0 is a fixed point and $f(x)$ is the density function of X_1. We still assume that $0 < \alpha < 1$ and $1 < \alpha\beta < 2$.

The kernel density estimator $\hat{f}(x_0)$ is defined by

$$\hat{f}(x_0) = \frac{1}{nh}\sum_{i=1}^{n} K\Big(\frac{X_i - x_0}{h}\Big), \tag{44}$$

where h is the bandwidth, $K(\xi)$ is a symmetric bounded density function, and X_i is defined in (1). As we mentioned in Section 1, the asymptotic distribution of $\hat{f}(x_0) - \mathrm{E}\{\hat{f}(x_0)\}$ is given in Honda (2006) in the cases of **a**, **b**, and **c** defined in Section 1. In Honda (2006), $K(\xi)$ has compact support. We just assume that $\int \xi^2 K(\xi)d\xi < \infty$ in this paper.

We derive the asymptotic distribution of $\hat{f}(x_0) - \mathrm{E}\{\hat{f}(x_0)\}$ by following the argument for Case 2 in Honda (2006) and exploiting Theorem 2.2 in this paper. The asymptotic distribution is the same as in Case 2 in Honda (2006).

We take $h = c_3 n^{-\gamma}$, where c_3 is a positive constant and $0 < \gamma < 1$. We recommend $\gamma = 1/5$ from a theoretical point of view. It is because this is the optimal order to estimate $f(x_0)$ when the effect of the heavy tail does not appear in the asymptotic distribution of $\hat{f}(x_0) - \mathrm{E}\{\hat{f}(x_0)\}$. The asymptotic distribution is given in Theorem 4.1 below. The asymptotic distribution is the same as in the case of independent and identically distributed observations when $2/(\alpha\beta) < 1 + \gamma$. On the other hand, the effect of the heavy tail appears when $2/(\alpha\beta) > 1 + \gamma$.

Theorem 4.1 *Suppose that assumptions* **A1-2** *hold. Then we have*

$2/(\alpha\beta) < 1+\gamma$*:* $\sqrt{nh}(\hat{f}(x_0) - \mathrm{E}\{\hat{f}(x_0)\}) \xrightarrow{d} \mathrm{N}(0, \kappa f(x_0))$,

$2/(\alpha\beta) > 1+\gamma$*:* $n^{1-1/(\alpha\beta)}(\hat{f}(x_0) - \mathrm{E}\{\hat{f}(x_0)\}) \xrightarrow{d} c_2^{1/(\alpha\beta)} c_f^+ L^+ + c_1^{1/(\alpha\beta)} c_f^- L^-$,

where $\xrightarrow{d}$ *means convergence in distribution,*

$$\kappa = \int K^2(\xi)d\xi, \qquad c_f^{\pm} = \sigma\int_0^\infty (f(x_0 \mp t) - f(x_0))t^{-(1+1/\beta)}dt,$$

and see Theorem 2.1 for the definitions of σ and $L^{\pm}$.

Proof) We decompose $\hat{f}(x_0)-\mathrm{E}\{\hat{f}(x_0)\}$ as in Wu and Mielniczuk (2002) and Honda (2006). We reproduce (2.2) in Honda (2006) here.

$$\hat{f}(x_0)-\mathrm{E}\{\hat{f}(x_0)\}=S_a+S_b, \tag{45}$$

where

$$\begin{aligned} S_a &= \frac{1}{nh}\sum_{i=1}^{n}\Big[K\Big(\frac{X_i-x_0}{h}\Big)-\mathrm{E}\Big\{K\Big(\frac{X_i-x_0}{h}\Big)\Big|\mathcal{F}_{i-s_0}\Big\}\Big] \qquad (46)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{h}K\Big(\frac{X_i-x_0}{h}\Big)-\int K(\xi)f_{s_0}(x_0+\xi h-\tilde{X}_{i,s_0})d\xi\Big]\\ S_b &= \frac{1}{nh}\sum_{i=1}^{n}\Big[\mathrm{E}\Big\{K\Big(\frac{X_i-x_0}{h}\Big)\Big|\mathcal{F}_{i-s_0}\Big\}-\mathrm{E}\Big\{K\Big(\frac{X_i-x_0}{h}\Big)\Big\}\Big] \qquad (47)\\ &= \frac{1}{n}\sum_{i=1}^{n}\Big[\int K(\xi)f_{s_0}(x_0+\xi h-\tilde{X}_{i,s_0})d\xi-\frac{1}{h}\mathrm{E}\Big\{K\Big(\frac{X_i-x_0}{h}\Big)\Big\}\Big]. \end{aligned}$$

The asymptotic distribution depends on which is stochastically larger, S_a and S_b. Note that $\int K(\xi)f_{s_0}(x_0+\xi h-x)d\xi$ in (47) is bounded and that

$$\int\Big\{\int K(\xi)f_{s_0}(x_0+\xi h-x)d\xi\Big\}dx=1.$$

We state two propositions about the asymptotic distributions of S_a and S_b. We omit the detailed proofs.

Proposition 4.1 *Suppose that assumptions* **A1-2** *hold. Then we have*

$$\sqrt{nh}S_a\xrightarrow{d}\mathrm{N}(0,\kappa f(x_0)).$$

Proposition 4.2 *Suppose that assumptions* **A1-2** *hold. Then we have*

$$n^{1-1/(\alpha\beta)}S_b\xrightarrow{d}c_2^{1/(\alpha\beta)}c_f^{+}L^{+}+c_1^{1/(\alpha\beta)}c_f^{-}L^{-}.$$

Proposition 4.1 follows from Lemma 4.1 and the proof of Theorem 2.2 in Honda (2006). As for Proposition 4.2, we should apply the argument for Theorem 2.2 in this paper to $\int K(\xi) f_{s_0}(x_0+\xi h - x)d\xi$. Then the argument for Theorem 2.2 here implies that

$$nS_b - W_n' = o_p(n^{1/(\alpha\beta)}), \tag{48}$$

where

$$\begin{aligned} W_n' &= \sum_{i=1}^{n}\sum_{j=s_0}^{\infty}(K_{n,\infty}(b_j\epsilon_i) - \mathrm{E}\{K_{n,\infty}(b_j\epsilon_i)\}), \\ K_{n,\infty}(z) &= \int K(\xi)(f(x_0+\xi h - z) - f(x_0+\xi h))d\xi. \end{aligned}$$

Since $\int \xi^2 K(\xi)d\xi < \infty$, the arugument in the proof of Proposition 4.2 in Honda (2006) carries over and we have

$$n^{-1/(\alpha\beta)}W_n' \xrightarrow{d} c_2^{1/(\alpha\beta)}c_f^{+}L^{+} + c_1^{1/(\alpha\beta)}c_f^{-}L^{-}. \tag{49}$$

Then Proposition 4.2 here follows from (48) and (49).

By combining (45) and Propositions 4.1-2 here, we obtain the asymptotic distribution in Theorem 4.1. Hence the proof of the theorem is complete.

5 Technical lemmas

All the technical lemmas and the proofs are given in this section. First we state the lemmas. Then we prove the lemmas.

Lemma 5.1 *Suppose that assumptions* **A1-2** *hold. Then*

$$\mathrm{P}(|b_j\epsilon_1| \geq 1) \leq C|b_j|^{\alpha} \quad and \quad \mathrm{P}(|\tilde{X}_{1,j}| \geq 1) \leq Cj^{-\alpha\beta+1} \quad for\ any\ j \geq 1.$$

Lemma 5.2 *Suppose that assumptions* **A1-2** *hold. Then for any $\gamma \geq 1$, there exists a positive constant C_γ such that*

$$\mathrm{E}\{|b_j\epsilon_1|^{\gamma}I(|b_j\epsilon_1|<1)\} \leq C_\gamma|b_j|^{\alpha} \quad and \quad \mathrm{E}\{|\tilde{X}_{1,j}|^{\gamma}I(|\tilde{X}_{1,j}|<1)\} \leq C_\gamma j^{-\alpha\beta+1}$$

for any $j \geq 1$.

Lemma 5.3 *Suppose that assumptions* **A1-2** *hold. Then*

$$\sup_x |F_j''(x) - F''(x)| \le Cj^{-\alpha\beta+1} \quad \textit{for any } j \ge s_0.$$

Lemma 5.4 *Suppose that assumptions* **A1-2** *hold and that* $K(x)$ *is a bounded function. Then*

$$\lim_{j\to\infty} \sup_x |K_j(x) - K_\infty(x)| = 0.$$

We prove the lemmas.

Proof of Lemma 5.1) The lemma follows from (3.35) in Pipiras and Taqqu (2003) with $j \sim c_0 j^{-\beta}$.

Proof of Lemma 5.2) We verify only the latter inequality with $\gamma = 1$. When j is sufficiently large, $2|b_l| < 1$ for any $l \ge j$. Then by exploiting (3.41) in Pipiras and Taqqu (2003), we get

$$\begin{aligned}
&\mathrm{E}\{|\tilde{X}_{1,j}|I(|\tilde{X}_{1,j}| < 1)\} \\
&\le\ C\sum_{l=j}^{\infty}\Big(\int_0^{2|b_l|} dx + |b_l|^{\alpha}\int_{2|b_l|}^{1} x^{-\alpha}dx\Big) \\
&\le\ C\sum_{l=j}^{\infty}[|b_l| + |b_l|^{\alpha}\{1-(2|b_l|)^{1-\alpha}\}] \\
&\le\ C\sum_{l=j}^{\infty}|b_l|^{\alpha}\ \le\ Cj^{-\alpha\beta+1}.
\end{aligned}$$

Proof of Lemma 5.3) Notice that

$$\begin{aligned}
&|F''(x) - F_j''(x)| \qquad (50)\\
&\le\ \int |F_j''(x-y) - F_j''(x)|d\tilde{F}_j(y) \\
&\le\ \int_{|y|<1} |F_j''(x-y) - F_j''(x)|d\tilde{F}_j(y) + C\int_{|y|\ge 1} d\tilde{F}_j(y) \\
&\le\ C\Big(\int_{|y|<1} |y|d\tilde{F}_j(y) + \mathrm{P}(|\tilde{X}_{1,j}| \ge 1)\Big).
\end{aligned}$$

The lemma follows from (50) and Lemmas 5.1-2.

Proof of Lemma 5.4) By using the differentiability and boundedness of $f_j(x)$ and Lemmas 5.1-2, we have

$$
\begin{aligned}
&|f(x) - f_j(x)| \\
&\le \int |f_j(x-y) - f_j(x)| d\tilde{F}_j(y) \\
&\le \int_{|y|<1} |f_j(x-y) - f_j(x)| d\tilde{F}_j(y) + C \int_{|y|\ge 1} d\tilde{F}_j(y) \\
&\le C\Big(\int_{|y|<1} |y| d\tilde{F}_j(y) + \mathrm{P}(|\tilde{X}_{1,j}| \ge 1) \Big) \le Cj^{-\alpha\beta+1}.
\end{aligned}
$$

Then

$$\lim_{j\to\infty} \sup_x |f(x) - f_j(x)| = 0. \tag{51}$$

(51) and Scheffé's theorem imply

$$\lim_{j\to\infty} \int |f(x) - f_j(x)| dx = 0.$$

Hence

$$|K_j(x) - K_\infty(x)| \le \int |K(x+y)||f_j(y) - f(y)| dy \to 0 \quad \text{as } j \to \infty.$$

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