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Multiplicity and Sensitivity of Stochastically Stable Equilibria in Coordination Games

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Abstract: We investigate the equilibrium selection problem in n-person binary coordination games by means of the adaptive play with mistakes (Young 1993). We show that whenever the difference between the deviation losses of respective equilibria is not overwhelming, the stochastic stability exhibits a notable dependence on payoff parameters associated with strategy profiles where the numbers of players for the respective strategies are nearly equal. This feature necessitates the existence of games that possess multiple stochastically stable equilibria.

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1 Introduction

In a unanimity game, each player receives a positive payoff only when everyone plays the same strategy. In any other strategy profile, the payoff is zero. When one of the two strict equilibria payoff-dominates the other, it may seem natural to expect the payoff dominant equilibrium to be the unique equilibrium selection outcome. For example, the theory of Harsanyi and Selten (1988) fulfills this expectation, as a payoff dominant equilibrium in a unanimity game is risk dominant. Young (1998) pointed out, however, that the expected result need not be true in equilibrium selection by stochastic evolution. Specifically, working in a multi-population random matching environment, Young (1998) shows that there are unanimity games with four players in which both equilibria are stochastically stable although one of them payoff-dominates the other in a wide margin.

At first sight, the source of the "counter-intuitive" result seems to be the non-generic payoffs in unanimity games. Working in the adaptive play with mistakes à la Young (1993), we identify the source of the multiplicity. What is responsible is not payoff ties *per se*, but a particular way the stochastically stable equilibrium depends on non-equilibrium payoffs that arise at strategy profiles where the numbers of players employing the respective strategies are nearly equal.

An example of such a dependence is the following. Consider symmetric binary four-person games, in which the payoff of a particular strategy, A or B, is determined by the number of players who take that strategy. In Figure 1, α and β are equilibrium payoffs. In G_1, each ε_k is a non-equilibrium payoff associated with A. It follows from our main result that if the difference between α and β is not so large, then $(A, \ldots, A)$ is a unique stochastically stable equilibrium for *every* $\varepsilon_k > 0$, *even if* $\alpha < \beta$. Specifically, our analysis shows that the claim is true whenever $2\alpha > \beta$. This is a consequence from the fact that, as long as the deviation losses are relatively close, payoffs in the middle range of the table have a decisive effect on the stochastic stability. Similarly in G_2, $(B, \ldots, B)$ is uniquely stochastically stable provided $2\beta > \alpha$ and $\varepsilon_k > 0$. By continuity, the multiplicity follows in the limiting unanimity game.

A related point can be made in terms of the "mistake-counting" argument in a simple adjustment scenario that is solely based on the stage game best response. In G_1, let the initial state be $(B, \ldots, B)$. If one of the players makes a mistake, all the others can switch to A. By contrast, just one mistake is not enough if the initial state is $(A, \ldots, A)$. This is just a rough idea, as it does not depend on the deviation losses whatsoever. Our analysis offers conditions under which the simple intuition agrees with the stochastic stability in the adaptive play.

	1	2	3	4
a_k	0	ε_2	ε_3	α
b_k	0	0	0	β

G_1

	1	2	3	4
a_k	0	0	0	α
b_k	0	ε_2	ε_3	β

G_2

Figure 1: Four-person symmetric binary coordination games.

In the next section, we specify the class of games under study and recall some relevant aspects of the adaptive play. The main analysis is given in section three. A sufficiency result for the stochastically stable equilibrium allows us to make the preceding argument both precise and more general. Concluding remarks are given in the final section. The appendix gives some of the details that are skipped in the main text.

2 Preliminaries

There are n players, denoted by $i \in I = \{1, \ldots, n\}$, $n \geq 3$. Each player chooses her strategy $\sigma^i \in \{A, B\}$. A generic strategy profile is denoted by $\sigma \in \Sigma = \{A, B\}^n$. Let $|\sigma|_X$ be the number of players employing $X \in \{A, B\}$ in σ. The payoff of player i is given as follows:

$$u^i(\sigma) = \begin{cases} a^i_{|\sigma|_A}, & \text{if } \sigma^i = A, \\ b^i_{|\sigma|_B}, & \text{if } \sigma^i = B, \end{cases}$$

where a^i_k and b^i_k are functions defined on $\{1, \ldots, n\}$ such that

(**G1**) a^i_k and b^i_k are nondecreasing in k,

(**G2**) $a^i_n > b^i_1$ and $b^i_n > a^i_1$.

The game thus defined is called a *binary coordination game*. The condition (**G1**) implies that the payoff associated with a particular strategy depends only on the number of players who adopt that strategy. By (**G2**), both $(A, \ldots, A)$ and $(B, \ldots, B)$ are strict equilibria. In order to ensure that the game has exactly two strict equilibria, we introduce an additional condition. For every $i \in I$, define $k^i = \max\{\, k \mid b^i_{n-k} > a^i_{k+1} \,\}$. Let Figure 2 depict payoff parameters of a particular player $i \in I$. Then the number k^i is $m - 1$. Let $k = |\sigma^{-i}|_A$ be the number of others adopting A and $BR^i(\cdot)$ be the pure best response correspondence. It follows that

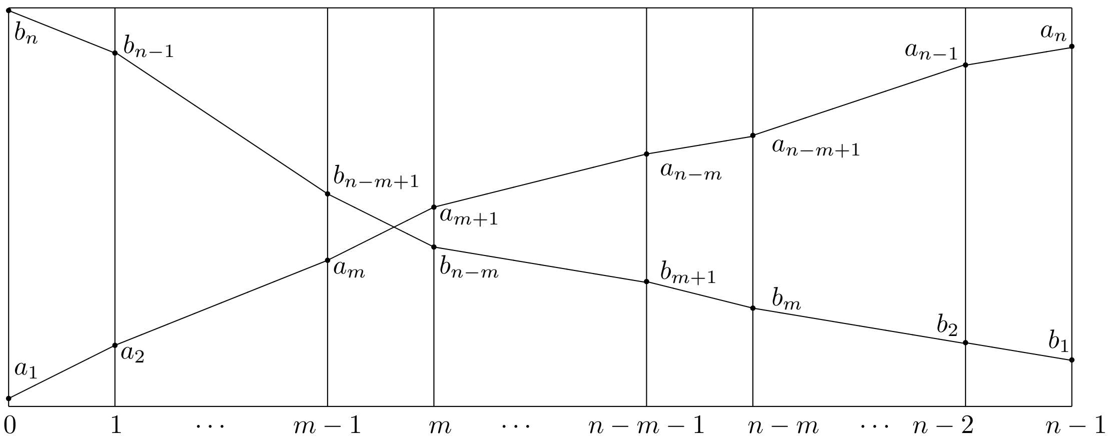

Figure 2: Payoff parameters in an n-person simple binary coordination game.

$BR^i(\sigma) = \{B\}$ if $k \leq k^i$ and $A \in BR^i(\sigma)$ otherwise. Note that $0 \leq k^i \leq n-2$ for every $\in I$. For every $k = 0, 1, \ldots, n-2$, let

$$\bar{I}(k) = \left\{ i \in I \mid k^i \leq k \right\}, \quad \underline{I}(k) = \left\{ i \in I \mid k^i \geq k \right\}, \quad I(k) = \left\{ i \in I \mid k^i = k \right\}.$$

$\bar{I}(k)$ is the set of players with threshold k^i less than or equal to k. Similarly for the other two. The additional condition concerns the distribution of these thresholds.

(**G3**) If there is k, $2 \leq k \leq n-2$, such that $|\bar{I}(k-2)| = k$, then $I(k-1) \neq \varnothing$.

Roughly speaking, (**G3**) implies that the thresholds may differ across players, but they can do so only in a "connected" way.[1] A *simple binary coordination game* is a binary coordination game that satisfies (**G3**). For every $\sigma \in \Sigma$ and $X \in \{A, B\}$, let $I_X(\sigma) = \left\{ i \in I \mid \sigma^i = X \right\}$.

Lemma 1. *A simple binary coordination game has exactly two strict equilibria.*

Proof. Let $\sigma \in \Sigma$ be a strategy profile such that $|\sigma|_A = k$. By (**G2**), if $k = 1$ or $k = n-1$, then σ is not an equilibrium. Thus assume that $2 \leq k \leq n-2$ and that σ is a strict equilibrium. Then one can verify that $I_A(\sigma) = \bar{I}(k-2)$ and that $I_B(\sigma) = \underline{I}(k)$. It follows that $|\bar{I}(k-2)| = k$, $2 \leq k \leq n-2$, and $I = \bar{I}(k-2) \cup \underline{I}(k)$, which contradict (**G3**). $\square$

It should be noted that for a binary coordination game without multiple best responses,[2] (**G3**) is a necessary condition for the game to possess exactly two equilibria. Take such a game

[1] It is clear that (**G3**) is a generalization of the symmetric payoff assumption.

[2] A binary coordination game involves no multiple best responses if and only if $BR^i(\sigma) = \{A\}$ whenever $|\sigma^{-i}|_A > k^i$.

and assume that there is k, $2 \leq k \leq n-2$, such that $|\bar{I}(k-2)| = k$ and $I = \bar{I}(k-2) \cup \underline{I}(k)$. Then the strategy profile σ such that $I_A(\sigma) = \bar{I}(k-2)$ is a strict equilibrium.

Following Harsanyi and Selten (1988), let us call $\alpha^i = a^i_n - b^i_1$ the *deviation loss* of $i \in I$ at equilibrium $(A, \dots, A)$. The deviation loss at $(B, \dots, B)$ is $\beta^i = b^i_n - a^i_1$.

As an equilibrium selection model, we employ the adaptive play with mistakes, introduced by Young (1993). We assume that the reader is familiar to the stochastic stability analysis in general, and the adaptive play with or without mistakes in particular. For details, the reader is referred to Young (1993). The sizes of a history and of a sample are denoted by T and s, respectively. Let $\mathbf{A}$ and $\mathbf{B}$ denote the T-fold concatenations of $(A, \dots, A)$ and $(B, \dots, B)$. We assume that $s \leq T/2$. In section A.1 in the appendix, we show that in a simple binary coordination game the adaptive play without mistakes converges to either $\mathbf{A}$ or $\mathbf{B}$ whenever $s \leq T/2$. Therefore the method of Young (1993) to identify the stochastically stable equilibrium is applicable in the simplest manner.[3] The resistance from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $r(A, B)$. $r(B, A)$ is the resistance for the other direction. $(A, \dots, A)$ is uniquely stochastically stable if and only if $r(A, B) > r(B, A)$.

3 Equilibrium Selection

3.1 The Relevant Linear Program and the Main Result

Consider the adaptive play with mistakes for a simple binary coordination game. The current state is $\mathbf{A}$. In any path from $\mathbf{A}$ to $\mathbf{B}$, there is a player who optimally chooses strategy B for the first time. Let us call that player a *first exitor*. The first exitor $i \in I$ must have a sample against which playing B is optimal. Such a sample must contain considerable number of Bs played by others. Since player i is a first exitor, all such Bs are mistakes. We are going to set up a linear program that gives us the minimum number of Bs that i must face. Its optimal solution not only gives us the number, but also reveals the way the mistakes occur. In many-person games, not only the number, but also the *distribution* of mistakes matters. The linear program introduced below takes care of the case in point.

Fix a player $i \in I$. Set

$$z^i_k = a^i_n - a^i_{n-k} + b^i_{k+1} - b^i_1$$

[3] If we drop (**G3**), then not only the game may have more than two strict equilibria, but also the adaptive play without mistakes may not converge to any strict equilibrium state. See the example given in section A.1. On the other hand, the results in the next section apply to any binary coordination game as long as it has exactly two strict equilibria, to which the adaptive play without mistakes converges.

for $k = 1, \dots, n-1$. Note that z_k^i is nonnegative and nondecreasing in k. The relevant linear program is given as follows:

$$(\mathbf{P^i_A}) \qquad \min x_1 + 2x_2 + \cdots + (n-1)x_{n-1}$$

$$s.t. \quad x_1 + \cdots + x_{n-1} \le s, \quad \textstyle\sum_{k=1}^{n-1} z_k^i x_k \ge s(a_n^i - b_1^i), \quad x_k \ge 0.$$

In this program, x_k is the number of profiles that contain exactly k mistakes. $\sum_k x_k$ is the number of profiles that contain at least one mistake. The first constraint comes from the fact that this number cannot exceed the sample size. The second constraint expands into

$$b_n^i x_{n-1} + \cdots + b_2^i x_1 + (s - \textstyle\sum_{k=1}^{n-1} x_k) b_1^i \ge a_1^i x_{n-1} + \cdots + a_{n-1}^i x_1 + (s - \sum_{k=1}^{n-1} x_k) a_n^i.$$

Thus it ensures that strategy B is a best response against the sample. The objective function gives the total number of mistakes in the sample. It is clear that $(\mathbf{P^i_A})$ has an optimal solution.

The stochastic stability analysis hinges on the number of mistakes. Do we need additional integer constraints? For our purposes, we do not need them, as we only need the following implications. By the definition of the first exitor, if the optimal value of $(\mathbf{P^i_A})$ is at least v for every $i \in I$, then v is a lower bound of the resistance $r(A, B)$. If the optimal value of $(\mathbf{P^i_A})$ is strictly greater than v for every $i \in I$, then the resistance $r(A, B)$ is strictly greater than v.[4]

We are ready to show the main result of the paper.

Theorem. *Consider an n-person simple binary coordination game. If there is a positive integer m such that*

$$(\mathbf{A1})\ m \in \arg\min_{\substack{k \\ z_k^i \neq 0}} \frac{k}{z_k^i}, \qquad (\mathbf{A2})\ m \le \frac{n-1}{2}, \qquad (\mathbf{A3})\ a^i{}_{m+1} \ge b_{n-m}^i$$

are satisfied for every $i \in I$, then $(A, \dots, A)$ is stochastically stable. If $(\mathbf{A3})$ is satisfied by a strict inequality, then $(A, \dots, A)$ is a unique stochastically stable equilibrium.

Proof. The result is a consequence of the following facts.

(1) Under $(\mathbf{A2})$ and $(\mathbf{A3})$, the resistance from $(B, \dots B)$ to $(A, \dots A)$ is at most sm.

(2) Under $(\mathbf{A1})$, $(\mathbf{A2})$ and $(\mathbf{A3})$, the optimal value of $(\mathbf{P^i_A})$ is at least sm for every $i \in I$. If $(\mathbf{A3})$ is strengthened to a strict inequality, the optimal value is strictly greater than sm.

[4]In general, the resistance $r(A, B)$ need not be the value of the optimal integer solution of some $(\mathbf{P^i_A})$.

	Phase 1 $\overbrace{\qquad}^{T}$			Phase 2 $\overbrace{\qquad}^{s}$			Phase 3 $\overbrace{\qquad}^{s}$			Phase 4 $\overbrace{\qquad}^{s}$		
σ_1	B	$\cdots$	B	A^*	$\cdots$	A^*	B	$\cdots$	B	A	$\cdots$	A
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_m	B	$\cdots$	B	A^*	$\cdots$	A^*	B	$\cdots$	B	A	$\cdots$	A
σ_{m+1}	B	$\cdots$	B	B	$\cdots$	B	A	$\cdots$	A	A	$\cdots$	A
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ_n	B	$\cdots$	B	B	$\cdots$	B	A	$\cdots$	A	A	$\cdots$	A

Figure 3: A path from **B** to **A**.

To prove (1), it suffices to construct a path from **B** to **A** in which there are exactly sm mistakes. See Figure 3. In Phase 2, let every player sample Phase 1. There are exactly sm mistakes in Phase 2. Let $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, n\}$. In Phase 3, let j sample Phase 2. Then A is a best response for j by (**A3**). In Phase 4, let i sample Phase 3. Then respective strategies yield a^i_{n-m+1} and b^i_m. By (**A2**), $n - m \geq m$. Therefore (**A3**) and (**G1**) imply that $a^i_{n-m+1} \geq a^i_{m+1} \geq b^i_{n-m} \geq b^i_m$, which allows i to choose A. Letting j sample the final available segment of Phase 2 and the initial segment of Phase 4, we make her choose A in Phase 4 as well. Finally, note that these sample assignments are possible as long as $s \leq T/2$.

To prove (2), pick $i \in I$ and let $(x_1, \dots, x_{n-1})$ be an optimal solution of $(\mathbf{P^i_A})$. By (**A1**),

$$mz^i_k x_k \leq kz^i_m x_k$$

for every $k = 1, \dots, n-1$. Assume for the contrary that the optimal value is smaller than sm. Then

$$m \sum_k z^i_k x_k \leq z^i_m \sum_k kx_k < z^i_m sm.$$

Therefore $\sum_k z^i_k x_k < sz^i_m$. On the other hand, the best response constraint in $(\mathbf{P^i_A})$ dictates that $\sum_k z^i_k x_k \geq s(a^i_n - b^i_1)$. Hence $a^i_n - b^i_1 < z^i_m$, or $a^i_{n-m} < b^i_{m+1}$. It follows from (**A2**) that $n - m \geq m + 1$. Thus the preceding strict inequality contradicts (**A3**) and (**G1**) since $a^i_{n-m} \geq a^i{}_{m+1} \geq b^i_{n-m} \geq b^i_{m+1}$. The claim for the unique selection can be proved similarly. □

Figure 2 should help us better understand the implications of the conditions. (**A3**) implies that $k^i \leq m - 1$ for every $i \in I$. (**A2**) and (**A3**) imply that the graph of A and the graph of B intersects in the left half of the domain. One might say that strategy A has the larger basin of attraction in this case. In many-person games, $(A, \dots, A)$ needs an additional condition to become stochastically stable.[5] The condition (**A1**) is deeply related to the relevant linear program. The fractions to be minimized appear as the relative cost coefficients in the simplex algorithm. If m is a unique minimizer in (**A1**), then (**A1**) through (**A3**) imply that in any optimal solution of ($\mathbf{P^i_A}$) the sample size constraint binds and the solution typically contains at least two non-zero entries.[6] This means that the program admits no simple solution that is analogous to those found in two-person games.

3.2 Sensitivity of Stochastic Stability on Payoffs in the "Middle"

It is clear that there are ranges for payoff parameters within which all the conditions in the preceding theorem are satisfied. A set of conditions below is particularly useful in revealing the sensitivity of the stochastic stability on payoffs at strategy profiles where the numbers of players for the respective strategies are nearly equal.

Proposition 1. *Consider an n-person simple binary coordination game. If there is a positive integer $m \leq (n-1)/2$ such that the following conditions are satisfied for every $i \in I$, then $(A, \ldots, A)$ is a unique stochastically stable equilibrium:*

(1.1) $\beta^i < \dfrac{(n-m-1)\alpha^i}{m}$.

(1.2) $a^i_{m+1} > b^i_{n-m}$.

(1.3) *There exists $\gamma^i \in (b^i_{n-m}, a^i_{m+1})$ such that for every $k \neq n-1$,*

$$a^i_{n-k} > \gamma^i + \frac{(m-k)\alpha^i}{2m} \quad \textit{and} \quad b^i_{k+1} < \gamma^i - \frac{(m-k)\alpha^i}{2m}.$$

A sketch of the proof follows. (1.1) and (1.3) imply that $mz^i_k < k\alpha^i$ for $k \neq m$. Pick sufficiently small $\varepsilon^i > 0$ such that $mz^i_k \leq k(\alpha^i - \varepsilon^i)$ for every k. Decreasing a^i_{m+1} and increasing b^i_{n-m} appropriately and then setting $\overline{a}^i_{m+1} = \cdots = \overline{a}^i_{n-m} = \overline{b}^i_{m+1} = \cdots = \overline{b}^i_{n-m}$, (1.2) allows us to construct a simple binary coordination game $\overline{G}$ in which

$$\overline{z}^i_k = \begin{cases} \alpha^i - \varepsilon^i, & \text{if } k = m, \ldots, n-m-1, \\ z^i_k, & \text{otherwise.} \end{cases}$$

[5] One can verify that by increasing b^i_n while keeping all the others fixed $(B, \ldots, B)$ eventually becomes a unique stochastically stable equilibrium, provided the sample size s is not extremely small. See section A.2. Hence the validity of the "intuition" that the stochastic stability selects an equilibrium with the largest basin of attraction is dubious outside the class of two-by-two games, *even* in models with state independent mistake (or mutation) rate.

[6] This follows from Proposition 3 in the appendix.

	$\left(\frac{n-m-1}{n-1}\right)s$			$\left(\frac{m}{n-1}\right)s$		
σ^1	A	$\cdots$	A	B^*	$\cdots$	B^*
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ^{n-1}	A	$\cdots$	A	B^*	$\cdots$	B^*
σ^n	A	$\cdots$	A	A	$\cdots$	A

Sample 1

	s		
σ^1	B^*	$\cdots$	B^*
$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ^m	B^*	$\cdots$	B^*
σ^{m+1}	A	$\cdots$	A
$\vdots$	$\vdots$	$\cdots$	$\vdots$
σ^n	A	$\cdots$	A

Sample 2

Figure 4: Samples in the adaptive play with mistakes.

It follows that $\overline{G}$ satisfies (**A1**) to (**A3**). By (2) in the proof of the main result, every feasible solution of $(\overline{\mathbf{P}}^{\mathbf{i}}_{\mathbf{A}})$ has a value greater than sm. Since $\overline{z}^i_k \geq z^i_k$, every feasible solution of $(\mathbf{P}^{\mathbf{i}}_{\mathbf{A}})$ is feasible in $(\overline{\mathbf{P}}^{\mathbf{i}}_{\mathbf{A}})$. Therefore every feasible solution of $(\mathbf{P}^{\mathbf{i}}_{\mathbf{A}})$ has a value greater than sm. On the other hand, m satisfies (**A2**) and (**A3**) in the original game. Thus it follows from (1) in the proof of the main theorem that the resistance from $(B, \ldots B)$ to $(A, \ldots A)$ is at most sm.

Note that as long as (1.2) is satisfied, (1.3) places no additional restrictions on the payoff parameters $a^i_{m+1}, \ldots, a^i_{n-m}$ and $b^i_{m+1}, \ldots, b^i_{n-m}$. Therefore the advantage of strategy A over strategy B can be arbitrarily small. The result shows that any slight advantage in the "middle" can compensate a disadvantage near equilibrium whenever the disadvantage is not overwhelming. It generalizes the argument given in the introduction.

Figure 4 illustrates how Proposition 1 works. By (1.1), sample 1 in Figure 4 does not have enough mistakes to make player n switch. Neither does sample 2, since $b^i_{m+1} < a^i_{n-m}$ by (1.2). Note that these samples contain sm mistakes. On the other hand, in the proof of the main theorem we saw that the phase 2 in Figure 3, which also contains sm mistakes, can reach **A**.

3.3 Multiplicity of Stochastically Stable Equilibria

Interchanging a^i_k and b^i_k and replacing z^i_k with $w^i_k = b^i_n - b^i_{n-k} + a^i_{k+1} - a^i_1$, the main theorem generates a set of conditions that ensures the stochastic stability of $(B, \ldots, B)$. If all the conditions for respective equilibria are satisfied within a single game, then multiple stochastically stable equilibria arise.

An n-person simple binary coordination game is said to possess *m-indifference property* if for every $i \in I$ both A and B are best responses against $\sigma \in \Sigma$ whenever there are at least m others who play A and at least m others who play B. An equivalent condition is given in (2.2) below. A *unanimity* game is a simple binary coordination game with 1-indifference property with additional conditions that $a^i_1 = a^i_{n-1}$ and $b^i_1 = b^i_{n-1}$ for every $i \in I$.

Proposition 2. *Consider an n-person simple binary coordination game. If there are a positive integer $m < (n-1)/2$ and real numbers $\gamma^i \in [b^i_1, a^i_n) \cap [a^i_1, b^i_n)$ such that the following conditions are satisfied for every $i \in I$, then both $(A, \ldots, A)$ and $(B, \ldots, B)$ are stochastically stable:*

(2.1) $\beta^i \leq \dfrac{(n-m-1)\alpha^i}{m}$ *and* $\alpha^i \leq \dfrac{(n-m-1)\beta^i}{m}$.

(2.2) $b^i_{m+1} = \cdots = b^i_{n-m} = a^i_{m+1} = \cdots = a^i_{n-m} = \gamma^i$.

(2.3) *For the remaining parameters* $k = 1, \ldots, m-1, n-m, \ldots, n-2$,

$$a^i_{n-k} \in \left[\gamma^i + \tfrac{(m-k)\alpha^i}{2m}, \gamma^i + \tfrac{(n-m-k-1)\beta^i}{2m}\right], \quad b^i_{k+1} \in \left[\gamma^i - \tfrac{(n-m-k-1)\beta^i}{2m}, \gamma^i - \tfrac{(m-k)\alpha^i}{2m}\right].$$

For games with 1*-indifference property,* (2.3) *can be simply dropped.*

In this result, payoff parameters should satisfy twice as many conditions as those in Proposition 1. Here again, (2.1) and (2.3) take care of conditions (**A1**) and (**B1**)[7]. A special care must be taken, however, for the consistency of (2.1) and (2.3). At this point, the condition that $m < (n-1)/2$ comes in. Without it, (2.1) may become trivial. Moreover, it implies that each interval in (2.3) has nonempty interior, and that the minimum of the interval for a^i_{n-k} (b^i_{k+1}, respectively) is strictly less than the maximum of the interval for a^i_{n-k+1} (b^i_{k+2}, respectively). The latter implication makes sure that the intervals are distributed in such a way that is consistent with (**G1**).

There are other classes of games in which non-trivial multiplicity may arise. An (n, l)-*coordination game*, where $l > n/2$, is an n-person simple binary coordination game with non-negative payoffs such that a particular strategy yields a positive payoff if and only if there are at least $l-1$ others who play the same. Such a game can be a natural model of collective decision making. Proposition 2 is not applicable to (n, l)-coordination games as payoff ties extend asymmetrically around the center. Invoking the main result directly, however, one can show that an (n, l)-coordination game can possess non-trivial multiple stochastically stable equilibria if and only if $l > (n+3)/2$.

[7]The counterpart condition of (**A1**) for the stochastic stability of $(B. \ldots, B)$.

4 Concluding Remarks

We have shown equilibrium selection results under which the multiplicity of stochastically stable equilibria can be understood as a consequence of its particular form of dependence on non-equilibrium payoffs. Given an m-indifferent simple binary coordination game with multiple stochastically stable equilibria, Proposition 1 implies that each neighborhood of the game contains a game in which $(A, \ldots, A)$ is uniquely stochastically stable and another in which $(B, \ldots, B)$ is uniquely stochastically stable. As a correspondence, the stochastically stable equilibrium is upper hemicontinuous but not lower hemicontinuous.[8]

In one of the rare studies that focus on the equilibrium selection in a many-person stage game, Kim (1996) obtains, among other things, a unique equilibrium selection for a symmetric n-person binary coordination game by means of the stochastic stability. Following Kandori, Mailath, and Rob (1993), he works in a single population random matching environment. The source of the difference between his result and ours should be found in the difference between the respective equilibrium selection dynamics.[9] In the adaptive play with mistakes, a state is a finite sequence of stage game strategy profiles. The whole state space can be naturally embedded into the mixed strategy space of the stage game. The crucial property is that the range is full dimensional, in that it includes every extreme point in the mixed strategy space. In the single population random matching model, by contrast, the range of the natural embedding is single dimensional. In particular, the range includes just two extreme points, the two unanimous pure strategy profiles. A consequence is that some features that the best response structure of the stage game possesses may simply disappear. For this reason, neither the presence of multiple best responses in unanimity games nor the mistake counting argument in Figure 4 has any implication in the model.[10,11] This observation should also convince us the reason why Young (1998) finds the multiplicity: the state space of the multi-population random matching is full dimensional.[12]

[8] This feature reminds us of the equilibrium refinement literature. See, for example, Okada (1981).

[9] Combining the result of Kim (1996) and ours, it follows that a simple binary coordination game that satisfies all the conditions of Proposition 2 is an example of games for which Kandori et al. (1993) and Young (1993) produce different selection outcomes. Jacobsen et al. (2000) find two-person coordination games on which the selection outcomes of the two models differ.

[10] As Kim (1996) suggests, it may well be the case that the "intuition" mentioned in footnote 5 is valid for single population models.

[11] Concerning the stability properties of versions of the replicator dynamics, Weibull (1995) discusses the difference between the single population model and the multi-population model.

[12] Thus, results analogous to ours should hold true in the multi-population random matching model.

Unexpected results in binary coordination games are not specific to the stochastic evolution literature. Morris and Ui (2005) and Oyama, Takahashi, and Hofbauer (2005) generalize previous results in equilibrium selection, respectively, by robustness with respect to incomplete information (Kajii and Morris 1997) and by perfect foresight dynamics (Matsui and Matsuyama 1995). Even for unanimity games, however, no clear-cut results have been shown. Meanwhile, Hofbauer (1999) offers a dynamic selection model in which the risk dominant equilibrium (in the sense of Harsanyi and Selten 1988) is selected in n-person unanimity games.

One might be inclined to think that binary coordination games are particularly simple class of games. As far as equilibrium selection is concerned, such a view may be ill-founded.

Appendix

A.1 Convergence in the adaptive play

For convenience, we collect relevant definitions and properties.

- $k^i = \max\left\{ k \mid b^i_{n-k} > a^i_{k+1} \right\}$. Recall that $0 \leq k^i \leq n-2$.
- For $\sigma \in \Sigma$ such that $|\sigma^{-i}|_A = k$, $BR^i(\sigma) = \{B\}$ if $k \leq k^i$ and $A \in BR^i(\sigma)$ if $k > k^i$.
- For every $k = 0, 1, \ldots, n-2$,

$$\bar{I}(k) = \left\{ i \in I \mid k^i \leq k \right\}, \quad \underline{I}(k) = \left\{ i \in I \mid k^i \geq k \right\}, \quad I(k) = \left\{ i \in I \mid k^i = k \right\}.$$

- For every $\sigma \in \Sigma$, $I_A(\sigma) = \left\{ i \in I \mid \sigma^i = A \right\}$ and $I_B(\sigma) = \left\{ i \in I \mid \sigma^i = B \right\}$.
- A simple binary coordination game is a binary coordination game satisfying

  (**G3**) If there is k, $2 \leq k \leq n-2$, such that $|\bar{I}(k-2)| = k$, then $I(k-1) \neq \varnothing$.

Recall that T and s are the history size and the sample size of the adaptive play. Adaptive play without mistakes is *absorbing* if, starting from any state, the play can reach a strict equilibrium state in a finite number of steps.

Lemma 2. *Adaptive play without mistakes for a simple binary coordination game is absorbing if $T = 2$ and $s = 1$.*

Proof. Consider the adaptive play with $(T, s) = (2, 1)$. Let $\sigma \in \Sigma$ be the sample given to $i \in I$ on a particular day. To avoid ambiguities caused by multiple best responses, let the player choose B if and only if B is a unique best response against σ. Formally,

$$br^i(\sigma) = \begin{cases} A, & \text{if } i \in \bar{I}(k-1), \\ B, & \text{if } i \in \underline{I}(k), \end{cases} \tag{$\star$}$$

where $k = |\sigma^{-i}|_A$, the number of others that adopt A. Clearly, $br^i(\sigma) \in BR^i(\sigma)$.

Pick $\sigma_1 \in \Sigma$. Starting from σ_1, we construct a path that leads to either $(A, \dots, A)$ or $(B, \dots, B)$. Set $|\sigma_1|_A = k_1$ so that

$$\sigma_1 = (\overbrace{A, \dots, A}^{k_1}, B, \dots, B).$$

We can assume that $1 \leq k_1 \leq n-1$. On day 2, let everyone sample σ_1. Following $(\star)$, they play $\sigma_2^i = br^i(\sigma_1)$. The outcome of day 2 is σ_2. By construction,

- For every $i \in \bar{I}(k_1 - 2)$, $i \in I_A(\sigma_2)$.
- For every $i \in \underline{I}(k_1)$, $i \in I_B(\sigma_2)$.
- For every $i \in I(k_1 - 1)$, $i \in I_A(\sigma_2)$ if and only if $i \in I_B(\sigma_1)$.

Thus σ_2 can be written as

$$\sigma_2 = (\overbrace{\overbrace{A, \dots, A}^{|\bar{I}(k_1-2)|},\ \overbrace{A, \dots\dots, A}^{|I(k_1-1)\cap I_B(\sigma_1)|}}^{k_2},\ \overbrace{B, \dots\dots, B}^{|I(k_1-1)\cap I_A(\sigma_1)|},\ \overbrace{B, \dots, B}^{|\underline{I}(k_1)|}).$$

Case 1. $k_2 > k_1$. Following $(\star)$, let $\sigma_3^i = br^i(\sigma_2)$ for every $i \in I$. We show that $k_3 = |\sigma_3|_A > k_2$. In σ_2, every $i \in I$ has at least k_1 others playing A. Therefore $i \in I_A(\sigma_3)$ for every $i \in \bar{I}(k_1 - 1)$. Hence $k_3 \geq k_2$. If $k_2 \geq n-1$, then $\sigma_3 = (A, \dots, A)$. Thus we can assume $2 \leq k_2 \leq n-2$.

Claim 1: If $I(k_1 - 1) \cap I_A(\sigma_1) = \varnothing$ then $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$.

Proof. Assume that $I(k_1 - 1) \cap I_A(\sigma_1) = \varnothing$. It follows that $|\bar{I}(k_1 - 1)| = k_2$. If $\underline{I}(k_1) \cap \bar{I}(k_2 - 2) \neq \varnothing$, then $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$. If $\underline{I}(k_1) \cap \bar{I}(k_2 - 2) = \varnothing$, then $\bar{I}(k_2 - 2) = \bar{I}(k_1 - 1)$. Hence $|\bar{I}(k_2 - 2)| = k_2$. Thus (**G3**) implies that $I(k_2 - 1) \neq \varnothing$. Since $k_2 > k_1$, $I(k_2 - 1) \subset \underline{I}(k_1)$. Therefore $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$. $\|$

It follows from Claim 1 that either $I(k_1 - 1) \cap I_A(\sigma_1) \neq \varnothing$ or $\underline{I}(k_1) \cap \bar{I}(k_2 - 1) \neq \varnothing$. Therefore $k_3 > k_2$. Under the sample assignment $\sigma^i_{t+1} = br^i(\sigma_t)$ for every $i \in I$, we have shown that $k_2 > k_1$ implies $k_3 > k_2$. By induction, the play eventually reaches $(A, \ldots, A)$ if $k_2 > k_1$.

Case 2. $k_2 < k_1$. Following $(\star)$, let $\sigma^i_3 = br^i(\sigma_2)$ for every $i \in I$. We show that $k_3 = |\sigma_3|_A < k_2$. In σ_2, every $i \in I$ has at most $k_1 - 1$ others playing A. Therefore $i \in I_B(\sigma_3)$ for every $i \in \underline{I}(k_1 - 1)$. Hence $k_3 \leq k_2$. If $k_2 \leq 1$, then $\sigma_3 = (B, \ldots, B)$. Thus we can assume $2 \leq k_2 \leq n - 2$.

> Claim 2: If $I(k_1 - 1) \cap I_B(\sigma_1) = \varnothing$ then $\bar{I}(k_1 - 2) \cap \underline{I}(k_2 - 1) \neq \varnothing$.
>
> *Proof.* Assume that $I(k_1-1) \cap I_B(\sigma_1) = \varnothing$. It follows that $|\bar{I}(k_1-2)| = k_2$. Assume that $R = \bar{I}(k_1-2) \cap \underline{I}(k_2-1) = \varnothing$. Then $\bar{I}(k_2-2) = \bar{I}(k_1-2)$. Hence $|\bar{I}(k_2-2)| = k_2$. Thus (**G3**) implies that $I(k_2 - 1) \neq \varnothing$. Since $k_2 < k_1$, $I(k_2 - 1) \subset \bar{I}(k_1 - 2)$. Therefore $R = \bar{I}(k_1 - 2) \cap \underline{I}(k_2 - 1) \neq \varnothing$. We have shown that $R = \varnothing$ implies $R \neq \varnothing$. This is equivalent to $R \neq \varnothing$. $\|$

It follows from Claim 2 that either $I(k_1 - 1) \cap I_B(\sigma_1) \neq \varnothing$ or $\bar{I}(k_1 - 2) \cap \underline{I}(k_2 - 1) \neq \varnothing$. Therefore $k_3 < k_2$. Under the sample assignment $\sigma^i_{t+1} = br^i(\sigma_t)$ for every $i \in I$, we have shown that $k_2 < k_1$ implies $k_3 < k_2$. By induction, the play eventually reaches $(B, \ldots, B)$ if $k_2 < k_1$.

Case 3. $k_2 = k_1$ and $2 \leq k_2 \leq n - 2$. In this case, σ_2 can be written as follows.

$$\sigma_2 = (\overbrace{\overbrace{A, \ldots, A}^{|\bar{I}(k_2-2)|}, \overbrace{A, \ldots\ldots, A}^{|I(k_2-1) \cap I_B(\sigma_1)|}}^{k_2}, \overbrace{B, \ldots\ldots, B}^{|I(k_2-1) \cap I_A(\sigma_1)|}, \overbrace{B, \ldots, B}^{|\underline{I}(k_2)|}).$$

If $I(k_2 - 1) = \varnothing$, then $|\bar{I}(k_2 - 2)| = k_2$. Thus (**G3**) implies $I(k_2 - 1) \neq \varnothing$.

Let $\sigma^i_3 = br^i(\sigma_2)$. Then

$$\sigma_3 = (\overbrace{\overbrace{A, \ldots, A}^{|\bar{I}(k_2-2)|}, \overbrace{A, \ldots\ldots, A}^{|I(k_2-1) \cap I_A(\sigma_1)|}}^{k_3}, \overbrace{B, \ldots\ldots, B}^{|I(k_2-1) \cap I_B(\sigma_1)|}, \overbrace{B, \ldots, B}^{|\underline{I}(k_2)|}).$$

If $k_3 \neq k_2$, then we can apply either Case 1 or Case 2. Thus we can assume that $k_3 = k_2 = k_1$. Then

$$|I(k_2 - 1) \cap I_A(\sigma_1)| = |I(k_2 - 1) \cap I_B(\sigma_1)| \geq 1. \tag{†}$$

The inequality follows from $I(k_2-1) \neq \varnothing$. For every $i \in I(k_2-1) \cap I_A(\sigma_1)$, let $\sigma_4^i = br^i(\sigma_2)$. For everyone else, let $\sigma_4^i = br^i(\sigma_3)$. Then

$$\sigma_4 = (\overbrace{\overbrace{A,\ldots,A}^{|\bar{I}(k_2-2)|},\ \overbrace{A,\ldots\ldots,A}^{|I(k_2-1)\cap I_A(\sigma_1)|},\ \overbrace{A,\ldots\ldots,A}^{|I(k_2-1)\cap I_B(\sigma_1)|}}^{k_4},\ \overbrace{B,\ldots,B}^{|\underline{I}(k_2)|}).$$

By (†), $k_4 > k_2$. Following ($\star$), let $\sigma_5^i = br^i(\sigma_4)$ for every $i \in I$. We show that $k_5 = |\sigma_5|_A > k_4$. In σ_4, every $i \in I$ has at least k_2 others playing A. Therefore $i \in I_A(\sigma_5)$ for every $i \in \bar{I}(k_2-1)$. Hence $k_5 \geq k_4$. If $k_4 \geq n-1$, then $\sigma_5 = (A,\ldots,A)$. Thus we can assume that $3 \leq k_4 \leq n-2$.

> Claim 3: If $\underline{I}(k_2) \cap \bar{I}(k_4-2) = \varnothing$ then $\underline{I}(k_2) \cap \bar{I}(k_4-1) \neq \varnothing$.
>
> *Proof.* Assume that $\underline{I}(k_2) \cap \bar{I}(k_4-2) = \varnothing$. It follows that $|\bar{I}(k_4-2)| = k_4$. Thus (**G3**) implies that $I(k_4-1) \neq \varnothing$. Since $k_4 > k_2$, $I(k_4-1) \subset \underline{I}(k_2)$. Therefore $\underline{I}(k_2) \cap \bar{I}(k_4-1) \neq \varnothing$. ‖

Noting that $\bar{I}(k_4-2) \subset \bar{I}(k_4-1)$, it follows from Claim 3 that $\underline{I}(k_2) \cap \bar{I}(k_4-1) \neq \varnothing$. Therefore $k_5 > k_4$, which allows us to apply Case 1 for the rest of the play.

Case 4. $k_2 = k_1 = 1$ or $k_2 = k_1 = n-1$. Consider the first case. Then σ_2 can be written as

$$\sigma_2 = (A, \overbrace{B,\ldots\ldots,B}^{|I(0)\cap I_A(\sigma_1)|}, \overbrace{B,\ldots,B}^{|\underline{I}(1)|}).$$

Letting $\sigma_3^1 = br^1(\sigma_2)$ and $\sigma_3^i = br^i(\sigma_1)$ for every $i \neq 1$, we have $\sigma_3 = (B,\ldots,B)$. In the second case,

$$\sigma_2 = (\overbrace{A,\ldots,A}^{|\bar{I}(n-3)|},\ \overbrace{A,\ldots\ldots,A}^{|I(n-2)\cap I_B(\sigma_1)|},\ B).$$

Letting $\sigma_3^n = br^n(\sigma_2)$ and $\sigma_3^i = br^i(\sigma_1)$ for every $i \neq n$, we have $\sigma_3 = (A,\ldots,A)$. □

Lemma 3. *Adaptive play without mistakes for a simple binary coordination game is absorbing if $s \leq T/2$.*

Proof. Consider the adaptive play in which $s \leq T/2$. Fix an initial state, an arbitrary sequence in Σ with length T. Let it be day 1. On day 1 through day s, let everyone sample the outcomes from day $-s+1$ through day zero and play best response. The assignment is possible since $s \leq T/2$. We can assume that players choose the same strategy throughout Phase 1, which consists of day 1 through day s. Phase 1 results in the s-run of $\sigma_1 \in \Sigma$.

In the adaptive play with $(T, s) = (2, 1)$, there is a path from σ_1 that eventually leads either $(A, \ldots, A)$ or $(B, \ldots, B)$. It suffices to replicate the sequence in the adaptive play with the current setting. Among cases appeared in the proof of Lemma 2, the replication is obvious unless $k_1 = k_2 = k_3$. Take $\sigma_1, \ldots, \sigma_4$ that appeared in Case 3 in the proof of Lemma 2, where $k_1 = k_2 = k_3$. Let Phase k be the s-run of σ_k, $k = 1, 2, 3$, and let Phase 3 follow Phase 2, which follows Phase 1.

Next s dates consist Phase 4. It suffices to show that it can be the s-run of σ_4. Throughout Phase 4, let every $i \in I(k_2 - 1) \cap I_A(\sigma_1)$ sample the final available segment of Phase 2 and the initial available segment of Phase 4. Let everyone else sample the entire Phase 3. These sample assignments are possible since $s \leq T/2$.

It is clear that every $i \notin I(k_2 - 1) \cap I_A(\sigma_1)$ plays σ_4^i throughout Phase 4. On first day in Phase 4, $i \in I(k_2 - 1) \cap I_A(\sigma_1)$ observes the s-run of σ_2. Thus she can play A. Hence the outcome of the first day in Phase 4 can be σ_4. On t-th day in Phase 4, inductively, i observes the $s - t + 1$-run of σ_2 and the $t - 1$-run of σ_4. Since $A \in BR^i(\sigma_2) \cap BR^i(\sigma_4)$, she plays A on day t. Consequently, Phase 4 can come out as the s-run of σ_4. □

In section 2 we remarked that a binary coordination game without (**G3**) may have a third strict equilibrium. Moreover, games without (**G3**) need not possess the desirable convergence property in the adaptive play.

Example 1. Let players $i = 1, 2$ have payoff function u in Figure 5. Let player $i = 3$ have payoff function w in the Figure. Let players $i = 4, 5$ have payoff function v in the Figure. It follows that $(k^1, k^2, k^3, k^4, k^5) = (0, 0, 0, 3, 3)$. Since $|\bar{I}(3 - 2)| = 3$, (**G3**) is violated. Nonetheless the five-person game has exactly two strict equilibria, because player 3 has a unique best response only if all the others are making a unanimous choice. Now consider the strategy profiles (A, A, A, B, B) and (A, A, B, B, B). Each is a non-strict equilibrium, in which all players but 3 are playing their unique best responses. It is clear that all the states consisting solely of these equilibria form a non-singleton recurrent class in the adaptive play.

A.2 Simple optimal solutions in the relevant program

The next result characterizes the conditions under which ($\mathbf{P^i_A}$) admits a simple optimal solution.

Proposition 3. *Consider the program* ($\mathbf{P^i_A}$) *of a player $i \in I$ in a binary coordination game. Denote by λ_1 and λ_2 the Lagrange multipliers for the best response constraint and the sample size constraint, respectively. The following conditions are equivalent:*

		1	2	3	4	5
u	a_k	0	ε_2	ε_3	ε_4	α
	b_k	0	0	0	0	β

		1	2	3	4	5
v	a_k	0	0	0	0	α
	b_k	0	ε_2	ε_3	ε_4	β

		1	2	3	4	5
w	a_k	0	0	0	0	α
	b_k	0	0	0	0	β

Figure 5: Payoffs in a five-person game, in which $0 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < \min\{\alpha, \beta\}$.

(1) *There is* $k^* \in \arg\min_{\substack{k \\ z_k^i \neq 0}} \dfrac{k}{z_k^i}$ *such that* $b_{k^*+1}^i \geq a_{n-k^*}^i$.

(2) *There is an optimal solution in which* $\lambda_2 = 0$.

(3) *The solution*

$$(x_1^*, \ldots, x_{n-1}^* : \lambda_1^*, \lambda_2^*) = \Big(\overbrace{0, \ldots, 0, \frac{s(a_n^i - b_1^i)}{z_{k^*}^i}}^{k^*}, 0, \ldots, 0 : \frac{k^*}{z_{k^*}^i}, 0\Big)$$

is optimal.

Proof. Since (3) implies (2), it suffices to show that (2) implies (1) and that (1) implies (3). In this proof, we omit superscripts for an arbitrarily chosen $i \in I$. We invoke the duality theorem. Note that the dual program of $(\mathbf{P_A^i})$ is given by

$$\max s(a_n - b_1)\lambda_1 - s\lambda_2$$
$$s.t. \quad z_1\lambda_1 - \lambda_2 \leq 1,\ \ldots,\ z_k\lambda_1 - \lambda_2 \leq k,\ \ldots,\ z_{n-1}\lambda_1 - \lambda_2 \leq n-1,$$

together with the nonnegativity condition $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

Assuming (2), we show (1). Take an optimal solution with $\lambda_2 = 0$. Then by dual feasibility, $z_k\lambda_1 \leq k$. Thus $\lambda_1 \leq k/z_k$ for every $z_k \neq 0$. By complementary slackness, $z_k\lambda_1 = k$ for every k such that $x_k \neq 0$. In particular, $z_k > 0$ for every such k. Therefore

$$\lambda_1 = \min_{\substack{k \\ z_k \neq 0}} \frac{k}{z_k} > 0.$$

Let

$$\arg\min_{\substack{k \\ z_k \neq 0}} \frac{k}{z_k} = \{k_1, \ldots k_l\}.$$

Assume that $b_{k_j+1}^i < a_{n-k_j}^i$ for every $j = 1, \ldots, l$. Then $z_{k_j} < a_n - b_1$. Thus

$$z_{k_1}x_{k_1} + \cdots + z_{k_l}x_{k_l} < (a_n - b_1)x_{k_1} + \cdots + (a_n - b_1)x_{k_l}$$

$$= (a_n - b_1)(x_1 + \cdots + x_n) \leq s(a_n - b_1).$$

Therefore $\sum_k z_k x_k < s(a_n - b_1)$ since $x_k > 0$ implies $k = k_j$ for some $j = 1, \ldots, l$. But this contradicts the complementary slackness since $\lambda_1 > 0$. Therefore there is k^* such that $k^*/z_{k^*} = \min_{k, z_k \neq 0} k/z_k$ and $b^i_{k^*+1} \geq a^i_{n-k^*}$.

Assuming (1), we show (3). Consider the solution given in (3). Nonnegativity constraints are all satisfied. Since $b_{k^*+1} \geq a_{n-k^*}$, $z_{k^*} \geq a_n - b_1$, which implies $x_{k^*} \leq s$. Thus $(x_1, \ldots, x_{n-1})$ is primal feasible. Since $\lambda^*_2 = 0$, the dual constraint is given by $z_k \lambda^*_1 \leq k$, which is satisfied by the definition of λ^*_1. Thus $(\lambda^*_1, \lambda^*_2)$ is dual feasible. It is straightforward to verify complementary slackness. Thus the solution is optimal by the duality theorem. $\square$

Interchanging a^i_k and b^i_k and replacing z^i_k with $w^i_k = b^i_n - b^i_{n-k} + a^i_{k+1} - a^i_1$, the result generates the conditions for the program $(\mathbf{P^i_B})$.

The result allows us to justify the claim made in footnote 5. Consider a simple binary coordination game that satisfies (**A1**), (**A2**), and (**A3**) with strict inequality. Let the parameters of the game be $(\overline{a}^i_k, \overline{b}^i_k)$. Consider another game (a^i_k, b^i_k) in which $b^i_n = \overline{b}^i_n + \xi$ but all the others remain unchanged. Let us verify that increasing $\xi > 0$ eventually results in a game in which $(B, \ldots, B)$ is a unique stochastically stable equilibrium, provided the sample size s is not extremely small.

If ξ is sufficiently large, then

$$\{n-1\} = \arg \min_{\substack{k \\ z^i_k \neq 0}} \frac{k}{z^i_k} \quad \text{and} \quad \{1\} = \arg \min_{\substack{k \\ w^i_k \neq 0}} \frac{k}{w^i_k}.$$

In fact, the former is obvious since b^i_n appears only in z^i_{n-1}. For the latter, it suffices to pick $\xi > \max \left\{ (\overline{w}^i_k - k\overline{w}^i_1)/(k-1) \mid k \neq 1, \overline{w}^i_k - k\overline{w}^i_1 > 0 \right\}$ whenever the set is non-empty.

By Proposition 3, the optimal value of $(\mathbf{P^i_A})$ is given by $s(n-1)(a^i_n - b^i_1)/z^i_{n-1}$. Taking the integer constraint and the non-first exitors into account, we obtain

$$r(A, B) \leq n \left\lceil \frac{(a^i_n - b^i_1)s}{z^i_{n-1}} \right\rceil = n \left\lceil \frac{s\alpha^i}{\alpha^i + \beta^i} \right\rceil.$$

Concerning $(\mathbf{P^i_B})$, Proposition 3 implies that

$$\min \left\{ \left\lceil \frac{s\beta^i}{\beta^i + a^i_2 - b^i_{n-1}} \right\rceil, s \right\} \leq r(B, A).$$

If ξ is sufficiently large so that

$$\frac{n\alpha^i}{\alpha^i + \beta^i} < \min \left\{ \frac{\beta^i}{\beta^i + a^i_2 - b^i_{n-1}}, 1 \right\},$$

then $r(A, B) < r(B, A)$ as long as s is large enough to preserve the strict inequality after the rounding. The argument also shows that if s is extremely small, such as $s = 1$, then the stochastic stability does not depend at all on the magnitudes of the deviation losses. In such a case, the model is essentially equivalent to the simple mistake counting model mentioned in the fourth paragraph of section 1.

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