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RATIONAL EXPECTATION CAN PRECLUDE TRADE

by

Takashi MATSUHISA and Ryuichiro ISHIKAWA

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Takashi Matsuhisa †

Department of Liberal Arts and Sciences, Ibaraki National College of Technology,

866 Nakane, Hitachinaka-shi, Ibaraki 312-8508, Japan.

e-mail: `mathisa@ge.ibaraki-ct.ac.jp`

Ryuichiro Ishikawa ‡

Graduate School of Economics, Hitotsubashi University,

Naka 2-1, Kunitachi-shi, Tokyo 186-8601, Japan.

e-mail: `pg01104@srv.cc.hit-u.ac.jp`

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Dedicated to

Professor Akira Yamazaki on the occasion of his Sixtieth birthday

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‡Corresponding author. He is supported by Grant-in-Aid for JSPS Fellows.

Abstract

We consider a pure exchange economy under uncertainty in which the traders have the non-partition structure of information. They willing to trade the amounts of state-contingent commodities and they know their own expectations. Common knowledge of these conditions among all the traders can preclude trade if the initial endowments allocation is ex-ante Pareto optimal. Furthermore we introduce rational expectations equilibrium under the non-partition information, and prove the existence theorem and the fundamental theorems of welfare economics.

Keywords: Economy with knowledge, Rational expectations equilibrium, No trade theorem, Ex-ante Pareto optimum, Common knowledge.

Journal of Economic Literature Classification: D51, C78, D61, D52.

1 Introduction

This paper investigates an exchange economy under uncertainty, in which the traders have asymmetric information. In our analysis we suppose that the traders may be imperfect rational; they may have non-partition information. Our purposes are two: The first is to show No trade theorem by Milgrom and Stokey (1982) in our circumstance. The second is to prove the existence theorem of rational expectations equilibrium under non-partition information, and to characterize welfare under the equilibrium in the economy.

Milgrom and Stokey (1982) show so-called No trade theorem as follows: Let us consider a pure exchange economy where traders face uncertain environment. Let Ω be a finite set of states. It is assumed here that the contingent commodities are ex-ante Pareto-optimally allocated, and that the traders receive information about the state of Ω representable by an information partition. It is also assumed that the traders' beliefs are *concordant*. Now, a trading process takes place where traders try to maximize their expected utilities. We assume that, in any equilibrium of this process, trades intended by traders are both jointly feasible and common knowledge among them. In this set-up Milgrom and Stokey show that if traders are strictly risk-averse, equilibrium trade is null.

The serious limitations of the analysis by Milgrom and Stokey are its use of the information partition structure by which the traders receive information and of the common prior assumption. From the epistemic point of view, the information partition structure represents the traders' knowledge: Precisely, the structure is equivalent to the *standard model of knowledge* that includes 'Truth' property Axiom **T** (what is known is true), the 'positive introspection' property Axiom **4** (that we know what we do) and the 'negative introspection' property Axiom **5** (that we know what we do not know). The postulate **5** is indeed so strong that describes the hyper-rationality of traders, and thus it is particularly objectionable. Also is the common knowledge assumption because the common knowledge operator is defined by an infinite recursion of the knowledge operators. The recent idea of 'bounded rationality' suggests dropping such assumptions since real people are not complete reasoners. As has already been pointed out in the literature,[1] this relaxation can potentially yield important results in a world with imperfectly Bayesian agents.

This raises the question to what extent results as No trade theorem depend on both

[1]E.g. Geanakoplos (1989).

common knowledge and the information partition structure (or the equivalent postulates of knowledge). The answer is that results which strengthen the Milgrom and Stokey's theorem have been obtained: Among other things Geanakoplos (1989) shows No trade theorem under the assumption that the information structure is reflexive, transitive and *nested*. Tanaka (2000) investigates the theorem on the information partition by iterated elimination reasoning instead of common knowledge.

This paper is in the same line of Geanakoplos (1989). We extend No trade theorem to the reflexive and transitive information structure. Without the 'nested' condition we show the results as follows: In a pure exchange economy under reflexive and transitive information structure, if the traders are assumed to have the subjective priors which are not concordant and to have strictly increasing preferences, then

Main Theorem 1. *Any price system for which the initial endowments allocation is a rational expectations equilibrium allocation can preclude trade if all the traders commonly know that they are willing to trade the amounts of state-contingent commodities and if they know their expectations everywhere with respect to the price system.*

To prove it we extend the notion of rational expectations equilibrium for an economy under uncertainty to that for an economy under reflexive and transitive information structure, and we establish the existence theorem for the equilibrium: The traders are further assumed to be strictly risk-averse.

Main Theorem 2. *There exists a rational expectations equilibrium allocation relative to a price system with respect to which the traders know their expectations everywhere.*

Moreover, we show a generalized version of fundamental theorem of welfare economics, a part of which plays an essential role in proving Main Theorem 1:

Main Theorem 3. *The initial endowments allocation is ex-ante Pareto optimal if and only if it is a rational expectations equilibrium allocation relative to a price system with respect to which the traders know their expectations everywhere.*

This paper is organized as follows: In Section 2 we first recall the reflexive and transitive information structure; the RT-information structure, and the knowledge operator model corresponding to it. Secondly we introduce the economy under RT-information structure, called an economy with knowledge, which is a generalization of an economy under uncertainty. In Section 3 we introduce the notion of rational expectations equilibrium for an economy with knowledge, and we establish the existence theorem and the fundamental theorem of welfare economics for the equilibrium. Main Theorem 3 is proved as a consequence of a part of the fundamental theorem. In Section 4 we show the extended No trade theorem and remark that the notion 'rationality about expectations' plays an essential role in No trade theorem. Finally Section 5 gives some remarks about our version of No trade theorem, and we shall compare it with the Geanakoplos's investigation.

2 The Model

Let Ω be a non-empty *finite* set called a *state space*, $N = \{1, 2, \cdots, n\}$ a set of finitely many *traders*, and let 2^{Ω} denote the field of all subsets of Ω. Each member of 2^{Ω} is called an *event* and each element of Ω called a *state*.

2.1 Information and Knowledge[2]

An *information structure* $(P_i)_{i \in N}$ is a class of mappings P_i of Ω into 2^{Ω}. It is said to be *reflexive* if the following property is true:

Ref $\quad \omega \in P_i(\omega) \quad$ for every $\omega \in \Omega$,

and it is said to be *transitive* if the following property is true:

Trn $\quad \xi \in P_i(\omega)$ implies $P_i(\xi) \subseteqq P_i(\omega)$ for all $\xi, \omega \in \Omega$.

Information structure $(P_i)_{i \in N}$ is called *RT-information structure* if it is reflexive and transitive.[3] Given our interpretation, a trader i for whom $P_i(\omega) \subseteqq E$ knows, when the state ω occurs, that some state in the event E has occurred. In this case we say that at the state ω the trader i knows E. i's *knowledge operator* K_i on 2^{Ω} is defined by

$$K_i E = \{\omega \in \Omega \mid P_i(\omega) \subseteqq E\ \}. \tag{1}$$

The set $P_i(\omega)$ will be interpreted as the set of all the states of nature that i knows to be possible at ω, and $K_i E$ will be interpreted as the set of states of nature for which i knows E to be possible. We will therefore call P_i i's *possibility operator* on Ω and also will call $P_i(\omega)$ i's *possibility set* at ω.

[2] See Bacharach (1985), Binmore (1992).

[3] *RT*-information structure stands for reflexive and transitive information structure.

It is noted that i's knowledge operator satisfies the following properties: For every E, F of 2^Ω,

N $\quad K_i\Omega = \Omega \quad$ and $\quad K_i\emptyset = \emptyset$;

K $\quad K_i(E \cap F) = K_iE \cap K_iF$;

T $\quad K_i(E) \subseteqq E \quad$ for every $E \in 2^\Omega$.

4 $\quad K_i(E) \subseteqq K_i(K_i(E)) \quad$ for every $E \in 2^\Omega$.

It is also noted that the possibility operator P_i is uniquely determined by the knowledge operator K_i such as $P_i(\omega) = \bigcap_{K_iE \ni \omega} E$.

The *mutual* knowledge operator K_E on 2^Ω is defined by $K_EF = \bigcap_{i \in N} K_iF$. The event K_EF is interpreted as that 'every trader knows F.' The *common knowledge* operator K_C is defined by the infinite recursion of knowledge operators:

$$K_CE := \bigcap_{k=1,2,\dots} \bigcap_{\{i_1,i_2,\dots,i_k\} \subset N} K_{i_1}K_{i_2}\cdots K_{i_k}E.$$

All traders *commonly know* E at ω if $\omega \in K_CE$; that is, when ω occurs then for all k and for all traders $i_1, i_2, \dots, i_k$, it is true that 'i_1 knows that [i_2 knows that [$\dots i_{k-1}$ knows that [i_k knows X]] $\dots$].' This is the *iterated notion* of common-knowledge.

The *communal* possibility operator is the mapping $M : \Omega \to 2^\Omega$ defined by $M(\omega) = \bigcap_{K_CE \ni \omega} E$. It is noted that $\omega \in K_CE$ if and only if $M(\omega) \subseteqq E$.

Remark 2.1. It can be observed from **K** that every knowledge operator $K_*(* = i, E, C)$ appeared as above satisfies the monotone property below:

M $\quad K_*E \subseteq K_*F \quad$ whenever $\quad E \subseteq F$.

2.2 Economy with knowledge

A *pure exchange economy under uncertainty* $\mathcal{E}$ is a tuple $\langle N, \Omega, (e_i)_{i \in N}, (U_i)_{i \in N}, (\mu_i)_{i \in N} \rangle$ consisting of the following structure and interpretations: There are l commodities at each state of the state space Ω, and it is assumed that Ω is *finite* and that the consumption set of trader i is $\mathbb{R}^l_+$;

- $N = \{1, 2, \cdots, n\}$ is the set of n traders;

- $e_i : \Omega \to \mathbb{R}^l_+$ is i's *endowment*;
- $U_i : \mathbb{R}^l_+ \times \Omega \to \mathbb{R}$ is i's von Neumann-Morgenstern utility function;
- μ_i is a subjective prior on Ω for i.

For simplicity it is assumed that (Ω, μ_i) is a finite probability space with μ_i *full support*[4] for every $i \in N$. When all the traders have a same prior,[5] we refer it as a *common prior.*

Definition 2.2. An *economy with knowledge* $\mathcal{E}^K$ is a structure $\langle \mathcal{E}, (P_i)_{i\in N}\rangle$, in which $\mathcal{E}$ is a pure exchange economy under uncertainty with reflexive and transitive information structure (P_i) on Ω.

We denote by $\mathcal{F}_i$ the field generated by $\{P_i(\omega) |\, \omega \in \Omega\}$ and by $\{A_i(\omega) |\, \omega \in \Omega\}$ the set of all atoms $A_i(\omega)$ containing ω. Let $\mathcal{F}$ denote the join of all $\mathcal{F}_i (i \in N)$; i.e. $\mathcal{F} = \vee_{i\in N}\mathcal{F}_i$. We denote by $\{A(\omega) |\, \omega \in \Omega\}$ the set of all atoms $A(\omega)$ containing ω of the field $\mathcal{F} = \vee_{i\in N}\mathcal{F}_i$.

Here we should note the following property:

Lemma 2.3. *Let Q be a mapping from Ω to 2^Ω satisfying the conditions* **Ref** *and* **Trn**. *For each $\omega \in \Omega$ the atom $\Pi(\omega)$ containing ω of the field generated by Q coincides with the component containing ω of the partition induced by Q; i.e.*

$$\Pi(\omega) = \{\xi \in \Omega |\; Q(\xi) = Q(\omega)\}.$$

Proof: It can be shown that

$$\{\xi \in \Omega |\; Q(\xi) = Q(\omega)\} = \cap_\xi \{Q(\omega) \setminus Q(\xi) |\; \xi \in Q(\omega), Q(\xi) \subsetneqq Q(\omega)\},$$

which also belongs to the field generated by Q. It follows that the field generated by Q is generated by all components of the partition induced by Q. Thus we can observe that the component containing ω is the atom $\Pi(\omega)$ of the field generated by Q for each ω, from which Lemma 2.3 immediately follows. □

By an *allocation* we mean a profile $\boldsymbol{a} = (a_i)_{i\in N}$ of $\mathcal{F}$-measurable functions a_i from Ω into $\mathbb{R}^l_+$ such that for every $\omega \in \Omega$,

$$\sum_{i\in N} a_i(\omega) \leqq \sum_{i\in N} e_i(\omega).$$

[4] I.e., $\mu_i(\omega) \gneqq 0$ for every $\omega \in \Omega$.

[5] I.e., there is a some prior μ such that $\mu_i = \mu$ for every $i \in N$.

We denote by $\mathcal{A}$ the set of all allocations and denote by $\mathcal{A}_i$ the set of all the i'th components: $\mathcal{A} = \times_{i \in N} \mathcal{A}_i$. A *trade* $\boldsymbol{t} = (t_i)_{i \in N}$ is a profile of $\mathcal{F}_i$-measurable functions t_i from Ω into $\mathbb{R}^l$. It is said to be *feasible* if for all $i \in N$ and for all $\omega \in \Omega$,

$$e_i(\omega) + t_i(\omega) \geqq 0; \quad \text{and} \quad \sum_{i \in N} t_i(\omega) \leqq 0.$$

It is natural that each allocation is realized by some feasible trade; $a_i = e_i + t_i$ for each i.

We shall often refer to the following conditions: For every $i \in N$,

A-1 The function $e_i(\cdot)$ is $\mathcal{F}_i$-measurable with $\sum_{i \in N} e_i(\omega) \gneqq 0$ for all $\omega \in \Omega$.

A-2 For each $x \in \mathbb{R}^l_+$, the function $U_i(x, \cdot)$ is $\mathcal{F}_i$-measurable.

A-3 For each $\omega \in \Omega$, the function $U_i(\cdot, \omega)$ is strictly increasing on $\mathbb{R}^l_+$.

A-4 For each $\omega \in \Omega$, the function $U_i(\cdot, \omega)$ is continuous, increasing, strictly quasi-concave and *non-saturated*[6] on $\mathbb{R}^l_+$.

Here it is noted that **A-4** implies **A-3**.

2.3 Pareto optimality and Acceptability

We set by $\mathbf{E}_i[U_i(a_i)]$ i's *ex-ante* expectation defined by

$$\mathbf{E}_i[U_i(a_i)] := \sum_{\omega \in \Omega} U_i(a_i(\omega), \omega)\mu_i(\omega)$$

for each $a_i \in \mathcal{A}_i$.

The endowments $(e_i)_{i \in N}$ are said to be *ex-ante Pareto-optimal* if there is no allocation $(a_i)_{i \in N}$ such that for all $i \in N$,

$$\mathbf{E}_i[U_i(a_i)] \geqq \mathbf{E}_i[U_i(e_i)];$$

and that for some $j \in N$,

$$\mathbf{E}_j[U_j(a_j)] \gneqq \mathbf{E}_j[U_j(e_j)].$$

Let $\mathbf{E}_i[U_i(a_i)|P_i](\omega)$ denote i's *interim* expectation defined by

$$\mathbf{E}_i[U_i(a_i)|P_i](\omega) := \sum_{\xi \in \Omega} U_i(a_i(\xi), \xi)\mu_i(\xi|P_i(\omega)).$$

[6] That is, for any $x \in \mathbb{R}^l_+$ there exists an $x' \in \mathbb{R}^l_+$ such that $U_i(x', \omega) \gneqq U_i(x, \omega)$.

Definition 2.4. Let $\mathcal{E}^K$ be an economy with knowledge and $\boldsymbol{t} = (t_i)_{i\in N}$ a feasible trade. We say that t_i is *acceptable* for i at state ω provided that

$$\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega) \geqq \mathbf{E}_i[U_i(e_i)|P_i](\omega).$$

Denote by $Acp(t_i)$ the set of all the states in which t_i is acceptable for i, and by $Acp(\boldsymbol{t})$ the intersection $\bigcap_{i\in N} Acp(t_i)$.

2.4 Rationality about expectation

We set the event:

$$[\mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)] := \{\xi \in \Omega \mid \mathbf{E}_i[U_i(e_i + t_i)|P_i](\xi) = \mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)\},$$

and the event

$$[\mathbf{E}_i[U_i(\cdot)|P_i](\omega)] := \bigcap_{t_i\in\mathcal{T}_i} \{\xi \in \Omega \mid \mathbf{E}_i[U_i(e_i + t_i)|P_i](\xi) = \mathbf{E}_i[U_i(e_i + t_i)|P_i](\omega)\},$$

where $\mathcal{T}_i$ is the subset of feasible trades assigned to i. This is interpreted as the event *'i's expectation at ω'*. We denote $R_i = \{\omega \in \Omega \mid P_i(\omega) \subseteqq [\mathbf{E}_i[U_i(\cdot)|P_i](\omega)]\ \}$ and $R = \bigcap_{i\in N} R_i$.

Definition 2.5. A trader i is *rational about his/her expectation* at ω if ω belongs to R_i; that is, $\omega \in K_i([\mathbf{E}_i[U_i(\cdot)|P_i](\omega))$, which means that i knows his/her own expectation at ω. He/she is rational *everywhere* about his expectation if $R_i = \Omega$.

Remark 2.6. The standard information structure involves what can be characterized as a 'partitional' framework. This structure is an information structure (P_i) with the additional condition: For each $i \in N$ and every $\omega \in \Omega$,

Sym $\quad \xi \in P_i(\omega)$ implies $P_i(\xi) \ni \omega$.

Each knowledge operator K_i induced by P_i satisfies the property:

5 $\quad \Omega \setminus K_i(E) \subseteqq K_i(\Omega \setminus K_i(E))$.

In this case it is plainly observed that every player i is rational everywhere; i.e., $R_i = \Omega$.

2.5 Fundamental Lemma

We show the result needed: The Fundamental lemma below plays an essential role in this paper.

Definition 2.7. Let Z be a set of decisions. A *decision function* is a mapping f of 2^{Ω} into Z. It is said to satisfy the *sure thing principle* if it is preserved under disjoint union; that is, for every pair of disjoint events S and T such that if $f(S) = f(T) = d$ then $f(S \cup T) = d$. The function f is said to be *preserved under difference* provided that if $f(S) = f(T) = d$ then $f(T \setminus S) = d$ for all events S and T with $S \subsetneqq T$.

Remark 2.8. For each $a_i \in \mathcal{A}_i$, the decision function $f_i(a_i) : 2^{\Omega} \to [0,1]$ defined by

$$f_i(a_i)(X) := \mathbf{E}_i[U_i(a_i)|X] = \sum_{\xi \in \Omega} U_i(a_i(\xi), \xi)\mu_i(\xi|X).$$

is preserved under difference and it satisfies the sure thing principle.

Lemma 2.9 (Fundamental lemma[7]). *Let Q be a reflexive and transitive information structure on Ω and $\Pi : \Omega \to 2^{\Omega}$ the partition induced by Q defined by*

$$\Pi(\omega) := \{\xi \in \Omega \mid Q(\xi) = Q(\omega)\ \}.$$

Suppose that f is a decision function which satisfies the sure thing principle and is preserved under difference. If $Q(\omega)$ is contained in $\{\xi \in \Omega \mid f(Q(\xi)) = f(Q(\omega))\ \}$ for a state $\omega \in \Omega$ then we obtain that for every $\xi \in Q(\omega)$,

$$f(\Pi(\xi)) = f(Q(\omega)).$$

Proof: See Matsuhisa and Kamiyama (1997). □

3 Rational Expectations Equilibrium

In this section we introduce the notion of rational expectations equilibrium for an economy with knowledge. We show the existence theorem of the equilibrium and fundamental theorem of welfare economics concerning the relationship between ex-ante Pareto optimal allocations and rational expectations equilibria.

[7] A similar result implicitly appeared in the proof of Theorem 7 of D. Samet (1990), and also it explicitly appeared in T. Matsuhisa and K. Kamiyama (1997, Fundamental lemma).

3.1 Price system and rational expectations equilibrium

Let $\mathcal{E}^K = \langle N, \Omega, (e_i)_{i \in N}, (U_i)_{i \in N}, (\mu_i)_{i \in N}, (P_i)_{i \in N} \rangle$ be an economy with knowledge. A *price system* is a non-zero $\mathcal{F}$-measurable function $p : \Omega \to \mathbb{R}^l_+$. We denote by $\sigma(p)$ the smallest field that p is measurable and by $\Delta(p)$ the set of all atoms of $\sigma(p)$ with $\Delta(p)(\omega)$ the component containing ω. The *budget set* of a trader i at a state ω for a price system p is defined by

$$B_i(\omega, p) = \{ \ a \in \mathbb{R}^l_+ \mid p(\omega) \cdot a \leqq p(\omega) \cdot e_i(\omega) \ \}.$$

Let $\Delta(p) \cap P_i : \Omega \to 2^\Omega$ be defined by $(\Delta(p) \cap P_i)(\omega) := \Delta(p)(\omega) \cap P_i(\omega)$; it is plainly observed that $\Delta(p) \cap P_i$ is reflexive and transitive information structure of trader i. We denote by $\sigma(p) \vee \mathcal{F}_i$ the field generated by $\{(\Delta(p) \cap P_i)(\omega) | \ \omega \in \Omega\}$ and by $A_i(p)(\omega)$ the atom containing ω. On noting that P_i satisfies **Ref** and **Trn**, it can be plainly observed that

$$A_i(p)(\omega) = (\Delta(p) \cap A_i)(\omega).$$

Definition 3.1. A *rational expectations equilibrium* for an economy $\mathcal{E}^K$ with knowledge is a pair $(p, \boldsymbol{x})$, in which p is a price system and $\boldsymbol{x} = (x_i)_{i \in N}$ is an allocation satisfying the following conditions:

RE 1 For every $i \in N$, x_i is $\sigma(p) \vee \mathcal{F}_i$-measurable.

RE 2 For every $i \in N$ and for every $\omega \in \Omega$, $x_i(\omega) \in B_i(\omega, p)$.

RE 3 For all $i \in N$, if $y_i : \Omega \to \mathbb{R}^l_+$ is $\sigma(p) \vee \mathcal{F}_i$-measurable with $y_i(\omega) \in B_i(\omega, p)$ for all $\omega \in \Omega$, then

$$\mathbf{E}_i[U_i(x_i) | \Delta(p) \cap P_i](\omega) \geqq \mathbf{E}_i[U_i(y_i) | \Delta(p) \cap P_i](\omega)$$

pointwise on Ω.

RE 4 For every $\omega \in \Omega$, $\sum_{i \in N} x_i(\omega) = \sum_{i \in N} e_i(\omega)$.

The profile $\boldsymbol{x} = (x_i)_{i \in N}$ is called a *rational expectations equilibrium allocation*. We permit i's trades to be $\sigma(p) \vee \mathcal{F}_i$-measurable under a price system.

We denote by $R_i(p)$ the event that i is rational about his/her expectation with respect to a price system; i.e.,

$$R_i(p) = \{\omega \in \Omega \mid (\Delta(p) \cap P_i)(\omega) \subseteqq [\mathbf{E}_i[U_i(\cdot) | \Delta(p) \cap P_i](\omega)]\}$$

and denote by $R(p)$ the event that all traders are rational with respect to a price system: i.e., $R(p) = \bigcap_{i \in N} R_i(p)$. The set $R_i(p)$ is interpreted as the event that i knows his/her interim expectation when he/she receives the information of the price system p, and $R(p)$ interpreted as everyone knows their expectation under the information of p.

Definition 3.2. A trader i is said to be *rational* about his expectation *with respect to a price system* p at ω if ω belongs to $R_i(p)$. And all traders are *rational everywhere* about their expectations with respect to p if $R(p) = \Omega$.

3.2 Existence Theorem

In this section we shall prove the existence theorem of rational expectations equilibrium for an economy with knowledge.

Theorem 3.3. *Suppose a pure exchange economy with knowledge satisfies the conditions A-1, A-2 and A-4. If the initial endowments allocation* $\mathbf{e} = (e_i)_{i \in N}$ *satisfies the additional condition that* $e_i(\omega) \gneqq 0$ *for all* $\omega \in \Omega$ *and for each* $i \in N$ *then there exists a rational expectations equilibrium for the economy such that all traders are rational everywhere about their expectations with respect to the price system.*

Proof: Let $\mathcal{E}^K$ be the economy with knowledge and $\mathcal{E}^K(\omega)$ the economy with complete information for each $\omega \in \Omega$. In view of the conditions **A-1**, **A-2** and **A-4**, it follows from the existence theorem of a competitive equilibrium for an economy with complete information (c.f.: Theorem 5 in Debreu (1982)) that there exists a competitive equilibrium $(p^*(\omega), (x_i^*(\omega))_{i \in N})$ for $\mathcal{E}^K(\omega)$. We take a sequence of strictly positive numbers $\{k_\omega\}_{\omega \in \Omega}$ such that $k_\omega p^*(\omega) \neq k_\xi p^*(\xi)$ for any $\omega \neq \xi$. We define the pair $(p, \boldsymbol{x})$ with $\boldsymbol{x} = (x_i)_{i \in N}$ such that for each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, $p(\xi) := k_\omega p^*(\omega)$ and $x_i(\xi) := x_i^*(\omega)$. We shall verify that $(p, \boldsymbol{x})$ is a rational expectations equilibrium for $\mathcal{E}^K$ with $\Delta(p)(\omega) = A(\omega)$: In fact, it is easily seen that p is $\mathcal{F}$-measurable with $\Delta(p)(\omega) = A(\omega)$ and that x_i is $\sigma(p) \vee \mathcal{F}_i$-measurable and so **RE 1** is valid. Because $(\Delta(p) \cap P_i)(\omega) = A(\omega)$ for every $\omega \in \Omega$, it can be plainly observed that $\boldsymbol{x} = (x_i)_{i \in N}$ satisfies **RE 2**, and it follows from **A-2** that for each $i \in N$,

$$\mathbf{E}_i[U_i(x_i)|(\Delta(p) \cap P_i)](\omega) = U_i(x_i(\omega), \omega) \tag{2}$$

Therefore we can plainly verify that $R_i(p) = \Omega$ for all $i \in N$. On noting that $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$ for any $\xi \in A(\omega)$, it is plainly observed that $(k_\omega p^*(\omega), (x_i^*(\omega))_{i \in N})$ is also a competitive equilibrium for $\mathcal{E}^K(\omega)$ for every $\omega \in \Omega$, and it can be observed by (2) that **RE 3** is valid for $(p, \boldsymbol{x})$, in completing the proof. $\square$

3.3 Fundamental Theorem in Welfare Economics

We can prove not only the existence theorem, but also guarantee welfare of a rational expectations equilibrium for an economy with knowledge as follows:

Theorem 3.4. *Let $\mathcal{E}^K$ be an economy with knowledge satisfying the conditions A-1, A-2 and A-4. The initial endowments allocation is ex-ante Pareto optimal if and only if it is a rational expectations equilibrium allocation relative to a price system with respect to which the traders are rational everywhere about their expectations.*

Proof: Follows immediately from the following Propositions 3.5 and 3.7.

Proposition 3.5. *Let $\mathcal{E}^K$ be an economy with knowledge satisfying the conditions A-1, A-2 and A-3. Then the initial endowments allocation $\boldsymbol{e} = (e_i)_{i \in N}$ is ex-ante Pareto optimal if it is a rational expectations equilibrium allocation relative to some price system p with respect to which all traders are rational everywhere about their expectations.*

Proof: Let ω be a state in Ω. On noting that $\Delta(p) \cap P_i$ is RT-information structure, it can be plainly observed by Lemma 2.3 that

$$A_i(p)(\omega) = \{\xi \in \Omega \mid (\Delta(p) \cap P_i)(\xi) = (\Delta(p) \cap P_i)(\omega)\ \}.$$

It is first noted that for all $\xi \in (\Delta(p) \cap P_i)(\omega)$ and all $a_i \in \mathcal{A}_i$,

$$\mathbf{E}_i[U_i(a_i)|(\Delta(p) \cap P_i)](\xi) = U_i(a_i(\xi), \xi).$$

In fact, since $\omega \in R_i(p) = \Omega$, it follows from the Fundamental lemma (Lemma 2.9) that for all $\xi \in (\Delta(p) \cap P_i)(\omega)$,

$$\mathbf{E}_i[U_i(a_i)|(\Delta(p) \cap P_i)](\xi) = \mathbf{E}_i[U_i(a_i)|A_i(p)](\xi).$$

In view of **A-2** it is observed that for every $\xi \in \Omega$,

$$\mathbf{E}_i[U_i(a_i)|A_i(p)](\xi) = U_i(a_i(\xi), \xi),$$

and thus the result follows as required.

Secondly it is noted that $(p(\omega), (e_i(\omega))_{i\in N})$ is a competitive equilibrium for the economy $\mathcal{E}^K(\omega) = \langle N, (e_i(\omega))_{i\in N}, (U_i(\cdot,\omega))_{i\in N}\rangle$ with complete information at each $\omega \in \Omega$. In fact, in view of **RE 3** it follows that $U_i(e_i(\omega),\omega) \geqq U_i(y_i(\omega),\omega)$ for all $y_i(\omega) \in B_i(\omega, p)$ and for all $i \in N$, as required.

Therefore it can be observed that for all $\omega \in \Omega$, $(e_i(\omega))_{i\in N}$ is Pareto optimal in $\mathcal{E}^K(\omega)$, and it can be easily verified that $(e_i)_{i\in N}$ is ex-ante Pareto optimal. □

The following remark has been already proved in the proof of Proposition 3.5:

Remark 3.6. Let $\mathcal{E}^K$ be a pure exchange economy with knowledge satisfying the conditions **A-1**, **A-2** and **A-3**. If the allocation of initial endowments $\boldsymbol{e} = (e_i)_{i\in N}$ is a rational expectations equilibrium allocation relative to some price system p with respect to which all traders are rational every where about their expectations then the pair $(p(\omega), (e_i(\omega))_{i\in N})$ constitutes an *ex-post* competitive equilibrium for the pure exchange economy $\mathcal{E}^K(\omega)$ with complete information for each $\omega \in \Omega$.

The next proposition states that the converse in Proposition 3.5 is also valid under the additional assumption that all the traders have continuous and strictly quasi-concave utilities:

Proposition 3.7. *Let $\mathcal{E}^K$ be an economy with knowledge satisfying the conditions A-1, A-2 and A-4. If the initial endowments allocation $\boldsymbol{e} = (e_i)_{i\in N}$ is ex-ante Pareto optimal then it is a rational expectations equilibrium allocation relative to some price system p with respect to which all traders are rational every where about their expectations.*

Proof: For each $\omega \in \Omega$ we denote by $G(\omega)$ the set of all vectors $\sum_{i\in N} e_i(\omega) - \sum_{i\in N} y_i$ such that $y_i \in \mathbb{R}^l_+$ and $U_i(y_i,\omega) \geqq U_i(e_i(\omega),\omega)$ for all $i \in N$.

First, in view of the conditions **A-1**, **A-2** and **A-4**, we note that that $G(\omega)$ is convex and closed in $\mathbb{R}^l_+$. It can be shown that

Claim 1: For each $\omega \in \Omega$ there exists $p^*(\omega) \in \mathbb{R}^l_+$ such that $p^*(\omega)\cdot v \leqq 0$ for all $v \in G(\omega)$.

Proof of Claim 1: By the separation theorem,[8] we can plainly observe that the assertion immediately follows from that $v \leqq 0$ for all $v \in G(\omega)$: Suppose to the contrary that there exist $\omega_0 \in \Omega$ and $v_0 \in G(\omega_0)$ with $v_0 \gneqq 0$. Take $\boldsymbol{y}^0 = (y^0_i)_{i\in N}$ with $y^0_i \in \mathbb{R}^l_+$ such that

[8] See Lemma 8 of Chapter 4 in Arrow and Hahn (1971, p.92).

for each i, $U_i(y_i, \omega_0) \geqq U_i(e_i(\omega_0), \omega_0)$ and $v_0 = \sum_{i \in N} e_i(\omega_0) - \sum_{i \in N} y_i^0$. Consider the allocation $\boldsymbol{z} = (z_i)_{i \in N}$ defined by

$$z_i(\xi) := \begin{cases} y_i^0 + \frac{v_0}{n} & \text{if } \xi \in A(\omega_0), \\ e_i(\xi) & \text{if not.} \end{cases}$$

It follows that for each $i \in N$,

$$\begin{aligned} \mathbf{E}_i[U_i(z_i)] &= \sum_{\xi \in A(\omega_0)} U_i(y_i^0 + \frac{v_0}{n}, \xi)\mu_i(\xi) \\ &\quad + \sum_{\xi \in \Omega \setminus A(\omega_0)} U_i(e_i(\xi), \xi)\mu_i(\xi) \\ &\gneqq \sum_{\xi \in A(\omega_0)} U_i(y_i^0, \xi)\mu_i(\xi) \\ &\quad + \sum_{\xi \in \Omega \setminus A(\omega_0)} U_i(e_i(\xi), \xi)\mu_i(\xi) \quad \text{because of } \mathbf{A\text{-}4} \\ &\geqq \mathbf{E}_i[U_i(e_i)]. \end{aligned}$$

This is in contradiction to which $\boldsymbol{e} = (e_i)_{i \in N}$ is ex-ante Pareto optimal as required.

Secondly, let p be the price system defined as follows: We take a sequence of strictly positive numbers $\{k_\omega\}_{\omega \in \Omega}$ such that $k_\omega p^*(\omega) \neq k_\xi p^*(\xi)$ for any $\omega \neq \xi$. We define the price system p such that for each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, $p(\xi) := k_\omega p^*(\omega)$. It can be observed that $\Delta(p)(\omega) = A(\omega)$. We shall show

Claim 2: The pair $(p, (e_i)_{i \in N})$ is a rational expectations equilibrium for $\mathcal{E}^K$.

Proof of Claim 2: We first note that

$$(\Delta(p) \cap P_i)(\omega) = \Delta(p)(\omega) \subseteqq A(\omega) \tag{3}$$

for every $\omega \in \Omega$. Therefore it follows from **A-2** that for every allocation $\boldsymbol{x} = (x_i)_{i \in N}$,

$$\mathbf{E}_i[U_i(x_i)|(\Delta(p) \cap P_i)](\omega) = U_i(x_i(\omega), \omega) \tag{4}$$

To prove Claim 2 it suffices to show that $\boldsymbol{e} = (e_i)_{i \in N}$ satisfies **RE 3**. Suppose to the contrary that there exist a trader $j \in N$ and an allocation $\boldsymbol{y} = (y_i)_{i \in N}$ with the two properties:

1. $y_i : \Omega \to \mathbb{R}^l_+$ is $\sigma(p) \vee \mathcal{F}_i$-measurable and $y_i(\omega) \in B_i(\omega, p)$ for every $i \in N$ and $\omega \in \Omega$;

2. $\mathbf{E}_i[U_i(y_j)|(\Delta(p) \cap P_j)](\omega_0) \gneqq \mathbf{E}_j[U_j(e_j)|(\Delta(p) \cap P_i)](\omega_0)$ for some $\omega_0 \in \Omega$.

In view of (4) it immediately follows from Property 2 that $U_j(y_j(\omega_0), \omega_0) \gneqq U_j(e_j(\omega_0), \omega_0)$, and thus $y_j(\omega_0) \gneqq e_j(\omega_0)$ by **A-4**. Therefore we obtain that $p(\omega_0) \cdot \boldsymbol{y}(\omega_0) \gneqq p(\omega_0) \cdot \boldsymbol{e}(\omega_0)$ in contradiction.

Finally in view of (3), we note that $\{(\Delta(p) \cap P_i)(\omega) | \ \omega \in \Omega\}$ makes a partition of Ω, and thus we can easily observe that $R_i(p) = \Omega$ for all i and so $R(p) = \Omega$. This means that all traders are rational everywhere with respect to the price system p, in completing the proof to Claim 2. $\square$

3.4 Remarks

It will well end this section in giving two remarks: First the suppression of each of the ancillary assumptions **A-1**, **A-2**, **A-3** and **A-4** renders Theorem 3.3 vulnerable. Secondly we give a remark about a fully revealing rational expectations equilibrium.

First we note that Theorem 3.3 does not hold without **A-3** and **A-4**. This is because we can easily observe the two points: First that an exchange economy with complete information coincides with an economy with knowledge admitting with the partitional information structure $(P_i)_{i \in N}$ defined by $P_i(\omega) = \{\omega\}$ for each $\omega \in \Omega$, and secondly that there does not necessarily exist a competitive equilibrium for the economy $\mathcal{E}_K(\omega)$ with perfect information when either **A-3** or **A-4** is not satisfied. The classical example provided by Kreps (1977) shows that Theorem 3.3 is not true without the assumption **A-2**.

The below example illustrates that **A-1** also plays a crucial role in the theorem.

Example 3.8. There are two traders $1, 2$ with a common prior μ and two commodities $(\boldsymbol{x}_1 = (x_{11}, x_{12}), \boldsymbol{x}_2 = (x_{21}, x_{22}))$. The economy with knowledge

$$\mathcal{E}^K = \langle N, \Omega, (e_i)_{i \in N}, (U_i)_{i \in N}, (\mu_i)_{i \in N}, (P_i)_{i \in N} \rangle.$$

consists of as following structures:

- $N = \{1, 2\}$
- $\Omega = \{\omega_1, \omega_2\}$;
- $\boldsymbol{e}_1(\omega) = \boldsymbol{e}_2(\omega) := \begin{cases} (1,1) & \text{if} \quad \omega = \omega_1 \\ (2,1) & \text{if} \quad \omega = \omega_2, \end{cases}$

- $U_i : \mathbb{R}^2_+ \times \Omega \to \mathbb{R}$ is defined by

$$U_1(\boldsymbol{x}, \omega) = \log x_{11} + x_{12} \quad \text{and} \quad U_2(\boldsymbol{x}, \omega) = \log x_{21} + x_{22};$$

- $\mu(\omega) := \frac{1}{2}$;
- $P_i : \Omega \to 2^\Omega$ is defined by

$$P_1(\omega) = P_2(\omega) := \Omega \quad \text{for each } \omega \in \Omega.$$

There exists no rational expectations equilibrium for $\mathcal{E}^K$.

In fact, it is first noted that each initial endowment e_i satisfies the condition $e_i(\omega) \gneqq 0$ for all $\omega \in \Omega$ but e_i is not $\mathcal{F}_i$ measurable; so it does not fulfill **A-1**. The other conditions **A-2** and **A-4** are true in the economy.

Suppose to the contrary that there is a rational expectations equilibrium $(p, \boldsymbol{x})$ with $p = (\boldsymbol{p}_1, \boldsymbol{p}_2)$ and $\boldsymbol{x} = ((x_{11}, x_{12}), (x_{21}, x_{22}))$. Without loss of generality we may assume here that $p = (\boldsymbol{p}_1, \mathbf{1})$ (i.e.; $\boldsymbol{p}_2(\omega) = (1, 1)$ for $\omega = \omega_1, \omega_2$.) Denote $p_1 = \boldsymbol{p}_1(\omega_1)$ and $p_2 = \boldsymbol{p}_1(\omega_2)$. On noting that p is $\mathcal{F}$-measurable it follows that $p_1 = p_2$.

Each trader i maximizes his/her expectation at ω:

$$\mathbf{E}_i[U_i((x_{i1}, x_{i2})|(\Delta(p) \cap P_i)](\omega) = \frac{1}{2}\{\log x_{i1}(\omega_1) + x_{i2}(\omega_1)\} + \frac{1}{2}\{\log x_{i1}(\omega_2) + x_{i2}(\omega_2)\}$$

subject to

$$p_1 x_{i1}(\omega_1) + x_{i2}(\omega_1) \leqq p_1 + 1 \quad \text{and} \quad p_2 x_{i1}(\omega_2) + x_{i2}(\omega_2) \leqq 2p_2 + 1.$$

It is observed that the demand functions of commodity 1 are

$$x_{i1}(\omega_j) = \frac{1}{p_j} \text{ for } i, j = 1, 2.$$

In view of the condition **RE 4**: $x_{11}(\omega) + x_{21}(\omega) = e_{11}(\omega) + e_{21}(\omega)$ for commodity 1 we obtain that

$$\begin{cases} \frac{1}{p_1} + \frac{1}{p_1} = 2 & \text{for } \omega = \omega_1; \\ \frac{1}{p_2} + \frac{1}{p_2} = 4 & \text{for } \omega = \omega_2, \end{cases}$$

and thus $p_1 \neq p_2$ in contradiction.

Finally we will extend the notion of fully revealing rational expectations equilibrium into an economy with knowledge.

Definition 3.9. A rational expectations equilibrium $(p, \boldsymbol{x})$ for an economy with knowledge $\mathcal{E}^K$ is called *fully revealing* if $\sigma(p) = \mathcal{F} := \vee_{i \in N} \mathcal{F}_i$.

In view of the proof of Theorem 3.3 we have shown that the rational expectations equilibrium $(p, \boldsymbol{x})$ constructed in the proof of Theorem 3.3 is fully revealing, thus

Corollary 3.10. *There exists a fully revealing rational expectations equilibrium for an economy with knowledge under the same assumptions in Theorem 3.3.*

4 No Trade Theorem

In this section we shall give two extensions of No trade theorem by Milgrom and Stokey (1982): First we prove the below theorem that directly extends No trade theorem to an economy with knowledge, and secondly we give the proof of Main Theorem 1 in our introduction.

4.1 Theorem of Milgrom and Stokey

Theorem 4.1. *Let $\mathcal{E}^K$ be an economy with knowledge satisfying the conditions A-1, A-2 and A-3, and let $\boldsymbol{t} = (t_i)_{i \in N}$ be a feasible trade. Suppose that the initial endowments allocation $(e_i)_{i \in N}$ is ex-ante Pareto optimal. Then the traders can never agree to any non null trade at each state where they commonly know both the acceptable trade $\boldsymbol{t} = (t_i)$ and rationality of their expectations; that is, $\boldsymbol{t}(\omega) = \mathbf{0}$ at every $\omega \in K_C(Acp(\boldsymbol{t}) \cap R)$.*

The next lemma is a key in the proof of Theorem 4.1:

Lemma 4.2. *Let $\mathcal{E}^K$, $\boldsymbol{t} = (t_i)_{i \in N}$ and $(e_i)_{i \in N}$ be the same as in Theorem 4.1. If $\omega \in K_C(Acp(t_i) \cap R_i)$ for each $i \in N$ then the equality is true:*

$$\mathbf{E}_i[U_i(t_i^* + e_i)|P_i](\omega) = \mathbf{E}_i[U_i(e_i)|P_i](\omega), \tag{5}$$

where the trade $\boldsymbol{t}^* = (t_i^*)_{i \in N}$ is defined by

$$t_i^*(\xi) := \begin{cases} t_i(\xi) & \text{if } \xi \in M(\omega), \\ 0 & \text{if not.} \end{cases} \tag{6}$$

Proof: It is noted by Lemma 2.3 that A_i coincides with the partition induced by P_i. We can observe the two points: First that $\boldsymbol{t}^* = (t_i^*)_{i \in N}$ is feasible because so is $\boldsymbol{t}$, and secondly that $M(\omega)$ is decomposed into the disjoint union of the components $A_i(\xi)$ for $\xi \in M(\omega)$; i.e.:

$$M(\omega) = A_i(\xi_1) \cup A_i(\xi_2) \cup \cdots \cup A_i(\xi_m).$$

It follows that

$$\begin{aligned}
\mathbf{E}_i[U_i(t_i^* + e_i)] &= \sum_{\xi \in M(\omega)} U_i(t_i^*(\xi) + e_i(\xi), \xi)\mu_i(\xi) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi) \\
&= \sum_{k=1}^{m} \sum_{\xi \in A_i(\xi_k)} U_i(t_i^*(\xi) + e_i(\xi), \xi)\mu_i(\xi) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi)
\end{aligned} \tag{7}$$

Let f be the decision function defined by $f(E) := \mathbf{E}_i[U_i(a_i)|E]$ for each $a_i \in \mathcal{A}_i$. It can be plainly observed that f satisfies the sure thing principle and is preserved under difference. Because $\omega \in K_C(Acp(t_i) \cap R_i)$ it follows that $M(\omega) \subseteqq R_i$. Therefore $P_i(\xi) \subseteqq M(\omega) \subseteqq R_i$ for every $\xi \in M(\omega)$. By the Fundamental lemma (Lemma 2.9), we obtain that

$$\mathbf{E}_i[U_i(a_i)|P_i](\xi) = \mathbf{E}_i[U_i(a_i)|A_i](\xi) \quad \text{for all } \xi \in M(\omega).[9] \tag{8}$$

Therefore, in view of (7), it follows that

$$\begin{aligned}
\mathbf{E}_i[U_i(t_i^* + e_i)] &= \sum_{k=1}^{m} \mu_i(A_i(\xi_k))\mathbf{E}_i[U_i(t_i^* + e_i)|A_i](\xi_k) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi) \qquad (9) \\
&= \sum_{k=1}^{m} \mu_i(A_i(\xi_k))\mathbf{E}_i[U_i(t_i^* + e_i)|P_i](\xi_k) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi)
\end{aligned}$$

On noting that $\xi_k \in M(\omega) \subseteqq Acp(t_i)$ for all $k = 1, 2, \ldots, m$, it immediately follows that

$$\begin{aligned}
\mathbf{E}_i[U_i(t_i^* + e_i)] &\geqq \sum_{k=1}^{m} \mu_i(A_i(\xi_k))\mathbf{E}_i[U_i(e_i)|P_i](\xi_k) \\
&\quad + \sum_{\xi \in \Omega \setminus M(\omega)} U_i(e_i(\xi), \xi)\mu_i(\xi)
\end{aligned} \tag{10}$$

[9] Where $\mathbf{E}_i[U_i(a_i)|A_i](\omega)$ is defined by

$$\mathbf{E}_i[U_i(a_i)|A_i](\omega) := \sum_{\xi \in \Omega} U_i(a_i(\xi), \xi)\mu_i(\xi|A_i(\omega)).$$

In view of (8) it can be obtained that

$$\begin{aligned}
\mathbf{E}_i[U_i(t_i^* + e_i)] &\geqq \sum_{k=1}^{m} \mu_i(A_i(\xi_k))\mathbf{E}_i[U_i(e_i)|A_i](\xi_k) \\
&\quad + \sum_{\xi\in\Omega\setminus M(\omega)} U_i(e_i(\xi),\xi)\mu_i(\xi) \\
&= \sum_{\xi\in M(\omega)} U_i(e_i(\xi),\xi)\mu_i(\xi) \\
&\quad + \sum_{\xi\in\Omega\setminus M(\omega)} U_i(e_i(\xi),\xi)\mu_i(\xi). \\
&\geqq \mathbf{E}_i[U_i(e_i)]
\end{aligned}$$

Therefore, if the equation (5) does not hold, the inequality (10) holds strictly. This yields that $\mathbf{E}_i[U_i(t_i^* + e_i)] \gneqq \mathbf{E}_i[U_i(e_i)]$, which contradicts the assumption that $(e_i)_{i\in N}$ is ex-ante Pareto optimal. □

Proof of Theorem 4.1. Now suppose to the contrary that $t_i(\omega)$ is not zero at some $\omega \in K_C(Acp(\boldsymbol{t}))$. Let $\boldsymbol{t}^* = (t_i^*)_{i\in N}$ be the feasible trade defined by (6). On noting by **M** that $K_C(Acp(\boldsymbol{t}) \cap R) \subseteqq K_C(Acp(t_i) \cap R_i)$, it immediately follows from **A-3** together with (8) that

$$\begin{aligned}
\mathbf{E}_i[U_i(t_i^* + e_i)|P_i](\omega) &= U_i(t_i^*(\omega) + e_i(\omega)) \\
&\neq U_i(e_i(\omega)) = \mathbf{E}_i[U_i(e_i)|P_i](\omega),
\end{aligned}$$

in contradiction to Lemma 4.2. □

4.2 Rational expectations equilibrium and No trade theorem

It is interesting to consider what can be said if we drop the hypothesis that the endowments are ex-ante Pareto optimal in Theorem 4.1. Is No trade theorem still true if the endowments allocation is a rational expectations equilibrium allocation? We shall give an affirmative answer.

To state it explicitly we introduce the knowledge operator $K_i^{(p)}$ associated with a price system p of an economy with knowledge $\mathcal{E}^K$. The knowledge operator $K_i^{(p)}$ is induced by the information structure $\Delta(p) \cap P_i$; that is,

$$K_i^{(p)}(E) = \{\omega \in \Omega \mid (\Delta(p) \cap P_i)(\omega) \subseteqq E\}.$$

Let $K_C^{(p)}$ be the common-knowledge operator defined by the infinite recursion of the operators $\{K_i^{(p)}\}_{i\in N}$; that is,

$$K_C^{(p)}E := \bigcap_{k=1,2,\dots} \bigcap_{\{i_1,i_2,\dots,i_k\}\subset N} K_{i_1}^{(p)} K_{i_2}^{(p)} \cdots K_{i_k}^{(p)} E.$$

We can now explicitly state Main Theorem 1 in our introduction as follows:

Theorem 4.3. *Let $\mathcal{E}^K$ be an economy with knowledge satisfying the conditions A-1, A-2 and A-3. If $\boldsymbol{e} = (e_i)_{i\in N}$ is a rational expectations equilibrium allocation relative to some price system p with respect to which all traders are rational everywhere about their expectations, then the traders can never agree to any non null trade at each state where they commonly know the acceptable feasible trade $\boldsymbol{t} = (t_i)_{i\in N}$; that is, $\boldsymbol{t}(\omega) = \mathbf{0}$ at every $\omega \in K_C^{(p)}(Acp(\boldsymbol{t}))$.*

Proof: Consider now the economy with knowledge

$$\mathcal{E}^{K(p)} = \langle N, \Omega, (e_i)_{i\in N}, (U_i)_{i\in N}, (\mu_i)_{i\in N}, (\Delta(p)\cap P_i)_{i\in N}\rangle.$$

It is noted that $R(p) = \Omega$. By the similar argument in the proof of Theorem 4.1 it can be plainly observed that $\boldsymbol{t}(\omega) = \mathbf{0}$ at every $\omega \in K_C^{(p)}(Acp(\boldsymbol{t}) \cap R(p)) = K_C^{(p)}(Acp(\boldsymbol{t}))$ if $\boldsymbol{e}$ is ex-ante Pareto optimal, and thus Theorem 4.3 follows from Proposition 3.5. □

The following corollary is another extended version of No trade theorem by Milgrom and Stokey.

Corollary 4.4. *Let $\mathcal{E}^K$ be an economy with knowledge satisfying the conditions A-1, A-2 and A-4 instead of A-3, and let $\boldsymbol{t} = (t_i)_{i\in N}$ be a feasible trade. If the initial endowments are ex-ante Pareto optimal then there exists a price system p with respect to which the traders can never agree to any non null trade at each state where they commonly know the acceptable feasible trade $\boldsymbol{t} = (t_i)_{i\in N}$. That is, $\boldsymbol{t}(\omega) = \mathbf{0}$ at every $\omega \in K_C^{(p)}(Acp(\boldsymbol{t}))$.*

Proof: Immediately follows from the combination of Theorems 3.4 and 4.3. □

4.3 Remarks

Could we prove the theorem under the generalized information structure jettisoning the reflexivity or the transitivity? The answer is no. The following two examples show that the reflexivity **Ref** and the transitivity **Trn** of the information structure (or the equivalent postulates Axioms **4** and **T**) do play an essential role.

Example 4.5. Let $\mathcal{E}^K = \langle N, \Omega, (e_i)_{i\in N}, (U_i)_{i\in N}, \mu, (P_i)_{i\in N}\rangle$ the economy with knowledge in which

- $N = \{1, 2\}$
- $\Omega = \{\omega_1, \omega_2\}$;
- $\mu(\omega) = \frac{1}{2}$ for each $\omega \in \Omega$;
- $e_1(\omega) = e_2(\omega) = 2$ for every $\omega \in \Omega$;
- $U_i : \mathbb{R}^1_+ \times \Omega \to \mathbb{R}$ is defined by, for each $\omega \in \Omega$,

$$U_1(x,\omega) = \sqrt{x+2} \quad \text{and} \quad U_2(x,\omega) = \sqrt{x};$$

- P_i is defined by

$$P_1(\omega) := \{\omega_2\} \quad \text{and} \quad P_2(\omega) := \{\omega_1\}$$

  for each $\omega \in \Omega$.

It is plainly observed the two points: First that both P_i $(i = 1, 2)$ are not reflexive but transitive, and second that the endowments $(e_i)_{i=1,2}$ are ex-ante Pareto optimal. Let $\boldsymbol{t} = (t_i)_{i=1,2}$ be the feasible trade defined by

$$t_1(\omega) := \begin{cases} -2 & \text{if} \quad \omega = \omega_1 \\ 0 & \text{if} \quad \omega = \omega_2 \end{cases} \quad \text{and} \quad t_2(\omega) := \begin{cases} 2 & \text{if} \quad \omega = \omega_1 \\ 0 & \text{if} \quad \omega = \omega_2. \end{cases}$$

Then it can be verified that $Acp(\boldsymbol{t}) = R = \Omega$ and thus $K_C(Acp(\boldsymbol{t}) \cap R) = \Omega$. However the trade $\boldsymbol{t}$ is not null at $\omega_1 \in K_C(Acp(\boldsymbol{t}) \cap R)$.

Example 4.6. Let $\mathcal{E}^K = \langle N, \Omega, (e_i)_{i\in N}, (U_i)_{i\in N}, \mu, (P_i)_{i\in N}\rangle$ the economy with knowledge in which

- $N = \{1, 2\}$;
- $\Omega = \{\omega_1, \omega_2, \omega_3\}$;
- $e_1(\omega) = e_2(\omega) = 2$ for every $\omega \in \Omega$;
- $\mu(\omega) = \frac{1}{3}$ for each $\omega \in \Omega$;

- $U_i : \mathbb{R}^1_+ \times \Omega \to \mathbb{R}$ is defined by

$$U_1(x,\omega) = (x+1)^2 \quad \text{and} \quad U_2(x,\omega) = \sqrt{x+3};$$

- P_i is defined by

$$P_1(\omega) := \begin{cases} \{\omega_1\} & \text{if } \omega = \omega_1 \\ \{\omega_2, \omega_3\} & \text{if } \omega = \omega_2 \text{ or } \omega_3 \end{cases} \quad \text{and} \quad P_2(\omega) := \begin{cases} \{\omega_1, \omega_3\} & \text{if } \omega = \omega_1 \\ \{\omega_2, \omega_3\} & \text{if } \omega = \omega_2 \text{ or } \omega_3. \end{cases}$$

It is plainly observed first that P_2 are reflexive and not transitive, and second that the endowments $(e_i)_{i=1,2}$ are ex-ante Pareto optimal. Let $\boldsymbol{t} = (t_i)_{i=1,2}$ be the feasible non-zero trade defined by

$$t_1(\omega) := \begin{cases} 1 & \text{if } \omega = \omega_1 \text{ or } \omega_2 \\ -1.5 & \text{if } \omega = \omega_3 \end{cases} \quad \text{and} \quad t_2(\omega) := \begin{cases} -1 & \text{if } \omega = \omega_1 \text{ or } \omega_2 \\ 1.5 & \text{if } \omega = \omega_3. \end{cases}$$

Then it follows that $Acp(\boldsymbol{t}) = \Omega$, $R = \{\omega_2, \omega_3\}$, and thus $K_C(Acp(\boldsymbol{t}) \cap R) = \{\omega_2, \omega_3\}$. However the trade $\boldsymbol{t}$ is not null at any $\omega \in K_C(Acp(\boldsymbol{t}) \cap R)$.

Nevertheless, common-knowledge of the acceptance of feasible trades seems a rather strong assumption. Could not we get away with less, say with mutual knowledge? The answer is no again: For the counter example see Fudenberg and Tirole (1991, p.552).

5 Concluding remarks

5.1 Extended notion of rational expectations equilibrium

We focus on the relaxation of the usual information structure involving what can be characterized as a 'partitional' framework in this article. One of the purposes is to show the existence theorem of the rational expectations equilibrium for an economy with knowledge and also to characterize welfare under the extended notion of equilibrium.

The main assumptions are **A-1**, **A-2** and **A-3** (occasionally replaced by **A-4**) as well as reflexivity and transitivity of the information structure. The almost all results in Section 4 crucially depend on these assumptions. Indeed the suppression of any conditions **A-1**, **A-2**, **A-3** and **A-4** renders Propositions 3.7 and Theorems 3.4 and 3.3 vulnerable to the discussion and the examples provided in Remarks 3.4.

5.2 Further relaxing the RT-information structure

Our real concern is to what extent No trade theorem of Milgrom and Stokey (1982) depends on the information partition and on the hypothesis that the initial endowments are ex-ante Pareto optimal. As we have observed, the reflexivity and transitivity of information structure can preclude trade if the traders commonly know that they are willing to trade the amounts of state-contingent commodities. Both the information partition and the strictly risk-aversion for the traders play no role in No trade theorem. If these assumptions (Reflexivity and Transitivity) fails then our No trade theorem (Theorem 4.1) is not true in viewing the above examples (Examples 4.5 and 4.6).

5.3 No trade theorem of Geanakoplos

The impact analysis of weakening the partition formulation on the usual No trade theorem of Milgrom and Stokey is done in different settings by Geanakoplos (1989), Morris (1994) and others. We will compare Geanakoplos's version of No trade theorem with ours.

In contrast to Theorem 4.1, Geanakoplos needs to impose the requirement that the information structure in an economy with knowledge is *nested*:

Definition 5.1. An information structure $(P_i)_{i \in N}$ in an economy with knowledge is said to be *nested* if for each $i \in N$ and for all states ω and ξ in Ω, either $P_i(\omega) \cap P_i(\xi) = \emptyset$, or else $P_i(\omega) \subseteqq P_i(\xi)$ or $P_i(\omega) \supseteqq P_i(\xi)$.

Under the circumstance that the information structure is nested he showed that: *If the initial endowments allocation $(e_i)_{i \in N}$ is ex-ante Pareto optimal then the traders can never agree to any non null trade at each state where they commonly know the acceptable trade $\boldsymbol{t} = (t_i)$; i.e., $\boldsymbol{t}(\omega) = \mathbf{0}$ at every $\omega \in K_C(Acp(\boldsymbol{t}))$.*[10]

The assumption 'nestedness' would appear to allow the modification of Theorem 4.1 such as in terms of states in the larger set $K_C(Acp(\boldsymbol{t}))$ instead of $K_C(Acp(\boldsymbol{t}) \cap R)$: Could we prove that under the same circumstances in Theorem 4.1; if the information structure is nested in addition then the feasible trades vanish on $K_C(Acp(\boldsymbol{t}))$? The following example shows that the hope is no vein under the assumption that the initial endowments are Pareto-optimal.

[10] This statement is a modified version of Corollary 5.2 in Geanakoplos (1989).

Example 5.2. There are one commodity and two traders with the common prior μ. The economy with knowledge

$$\mathcal{E}^K = \langle N, \Omega, (e_i)_{i\in N}, (U_i)_{i\in N}, (\mu_i)_{i\in N}, (P_i)_{i\in N}\rangle.$$

consists of:

- $N = \{1, 2\}$;
- $\Omega = \{\omega_1, \omega_2, \omega_3\}$;
- $e_1(\omega) = e_2(\omega) = 1$ for every $\omega \in \Omega$;
- Let $\boldsymbol{t} = (t_i)_{i=1,2}$ be the trade defined by

$$t_1(\omega) := \begin{cases} -\frac{3}{10} & \text{if} \quad \omega = \omega_1 \\ 1 & \text{if} \quad \omega = \omega_2 \\ 0 & \text{if} \quad \omega = \omega_3 \end{cases} \quad \text{and} \quad t_2(\omega) := \begin{cases} \frac{3}{10} & \text{if} \quad \omega = \omega_1 \\ -1 & \text{if} \quad \omega = \omega_2 \\ 0 & \text{if} \quad \omega = \omega_3, \end{cases}$$

- $U_i : \mathbb{R}^1_+ \times \Omega \to \mathbb{R}$ is defined by $U_i(x, \omega) = x_i^2$ for every $i = 1, 2$;
- $\mu_1(\omega) = \mu_2(\omega) = \mu(\omega) := \begin{cases} \frac{1}{2} & \text{for } \omega = \omega_1 \\ \frac{1}{4} & \text{for } \omega = \omega_2, \omega_3, \end{cases}$
- $P_i : \Omega \to 2^\Omega$ is defined by

$$P_1(\omega) := \begin{cases} \{\omega_1, \omega_2\} & \text{for } \omega = \omega_1 \\ \{\omega_2\} & \text{for } \omega = \omega_2 \\ \{\omega_3\} & \text{for } \omega = \omega_3 \end{cases} \quad \text{and} \quad P_2(\omega) := \begin{cases} \{\omega_1\} & \text{for } \omega = \omega_1 \\ \{\omega_1, \omega_2\} & \text{for } \omega = \omega_2 \\ \{\omega_3\} & \text{for } \omega = \omega_3. \end{cases}$$

It is observed that the endowments $(e_i)_{i=1,2}$ are ex-ante Pareto optimal and that $\mathcal{E}^K$ is an economy under reflexive, transitive and nested information structure. It can be seen that the feasible trade $\boldsymbol{t}$ is not null at both states ω_1 and ω_2 in $K_C(Acp(\boldsymbol{t})) = \Omega$, but it vanishes on $K_C(Acp(\boldsymbol{t}) \cap R) = \{\omega_3\}$.

Nonetheless, such modification of No trade theorem in our version can be obtained by Theorem 4.3 and its corollary when the initial endowments allocation is assumed to be a rational expectations equilibrium allocation relative to some price system with respect to which all traders are rational everywhere about their expectations.

This observation together with the following Example 5.3 shows the essential role of the notion 'rationality about expectations', which has not been appeared in the partitional formulation.

Example 5.3. There are one commodity and two traders with the common prior μ. The economy with knowledge

$$\mathcal{E}^K = \langle N, \Omega, (e_i)_{i\in N}, (U_i)_{i\in N}, (\mu_i)_{i\in N}, (P_i)_{i\in N}\rangle.$$

consists of:

- $N = \{1, 2\}$
- $\Omega = \{\omega_1, \omega_2, \omega_3\}$;
- $e_1(\omega) = e_2(\omega) = 1$ for every $\omega \in \Omega$;
- Let $\boldsymbol{t} = (t_i)_{i=1,2}$ be the trade defined by

$$t_1(\omega) := \begin{cases} -\frac{3}{10} & \text{if} \quad \omega = \omega_1 \\ 0 & \text{if} \quad \omega = \omega_2 \\ 1 & \text{if} \quad \omega = \omega_3 \end{cases} \quad \text{and} \quad t_2(\omega) := \begin{cases} \frac{3}{10} & \text{if} \quad \omega = \omega_1 \\ 0 & \text{if} \quad \omega = \omega_2 \\ -1 & \text{if} \quad \omega = \omega_3, \end{cases}$$

- $U_i : \mathbb{R}^1_+ \times \Omega \to \mathbb{R}$ is defined by $U_i(x, \omega) = \sqrt{x+4}$;
- $\mu_1(\omega) = \mu_2(\omega) = \mu(\omega) := \begin{cases} \frac{3}{7} & \text{if} \quad \omega = \omega_1,\ \omega_3 \\ \frac{1}{7} & \text{if} \quad \omega = \omega_2, \end{cases}$
- $P_i : \Omega \to 2^\Omega$ is defined by

$$P_1(\omega) := \begin{cases} \{\omega_1\} & \text{for } \omega = \omega_1 \\ \{\omega_2\} & \text{for } \omega = \omega_2 \\ \{\omega_2, \omega_3\} & \text{for } \omega = \omega_3 \end{cases} \quad \text{and} \quad P_2(\omega) := \begin{cases} \{\omega_1, \omega_2\} & \text{for } \omega = \omega_1 \\ \{\omega_2\} & \text{for } \omega = \omega_2 \\ \{\omega_2, \omega_3\} & \text{for } \omega = \omega_3. \end{cases}$$

It is observed that the endowments $(e_i)_{i=1,2}$ are ex-ante Pareto optimal and that $\mathcal{E}^K$ is an economy under RT-information structure. Note that P_2 is not nested. $Acp(\boldsymbol{t}) = R = \{\omega_2\}$, and the feasible trade $\boldsymbol{t}$ is indeed null on $\{\omega_2\} = K_C(Acp(\boldsymbol{t}) \cap R)$.

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