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A Note on Pricing Derivatives
in an Incomplete Market

by

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August 2000

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Abstract

The assumption of the complete market simplifies the whole theory of arbitrage pricing theory since the pioneering work of Black and Scholes. The martingale approach is one of the most powerful tools for pricing derivative securities in the complete market. The existence of such a market, however, may not be guaranteed in the real world, where the martingale measure is not unique. Therefore, it is more natural to work on the incomplete market. We consider this problem in the simple discrete time and state model. We will present new approach of choosing the martingale measure, which is the combination of the method of least squares and the embedded complete markets.

Key Words and Phrases: Incomplete market, method of least squares, martingale measures, and embedded complete markets.

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1 Introduction

In the last 3 decades, motivated by the pioneering work of Black and Scholes (1973), the no arbitrage pricing theory has been a major tool of determining the price of derivative securities. The basic assumption of this beautiful method lies in the concept of the complete market, where every contingent claim can be reproduced by the self financing portfolio of the underlying basic assets in the market. However, it is not easy to tell if a given market is complete, or even more we can ask if there is a complete market in the real world. Therefore, more efficient and practically tractable method of pricing derivatives in an incomplete market is necessary for the practitioners. This problems has been considered by many authors such as Duffie and Richardson (1991), Shweizer (1992, 1995), and Berstimas, Kogan and Lo (1997). Our approach is basically
the same as these authors to the extent that the method of the least squares is the central tool. We will supplement the previous results by introducing the natural martingale measure in an incomplete market. The different approach to this problem is discussed by Karatzas, Leheczy, Shreve and Xu (1991), which we do not consider in this note.

In this note we will assume the discrete time and discrete time model and we will propose a method of determining the price of derivatives in an incomplete market, and our method is reduced to the usual no arbitrage pricing theory if the market is complete. Although we may lose some generality by assuming the discrete state models, it helps us to clarify the basic idea more transparently. The discrete state and time incomplete market considered in this note may appear when the prediction of future volatility in the stock market and the option market are different (cf. Takahashi (2000)). Since the magnitude of volatility in the binomial model may be expressed by the values of the stock price at the maturity, the possible states which the stock price assumes at maturity may differ across the markets (stock market and option market). Hence the two period trinomial model may be the simplest stochastic model which describes this situation. On the other hand, an example of incomplete market in the continuous time and state model is discussed, for example by Duffie and Richardson (1991). We will first review a method of pricing a option in a complete market in the rest of this section, where we will assume two period binomial stock price model.

Suppose the risk free bond \( \{ S^0_t \} = (1 + r)^t, t = 0, 1 \} \) and the stock \( \{ S_t, t = 0, 1 \} \) are the only assets in the market, where \( r \) is the risk free interest rate. Let \( C_0 \) be the price of European call option on the stock with the exercise price \( K \) and the maturity at \( T = 1 \). If the stock price process follows Bernoulli two period process, we can show that the market is complete and the price of the option can be determined uniquely by the method of Harrison and Kreps (1979), and Harrison and Pliska (1981). To be more specific, at \( t = 0 \) we suppose the stock price is \( S_0 = S \) for some \( S \), and at day \( T = 1 \), it goes up to \( S_1 = uS \) or goes down to \( S_1 = dS \) with probability \( q^* \) or \( 1 - q^* \) respectively. We will also assume that

\[
d < 1 + r < u
\]  

(1)

to eliminate the arbitrage opportunity. The value of the option at the maturity \( T = 1 \) equals either \( C_1^{(u)} = \max\{uS - K, 0\} \) or \( C_1^{(d)} = \max\{dS - K, 0\} \). Now, an equivalent portfolio (EP for short) to this call option is a portfolio whose time \( T = 1 \) value is the same as that of the option. If time \( t = 0 \) price of the EP differs from that of the option, there is an arbitrage opportunity. Therefore the time 0 price of the option \( C_0 \) should be equal to the time 0 price of the EP. Suppose the equivalent portfolio consists of \( B \) yen of the bond and \( \alpha \) unit of the stock, then the following system of linear equations must hold.

\[
\alpha(uS) + (1 + r)B = C_1^{(u)}
\]  

(2)
If \( u \neq d \), then the above equations have the unique root \((\alpha, B)\) and the market becomes complete. Of course, this simply follows from the fact that the number of distinctive equations and unknowns \((\alpha, B)\) are the same. If the equations (2) and (3) reduce to one; this is the case when \( u = d \), then there is an arbitrage opportunity in the market unless the rate of return for the stock \((u + 1 = d + 1)\) and the bond are both equal to \( r \). If, on the other hand, there are more than two equations; the possible states at \( T = 1 \) is more than or equal to 3, we have an incomplete market. We shall discuss the later case more extensively in the next section.

Now, by solving (2) and (3), we have

\[
C_0 = \frac{1}{1+r} \{qC_1^{(u)} + (1-q)C_1^{(d)}\} \tag{4}
\]

where,

\[
q = \frac{1 + r - d}{u - d}. \tag{5}
\]

Note that under the condition (1), it follows that \( 0 < q < 1 \), and \( q \) can be interpreted as a probability (see equation (6) below). With this in mind, \( C_0 \) may be interpreted as the (conditional) expectation of \( C_1 \) (given \( S_0 = S \)) discounted by the risk free interest rate \( r \), where the expectation is taken under the probability measure \( \{q, 1-q\} \), where

\[
P\{S_1 = uS\} = q = 1 - P\{S_1 = dS\} \tag{6}
\]

It is also known that under this probability, the process \( \{\frac{S_t}{1+r}, F_t, \ t=0,1\} \) is a martingale, where \( F_t = \beta\{S_u, 0 \leq u \leq t\} \) is a sigma algebra of information carried by \( \{S_u\} \) up to and including time \( t \). This is the simplest case of the general martingale method proposed by Harrison and Kreps (1979) (also see Harrison and Pliska (1981)), who proved that the necessary and sufficient condition for no arbitrage opportunity in a frictionless market is the existence of a equivalent martingale measure \( P \).
2 Incomplete Market and Embedded Complete Markets

We will consider a problem of pricing derivatives in an incomplete market. We suppose that there are two securities, the stock (risky asset) and the bond (riskless asset), in the market as in Section 1. We will, however, assume that the stock price follows the trinomial and two period stochastic process, where time $t = 0$ and time $T = 1$ are the present and the maturity respectively. Our problem is to determine the time 0 theoretical price $C_0$ of the European call option on the stock with exercise price $K (> 0)$ which matures at $T = 1$. We also assume that the risk free interest rate is $r (> 0)$. To be more rigorous, we should write $C(S_t, t, T, r, K, \sigma)$, rather than $C_t$ to denote the time $t$ price of European Call option, where $S_t = \text{present stock price}$, $t = \text{present day}$, $T = \text{maturity}$, $r = \text{risk free interest rate}$, $K = \text{exercise price}$, and $\sigma = \text{volatility of the stock return}$. We will, however, use the simplified notation whenever there is no confusion.

Let $S_0 = S$ be the time $t = 0$ stock price as before. We suppose that, at $T = 1$, the random variable $S_1^{(1)}$ takes one of the following three values; $S_1^{(u)} = uS$, $S_1^{(m)} = mS$, or $S_1^{(d)} = dS$, with probabilities $p_u$, $p_m$, and $p_d$ respectively, where $0 < p_u^T < 1$ ($y = u, m, d$) and $p_u + p_m + p_d = 1$. The values of $S_1^{(1)}$ are determined by the market's prediction of the future volatility of the stock return. Also we will assume that

$$d < m < u \quad \text{(7)}$$

and

$$d < 1 + r < u \quad \text{(8)}$$

to exclude arbitrage opportunities in the market. We will call the probability measure $P^* = \{p_u, p_m, p_d\}$ the real probability measure. Note that as far as we are concerned with the pricing of derivatives, the measure $P^*$ may be quite arbitrary except that it assigns positive probabilities to all the states in the space of $S_1^{(1)}$. Also note that at $T = 1$, the value of the option becomes either $C_1^{(u)} = \max\{S_1^{(u)} - K, 0\}$, $C_1^{(m)} = \max\{S_1^{(m)} - K, 0\}$, or $C_1^{(d)} = \max\{S_1^{(d)} - K, 0\}$, in accordance with the stock price at $T = 1$.

Since the necessary and sufficient condition that there is no arbitrage opportunities in the market is the existence of martingale measure (cf. Harrison and Kreps (1979), and Harrison and Pliska (1981)), our first step toward the pricing problem is a search of martingale measures. Hence, in the following simple lemma, we will give the necessary and sufficient condition for martingale measures.

Lemma 1 Suppose $\{S_t, t = 0, 1\}$ follows two period trinomial process. Then a probability measure $P = \{p_u, p_m, p_d\}$ is an equivalent martingale measure for
\{ \frac{S_t}{(1+r)^t}, F_t, t=0,1 \}, if and only if
\begin{equation}
p_y > 0 \quad \text{for all} \quad y = u, m, d
\end{equation}
and
\begin{equation}
(u - d)p_u + (m - d)p_m = 1 + r - d
\end{equation}

Proof. If \( \{ \frac{S_t}{(1+r)^t}, F_t, t=0,1 \} \) is a martingale under \( P \), we have
\begin{equation}
\frac{1}{1+r} [uS \cdot p_u + mS \cdot p_m + dS \cdot p_u] = E\{ \frac{S_1}{(1+r)^1} | S_0 = S \} = \frac{S_0}{(1+r)^0} = S
\end{equation}

It follows that \( up_u + mp_m + dp_u = (1 + r) \), and (10) follows readily. Since \( p_y > 0 \) for all \( y = u, m, d \), (9) is also true. The converse is proved in the same manner.

Since there are infinitely many \( P = \{ p_u, p_m, p_d \} \)'s which satisfy (9) and (10) in our trinomial model, the martingale measure is not unique. Therefore, it is not possible to determine the value of \( C_0 \) uniquely by the martingale method alone. This can be understood more transparently by the following elementary argument.

Now, there are only two basic assets in this economy; the stock and the riskless bond, and we consider the portfolio \( PF \) between time 0 and 1, which consists of \( a \) units of the stock and \( B \) units of the risk-free bond. The number of states which \( C_1 \) can assume is, however, three. Then the necessary condition for the portfolio \( PF \) to be an equivalent portfolio to \( C \) is expressed by the three equations,
\begin{align}
\alpha(uS) + (1 + r)B &= C_1^{(u)} \\
\alpha(mS) + (1 + r)B &= C_1^{(m)} \\
\alpha(dS) + (1 + r)B &= C_1^{(d)}
\end{align}

Then there will be three sets of solutions \((\alpha, B)\) for the above system of equations, and there is no equivalent portfolio, unless these solutions are identical. Therefore, the method of the previous section fails to give the unique price to the time 0 option value. Here, the simple geometry gives us a clear picture of what is going on. The value vector \( C_1 = (C_1^{(u)}, C_1^{(m)}, C_1^{(d)}) \) does not lay in the space \( sp \{ S_1, r \} \) spanned by the vectors \( S_1 = (uS, mS, dS) \) and \( r = (1 + r)1 = (1 + r, 1 + r, 1 + r) \). The problem of finding an equivalent portfolio is to express the vector \( C_1 \) by the elements of \( sp \{ S_1, r \} \), which is to find constants \((\alpha, B)\) such that \( C_1 = \alpha S_1 + (1 + r)B1 \). Clearly this is by no means possible in this case.
Now, another way of looking at this problem is to consider the embedded complete market models inside this incomplete market model. For example, we can imagine a sub-market where $C_1$ takes only $(C_1^{(u)}, C_1^{(m)})$ at $T = 1$. Then, this sub-market is shown to be a complete market as in Section 1. This is equivalent to consider the system of equation $\{(12), (13)\}$ out of $\{(12),(13),(14)\}$. Together with $\{(13),(14)\}$, and $\{(14),(12)\}$, we can altogether form three embedded "complete markets" from the original incomplete market. Except for some technical problems which we will discuss later, we obtain martingale measures in each of the three embedded markets as in Section 1.

Form (12), and (13), we have,

$$\alpha^{(u,m)}(uS) + (1 + r)B^{(u,m)} = C_1^{(u)}$$
$$\alpha^{(u,m)}(mS) + (1 + r)B^{(v,m)} = C_1^{(m)}$$

(15)

It follows that the martingale measure in this sub-market is given by,

$$Q^{(u,m)} = \{p^{(u,m)}, q^{(u,m)} = 1 - p^{(u,m)}\},$$

where,

$$p^{(u,m)} = P^{(u,m)}\{S_1 = uS|S_0 = S\} = \frac{(1 + r) - m}{u - m}$$

(16)

Next, (13), and (14) gives us,
\[
\alpha^{(m,d)}(mS) + (1 + r)B^{(m,d)} = C_1^{(m)}
\]
\[
\alpha^{(d,u)}(dS) + (1 + r)B^{(d,u)} = C_1^{(d)}
\]

Then the martingale measure is
\[
Q^{(m,d)} = \{ p^{(m,d)}, q^{(m,d)} = 1 - p^{(m,d)} \},
\]
where,
\[
p^{(m,d)} = P^{(m,d)} \{ S_1 = mS | S_0 = S \} = \frac{(1 + r) - d}{m - d}
\]

Finally from (14) and (12)
\[
\alpha^{(d,u)}(dS) + (1 + r)B^{(d,u)} = C_1^{(d)}
\]
\[
\alpha^{(d,u)}(uS) + (1 + r)B^{(d,u)} = C_1^{(u)}
\]

we have the martingale measure
\[
Q^{(d,u)} = \{ p^{(d,u)}, q^{(d,u)} = 1 - p^{(d,u)} \},
\]
where,
\[
p^{(d,u)} = P^{(d,u)} \{ S_1 = dS | S_0 = S \} = \frac{(1 + r) - u}{d - u}
\]

Here, we have set \( P^{(u,v)} \) denotes the martingale probability measure in the space of \( S_1 = (x_S, y_S) \). Note that the measures \( Q^{(u,m)} \), \( Q^{(m,d)} \), \( Q^{(d,u)} \) are neither equivalent to each other, nor they are equivalent to the original probability measure \( Q^* \). Let \( \Theta = (\theta_1, \theta_2, \theta_3) \) be a vector such that,
\[
\theta_1 + \theta_2 + \theta_3 = 1, \quad \text{for all} \quad i = 1, 2, 3.
\]

We will define a strong convex combination \( Q(\Theta) \) of these measures \( \{ Q^{(u,m)}, Q^{(m,d)}, Q^{(d,u)} \} \), where
\[
Q(\Theta) = \theta_1 Q^{(u,m)} + \theta_2 Q^{(m,d)} + \theta_3 Q^{(d,u)}
\]
\[
= \{ \hat{p}(u), \hat{p}(m), \hat{p}(d) \}
\]
and
\[
\hat{p}(u) = \hat{p}(u) = P\{ S_1 = uS | S_0 = S \} = \theta_1 p^{(u,m)} + \theta_2 q^{(d,u)}
\]
\[
\hat{p}(m) = \hat{p}(m) = P\{ S_1 = mS | S_0 = S \} = \theta_2 p^{(m,d)} + \theta_3 q^{(u,m)}
\]
\[
\hat{p}(d) = \hat{p}(d) = P\{ S_1 = dS | S_0 = S \} = \theta_3 p^{(d,u)} + \theta_2 q^{(m,d)}
\]

Then, \( Q(\Theta) \) is equivalent to \( Q^* \). We note that when \( d < 1 + r < m \), \( Q^{(u,m)} \) is a signed measure and it is by no means a probability measure. However, by choosing an appropriate weigh \( \Theta = (\theta_1, \theta_2, \theta_3) \) the combined measure \( Q(\Theta) \) becomes a probability measure. Moreover, in the next lemma, we will show that as long as it is a probability measure, it is martingale measure.
Lemma 2 For any $\theta_i > 0$ ($i = 1, 2, 3$) with $\theta_1 + \theta_2 + \theta_3 = 1$, if $Q(\Theta)$ is a probability measure, then it is a martingale measure for $\{\frac{S_{t+i}}{S_t}, F_t, t = 0, 1\}$.

Proof. It is straightforward that for any $\theta_i > 0$ ($i = 1, 2, 3$) with $\theta_1 + \theta_2 + \theta_3 = 1$,

$$(u - m)\hat{p}(u) + (m - d)\hat{p}(m) = 1 + r - d$$

holds. The lemma follows readily from Lemma 1.

Note that, with any martingale measure $Q(\Theta) = \{\hat{p}(u), \hat{p}(m), \hat{p}(d)\}$ defined for any $\Theta = (\theta_1, \theta_2, \theta_3)$, an arbitrage free price of the option will be given by

$$C_0(\Theta) = \left(\frac{1}{1 + r}\right)[\hat{p}(u)C_1^{(u)} + \hat{p}(m)C_1^{(m)} + \hat{p}(d)C_1^{(d)}]$$

The option price is not unique in general, for there are infinitely many weight $\Theta = (\theta_1, \theta_2, \theta_3)$ which makes $Q(\Theta)$ a probability. The problem of which measure should be selected has been considered by several authors. For example, Miyahara (1996) obtained the supreme or infimum of $C_0(\Theta)$ over the all possible $\Theta$'s, which, however, does not give us a unique price of the option. In the next section, we will see that the weight obtained from the method of least squares may be the most natural choice in this problem.
3 The Methods of Least Squares

We will start this section with the trivial statement. "If the option is traded in the market, the price of which is determined by the market and the unique martingale measure for all assets, including the option, exists." If the random fluctuation of the derivative security and the underlying asset are completely correlated, there is a chance that we have a complete hedging against the derivative (cf. Duffie and Richardson (1991), Karatzas, Lehoczky, Shreve and Xu (1991)). In the incomplete market, however, our first question is how to obtain the portfolio which reproduces the given derivative as close as possible in some sense to be stated next. Here, we will use the average mean squared error as a criterion of the "best" approximation. Although the choice of this criterion seems very subjective, we will see later in this section that the mean squared error criterion gives us a martingale measure which is consistent with both the complete market and the market where the derivative are actually traded. Therefore, the weight obtained from the method of least squares is the natural choice with respect to the dogma of martingale method, or general equilibrium, where the price of every good in the economy is expressed by the linear functional (cf. Harrison and Kreps (1979)).

Let \( \hat{C}_1 \) be the orthogonal projection of \( C_1 \) on to the space \( sp\{S_1, r\} \). Then, there is a pair of constants \( (\hat{\alpha}, \hat{B}) \) and it can be written as
\[
\hat{C}_1 = \hat{\alpha}S_1 + (1 + r)\hat{B}1
\]
(25)

We will obtain the constants \( (\hat{\alpha}, \hat{B}) \) by the method of least squares. It is well known that the least square estimates for \( \alpha \) and \( (1 + r)B \) are given by
\[
\hat{\alpha} = \frac{(uS - \hat{S})C_{1(u)} + (mS - \hat{S})C_{1(m)} + (dS - \hat{S})C_{1(d)}}{(uS - \hat{S})^2 + (mS - \hat{S})^2 + (dS - \hat{S})^2}
\]
and,
\[
(1 + r)\hat{B} = \hat{C} - \hat{\alpha}S,
\]
where
\[
\hat{C} = \frac{1}{3}(C_{1(u)} + C_{1(m)} + C_{1(d)}),
\]
and
\[
\hat{S} = \frac{1}{3}(uS + mS + dS)
\]

The following representation of \( \hat{\alpha} \) and \( (1 + r)\hat{B} \) is the key to our analysis (cf. Wu (1986)). It follows from the straightforward algebra that
\[
\hat{\alpha} = \frac{(uS - mS)(C_{1(u)} - C_{1(m)}) + (mS - dS)(C_{1(m)} - C_{1(d)}) + (dS - uS)(C_{1(d)} - C_{1(u)})}{(uS - mS)^2 + (mS - dS)^2 + (dS - uS)^2}
\]
\[
= w_{u(m)}\frac{(C_{1(u)} - C_{1(m)})}{(uS - mS)} + w_{(m,d)}\frac{(C_{1(m)} - C_{1(d)})}{(mS - dS)} + w_{(d,u)}\frac{(C_{1(d)} - C_{1(u)})}{(dS - uS)}
\]
(26)
(1+r)\hat{B} = \sum_{(s,m)} w^{(s,m)} \left( \frac{uC_1^{(m)} - mC_1^{(u)}}{u - m} \right) + \sum_{(m,d)} w^{(m,d)} \left( \frac{mC_1^{(d)} - dC_1^{(m)}}{m - d} \right) + \sum_{(d,u)} w^{(d,u)} \left( \frac{dC_1^{(d)} - uC_1^{(u)}}{d - u} \right) \tag{27}

where, the weights are given by

w^{(x,y)} = \frac{(xS - yS)^2}{(uS - mS)^2 + (mS - dS)^2 + (uS - dS)^2}, \quad (x, y) = \{(u, m), (m, d), (d, u)\}, \tag{28}

The interpretation of the above expressions (26) and (27) of (\hat{\alpha}, (1+r)\hat{B}) is interesting. Note that,

\alpha^{(x,y)} = \frac{C_1^{(x)} - C_1^{(y)}}{xS - yS} \quad \text{and} \quad (1+r)B^{(x,y)} = \frac{(xC_1^{(y)} - yC_1^{(x)})}{x - y}

are the slope and intercept of the line connecting the points (xS, C_1^{(x)}) and (yS, C_1^{(y)}), where ((x, y) = (u, m), (m, d), (d, u)). Therefore, the over all least square estimator (\hat{\alpha}, (1+r)\hat{B}) is the weighted average of the slope of the every line determined by every pair in the sample. This should be compared with (15), (17), and (19) in the previous section. Since the martingale measures in each sub markets are given in (16), (18), and (20), the only issue is to find a reasonable weights \Theta's. Here we claim that the weight \{w^{(x,y)}\} appeared in the least square estimates of \alpha and (1+r)B is the most natural candidate in a sense it minimizes the amount you fail to hedge (cf. Berstimas et. al (1997)). Also, we note that the method of least squares contains the theory of complete market as a special case. Therefore, by taking \Theta = W = \{w^{(u,m)}, w^{(m,d)}, w^{(d,u)}\}, we have the martingale measure \{\hat{\rho}^{(W)}(u), \hat{\rho}^{(W)}(m), \hat{\rho}^{(W)}(d)\}, where

\begin{align*}
\hat{\rho}^{(W)}(u) &= w^{(u,m)} \hat{\rho}^{(u,m)} + w^{(d,u)} \hat{\rho}^{(d,u)} \\
\hat{\rho}^{(W)}(m) &= w^{(m,d)} \hat{\rho}^{(m,d)} + w^{(u,m)} \hat{\rho}^{(u,m)} \\
\hat{\rho}^{(W)}(d) &= w^{(d,u)} \hat{\rho}^{(d,u)} + w^{(m,d)} \hat{\rho}^{(m,d)}
\end{align*} \tag{29}

And we have shown in Lemma 2 that the weight also preserves the martingale nature if \{\hat{\rho}^{(W)}(u), \hat{\rho}^{(W)}(m), \hat{\rho}^{(W)}(d)\} is a probability measure. The sufficient condition of the weight becomes probability depends on the value of u, m, d and r. We will leave this question to the numerical analysis.
4 Multiperiod Models

We will extend our trinomial two period model to the n period model. The idea is the same as the extension of the usual two period binomial option price model to the multiperiod models. We will first consider the Markovian three periods trinomial model for simplicity. The model may be expressed in the following picture (See Picture 2). In order the model to be Markov, it is necessary that \( m^2 = ud \) must hold.

Table 1

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>( S )</td>
<td>( uS )</td>
<td>( mS )</td>
<td>( dS )</td>
<td>( u^2S )</td>
</tr>
<tr>
<td>Probability</td>
<td>1</td>
<td>( \hat{p}^{(u)}(u) )</td>
<td>( \hat{p}^{(m)}(m) )</td>
<td>( \hat{p}^{(d)}(d) )</td>
<td>( \hat{p}^{(u)}(u)^2 )</td>
</tr>
</tbody>
</table>

\[
\begin{array}{|c|c|c|c|c|}
\hline
F & G & H & I \\
\hline
u mS = muS & m^2S = u dS = d u S & dmS = m dS & d^2S \\
\hline
\hat{p}^{(u)}(u) \hat{p}^{(m)}(m) & \hat{p}^{(m)}(m)^2 + 2 \hat{p}^{(m)}(u) \hat{p}^{(d)}(d) & 2 \hat{p}^{(m)}(m) \hat{p}^{(d)}(d) & \hat{p}^{(d)}(d)^2 \\
\hline
\end{array}
\]
In each node at time \( t = 1 \), we will assign the probability \( \{ \hat{\beta}^{(W)} (u), \hat{\beta}^{(W)} (m), \hat{\beta}^{(W)} (d) \} \)

which is defined in (29). From the node B, for example, the process moves up to the node E with probability \( \hat{\beta}^{(W)} (u) \). Hence the process reaches to the node E with probability \( \hat{\beta}^{(W)} (u)^2 \). The node G will be reached either from the nodes B, C, or D, with probabilities \( \hat{\beta}^{(W)} (d), \hat{\beta}^{(W)} (m), \) and \( \hat{\beta}^{(W)} (u) \) respectively. Hence the probability of getting the state G is given by \( \hat{\beta}^{(W)} (m)^2 + 2\hat{\beta}^{(W)} (u)\hat{\beta}^{(W)} (d) \), which complicates obtaining the whole tree structures and as we will see below, makes it difficult to present the general case. We summarize this three period model in Picture 2 and Table 1.

The general n-period model may be described as follows. The total number of states at the \((n-1)\)st stage is given by the number of terms in the expansion of \((u + m + d)^n S = (u + \sqrt{ud} + d)^n S\), where we have set \( m^2 = ud \). The typical state may be expressed as \( \sqrt{m} - k \sqrt{d} \), \( k = 0, 1, ..., 2n \), and the coefficient of which gives us the number of routes leading to that state. The coefficient is proved to be

\[
\sum_{p=0}^{[k/3]} (-1)^p \binom{n}{p} \binom{n + k - 3p - 1}{n-1} \tag{30}
\]

(This formula was personally communicated to the author by Professors Nabeya and Hayakawa, to whom the author expresses his deep appreciation.) The associated probability, however, is not easily calculated, for the probability of obtaining each route may differ which results from the fact that \( \hat{\beta}^{(W)}(d) \hat{\beta}^{(W)}(u) \) need not be equal to \( \hat{\beta}^{(W)}(m)^2 \). Therefore, the numerical calculation will be necessary to construct the general multi period models.
References


