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Estimation in Partial Linear Models under Long-Range Dependence

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Abstract

We consider estimation of the linear component of a partial linear model when errors and regressors have long-range dependence. Assuming that errors and the stochastic component of regressors are linear processes with i.i.d. innovations, we closely examine the asymptotic properties of the OLS estimator calculated from nonparametric regression residuals. Especially we derive the asymptotic distribution when the combined long-range dependence of errors and the stochastic component of regressors exceeds a level. The case is not considered in any previous works on partial linear models. We also improve the existing results when the combined long-range dependence is less than the level.

Keywords: asymptotic normality; linear process; long-range dependence; kernel estimator; nonparametric regression; partial linear models

*Running head: Partial Linear Models

1 Introduction

We consider estimation of β in the partial linear model defined in (1) below. The regressor X_i is a $k-$dimensional random vector and v^T denotes the transpose of v.

$$Y_i = X_i^T\beta + m(t_i) + \epsilon_i \quad \text{and} \quad X_i = g(t_i) + \delta_i, \tag{1}$$

where $t_i = (i - 0.5)/n$, $i = 1, \ldots, n$, $m(\cdot)$ and $g(\cdot)$ are unknown functions, and $\{\epsilon_i\}$ and $\{\delta_i\}$ are 1-dimensional and $k-$dimensional long-range dependent linear processes, respectively. We assume that $\{\epsilon_i\}$ and $\{\delta_i\}$ are mutually independent. Assumptions on $m(\cdot)$, $g(\cdot)$, $\{\epsilon_i\}$, and $\{\delta_i\}$ are given in Section 2.

We observe (Y_i, X_i), $i = 1, \ldots, n$, and estimate β by the OLS estimator calculated from nonparametric regression residuals. The estimator is denoted by $\hat{\beta}_h$, where h is the bandwidth of the nonparametric regression. We consider asymptotics when the sample size n tends to infinity.

[4] and [2] consider the same model and estimate β by the same kind of estimator. Especially our estimator of β, $\hat{\beta}_h$, coincides with that of [2].

The results on estimation of β in the two papers are theoretically interesting and practically important. However, those are limited since they consider only the case where the combined long-range dependence is less than a level, "$d_1 + d_2 < 1/2$" in the notation defined in Section 2. See **A1** in Section 2 for the definitions of d_1 and d_2. There are some unnecessary conditions on the bandwidth h and no general condition for the conditional asymptotic normality is explicitly given in the two papers.

The main purpose of this paper is to derive the asymptotic distribution of $\hat{\beta}_h$ when the combined long-range dependence exceeds a level, "$d_1 + d_2 > 1/2$" in the notation defined in Section 2. Besides, we improve the existing results when $d_1 + d_2 < 1/2$. We refer to nonparametric estimation of $m(\cdot)$ at the end of Section 2.

We can say that the asymptotic distribution of $n^{1/2}(\hat{\beta}_h - \beta)$ is the same up to a matrix multiplication as that of $n^{-1/2}\sum_{i=1}^n \delta_i\epsilon_i$ when $d_1 + d_2 < 1/2$. When $d_1+d_2 > 1/2$, $\hat{\beta}_h$ is not root-n-consistent and $n^{1/2}(nh)^{-(d_1+d_2)+1/2}(\hat{\beta}_h-\beta)$ converges in distribution to a k-variate normal distribution under the assumptions in Theorem 2.2. In contrast to the case where $d_1 + d_2 < 1/2$, the asymptotic distribution of $n^{1/2}(nh)^{-(d_1+d_2)+1/2}(\hat{\beta}_h - \beta)$ is completely different from that of $n^{-(d_1+d_2)}\sum_{i=1}^n \delta_i\epsilon_i$ and the rate of convergence is better than that related to $n^{-(d_1+d_2)}\sum_{i=1}^n \delta_i\epsilon_i$. It is because nonparametric regression effectively works on δ_i and ϵ_i. Then some of the long-range dependence is cancelled out. See Corollary 4.1 in [6] for the asymptotic distribution of $n^{-(d_1+d_2)}\sum_{i=1}^n \delta_i\epsilon_i$.

There has been a lot of research on partial linear models since [15] and [18]. See [12] for the research up to year 2000. In [15], the partial linear model is different from (1) since

$$Y_i = X_i^T\beta + m(Z_i) + \epsilon_i, \tag{2}$$

where Z_i is a random variable. In (2), nonparametric regression with random-design is applied to obtain the estimate of β. On the other hand, nonparametric regression with fixed-design is applied in the setup of (1) as in [18]. We concentrate on the setup of (1). It will be extremely difficult to study partial linear models in the setup of (2) under long-range dependence.

Recently, partial linear models have been studied for short-range dependent and long-range dependent time series. See [1], [11], and the references therein for the results under short-range dependence. See [4] and [2] for the results under long-range dependence. In those papers, the estimators of β are root-n-consistent. [7] studied a similar model, where there is no δ_i, under long-range dependence.

There are a few different definitions of long-range dependence for time series models, for example, poles of spectral densities, slowly-decaying autocovariances, and so on. There is a lot of theoretical and practical research on time series with long-range dependence because of its practical and theoretical importance. See [3], [17], and [5] for the definitions, properties, and surveys of the research on long-range dependent time series. In this paper we assume that $\{\delta_i\}$ and $\{\epsilon_i\}$ are long-range dependent linear processes as defined in **A1** below. See [13], [14], [19], and [20] for theoretical results on linear processes with long-range dependence.

This paper is organized as follows. We define the assumptions and the estimator $\hat{\beta}_h$ in Section 2. The asymptotic properties are stated in Theorems 2.1 and 2.2. Some important propositions are given and the theorems are proved in Section 3. The proofs of the propositions are deferred to Section 4. Technical lemmas and the proofs are confined to Section 5.

In this paper $\lfloor a \rfloor$ stands for the largest integer which is less than or equal to a. C's are generic positive constants and $a_j \sim a'_j$ means that $a_j/a'_j \to 1$ as $j \to \infty$ or $n \to \infty$. We denote convergence in distribution and in probability by $\stackrel{d}{\to}$ and $\stackrel{p}{\to}$, respectively, and $N(\mu, \Sigma)$ stands for a normal distribution with mean μ and covariance matrix Σ. Let $\mathrm{Var}(W)$ stand for the covariance matrix of a random vector W. The Euclidean norm of a vector v is denoted by $|v|$.

2 The estimator and the asymptotic distributions

In this section, we describe the assumptions, **A1-A5**, define $\hat{\beta}_h$, and present the asymptotic distributions in Theorems 2.1 and 2.2. Some notation is also introduced.

A1 and **A2** below are about the properties of $\{\delta_i\}$ and $\{\epsilon_i\}$. **A1** is a standard assumption to examine the asymptotic properties of statistics and estimators under long-range dependence. For example, see [13], [14], and [20].

A1:

(a)

$$\delta_i = \sum_{j=0}^{\infty} A_j \zeta_{i-j} \quad \text{and} \quad \epsilon_i = \sum_{j=0}^{\infty} b_j \eta_{i-j},$$

where $A_j, j = 0, 1, \ldots,$ is a sequence of $k \times k$ constant matrices and we denote the (l, m) element of A_j by $a_j^{(l,m)}$. In addition each element of A_j and b_j satisfy

$$a_j^{(l,m)} \sim \frac{\alpha^{(l,m)}}{(j+1)^{1-d_1}} \quad \text{and} \quad b_j \sim \frac{\gamma}{(j+1)^{1-d_2}},$$

where $0 < d_1 < 1/2$ and $0 < d_2 < 1/2$.

(b) $\{\zeta_i\}$ and $\{\eta_i\}$ are i.i.d. processes with $\mathrm{E}(\zeta_1) = 0$ and $\mathrm{E}(\eta_1) = 0$. Each element of ζ_1 and η_1 have the finite fourth moment. Besides, $\{\zeta_i\}$ and $\{\eta_i\}$ are mutually independent.

Incorporating slowly varying functions into the definitions of A_j and b_j is not so difficult when $d_1 + d_2 < 1/2$. However, it will be tough and notationally quite complicated when $d_1 + d_2 > 1/2$ and our main purpose is to consider the latter case. Therefore we do not incorporate slowly varying functions into the definitions of A_j and b_j to make the proofs simpler and this paper more readable. We assume that d_1 is common to each element of A_j just for simplicity of presentation.

When **A1** holds, we have

$$\mathrm{E}(\delta_1 \delta_{j+1}^T) \sim \bar{A}_\infty j^{-1+2d_1} \quad \text{and} \quad \mathrm{E}(\epsilon_1 \epsilon_{j+1}) \sim \bar{b}_\infty j^{-1+2d_2}, \tag{3}$$

where $\bar{A}_\infty$ is a $k \times k$ constant matrix and $\bar{b}_\infty$ is a constant. We write $r_{lm}(j)$ and $r_\epsilon(j)$ for the (l, m) element of $\mathrm{E}(\delta_1 \delta_{j+1}^T)$ and $\mathrm{E}(\epsilon_1 \epsilon_{j+1})$, respectively. Since $d_1 > 0$ and $d_2 > 0$, they are not absolutely summable. This is one of the characteristics of time series with long-range dependence.

As for the second moments of ζ_1 and η_1, we use the following notation.

$$\mathrm{E}(\zeta_1 \zeta_1^T) = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1k} \\ \vdots & \ddots & \vdots \\ \sigma_{k1} & \cdots & \sigma_{kk} \end{pmatrix} \quad \text{and} \quad \mathrm{E}(\eta_1^2) = \sigma_b^2. \tag{4}$$

We often write σ_j^2 for σ_{jj} in the proofs for notational convenience.

A2: In addition to **(a)** of **A1**,

$$a_j^{(l,m)} = \frac{\alpha^{(l,m)}}{(j+1)^{1-d_1}}, \ 1 \le l, m \le k, \quad \text{and} \quad b_j = \frac{\gamma}{(j+1)^{1-d_2}}, \ j > j_0,$$

where j_0 is a positive integer.

A2 is a restrictive assumption. However, it is necessary to examine the asymptotic properties of $\hat{\beta}_h$ when $d_1 + d_2 > 1/2$. See Lemma 5.1 for the details.

We can relax **A2** to some extent. For example,

$$b_j = \sum_{s=0}^{\infty} \frac{\gamma_s}{(j+1)^{1+s-d_2}} \quad \text{and} \quad |\gamma_s| \le C \tag{5}$$

This is also true of A_j. Some more generalization such as $b_j = \log(j+1)j^{-1+d_2}$ may be possible since what we need in Lemma 5.1 is a kind of smoothness.

A3 below is about the smoothness of $g(\cdot)$ and $m(\cdot)$ and it is a usual assumption in the literature of partial linear models. For example, See [2].

A3: Each element of $g(\cdot)$ and $m(\cdot)$ is twice continuously differentiable on $[0,1]$.

We give definitions and assumptions about nonparametric regression before we define $\hat{\beta}_h$. Our $\hat{\beta}_h$ is calculated from nonparametric regression residuals based on the Gasser-Müller weights as in [2]. See [8] and [9] for the properties of nonparametric regression estimators with the Gasser-Müller weights.

We define the $n \times n$ weight matrix $W_h = (w_{ij})$ by

$$w_{ij} = \begin{cases} h^{-1} \int_{(j-1)/n}^{j/n} K\Big(\frac{t_i - u}{h}\Big) du, & t_i \in [h, 1-h] \\ h^{-1} \int_{(j-1)/n}^{j/n} K_q\Big(\frac{t_i - u}{h}\Big) du, & t_i = qh =\in [0, h) \\ h^{-1} \int_{(j-1)/n}^{j/n} K_q^*\Big(\frac{t_i - u}{h}\Big) du, & t_i = 1 - qh \in (1-h, 1] \end{cases}, \tag{6}$$

where h is the bandwidth and $K(\cdot)$, $K_q(\cdot)$, $0 \le q \le 1$, and $K_q^*(\cdot)$, $0 \le q \le 1$, are kernel functions. They satisfy **A4** and **A5** below. We usually call $K_q(\cdot)$ and $K_q^*(\cdot)$ boundary kernels. Under **A4**, we have

$$w_{ij} = 0 \text{ if } |i-j| \ge \lfloor nh \rfloor + 2 \quad \text{and} \quad |w_{ij}| \le \frac{C}{nh}. \tag{7}$$

Then our nonparametric estimator of $g(t_i)$ is given by $\sum_{j=1}^n w_{ij} X_j$ and the nonparametric regression residual is $X_i - \sum_{j=1}^n w_{ij} X_j$.

We define an $n \times k$ matrix $\tilde{X}$ and an n-dimensional vector $\tilde{Y}$ by

$$\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)^T = (I - W_h)X \quad \text{and} \quad \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_n)^T = (I - W_h)Y, \qquad (8)$$

where I is the $n \times n$ identity matrix, $X = (X_1, \ldots, X_n)^T$, and $Y = (Y_1, \ldots, Y_n)^T$. The elements of $\tilde{X}$ and $\tilde{Y}$ are nonparametric regression residuals.

When we use kernels satisfying **A4** below, we have uniformly in elements

$$(I - W_h)G = O(h^2 + n^{-1}) \quad \text{and} \quad (I - W_h)M = O(h^2 + n^{-1}), \qquad (9)$$

where $G = (g(t_1), \ldots, g(t_n))^T$ and $M = (m(t_1), \ldots, m(t_n))^T$. Hence the unknown function $m(\cdot)$ is cancelled out in $\tilde{Y}$ and we can regard $\tilde{Y}$ as an estimate of $\tilde{X}^T\beta$. Therefore we estimate β by

$$\hat{\beta}_h = (\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T\tilde{Y}. \qquad (10)$$

A4 and **A5** below are about kernel functions and h, respectively. Some more conditions on h are imposed in Theorems 2.1 and 2.2.

A4:

(a) $K(\cdot)$ is a bounded and symmetric density function with support $[-1, 1]$.

(b) $K_q(\cdot)$ and $K_q^*(\cdot)$, $0 \le q \le 1$, are uniformly bounded, the support of $K_q(\cdot)$ is $[-1, q]$, and the support of $K_q^*(\cdot)$ is $[-q, 1]$. They also satisfy

$$\int K_q(u)du = 1, \ \int K_q^*(u)du = 1, \ \int uK_q(u)du = 0, \text{ and } \int uK_q^*(u)du = 0.$$

A5: $h \to 0$ and $nh^2 \to \infty$ as $n \to \infty$.

We state Theorem 2.1. Note that this theorem can cover the case of short-range dependence with minor modifications of the proof.

Theorem 2.1 *Suppose that* $0 < d_1 + d_2 < 1/2$ *and that* **A1**, **A3**, *and* **A4** *hold. In addition, we assume* $h = c_h n^{-d_h}$, *where* $c_h > 0$ *and* $1/8 \vee d_1/2 \vee d_2/2 < d_h < 1/2$, *and that* $\mathrm{E}(\delta_1\delta_1^T)$ *is positive definite. Then we have*

$$\sqrt{n}(\hat{\beta}_h - \beta) \xrightarrow{d} N(0, U^{-1}V_1U^{-1}),$$

where $U = \mathrm{E}(\delta_1\delta_1^T)$ *and the* (l,m) *element of* V_1 *is* $\sum_{j=-\infty}^{\infty} r_{lm}(j)r_\epsilon(j)$*.*

We find the asymptotic normality of $\sqrt{n}(\hat\beta_h - \beta)$ in 1 of Theorem 1 in [4] and (c) of Theorem 1 in [2]. The latter paper relaxes the conditions of the former. However, in the latter, d_h in the definition of h still has to satisfy

$$\frac{1}{8} \vee \frac{2d_1}{3+2d_1} \vee \frac{d_2}{2} < d_h < 1 - \frac{2d_2}{1-2d_1}. \tag{11}$$

If $1-2d_1 = \nu$ and $2d_2 = 0.9\nu$, there is no d_h satisfying (11). We have shown that we can always take $h = cn^{-1/4}$.

The asymptotic normality in the two papers is conditional on $\{X_i\}$. When the asymptotic normality of $n^{-1/2}\sum_{i=1}^n \delta_i\epsilon_i$ holds conditionally, it obviously holds unconditionally, too. However, no general sufficient condition of the conditional asymptotic normality that can be applied to (1) under long-range dependence is not presented in the two papers or the papers cited in Section 5 of [2]. We believe it is much easier to treat the asymptotic distribution unconditionally as in this paper. We will be able to carry out statistical inference on β since the asymptotic covariance matrix is explicitly given in Theorem 2.1.

Theorem 2.2 deals with the case where $d_1 + d_2 > 1/2$.

Theorem 2.2 *Suppose that* $d_1 + d_2 > 1/2$ *and that* **A1**, **A2**, **A3**, *and* **A4** *hold. In addition, we assume* $h = c_h n^{-d_h}$, *where* $c_h > 0$ *and* $1/8 \vee d_1/2 \vee d_2/2 < d_h < 1/2$, *and that* $\mathrm{E}(\delta_1\delta_1^T)$ *is positive definite. Then we have*

$$n^{1/2}(nh)^{-(d_1+d_2)+1/2}(\hat\beta_h - \beta) \xrightarrow{d} N(0, U^{-1}V_2U^{-1}),$$

where $U = \mathrm{E}(\delta_1\delta_1^T)$ *and the* (l,m) *element of* V_2, $v_2^{(l,m)}$, *is defined by*

$$\begin{aligned} v_2^{(l,m)} &= \sigma_b^2 \sum_{1\le e,f\le k} \sigma_{ef} \int_0^\infty \Big(\int_{-1}^\infty \frac{\psi_1^{(l,e)}(s)\psi_1^{(m,f)}(s+u) + \psi_1^{(m,f)}(s)\psi_1^{(l,e)}(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}} ds \\ &\quad \times \int_{-1}^\infty \frac{\psi_2(t)\psi_2(t+u) + \psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}} dt \Big) du, \end{aligned} \tag{12}$$

$$\psi_1^{(p,q)}(s) \;=\; \alpha^{(p,q)}\Big(I\{s \geq 0\} - |s|^{1-d_1}\int_{(-1)\vee(-s)}^{1} \frac{K(z)}{(s+z)^{1-d_1}}dz\Big), \tag{13}$$

$$\psi_2(s) \;=\; \gamma\Big(I\{s \geq 0\} - |s|^{1-d_2}\int_{(-1)\vee(-s)}^{1} \frac{K(z)}{(s+z)^{1-d_2}}dz\Big). \tag{14}$$

Remark 2.1 *We are unable to verify Proposition 3.4 below when $d_1 + d_2 = 1/2$. The proposition is about the asymptotic covariance matrix. However, it is not so tough to show that*

$$\mathrm{Var}(\sum_{i=1}^{n} \tilde{\delta}_i \tilde{\epsilon}_i) = O(n \log n).$$

See the beginning of Section 3 for the definitions of $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$. If we have

$$\lim_{n\to\infty} \frac{1}{n \log n}\mathrm{Var}(\sum_{i=1}^{n} \tilde{\delta}_i \tilde{\epsilon}_i) = V_3,$$

we will be able to establish

$$\sqrt{n \log n}(\hat{\beta}_h - \beta) \stackrel{d}{\to} N(0, U^{-1}V_3U^{-1}).$$

See Remark 5.1 just after Lemma 5.5. Some of the propositions and lemmas for Theorem 2.2 deal with the case of $d_1 + d_2 = 1/2$ when they can easily include the case.

The effect of using the boundary kernels is negligible in the asymptotic distribution. The asymptotic covariance matrix in Theorem 2.2 is finite from Lemma 5.1(2) and (15) below.

$$\begin{aligned} &\int_0^M \Big(\int_0^M \frac{ds}{s^{1-d_1}(s+u)^{1-d_1}} \int_0^M \frac{dt}{t^{1-d_2}(t+u)^{1-d_2}}\Big)du \qquad (15)\\ &= \int_0^M \frac{1}{u^{2-2(d_1+d_2)}}\Big(\int_{u/M}^{\infty} \frac{dv}{v^{2d_1}(1+v)^{1-d_1}} \int_{u/M}^{\infty} \frac{dw}{w^{2d_2}(1+w)^{1-d_2}}\Big)du, \end{aligned}$$

where M is an arbitrary positive number. The asymptotic covariance matrix has a complicated form and statistical inference on β is a subject of future research. As we mentioned in Section 1, the asymptotic distribution is completely different from that of $n^{-(d_1+d_2)}\sum_{i=1}^{n} \delta_i \epsilon_i$. See a comment around (30) and Lemma 5.1 below.

Finally we see how estimating β affects nonparametric estimation of $m(t_i)$. Suppose that **A1**, **A2**, **A3**, and **A4** hold and that $h = c_h n^{-1/4}$.

We estimate $m(t_i)$ by $\hat{m}(t_i)$ in (16) below.

$$\hat{m}(t_i) = \sum_{j=1}^{n} w_{ij}(Y_j - X_j^T \hat{\beta}_h). \tag{16}$$

We take $b = c_b n^{-d_b}$ for the bandwidth of (16). Then we have

$$\begin{aligned} &\hat{m}(t_i) - m(t_i) \\ =\ & (\sum_{j=1}^{n} w_{ij} m(t_j) - m(t_i)) + \sum_{j=1}^{n} w_{ij}\epsilon_j + \sum_{j=1}^{n} w_{ij} X_j^T(\beta - \hat{\beta}_h). \end{aligned} \tag{17}$$

Some routine calculations show that

$$\sum_{i=1}^{n} w_{ij} m(t_j) - m(t_i) = O(b^2) \tag{18}$$

$$\sum_{i=1}^{n} w_{ij}\epsilon_j = O_p((nb)^{d_2 - 1/2}) \tag{19}$$

$$\sum_{i=1}^{n} w_{ij} X_j = g(t_i) + O_p(b^2 + (nb)^{d_1 - 1/2}). \tag{20}$$

See [16] for the asymptotic normality of (19). The optimal d_b for the first two terms of the RHS of (17) is $d_b = (1 - 2d_2)/(5 - 2d_2)$. Then

$$b^2 = O(n^{-(2-4d_2)/(5-2d_2)}) \quad \text{and} \quad (nb)^{d_2 - 1/2} = O(n^{-(2-4d_2)/(5-2d_2)}).$$

When $d_1 + d_2 < 1/2$, $\hat{\beta}_h - \beta = O_p(n^{-1/2})$ and the effect of the last term of the RHS of (17) is asymptotically negligible compared to the first two terms.

Next we consider the case where $d_1 + d_2 > 1/2$. Some elementary calculation shows that if $(2 - 4d_2)/(5 - 2d_2) < 7/8 - 3(d_1 + d_2)/4$, the last term of the RHS of (17) is asymptotically negligible compared to the first two terms. The inequality is equivalent to

$$\frac{9}{8} + \frac{3}{4}d_1 < \frac{8}{5 - 2d_2} - \frac{3}{4}d_2. \tag{21}$$

The LHS is smaller than $3/2$ and the RHS is larger than $3/2$. Hence we have shown that estimating β does not have any asymptotic effects on nonparametric estimation of $m(\cdot)$ if we take $h = cn^{-1/4}$.

3 Proof of theorems

In this section we prove Theorems 2.1 and 2.2. We introduce some notation and state propositions. Then we prove the theorems by using the propositions. The proofs of the propositions are given in Section 4.

We can represent $\tilde{X}$ and $\tilde{Y}$ as

$$\tilde{X} = (I - W_h)G + (I - W_h)D = \tilde{G} + \tilde{D}, \tag{22}$$

$$\tilde{Y} = \tilde{X}\beta + (I - W_h)M + (I - W_h)E = \tilde{X}\beta + \tilde{M} + \tilde{E}, \tag{23}$$

where $D = (\delta_1, \ldots, \delta_n)^T$, $E = (\epsilon_1, \ldots, \epsilon_n)^T$,

$$\tilde{G} = (I - W_h)G = (\tilde{g}(t_1), \ldots, \tilde{g}(t_n))^T, \tag{24}$$

$$\tilde{M} = (I - W_h)M = (\tilde{m}(t_1), \ldots, \tilde{m}(t_n))^T, \tag{25}$$

$$\tilde{D} = (I - W_h)D = (\tilde{\delta}_1, \ldots, \tilde{\delta}_n)^T, \tag{26}$$

$$\tilde{E} = (I - W_h)E = (\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n)^T. \tag{27}$$

We can say that elements of $\tilde{X}$ and $\tilde{Y}$ are nonparametric regression residuals.

We define the coefficients of $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$, $\tilde{A}_j$ and $\tilde{b}_j$, by

$$\tilde{\delta}_i = \delta_i - \sum_{j=1}^{n} w_{ij}\delta_j = \sum_{j=-\lfloor nh \rfloor -1}^{\infty} \tilde{A}_j \zeta_{i-j}, \tag{28}$$

$$\tilde{\epsilon}_i = \epsilon_i - \sum_{j=1}^{n} w_{ij}\epsilon_j = \sum_{j=-\lfloor nh \rfloor -1}^{\infty} \tilde{b}_j \eta_{i-j}. \tag{29}$$

We denote the (l, m) element of A_j by $\tilde{a}_j^{(l,m)}$. We prove in Lemma 5.1 below that there is a large positive integer $\bar{M}$ such that

$$\tilde{A}_j = O((nh)^2 (j+1)^{d_1 - 3}) \quad \text{and} \quad \tilde{b}_j = O((nh)^2 (j+1)^{d_2 - 3}), \quad \text{for } j \geq \bar{M}\lfloor nh \rfloor. \tag{30}$$

(30) means that some of the long-range dependence is cancelled out in $\tilde{\delta}_i$ between δ_i and $\sum_{j=1}^{n} w_{ij}\delta_j$. This is also true of $\tilde{\epsilon}_i$. The effect of this cancellation appears when $d_1 + d_2 > 1/2$.

Recalling that $\hat{\beta}_h = (\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T\tilde{Y}$, we examine $\tilde{X}^T\tilde{Y}$ and $n^{-1}\tilde{X}^T\tilde{X}$ in Propositions 3.1 and 3.2, respectively. We do not use **A1** fully in the proof of Proposition 3.1 and we use only properties of autocovariances in (3).

Proposition 3.1 *Suppose that* **A1**, **A3**, **A4**, *and* **A5** *hold. Then we have*

$$\begin{aligned}
&\tilde{X}^T\tilde{Y} - \tilde{X}^T\tilde{X}\beta \\
&= \sum_{i=1}^n \delta_i\epsilon_i + O(nh^4) + O_p(n^{d_1+1/2}h^2) + O_p(n^{d_2+1/2}h^2) \\
&\qquad +O_p(n^{1/2}(nh)^{d_1+d_2-1/2}), \ \textit{when } 0 < d_1 + d_2 < 1/2
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{X}^T\tilde{Y} - \tilde{X}^T\tilde{X}\beta \\
&= \sum_{i=1}^n \tilde{\delta}_i\tilde{\epsilon}_i + O(nh^4) + O_p(n^{d_1+1/2}h^2) + O_p(n^{d_2+1/2}h^2), \ \textit{when } d_1 + d_2 \geq 1/2.
\end{aligned}$$

Proposition 3.2 *Suppose that* **A1**, **A3**, **A4**, *and* **A5** *hold. Then we have*

$$\frac{1}{n}\tilde{X}^T\tilde{X} \xrightarrow{p} \mathrm{E}(\delta_1\delta_1^T).$$

We establish the central limit theorem (CLT) of $n^{-1/2}\sum_{i=1}^n \delta_i\epsilon_i$ by appealing to the truncation argument and the CLT for stationary m-dependent sequences in Proposition 3.3 below. Some CLT's concerning long-range dependent linear processes and similar arguments are found in the literature, for example, in [10]. Thus Proposition 3.3 below may not be new. However, we present it together with the proof to complement the previous results on estimation of β.

Proposition 3.3 *Suppose that* $0 < d_1 + d_2 < 1/2$ *and that* **A1** *hold. Then we have*

$$\frac{1}{n}\sum_{i=1}^n \delta_i\epsilon_i \xrightarrow{d} N(0, V_1),$$

where V_1 *is defined in Theorem 2.1.*

We calculate the covariance matrix of $\sum_{i=1}^n \tilde{\delta}_i\tilde{\epsilon}_i$ in Proposition 3.4 and establish the asymptotic normality in Proposition 3.5 when $d_1 + d_2 > 1/2$.

We define $\tilde{n}$ for the propositions by

$$\tilde{n} = n(nh)^{-1+2(d_1+d_2)}. \tag{31}$$

Note that $n\tilde{n}^{-1/2}(\hat{\beta}_h - \beta)$ converges in distribution to a k-variate normal distribution.

Proposition 3.4 *Suppose that $d_1 + d_2 > 1/2$ and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\frac{1}{\tilde{n}}\mathrm{Var}(\sum_{i=1}^{n} \tilde{\delta}_i \tilde{\epsilon}_i) \to V_2,$$

where V_2 is defined in Theorem 2.2.

Proposition 3.5 *Suppose that $d_1 + d_2 > 1/2$ and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\frac{1}{\sqrt{\tilde{n}}}\sum_{i=1}^{n} \tilde{\delta}_i \tilde{\epsilon}_i \xrightarrow{d} N(0, V_2).$$

We prove Theorems 2.1 and 2.2.

Proof of Theorems 2.1 and 2.2. We prove only Theorem 2.2 since we can prove Theorem 2.1 in the same way.

Proposition 3.1 and the conditions on the bandwidth h yield that

$$\begin{aligned} & \frac{n}{\sqrt{\tilde{n}}}(\hat{\beta}_h - \beta) \qquad (32) \\ = \ & \Big(\frac{1}{n}\tilde{X}^T\tilde{X}\Big)^{-1}\frac{1}{\sqrt{\tilde{n}}}\sum_{i=1}^{n} \tilde{\delta}_i \tilde{\epsilon}_i + \frac{1}{\sqrt{\tilde{n}}}(O(nh^4) + O_p(n^{d_1+1/2}h^2) + O_p(n^{d_2+1/2}h^2)) \\ = \ & \Big(\frac{1}{n}\tilde{X}^T\tilde{X}\Big)^{-1}\frac{1}{\sqrt{\tilde{n}}}\sum_{i=1}^{n} \tilde{\delta}_i \tilde{\epsilon}_i + o_p(1). \end{aligned}$$

Theorem 2.2 follows from (32) and Propositions 3.2 and 3.5. Hence the proof is complete. □

4 Proofs of propositions

In this section we prove Propositions 3.1–3.5.

We elaborate the evaluation of the second moments given in [2] in the second half of the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $k = 1$ be in the proof for notational simplicity. Recall the definitions of $r_{11}(j)$ and $r_\epsilon(j)$ in (3) and the properties of w_{ij} in (7).

First we write $\tilde{X}^T\tilde{Y} - \tilde{X}^T\tilde{X}\beta$ by using $\tilde{G}$, $\tilde{M}$, $\tilde{D}$, and $\tilde{E}$ in (24)-(27).

$$\begin{aligned} & \tilde{X}^T\tilde{Y} - \tilde{X}^T\tilde{X}\beta \\ = \ & \tilde{G}^T\tilde{M} + \tilde{G}^T\tilde{E} + \tilde{D}^T\tilde{M} + \tilde{D}^T\tilde{E}. \end{aligned} \tag{33}$$

It follows from (9) that

$$\tilde{G}^T\tilde{M} = O(nh^4). \tag{34}$$

Next we deal with $\tilde{G}^T\tilde{E} = \tilde{G}^T E - \tilde{G}^T W_h E$. A standard argument implies that

$$\mathrm{Var}(\tilde{G}^T E) \leq \sum_{1\leq i,j\leq n} |\tilde{g}_i||\tilde{g}_j||r_\epsilon(i-j)| \leq Cn^{1+2d_2}h^4. \tag{35}$$

Noticing that $\tilde{G}^T W_h E = \sum_{i=1}^n \tilde{g}_i(\sum_{j=1}^n w_{ij}\epsilon_j)$, we obtain

$$\begin{aligned} & \mathrm{Var}(\tilde{G}^T W_h E) \\ \leq \ & \sum_{1\leq i_1,i_2\leq n}\sum_{1\leq j_1,j_2\leq n} |\tilde{g}_{i_1}||\tilde{g}_{i_2}||w_{i_1j_1}||w_{i_2j_2}||r_\epsilon(j_1-j_2)| \\ \leq \ & Ch^4 \sum_{1\leq j_1,j_2\leq n} |r_\epsilon(j_1-j_2)| \sum_{1\leq i_1,i_2\leq n} |w_{i_1j_1}||w_{i_2j_2}| \\ \leq \ & Cn^{1+2d_2}h^4. \end{aligned} \tag{36}$$

Here we used the fact that $\sum_{1\leq i_1,i_2\leq n}|w_{i_1j_1}||w_{i_2j_2}| = O(1)$ uniformly in j_1 and j_2. This fact is due to (7).

From (35) and (36), we obtain

$$\mathrm{Var}(\tilde{G}^T\tilde{E}) \leq Cn^{1+2d_2}h^4. \tag{37}$$

Similarly, we can show that

$$\mathrm{Var}(\tilde{D}^T \tilde{M}) \le C n^{1+2d_1} h^4. \tag{38}$$

The proof is complete when $d_1 + d_2 \ge 1/2$.

Hereafter we assume that $0 < d_1 + d_2 < 1/2$ in the proof and examine $\tilde{D}^T \tilde{E}$ closely. It is written as

$$\tilde{D}^T \tilde{E} = D^T E - D^T W_h E - D^T W_h^T E + D^T W_h^T W_h E. \tag{39}$$

Provided that

$$D^T W_h E = O_p(n^{1/2}(nh)^{d_1+d_2-1/2}), \tag{40}$$

$$D^T W_h^T E = O_p(n^{1/2}(nh)^{d_1+d_2-1/2}), \tag{41}$$

$$D^T W_h^T W_h E = O_p(n^{1/2}(nh)^{d_1+d_2-1/2}), \tag{42}$$

we have

$$\tilde{D}^T \tilde{E} - D^T E = O_p(n^{1/2}(nh)^{d_1+d_2-1/2}). \tag{43}$$

Then the proposition follows from (33), (34), (37), (38), and (43).

We will establish (40) and (42). (41) can be treated in the same way as (40).

The variance of $D^T W_h E$ is written as

$$\mathrm{Var}(D^T W_h E) = \sum_{1 \le i_1, i_2 \le n} \sum_{1 \le j_1, j_2 \le n} w_{i_1 j_1} w_{i_2 j_2} r_{11}(i_1 - i_2) r_\epsilon(j_1 - j_2). \tag{44}$$

We fix a sufficiently large positive integer M. When $i_1 - i_2 > Mnh$,

$$\begin{aligned} & \sum_{1 \le j_1, j_2 \le n} |w_{i_1 j_1} w_{i_2 j_2} r_\epsilon(j_1 - j_2)| \\ \le\ & \frac{C}{(nh)^2} \sum_{l=1}^{\lfloor nh \rfloor} \frac{nh + 1 - l}{(i_1 - i_2 + 1 - l)^{1-2d_2}} \\ \le\ & C(nh)^{2d_2 - 1} \int_0^1 \frac{1 - t}{\{(i_1 - i_2)/(nh) - t\}^{1-2d_2}} dt \\ \le\ & \frac{C(nh)^{2d_2-1}}{(|i_1 - i_2|/(nh))^{1-2d_2}} \le \frac{C}{|i_1 - i_2|^{1-2d_2}}. \end{aligned} \tag{45}$$

Put $\mathcal{U}_M = \{(i_1, i_2) \,|\, 1 \le i_1, i_2 \le n$ and $|i_1 - i_2| > Mnh\}$ and $\mathcal{L}_M = \{(i_1, i_2) \,|\, 1 \le i_1, i_2 \le n$ and $|i_1 - i_2| \le Mnh\}$. Then the summation over $\mathcal{U}_M$ is bounded by $Cn(nh)^{2(d_1+d_2)-1}$ since

$$\begin{aligned} &\sum_{\mathcal{U}_M} \sum_{1 \le j_1, j_2 \le n} |w_{i_1 j_1} w_{i_2 j_2} r_{11}(i_1 - i_2) r_\epsilon(j_1 - j_2)| \\ &\le \; C \sum_{\mathcal{U}_M} \frac{1}{|i_1 - i_2|^{2-2(d_1+d_2)}} \le Cn(Mnh)^{2(d_1+d_2)-1}. \end{aligned} \tag{46}$$

The summation over $\mathcal{L}_M$ is also bounded by $Cn(nh)^{2(d_1+d_2)-1}$ since

$$\begin{aligned} &\sum_{\mathcal{L}_M} \sum_{1 \le j_1, j_2 \le n} |w_{i_1 j_1} w_{i_2 j_2} r_{11}(i_1 - i_2) r_\epsilon(j_1 - j_2)| \\ &\le \; \frac{Cn}{(nh)^2} \sum_{k=1}^{M\lfloor nh \rfloor} \frac{(Mnh)^{1+2d_2}}{k^{1-2d_1}} \le Cn(nh)^{2(d_1+d_2)-1} M^{1+2d_1+d_2}. \end{aligned} \tag{47}$$

By combining (46) and (47), we obtain

$$\mathrm{Var}(D^T W_h E) \le Cn(nh)^{2(d_1+d_2)-1}.$$

Hence (40) is established.

Finally we verify (42). The variance is given by

$$\begin{aligned} &\mathrm{Var}(D^T W_h W_h E) \\ &= \sum_{1 \le i_1, i_2 \le n} \sum_{1 \le j_1, j_2 \le n} \sum_{1 \le l_1, l_2 \le n} w_{i_1 j_1} w_{i_2 j_2} w_{i_1 l_1} w_{i_2 l_2} r_{11}(j_1 - j_2) r_\epsilon(l_1 - l_2). \end{aligned} \tag{48}$$

As in the proof of (40), we fix a large positive integer M and define $\mathcal{U}_M$ and $\mathcal{L}_M$. Then when $|i_1 - i_2| > Mnh$, we have as in (45) that

$$\Big| \sum_{1 \le j_1, j_2 \le n} w_{i_1 j_1} w_{i_2 j_2} r_{11}(j_1 - j_2) \Big| \le \frac{C}{|i_1 - i_2|^{1-2d_1}}, \tag{49}$$

$$\Big| \sum_{1 \le l_1, l_2 \le n} w_{i_1 l_1} w_{i_2 l_2} r_\epsilon(l_1 - l_2) \Big| \le \frac{C}{|i_1 - i_2|^{1-2d_2}}. \tag{50}$$

By substituting (49) and (50) into (48), we get

$$\begin{aligned} &\sum_{\mathcal{U}_M} \sum_{1 \le j_1, j_2 \le n} \sum_{1 \le l_1, l_2 \le n} |w_{i_1 j_1} w_{i_2 j_2} w_{i_1 l_1} w_{i_2 l_2} r_{11}(j_1 - j_2) r_\epsilon(l_1 - l_2)| \\ &\le \sum_{\mathcal{U}_M} \frac{C}{|i_1 - i_2|^{2-2(d_1+d_2)}} \le Cn(Mnh)^{2(d_1+d_2)-1}. \end{aligned} \tag{51}$$

The summation over $\mathcal{L}_M$ is bounded by $Cn(nh)^{2(d_1+d_2)-1}M^{3+2(d_1+d_2)}$ since

$$\begin{aligned} & \sum_{\mathcal{L}_M} \sum_{1\le j_1,j_2\le n} \sum_{1\le l_1,l_2\le n} |w_{i_1j_1}w_{i_2j_2}w_{i_1l_1}w_{i_2l_2}r_{11}(j_1-j_2)r_\epsilon(l_1-l_2)| \\ & \le \frac{Cn}{(nh)^4}\sum_{k=1}^{M\lfloor nh\rfloor}(Mnh)^{1+2d_1}(Mnh)^{1+2d_2} \le Cn(nh)^{2(d_1+d_2)-1}M^{3+2(d_1+d_2)}. \end{aligned} \tag{52}$$

(42) follows from (51) and (52). Hence the proof of Proposition 3.1 is complete. □

Proof of Proposition 3.2. Let $k=1$ for notational simplicity. Note that $\tilde{X}^T\tilde{X}$ is written as

$$\tilde{X}^T\tilde{X} = \tilde{G}^T\tilde{G} + \tilde{G}^T\tilde{D} + \tilde{D}^T\tilde{G} + D^TD - D^TW_h^TD - D^TW_hD + DW_h^TW_hD.$$

It is obvious from the proof of Proposition 3.1 that

$$\tilde{G}^T\tilde{G} = O(nh^4) \quad \text{and} \quad \tilde{G}^T\tilde{D} = O_p(n^{d_1+1/2}h^2).$$

We examine the other terms of $\tilde{X}^T\tilde{X}$.

Since

$$\begin{aligned} \mathrm{Var}(\sum_{j=1}^n w_{ij}\delta_j) &= \sum_{1\le j_1,j_2\le n} w_{ij_1}w_{ij_2}r_{11}(j_1-j_2) \\ &\le \frac{C}{(nh)^2}\sum_{-\lfloor nh\rfloor-1\le j_1-i,j_2-i\le\lfloor nh\rfloor+1}\frac{1}{(|j_1-j_2|+1)^{1-2d_1}} \\ &\le C(nh)^{2d_1-1} \quad \text{uniformly in } i, \end{aligned}$$

we have

$$\mathrm{E}(|D^TW_h^TD|) \le Cn(nh)^{d_1-1/2} \quad \text{and} \quad \mathrm{E}(|D^TW_h^TW_hD|) \le Cn(nh)^{2d_1-1}.$$

Hence the proposition follows from the ergodicity of $\{\delta_i\}$ and the proof of the proposition is complete. □

Proof of Proposition 3.3. Let $k=1$ for notational simplicity. We appeal to the Cramér-Wold device when $k>1$. We write a_j for A_j and $a_j^{(1,1)}$ since $k=1$.

We decompose δ_i and ϵ_i as

$$\delta_i = \sum_{j=0}^{m} a_j \zeta_{i-j} + \sum_{j=m+1}^{\infty} a_j \zeta_{i-j} = \delta_i^{(m)} + \bar{\delta}_i^{(m)}, \tag{53}$$

$$\epsilon_i = \sum_{j=0}^{m} a_j \eta_{i-j} + \sum_{j=m+1}^{\infty} a_j \eta_{i-j} = \epsilon_i^{(m)} + \bar{\epsilon}_i^{(m)}, \tag{54}$$

where $\delta_i^{(m)}$, $\bar{\delta}_i^{(m)}$, $\epsilon_i^{(m)}$, and $\bar{\epsilon}_i^{(m)}$ are defined by the above expressions.

We denote the autocovariances of $\delta_i^{(m)}$, $\epsilon_i^{(m)}$, $\bar{\delta}_i^{(m)}$, and $\bar{\epsilon}_i^{(m)}$ by $r_\delta^{(m)}(j)$, $r_\epsilon^{(m)}(j)$, $\bar{r}_\delta^{(m)}(j)$, and $\bar{r}_\epsilon^{(m)}(j)$, respectively. It is easy to see that

$$r_\delta^{(m)}(j) \to r_{11}(j) \quad \text{and} \quad r_\epsilon^{(m)}(j) \to r_\epsilon(j) \text{ as } m \to \infty, \tag{55}$$

$$|r_\delta^{(m)}(j)| \le \frac{C}{(|j|+1)^{1-2d_1}} \quad \text{and} \quad |r_\epsilon^{(m)}(j)| \le \frac{C}{(|j|+1)^{1-2d_2}}. \tag{56}$$

We apply the CLT for stationary m-dependent sequences to $n^{-1/2}\sum_{i=1}^n \delta_i^{(m)}\epsilon_i^{(m)}$ and obtain

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \delta_i^{(m)}\epsilon_i^{(m)} \xrightarrow{d} N(0, V_{1m}),$$

where $V_{1m} = \sum_{j=-\infty}^{\infty} r_\delta^{(m)}(j) r_\epsilon^{(m)}(j)$. In addition, (55) and (56) imply that $V_{1m} \to V_1$ as $m \to \infty$.

We have only to show that the effect of truncation in (53) and (54) becomes negligible as $m \to \infty$, that is

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^n \delta_i^{(m)}\bar{\epsilon}_i^{(m)} + \sum_{i=1}^n \bar{\delta}_i^{(m)}\epsilon_i^{(m)} + \sum_{i=1}^n \bar{\delta}_i^{(m)}\bar{\epsilon}_i^{(m)}\right) = o_p(1) \text{ as } m \to \infty. \tag{57}$$

We give the details of only the first term of (57).

We examine $\bar{r}_\epsilon^{(m)}(j)$ and use the result to bound $\mathrm{Var}(\sum_{i=1}^n \delta_i^{(m)}\bar{\epsilon}_i^{(m)})$. When i is positive and sufficiently large,

$$\begin{aligned} |\bar{r}_\epsilon^{(m)}(i)| &\le C\sum_{j=m+1}^{\infty} \frac{1}{j^{1-d_2}(i+j)^{1-d_2}} \\ &\le \frac{C}{i^{1-2d_2}} \sum_{j=m+1}^{\infty} \frac{1}{(j/i)^{1-d_2}(1+j/i)^{1-d_2} i}. \end{aligned} \tag{58}$$

The last summation in (58) tends to 0 as $m \to \infty$. From (58), we get

$$
\begin{aligned}
&\frac{1}{n}\mathrm{Var}(\sum_{i=1}^{n} \delta_i^{(m)}\bar{\epsilon}_i^{(m)}) \\
&\le \frac{C}{n}\sum_{1\le i_1,i_2\le n} |r_\delta^{(m)}(i_1-i_2)\bar{r}_\epsilon^{(m)}(i_1-i_2)| \\
&\le C\sum_{i=1}^{n}(i+1)^{2(d_1+d_2)-2}\sum_{j=m+1}^{\infty}\frac{1}{(j/i)^{1-d_2}(1+j/i)^{1-d_2}i} \quad \to 0 \text{ as } m\to\infty.
\end{aligned}
$$

Hence the proof of Proposition 3.3 is complete. □

We need some elaborate results on the coefficients of $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$ in the proofs of Propositions 3.4 and 3.5. The results are given in Lemma 5.1. Exactly speaking, the coefficients of $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$ depend on i since the kernel functions for w_{ij} depend on i. However, Lemma 5.2 implies that the effect of using the boundary kernels $K_q(\cdot)$ and $K_q^*(\cdot)$ is negligible in the propositions.

Proof of Proposition 3.4. Let $k=1$ for notational simplicity and we also suppress the superscript $(1,1)$ since $k=1$. For example, we write $\tilde{a}_j$ and $\psi_{1n}(\cdot)$ for $\tilde{a}_j^{(1,1)}$ and $\psi_{1n}^{(1,1)}(\cdot)$, respectively. Lemm 5.2 ensures that we can neglect the effect of using the boundary kernels and prove the propositions as if we did not use the boundary kernels $K_q(\cdot)$ and $K_q^*(\cdot)$.

We decompose $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$ as

$$
\begin{aligned}
\tilde{\delta}_i &= \sum_{j=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor}\tilde{a}_j\zeta_{i-j} + \sum_{j=\bar{M}\lfloor nh\rfloor+1}^{\infty}\tilde{a}_j\zeta_{i-j} = \tilde{\delta}_{1i}+\tilde{\delta}_{2i}, \\
\tilde{\epsilon}_i &= \sum_{j=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor}\tilde{b}_j\eta_{i-j} + \sum_{j=\bar{M}\lfloor nh\rfloor+1}^{\infty}\tilde{b}_j\eta_{i-j} = \tilde{\epsilon}_{1i}+\tilde{\epsilon}_{2i},
\end{aligned}
$$

where $\bar{M}$ is defined in Lemma 5.1 and $\tilde{\delta}_{1i}$, $\tilde{\delta}_{2i}$, $\tilde{\epsilon}_{1i}$, and $\tilde{\epsilon}_{2i}$ are defined by the above expressions.

We write $\mathrm{Var}(\sum_{i=1}^{n}\tilde{\delta}_i\tilde{\epsilon}_i)$ in covariances of $\tilde{\delta}_{1i}$, $\tilde{\delta}_{2i}$, $\tilde{\epsilon}_{1i}$, and $\tilde{\epsilon}_{2i}$.

$$
\mathrm{Var}(\sum_{i=1}^{n}\tilde{\delta}_i\tilde{\epsilon}_i) \tag{59}
$$

$$= 2\sum_{i<j}\mathrm{E}(\tilde{\delta}_i\tilde{\delta}_j)\mathrm{E}(\tilde{\epsilon}_i\tilde{\epsilon}_j)+O(n),$$

$$\sum_{i<j}\mathrm{E}(\tilde{\delta}_i\tilde{\delta}_j)\mathrm{E}(\tilde{\epsilon}_i\tilde{\epsilon}_j) \tag{60}$$

$$\begin{aligned}
&= \sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\mathrm{E}(\tilde{\epsilon}_{2i}\tilde{\epsilon}_{2j})+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{2j})\\
&\quad+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{2i}\tilde{\epsilon}_{2j})+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{2j})\\
&\quad+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{2i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{2i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{2i}\tilde{\epsilon}_{2j})+\sum_{i<j}\mathrm{E}(\tilde{\delta}_{2i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{2j}).
\end{aligned}$$

We do not present every detail since it will be long and tedious. We closely look at only the first term of the RHS of (60) in the proof.

When $j-i=\lfloor unh\rfloor$ and $0<u\le\bar{M}+1$, we use the expression of $\tilde{a}_j$ in Lemma 5.1 to obtain

$$\begin{aligned}
\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j}) &= \sigma_1^2\sum_{l=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor-(j-i)}\tilde{a}_l\tilde{a}_{l+(j-i)} \qquad (61)\\
&= \sigma_1^2\sum_{l=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor-(j-i)}\frac{\psi_{1n}(l/\lfloor nh\rfloor)\psi_{1n}((l+j-i)/\lfloor nh\rfloor)}{(|l|+1)^{1-d_1}(|l+(j-i)|+1)^{1-d_1}}\\
&\sim \frac{\sigma_1^2}{(nh)^{1-2d_1}}\int_{-1}^{\bar{M}-u}\frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}}ds.
\end{aligned}$$

Here we put $l=\lfloor snh\rfloor$. We also have

$$\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})\sim\frac{\sigma_b^2}{(nh)^{1-2d_2}}\int_{-1}^{\bar{M}-u}\frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}}dt. \tag{62}$$

Noticing that $\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})=0$ and $\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})=0$ when $j-i>(\bar{M}+1)\lfloor nh\rfloor+1$, we fix a small positive number ν and use (61) and (62) to give an expression of the first term of the RHS of (60). Recall that $\tilde{n}=n(nh)^{-1+2(d_1+d_2)}$.

$$\begin{aligned}
&\sum_{i<j}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})\\
&\sim \sum_{0<j-i<\nu\lfloor nh\rfloor}\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\\
&\quad+\tilde{n}\sigma_1^2\sigma_b^2\int_{\nu}^{\bar{M}+1}\Big(\int_{-1}^{\bar{M}-u}\frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}}ds\int_{-1}^{\bar{M}-u}\frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}}dt\Big)du.
\end{aligned}$$

The first term of the above expression becomes negligible as $\nu\to 0$ since

$$\sum_{0<j-i<\nu\lfloor nh\rfloor}|\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})|\le\sum_{0<j-i<\nu\lfloor nh\rfloor}\frac{C}{(|i-j|+1)^{2-2(d_1+d_2)}}\le C\nu^{2(d_1+d_2)-1}\tilde{n}.$$

Hence we have shown that

$$\lim_{n\to\infty} \tilde{n}^{-1} \sum_{0<j-i\le \bar{M}\lfloor nh\rfloor+1} \mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{1j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j}) \tag{63}$$

$$= \sigma_1^2\sigma_b^2 \int_0^{\bar{M}+1} \Big(\int_{-1}^{\bar{M}-u} \frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}} ds \int_{-1}^{\bar{M}-u} \frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}} dt \Big) du.$$

The other terms appearing in (60) have expressions similar to (61) and (62) when $j-i=\lfloor unh\rfloor$ and $u>0$. The expressions are

$$\begin{aligned}
\mathrm{E}(\tilde{\delta}_{2i}\tilde{\delta}_{2j}) &\sim \frac{\sigma_1^2}{(nh)^{1-2d_1}} \int_{\bar{M}}^{\infty} \frac{\phi_1(s)\phi_1(s+u)}{s^{3-d_1}(s+u)^{3-d_1}} ds,\\
\mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{2j}) &\sim \frac{\sigma_1^2}{(nh)^{1-2d_1}} \int_{\bar{M}-u}^{\bar{M}} \frac{\psi_1(s)\phi_1(s+u)}{|s|^{1-d_1}(s+u)^{3-d_1}} ds I\{0<u\le \bar{M}+1\}\\
&\quad + \frac{\sigma_1^2}{(nh)^{1-2d_1}} \int_{-1}^{\bar{M}} \frac{\psi_1(s)\phi_1(s+u)}{|s|^{1-d_1}(s+u)^{3-d_1}} ds I\{\bar{M}+1<u\},\\
\mathrm{E}(\tilde{\epsilon}_{2i}\tilde{\epsilon}_{2j}) &\sim \frac{\sigma_b^2}{(nh)^{1-2d_2}} \int_{\bar{M}}^{\infty} \frac{\phi_2(t)\phi_2(t+u)}{t^{3-d_2}(t+u)^{3-d_2}} dt,\\
\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{2j}) &\sim \frac{\sigma_b^2}{(nh)^{1-2d_2}} \int_{\bar{M}-u}^{\bar{M}} \frac{\psi_2(t)\phi_2(t+u)}{|t|^{1-d_2}(t+u)^{3-d_2}} dt I\{0<u\le \bar{M}+1\}\\
&\quad + \frac{\sigma_1^2}{(nh)^{1-2d_2}} \int_{-1}^{\bar{M}} \frac{\psi_2(t)\phi_2(t+u)}{|t|^{1-d_2}(t+u)^{3-d_2}} dt I\{\bar{M}+1<u\}.
\end{aligned}$$

By following the derivation of (63) and using the fact that $\phi_l(s)=s^2\psi_l(s)$, $l=1,2$, in Lemma 5.1, we consider the 4th and 7th terms of the RHS of (60) and obtain

$$\sum_{i<j} \mathrm{E}(\tilde{\delta}_{1i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j}) \tag{64}$$

$$\sim \tilde{n}\sigma_1^2\sigma_b^2 \int_0^{\bar{M}+1} \Big(\int_{\bar{M}-u}^{\bar{M}} \frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}} ds \int_{-1}^{\bar{M}-u} \frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}} dt \Big) du,$$

$$\sum_{i<j} \mathrm{E}(\tilde{\delta}_{2i}\tilde{\delta}_{2j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j}) \tag{65}$$

$$\sim \tilde{n}\sigma_1^2\sigma_b^2 \int_0^{\bar{M}+1} \Big(\int_{\bar{M}}^{\infty} \frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}} ds \int_{-1}^{\bar{M}-u} \frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}} dt \Big) du.$$

Summing up (63), (64), and (65), we obtain

$$\sum_{i<j} \mathrm{E}(\tilde{\delta}_{i}\tilde{\delta}_{j})\mathrm{E}(\tilde{\epsilon}_{1i}\tilde{\epsilon}_{1j})$$

$$\sim \tilde{n}\sigma_1^2\sigma_b^2 \int_0^{\bar{M}+1} \Big(\int_{-1}^{\infty} \frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}} ds \int_{-1}^{\bar{M}-u} \frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}} dt \Big) du.$$

We treat the other terms in the same way and sum up all the terms of (60). Then we have

$$\sum_{i<j} \mathrm{E}(\tilde{\delta}_i\tilde{\delta}_j)\mathrm{E}(\tilde{\epsilon}_i\tilde{\epsilon}_j)$$
$$\sim \tilde{n}\sigma_1^2\sigma_b^2\int_0^\infty \Big(\int_{-1}^\infty \frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}}ds\int_{-1}^\infty \frac{\psi_2(t)\psi_2(t+u)}{|t|^{1-d_2}|t+u|^{1-d_2}}dt\Big)du.$$

Hence the proof of Proposition 3.4 is complete. □

The proof of Proposition 3.5 is based on the truncation argument and the large and small block argument. The proof is much more complicated than that of Proposition 3.3.

Proof of Proposition 3.5. Let $k=1$ for notational simplicity and we also suppress the superscript $(1,1)$ since $k=1$ as in the proof of Proposition 3.4. When $k>1$, we appeal to the Cramér-Wold device.

First we choose two sequences of positive integers $\{l_n\}$ and $\{s_n\}$ such that $l_n\to\infty$, $s_n\to\infty$, $s_n/l_n\to 0$, and $l_nh\to 0$. In the proof, $l_n\lfloor nh\rfloor$ and $(s_n+2)\lfloor nh\rfloor$ are the large block size and the small block size, respectively. The number of the blocks is denoted by c_n and

$$c_n = \lfloor n/\{(l_n+s_s+2)\lfloor nh\rfloor\}\rfloor \sim \frac{1}{l_nh}\to\infty.$$

We define $\hat{\delta}_{i1}$, $\hat{\delta}_{i2}$, $\hat{\delta}_{i3}$, $\hat{\epsilon}_{i1}$, $\hat{\epsilon}_{i2}$, and $\hat{\epsilon}_{i3}$ by

$$\hat{\delta}_{i1} = \tilde{\delta}_{i1},\quad \hat{\delta}_{i2} = \sum_{i=\bar{M}\lfloor nh\rfloor+1}^{s_n\lfloor nh\rfloor}\tilde{a}_j\zeta_{i-j},\quad \hat{\delta}_{i3} = \sum_{i=s_n\lfloor nh\rfloor+1}^{\infty}\tilde{a}_j\zeta_{i-j}, \tag{66}$$

$$\hat{\epsilon}_{i1} = \tilde{\epsilon}_{i1},\quad \hat{\epsilon}_{i2} = \sum_{i=\bar{M}\lfloor nh\rfloor+1}^{s_n\lfloor nh\rfloor}\tilde{b}_j\eta_{i-j},\quad \hat{\epsilon}_{i3} = \sum_{i=s_n\lfloor nh\rfloor+1}^{\infty}\tilde{b}_j\eta_{i-j}. \tag{67}$$

Then

$$\tilde{\delta}_i = \hat{\delta}_{1i}+\hat{\delta}_{2i}+\hat{\delta}_{3i}\quad\text{and}\quad \tilde{\epsilon}_i = \hat{\epsilon}_{1i}+\hat{\epsilon}_{2i}+\hat{\epsilon}_{3i}.$$

In Lemma 5.3 below, we show that

$$\lim_{n\to\infty}\frac{1}{\tilde{n}}\mathrm{Var}(\sum_{i=1}^n(\hat{\delta}_{1i}+\hat{\delta}_{2i})\hat{\epsilon}_{3i}) = 0,\ \lim_{n\to\infty}\frac{1}{\tilde{n}}\mathrm{Var}(\sum_{i=1}^n\hat{\delta}_{3i}(\hat{\epsilon}_{1i}+\hat{\epsilon}_{2i})) = 0,$$

$$\lim_{n\to\infty}\frac{1}{\tilde{n}}\mathrm{Var}(\sum_{i=1}^{n}\hat{\delta}_{3i}\hat{\epsilon}_{3i}) = 0.$$

Thus we have only to deal with $\sum_{i=1}^{n}(\hat{\delta}_{1i}+\hat{\delta}_{2i})(\hat{\epsilon}_{1i}+\hat{\epsilon}_{2i})$. We rewrite it as in (68) below.

$$\sum_{i=1}^{n}(\hat{\delta}_{1i}+\hat{\delta}_{2i})(\hat{\epsilon}_{1i}+\hat{\epsilon}_{2i}) = \sum_{i=1}^{c_n} W_i + \sum_{i=1}^{c_n} Z_i + R_n, \tag{68}$$

where

$$\begin{aligned} W_i &= \sum_{j=1+(i-1)(l_n+s_n+2)\lfloor nh\rfloor}^{l_n\lfloor nh\rfloor+(i-1)(l_n+s_n+2)\lfloor nh\rfloor}(\hat{\delta}_{1j}+\hat{\delta}_{2j})(\hat{\epsilon}_{1j}+\hat{\epsilon}_{2j}), \\ Z_i &= \sum_{j=1+l_n\lfloor nh\rfloor+(i-1)(l_n+s_n+2)\lfloor nh\rfloor}^{i(l_n+s_n+2)\lfloor nh\rfloor}(\hat{\delta}_{1j}+\hat{\delta}_{2j})(\hat{\epsilon}_{1j}+\hat{\epsilon}_{2j}), \end{aligned}$$

and R_n consists of the remainder terms.

It is easy to show that $\mathrm{Var}(R_n) = o(\tilde{n})$ as in the proof of Lemma 5.2 and the details are omitted. Besides, $\{W_i\}_{i=1}^{c_n}$ and $\{Z_i\}_{i=1}^{c_n}$ are two sequences of i.i.d. random variables due to the definitions of $\hat{\delta}_{1j}$, $\hat{\delta}_{2j}$, $\hat{\epsilon}_{1j}$, and $\hat{\epsilon}_{2j}$.

From Lemm 5.4, we have

$$\mathrm{Var}(\sum_{i=1}^{c_n} Z_i) = c_n O(s_n(nh)^{2(d_1+d_2)}) = O\Big(\frac{s_n}{l_n}\tilde{n}\Big) = o(\tilde{n}), \tag{69}$$

which implies that $\sum_{i=1}^{c_n} Z_i/\sqrt{\tilde{n}} = o_p(1)$.

Finally we consider $\sum_{i=1}^{c_n} W_i/\sqrt{\tilde{n}}$. From Lemma 5.5 and the definition of $\{l_n\}$, we have

$$c_n\mathrm{E}\Big\{I\{|W_1|/\sqrt{\tilde{n}}>\epsilon\}\frac{W_1^2}{\tilde{n}}\Big\} \le \frac{C}{l_n h}\mathrm{E}\Big(\frac{W_1^4}{\epsilon^2\tilde{n}^2}\Big) = o\Big(\frac{(nh)^{4(d_1+d_2)}}{h^2n^2(nh)^{4(d_1+d_2)-2}}\Big) = o(1). \tag{70}$$

The proposition follows from (68), (69), (70), Proposition 3.4, and the Lindeberg CLT. Hence the proof of Proposition 3.5 is complete. □

5 Technical lemmas and the proofs

We state the technical lemmas used in Section 4 and give the proofs at the end of this section.

We examine the coefficients of $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$ in Lemma 5.1. See a comment before the proof of Proposition 3.4 and Lemma 5.2 about the effect of the boundary kernels.

Lemma 5.1 *Suppose that* **A1**, **A2**, **A4**, *and* **A5** *hold. We take a sufficiently large positive integer* $\bar{M}$ *and represent* $\tilde{b}_j$ *as*

$$\begin{aligned} \tilde{b}_j &= \psi_{2n}(j/\lfloor nh \rfloor)(1+|j|)^{-1+d_2}, \quad -\lfloor nh \rfloor - 1 \le j \le \bar{M}\lfloor nh \rfloor, \\ \tilde{b}_j &= (nh)^2 \phi_{2n}(j/\lfloor nh \rfloor)(1+j)^{-3+d_2}, \quad \bar{M}\lfloor nh \rfloor < j. \end{aligned}$$

Then $\psi_{2n}(\cdot)$ *and* $\phi_{2n}(\cdot)$ *satisfy the following properties.*

(1) $\psi_{2n}(\cdot)$ *and* $\phi_{2n}(\cdot)$ *are uniformly bounded in* n.

(2) When $j = \lfloor snh \rfloor$, *we have*

$$\begin{aligned} \lim_{n\to\infty} \psi_{2n}(j/\lfloor nh \rfloor) &= \psi_2(s), \quad -1 \le s \le \bar{M}, \\ \lim_{n\to\infty} \phi_{2n}(j/\lfloor nh \rfloor) &= \phi_2(s), \quad \bar{M} \le s, \end{aligned}$$

where

$$\psi_2(s) = \gamma\Big(I\{s \ge 0\} - |s|^{1-d_2} \int_{(-1)\vee(-s)}^{1} \frac{K(s)}{(s+z)^{1-d_2}} dz\Big)$$

and $\phi_2(s) = s^2\psi_2(s)$. *Note that* $\psi_2(s)$ *can be defined for any large positive* s *and that* $\psi_2(s)$ *and* $\phi_2(s)$ *are bounded.*

(3) When $j = \lfloor snh \rfloor < 0$, $-1 \le s < 0$, *we have*

$$|\psi_{2n}(j/\lfloor nh \rfloor)||s|^{d_2-1} \le C \quad \text{and} \quad |\psi_2(j/\lfloor nh \rfloor)||s|^{d_2-1} \le C.$$

(4) $|\tilde{b}_j| \le C(nh)^{-1+d_2}$, $-\lfloor nh \rfloor - 1 \le j < 0$.

When we represent $\tilde{a}_j^{(l,m)}$ *as*

$$\begin{aligned} \tilde{a}_j^{(l,m)} &= \psi_{1n}^{(l,m)}(j/\lfloor nh \rfloor)(1+|j|)^{-1+d_1}, \quad -\lfloor nh \rfloor - 1 \le j \le \bar{M}\lfloor nh \rfloor, \\ \tilde{a}_j^{(l,m)} &= (nh)^2 \phi_{1n}^{(l,m)}(j/\lfloor nh \rfloor)(1+j)^{-3+d_1}, \quad \bar{M}\lfloor nh \rfloor < j, \end{aligned}$$

$\psi_{1n}^{(l,m)}(\cdot)$ *and* $\phi_{1n}^{(l,m)}(\cdot)$ *have the same properties as (1)-(4) with obvious changes. Especially we denote the limits in (2) by* $\psi_1^{(l,m)}(\cdot)$ *and* $\phi_1^{(l,m)}(\cdot)$.

Lemma 5.2 *Suppose that* $d_1 + d_2 \geq 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\sum_{i=1}^{\lfloor nh \rfloor +1}(\delta_i - \sum_{j=1}^n w_{ij}\delta_j)(\epsilon_i - \sum_{j=1}^n w_{ij}\epsilon_j) = o_p(\sqrt{\tilde{n}}),$$
$$\sum_{i=n-\lfloor nh \rfloor}^{n}(\delta_i - \sum_{j=1}^n w_{ij}\delta_j)(\epsilon_i - \sum_{j=1}^n w_{ij}\epsilon_j) = o_p(\sqrt{\tilde{n}}),$$

where we put $\tilde{n} = n \log n$ *when* $d_1 + d_2 = 1/2$. *This implies that the effect of using boundary kernels* $K_q(\cdot)$ *and* $K_q^*(\cdot)$ *is negligible in Propositions 3.4 and 3.5.*

Lemma 5.3 *Suppose that* $d_1 + d_2 > 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\mathrm{Var}(\sum_{i=1}^n(\hat{\delta}_{1i} + \hat{\delta}_{2i})\hat{\epsilon}_{3i}) = o(\tilde{n}),\ \mathrm{Var}(\sum_{i=1}^n \hat{\delta}_{3i}(\hat{\epsilon}_{1i} + \hat{\epsilon}_{2i})) = o(\tilde{n}),\ \mathrm{Var}(\sum_{i=1}^n \hat{\delta}_{3i}\hat{\epsilon}_{3i}) = o(\tilde{n}).$$

Lemma 5.4 *Suppose that* $d_1 + d_2 \geq 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\mathrm{E}(|Z_1|^2) = \begin{cases} O(s_n nh \log n), & d_1 + d_2 = 1/2 \\ O(s_n (nh)^{2(d_1+d_2)}), & d_1 + d_2 > 1/2 \end{cases}$$

Lemma 5.5 *Suppose that* $d_1 + d_2 > 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\mathrm{E}(|W_1|^4) = O(l_n^2 (nh)^{4(d_1+d_2)}) = o\Big(\frac{l_n}{h}(nh)^{4(d_1+d_2)}\Big).$$

Remark 5.1 *When* $d_1 + d_2 = 1/2$, *we should be careful in dealing with the integration over the regions containing* 0 *in the proofs. We will be able to proceed with some modifications and establish*

$$\mathrm{E}(|W_1|^4) = O(l_n^2 \log n (nh)^{4(d_1+d_2)}) = o\Big(\frac{l_n}{h}\log n (nh)^{4(d_1+d_2)}\Big).$$

However, the notation and the details are extremely complicated and we have not checked all the details yet.

Lemma 5.6 *Suppose that* $d_1 + d_2 > 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\mathrm{E}\{|\sum_{i=1}^{l_n\lfloor nh\rfloor} \hat{\delta}_{2i}\hat{\epsilon}_{2i}|^4\} = O(l_n^2(nh)^{4(d_1+d_2)}).$$

Lemma 5.7 *Suppose that* $d_1 + d_2 > 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\mathrm{E}\{|\sum_{i=1}^{l_n\lfloor nh\rfloor} \hat{\delta}_{1i}\hat{\epsilon}_{1i}|^4\} = O(l_n^2(nh)^{4(d_1+d_2)}).$$

Lemma 5.8 *Suppose that* $d_1 + d_2 > 1/2$ *and that* **A1**, **A2**, **A4**, *and* **A5** *hold. Then we have*

$$\mathrm{E}\{|\sum_{i=1}^{l_n\lfloor nh\rfloor} \hat{\delta}_{1i}\hat{\epsilon}_{2i}|^4\} = O(l_n^2(nh)^{4(d_1+d_2)}) \text{ and } \mathrm{E}\{|\sum_{i=1}^{l_n\lfloor nh\rfloor} \hat{\delta}_{2i}\hat{\epsilon}_{1i}|^4\} = O(l_n^2(nh)^{4(d_1+d_2)}).$$

Lemma 5.9 *Suppose that* **A4** *and* **A5** *hold. Then we have*

$$\sum_{i=-\lfloor nh\rfloor-1}^{\lfloor nh\rfloor+1} \int_{(-0.5-i)/(nh)}^{(0.5-i)/(nh)} K(z)dz = 1, \qquad \sum_{i=-\lfloor nh\rfloor-1}^{\lfloor nh\rfloor+1} i\int_{(-0.5-i)/(nh)}^{(0.5-i)/(nh)} K(z)dz = 0,$$
$$\sum_{i=-\lfloor nh\rfloor-1}^{\lfloor nh\rfloor+1} i^2\int_{(-0.5-i)/(nh)}^{(0.5-i)/(nh)} K(z)dz = (nh)^2\Big(\int_{-1}^{1} z^2K(z)dz + O((nh)^{-1})\Big),$$
$$\sum_{i=-\lfloor nh\rfloor-1}^{\lfloor nh\rfloor+1} |i|^3\int_{(-0.5-i)/(nh)}^{(0.5-i)/(nh)} K(z)dz = (nh)^3\Big(\int_{-1}^{1} |z|^3K(z)dz + O((nh)^{-1})\Big).$$

The proof of Lemma 5.9 is trivial and omitted.

Proof of Lemma 5.1. We prove only the results for $\tilde{b}_j$. We put $b_j = 0$ for $j < 0$ and represent $\tilde{b}_j$ as

$$\tilde{b}_j = b_j - \sum_{i=-\lfloor nh\rfloor-1}^{\lfloor nh\rfloor+1} w_{j,i+j}b_{i+j}. \tag{71}$$

Obviously $\tilde{b}_j = 0$ when $j < -\lfloor nh\rfloor - 1$.

We take and fix a sufficiently large positive integer $\bar{M}$. We need **A2** only when $j > \bar{M}\lfloor nh\rfloor$.

First we deal with the case where $j < 0$ and $j = \lfloor snh\rfloor$, $-1 \le s < 0$. Noticing that

$$w_{j,i+j} = \int_{(-0.5-i)/(nh)}^{(0.5-i)/(nh)} K(z)dz, \tag{72}$$

we have

$$\begin{aligned} \tilde{b}_j &= -\sum_{i=(-j)\vee(-\lfloor nh\rfloor-1)}^{\lfloor nh\rfloor+1} \frac{\gamma}{(i+j+1)^{1-d_2}} \int_{(0.5-i)/(nh)}^{(-0.5-i)/(nh)} K(z)dz \qquad (73)\\ &\sim \frac{\gamma}{(nh)^{1-d_2}} \int_{(-s)\vee(-1)}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz \\ &\sim -\frac{\{(|j|+1)(nh)^{-1}\}^{1-d_2}}{(|j|+1)^{1-d_2}} \gamma \int_{(-s)\vee(-1)}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz. \end{aligned}$$

(73) yields that

$$\begin{aligned} \psi_{2n}(j/\lfloor nh\rfloor) &\sim -\gamma\Big(\frac{|j|+1}{nh}\Big)^{1-d_2} \int_{(-s)\vee(-1)}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz, \\ \lim_{n\to\infty} \psi_{2n}(j/\lfloor nh\rfloor) &= -\gamma|s|^{1-d_2} \int_{(-s)\vee(-1)}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz. \qquad (74) \end{aligned}$$

We define $\psi_2(s)$, $-1 \le s < 0$, by the RHS of (74). The proof of (3) and (4) is complete. We have also established (1) and (2) when $-1 \le s < 0$.

Next we deal with the case where $0 \le j \le \bar{M}\lfloor nh\rfloor$ and $j = \lfloor snh\rfloor$, $0 \le s \le \bar{M}$. Then we have as in (73) that

$$\begin{aligned} \tilde{b}_j &\sim \frac{\gamma}{(j+1)^{1-d_2}} - \frac{\gamma}{(nh)^{1-d_2}} \int_{-1}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz \qquad (75)\\ &\sim \frac{\gamma}{(j+1)^{1-d_2}} \Big\{1 - \Big(\frac{j+1}{nh}\Big)^{1-d_2} \int_{-1}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz\Big\}. \end{aligned}$$

(75) yields that

$$\begin{aligned} \psi_{2n}(j/\lfloor nh\rfloor) &\sim \gamma\Big\{1 - \Big(\frac{j+1}{nh}\Big)^{1-d_2} \int_{-1}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz\Big\}, \\ \lim_{n\to\infty} \psi_{2n}(j/\lfloor nh\rfloor) &= \gamma\Big(1 - s^{1-d_2} \int_{-1}^{1} \frac{K(z)}{(s+z)^{1-d_2}} dz\Big). \qquad (76) \end{aligned}$$

We define $\psi_2(s)$, $0 \le s \le \bar{M}$, by the RHS of (76). We have established (1) and (2) when $0 \le s \le \bar{M}$.

Finally we deal with the case where $j > \bar{M}\lfloor nh\rfloor$ and $j = \lfloor snh\rfloor$. Then we have from **A2** that

$$b_{i+j} = \frac{\gamma}{(j+1)^{1-d_2}} \Big\{1 + (d_2-1)\frac{i}{j+1} + \frac{(d_2-1)(d_2-2)}{2}\Big(\frac{i}{j+1}\Big)^2 + O\Big(\Big|\frac{i}{j+1}\Big|^3\Big)\Big\}. \qquad (77)$$

By combining (77) and Lemma 5.9, we have

$$\sum_{i=-\lfloor nh\rfloor-1}^{\lfloor nh\rfloor+1} w_{j,i+j}b_{i+j} = \frac{\gamma}{(j+1)^{1-d_2}} + \frac{\gamma(d_2-1)(d_2-2)(nh)^2}{2(j+1)^{3-d_2}} \times\Big(\int_{-1}^{1} z^2K(z)dz + O((nh)^{-1})\Big) + \Big(\frac{(nh)^3}{(j+1)^{4-d_2}}\Big). \tag{78}$$

We substitute (78) into (71) and obtain

$$\tilde{b}_j = -\frac{(nh)^2}{(j+1)^{3-d_2}}\frac{\gamma(d_2-1)(d_2-2)}{2} \times\Big(\int_{-1}^{1} z^2K(z)dz + O\Big(\frac{nh}{j+1}\Big) + O\Big(\frac{1}{nh}\Big)\Big).$$

This implies that

$$\phi_{2n}(j/\lfloor nh\rfloor) = -\frac{\gamma(d_2-1)(d_2-2)}{2} \times\Big(\int_{-1}^{1} z^2K(z)dz + O\Big(\frac{nh}{j+1}\Big) + O\Big(\frac{1}{nh}\Big)\Big).$$

and that $\phi_{2n}(\cdot)$ are uniformly bounded.

We can take any large $\bar{M}$. Then we have

$$\phi_{2n}(j/\lfloor nh\rfloor) = (j/\lfloor nh\rfloor)^2\psi_{2n}(j/\lfloor nh\rfloor).$$

Obviously from the above expression,

$$\lim_{n\to\infty}\phi_{2n}(j/\lfloor nh\rfloor) = s^2\psi_2(s). \tag{79}$$

We define $\phi_2(s)$ by $s^2\psi_2(s)$. Since $\phi_{2n}(\cdot)$ are uniformly bounded, $\phi_2(s)$ is also bounded. (1) and (2) is established when $s > \bar{M}$. Hence the proof of Lemma 5.1 is complete. □

Proof of Lemma 5.2. Suppose that $k = 1$ and $d_1 + d_2 > 1/2$ for notational simplicity. We illustrate only the former result, the summation from 1 to $\lfloor nh\rfloor + 1$. By carrying out routine calculations and using (3) and (7), we obtain

$$\mathrm{Var}\Big(\sum_{i=1}^{\lfloor nh\rfloor+1}\delta_i\epsilon_i\Big) = O((nh)^{2(d_1+d_2)}),$$

$$\begin{aligned}
&\mathrm{Var}\Big(\sum_{i=1}^{\lfloor nh\rfloor+1}\Big(\sum_{j=1}^{n} w_{ij}\delta_j\Big)\epsilon_i\Big)\\
&= \sum_{1\le i_1,i_2\le \lfloor nh\rfloor+1}\ \sum_{1\le j_1,j_2\le n} w_{i_1j_1}w_{i_2j_2}r_{11}(j_1-j_2)r_\epsilon(i_1-i_2)\\
&\le C(nh)^{-1+2d_2}\sum_{1\le i_1,i_2\le \lfloor nh\rfloor+1} r_\epsilon(i_1-i_2) = O((nh)^{2(d_1+d_2)}),\\
&\mathrm{Var}\Big(\sum_{i=1}^{\lfloor nh\rfloor+1}\delta_i\Big(\sum_{j=1}^{n} w_{ij}\epsilon_j\Big)\Big) = O((nh)^{2(d_1+d_2)}),\\
&\mathrm{Var}\Big(\sum_{i=1}^{\lfloor nh\rfloor+1}\Big(\sum_{j=1}^{n} w_{ij}\delta_j\Big)\Big(\sum_{l=1}^{n} w_{il}\epsilon_l\Big)\Big)\\
&= \sum_{1\le i_1,i_2\le \lfloor nh\rfloor+1}\ \sum_{1\le j_1,j_2\le n}\ \sum_{1\le l_1,l_2\le n} w_{i_1j_1}w_{i_2j_2}w_{i_1l_1}w_{i_2l_2}r_{11}(j_1-j_2)r_\epsilon(l_1-l_2)\\
&\le C(nh)^2\frac{1}{(nh)^4}(nh)^{1+2d_1}(nh)^{1+2d_2} = O((nh)^{2(d_1+d_2)}).
\end{aligned}$$

The former result follows from the above expressions.

When $d_1+d_2 = 1/2$, we should replace $(nh)^{2(d_1+d_2)}$ by $nh\log n$. Hence the proof of Lemma 5.2 is complete. □

Proof of Lemma 5.3. Let $k = 1$ for notational simplicity and we also suppress the superscript $(1,1)$ since $k = 1$. We only outline the proof of the first and last expressions.

By following the proof of Proposition 3.4, we obtain

$$\sum_{1\le i,j\le n}\mathrm{E}(\hat\delta_{1i}\hat\delta_{1j})\mathrm{E}(\hat\epsilon_{3i}\hat\epsilon_{3j}) \tag{80}$$
$$\sim 2\tilde n\sigma_1^2\sigma_b^2\int_0^{\bar M+1}\Big(\int_{-1}^{\bar M-u}\frac{\psi_1(s)\psi_1(s+u)}{|s|^{1-d_1}|s+u|^{1-d_1}}ds\int_{s_n}^{\infty}\frac{\phi_2(t)\phi_2(t+u)}{t^{3-d_2}(t+u)^{3-d_2}}dt\Big)du$$
$$= O\Big(\frac{\tilde n}{s_n^{2-d_2}}\Big),$$
$$\sum_{1\le i,j\le n}\mathrm{E}(\hat\delta_{2i}\hat\delta_{2j})\mathrm{E}(\hat\epsilon_{3i}\hat\epsilon_{3j}) \tag{81}$$
$$\sim 2\tilde n\sigma_1^2\sigma_b^2\int_0^{\infty}\Big(\int_{\bar M}^{s_n}\frac{\phi_1(s)\phi_1(s+u)}{s^{3-d_1}(s+u)^{3-d_1}}ds\int_{s_n}^{\infty}\frac{\phi_2(t)\phi_2(t+u)}{t^{3-d_2}(t+u)^{3-d_2}}dt\Big)du$$
$$= O\Big(\frac{\tilde n}{s_n^{2-d_2}}\Big),$$
$$\sum_{1\le i<j\le n}\mathrm{E}(\hat\delta_{1i}\hat\delta_{2j})\mathrm{E}(\hat\epsilon_{3i}\hat\epsilon_{3j}) \tag{82}$$

$$
\begin{aligned}
&\sim \quad \tilde{n}\sigma_1^2\sigma_b^2\int_0^{\bar{M}+1}\Big(\int_{\bar{M}-u}^{\bar{M}}\frac{\psi_1(s)\phi_1(s+u)}{|s|^{1-d_1}(s+u)^{3-d_1}}ds\int_{s_n}^{\infty}\frac{\phi_2(t)\phi_2(t+u)}{t^{3-d_2}(t+u)^{3-d_2}}dt\Big)du \\
&\quad +\tilde{n}\sigma_1^2\sigma_b^2\int_{\bar{M}+1}^{s_n+1}\Big(\int_{-1}^{\bar{M}\wedge(s_n-u)}\frac{\psi_1(s)\phi_1(s+u)}{|s|^{1-d_1}(s+u)^{3-d_1}}ds\int_{s_n}^{\infty}\frac{\phi_2(t)\phi_2(t+u)}{t^{3-d_2}(t+u)^{3-d_2}}dt\Big)du \\
&= \quad O\Big(\frac{\tilde{n}}{s_n^{2-d_2}}\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{1\le i,j\le n}\mathrm{E}(\hat{\delta}_{3i}\hat{\delta}_{3j})\mathrm{E}(\hat{\epsilon}_{3i}\hat{\epsilon}_{3j}) \qquad (83)\\
&\sim \quad 2\tilde{n}\sigma_1^2\sigma_b^2\int_0^{\infty}\Big(\int_{s_n}^{\infty}\frac{\phi_1(s)\phi_1(s+u)}{s^{3-d_1}(s+u)^{3-d_1}}ds\int_{s_n}^{\infty}\frac{\phi_2(t)\phi_2(t+u)}{t^{3-d_2}(t+u)^{3-d_2}}dt\Big)du \\
&= \quad O\Big(\frac{\tilde{n}}{s_n^{4-(d_1+d_2)}}\Big).
\end{aligned}
$$

The first and last expressions follow from (80)–(83). Hence the proof of Lemma 5.3 is complete. □

Proof of Lemma 5.4. We can verify this lemma in a fashion similar to that of Proposition 3.4 by just noticing that the summation of $\mathrm{E}\{(\hat{\delta}_{1i}+\hat{\delta}_{2i})(\hat{\delta}_{1j}+\hat{\delta}_{2j})\}\mathrm{E}\{(\hat{\epsilon}_{1i}+\hat{\epsilon}_{2i})(\hat{\epsilon}_{1j}+\hat{\epsilon}_{2j})\}$ is over $\{(i,j)\,|\,1\le i,j\le (s_n+1)\lfloor nh\rfloor\}$. The details are omitted. When $d_1+d_2=1/2$, we should replace $(nh)^{2(d_1+d_2)-1}$ with $\log n$. Hence the proof of Lemma 5.4 is complete. □

Proof of Lemma 5.5. Let $k=1$ for notational simplicity and we also suppress the superscript $(1,1)$ since $k=1$ in the proofs of Lemmas 5.5-5.8.

Since W_1 has the same distribution as

$$
W_1=\sum_{i=1}^{l_n\lfloor nh\rfloor}(\hat{\delta}_{1i}\hat{\epsilon}_{1i}+\hat{\delta}_{2i}\hat{\epsilon}_{1i}+\hat{\delta}_{1i}\hat{\epsilon}_{2i}+\hat{\delta}_{2i}\hat{\epsilon}_{2i}),
$$

the lemma follows from Lemmas 5.6-5.8. Hence the proof of the Lemma 5.5 is complete. □

Proof of Lemma 5.6. We represent $\hat{\delta}_{2i}$ and $\hat{\epsilon}_{2i}$ in ζ_l and η_j as

$$
\hat{\delta}_{2i} = \sum_{j=\bar{M}\lfloor nh\rfloor+1}^{s_n\lfloor nh\rfloor}\tilde{a}_j\zeta_{i-j} = \sum_{l=i-s_n\lfloor nh\rfloor}^{i-\bar{M}\lfloor nh\rfloor-1}\tilde{a}_{i-l}\zeta_l \qquad (84)
$$

$$
\hat{\epsilon}_{2i} = \sum_{j=\bar{M}\lfloor nh\rfloor+1}^{s_n\lfloor nh\rfloor}\tilde{b}_j\eta_{i-j} = \sum_{j=i-s_n\lfloor nh\rfloor}^{i-\bar{M}\lfloor nh\rfloor-1}\tilde{b}_{i-j}\eta_j. \qquad (85)
$$

Since the purpose of this lemma is to find an upper bound of $\mathrm{E}\{|\sum_{i=1}^{l_n\lfloor nh\rfloor} \hat{\delta}_{2i}\hat{\epsilon}_{2i}|^4\}$, we use $|\tilde{a}_j|$ and $|\tilde{b}_j|$ instead of $\tilde{a}_j$ and $\tilde{b}_j$, respectively in calculating the autocovariances of $\hat{\delta}_{2i}$ and $\hat{\epsilon}_{2i}$.

Because of the symmetry, we have only to take into account the summation of

$$|\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})| \tag{86}$$

over $\mathcal{S}_j$, $j = 1, \ldots, 8$, where

$$\begin{aligned}
\mathcal{S}_1 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 < i_2 < i_3 < i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_2 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 = i_2 < i_3 < i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_3 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 < i_2 = i_3 < i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_4 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 < i_2 < i_3 = i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_5 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 < i_2 = i_3 = i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_6 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 = i_2 < i_3 = i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_7 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 = i_2 = i_3 < i_4 \le l_n\lfloor nh\rfloor\},\\
\mathcal{S}_8 &= \{(i_1, i_2, i_3, i_4) \,|\, 1 \le i_1 = i_2 = i_3 = i_4 \le l_n\lfloor nh\rfloor\}.
\end{aligned}$$

Hereafter $\sum_{\mathcal{S}_p}$ means the summation over $\mathcal{S}_p$.

We put

$$i_2 - i_1 = \lfloor \xi_1 nh\rfloor, \quad i_3 - i_2 = \lfloor \xi_2 nh\rfloor, \quad \text{and} \quad i_4 - i_3 = \lfloor \xi_3 nh\rfloor, \tag{87}$$

where $0 \le \xi_1 \le s_n + 1$, $0 \le \xi_2 \le l_n$, and $0 \le \xi_3 \le s_n + 1$ from (84) and (85).

First we consider $\mathcal{S}_5 - \mathcal{S}_8$. Then for $j = 5, \ldots, 8$,

$$\sum_{\mathcal{S}_j} |\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})| \le Cl_n^2(nh)^2 \le Cl_n^2(nh)^{4(d_1+d_2)} \tag{88}$$

since

$$|\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})| \le C \quad \text{and} \quad |\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})| \le C.$$

Next we consider $\mathcal{S}_1$. From (84) and (85), $\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})$ and $\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})$ are given by

$$\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4}) = \sum_{l_1,l_2,l_3,l_4} \tilde{a}_{i_1-l_1}\tilde{a}_{i_2-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4}), \tag{89}$$

$$\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4}) = \sum_{j_1,j_2,j_3,j_4} \tilde{b}_{i_1-j_1}\tilde{b}_{i_2-j_2}\tilde{b}_{i_3-j_3}\tilde{b}_{i_4-j_4}\mathrm{E}(\eta_{j_1}\zeta_{j_2}\eta_{j_3}\eta_{j_4}). \tag{90}$$

In (89) and (90), we put $\tilde{a}_{i_p-l_p}=0$ and $\tilde{b}_{i_q-j_q}=0$, when i_p-l_p and i_q-j_q are not included in the definitions of $\hat{\delta}_{2i_p}$ and $\hat{\epsilon}_{2i_q}$, respectively.

As for (89), there are 4 subsets of $\{(l_1,l_2,l_3,l_4)\,|\,-\infty < l_1,l_2,l_3,l_4 < \infty\}$ we should take into consideration and they are

$$\begin{aligned}
\mathcal{L}_1 &= \{(l_1,l_2,l_3,l_4)\,|\,l_1=l_2\neq l_3=l_4\},\\
\mathcal{L}_2 &= \{(l_1,l_2,l_3,l_4)\,|\,l_1=l_3\neq l_2=l_4\},\\
\mathcal{L}_3 &= \{(l_1,l_2,l_3,l_4)\,|\,l_1=l_4\neq l_2=l_3\},\\
\mathcal{L}_4 &= \{(l_1,l_2,l_3,l_4)\,|\,l_1=l_2=l_3=l_4\}.
\end{aligned}$$

Hereafter $\sum_{\mathcal{L}_p}$ means the summation over $\mathcal{L}_p$.

We have for $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$,

$$\sum_{\mathcal{L}_p}|\tilde{a}_{i_1-l_1}\tilde{a}_{i_2-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}|\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4}) \leq \begin{cases} \mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2})\mathrm{E}(\hat{\delta}_{2i_3}\hat{\delta}_{2i_4}), & p=1 \\ \mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_3})\mathrm{E}(\hat{\delta}_{2i_2}\hat{\delta}_{2i_4}), & p=2 \\ \mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}), & p=3 \end{cases}. \tag{91}$$

In the same way as in the proof of Proposition 3.4, we can show that for $1 \leq p \leq q \leq 4$,

$$|\mathrm{E}(\hat{\delta}_{2i_p}\hat{\delta}_{2i_q})| \leq \frac{C}{(nh)^{1-2d_1}}\int_{\bar{M}}^{\infty}\frac{d\xi_a}{\xi_a^{3-d_1}(\xi_a+\xi_p+\cdots+\xi_{q-1})^{3-d_1}}. \tag{92}$$

In the case of $\mathcal{L}_4$, we have

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty}|\tilde{a}_{i_1-l}\tilde{a}_{i_2-l}\tilde{a}_{i_3-l}\tilde{a}_{i_4-l}|\mathrm{E}(\zeta_1^4) &(93)\\
\leq\ &\frac{C}{(nh)^{3-4d_1}}\int_{\bar{M}}^{\infty}\frac{d\xi_a}{\xi_a^{3-d_1}(\xi_a+\xi_1)^{3-d_1}(\xi_a+\xi_1+\xi_2)^{3-d_1}(\xi_a+\xi_1+\xi_2+\xi_3)^{3-d_1}}.
\end{aligned}$$

We treat $\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})$ in the same way as $\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})$ and we define $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, and $\mathcal{J}_4$ by

$$\begin{aligned}
\mathcal{J}_1 &= \{(j_1,j_2,j_3,j_4)\,|\,j_1=j_2\neq j_3=j_4\},\\
\mathcal{J}_2 &= \{(j_1,j_2,j_3,j_4)\,|\,j_1=j_3\neq j_2=j_4\},\\
\mathcal{J}_3 &= \{(j_1,j_2,j_3,j_4)\,|\,j_1=j_4\neq j_2=j_3\},\\
\mathcal{J}_4 &= \{(j_1,j_2,j_3,j_4)\,|\,j_1=j_2=j_3=j_4\}.
\end{aligned}$$

We have for $\mathcal{J}_1$, $\mathcal{J}_2$, and $\mathcal{J}_3$,

$$\sum_{\mathcal{J}_p}|\tilde{b}_{i_1-j_1}\tilde{b}_{i_2-j_2}\tilde{b}_{i_3-j_3}\tilde{b}_{i_4-j_4}|\mathrm{E}(\eta_{j_1}\eta_{j_2}\eta_{j_3}\eta_{j_4})\le\begin{cases}\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2})\mathrm{E}(\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4}), & p=1\\ \mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_3})\mathrm{E}(\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_4}), & p=2\\ \mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}), & p=3\end{cases}. \quad (94)$$

In the same way as in the proof of Proposition 3.4, we can show that for $1\le p\le q\le 4$,

$$|\mathrm{E}(\hat{\epsilon}_{2i_p}\hat{\epsilon}_{2i_q})|\le\frac{C}{(nh)^{1-2d_2}}\int_{\bar{M}}^{\infty}\frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_p+\cdots+\xi_{q-1})^{3-d_2}}. \quad (95)$$

In the case of $\mathcal{J}_4$, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}|\tilde{b}_{i_1-j}\tilde{b}_{i_2-j}\tilde{b}_{i_3-j}\tilde{b}_{i_4-j}|\mathrm{E}(\eta_1^4) \quad (96)\\
&\le\frac{C}{(nh)^{3-4d_2}}\int_{\bar{M}}^{\infty}\frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_1)^{3-d_2}(\xi_b+\xi_1+\xi_2)^{3-d_2}(\xi_b+\xi_1+\xi_2+\xi_3)^{3-d_2}}.
\end{aligned}$$

Here we define $\mathcal{T}(r,p,q)$ by

$$\begin{aligned}
&\mathcal{T}(r,p,q) \quad (97)\\
&=\sum_{\mathcal{S}_r}\sum_{\mathcal{L}_p}\sum_{\mathcal{J}_q}|\tilde{a}_{i_1-l_1}\tilde{a}_{i_2-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}\tilde{b}_{i_1-j_1}\tilde{b}_{i_2-j_2}\tilde{b}_{i_3-j_3}\tilde{b}_{i_4-j_4}|\\
&\qquad\times\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4})\mathrm{E}(\eta_{j_1}\eta_{j_2}\eta_{j_3}\eta_{j_4}).
\end{aligned}$$

It is easy to see that for any r,

$$\sum_{\mathcal{S}_r}|\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})|\le\sum_{1\le p,q\le 4}\mathcal{T}(r,p,q). \quad (98)$$

By exploiting (91)–(96), we evaluate $\mathcal{T}(1,p,q)$ for every pair of $\{(p,q)\,|\,1 \le p,q \le 4\}$. We have for $(1,1)$,

$$
\begin{aligned}
&\mathcal{T}(1,1,1) \qquad (99)\\
&\le \; Cl_n nh(nh)^{4(d_1+d_2)-4}(nh)^3 \int_{\Omega_{11}} \Big(\int_{\bar{M}}^{\infty} \frac{d\xi_a}{\xi_a^{3-d_1}(\xi_a+\xi_1)^{3-d_1}} \int_{\bar{M}}^{\infty} \frac{d\xi_a}{\xi_a^{3-d_1}(\xi_a+\xi_3)^{3-d_1}} \\
&\quad \times \int_{\bar{M}}^{\infty} \frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_1)^{3-d_2}} \int_{\bar{M}}^{\infty} \frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_3)^{3-d_2}} \Big) d\xi_1 d\xi_2 d\xi_3 \\
&\le \; Cl_n(nh)^{4(d_1+d_2)},
\end{aligned}
$$

where $\Omega_{11} = \{(\xi_1,\xi_2,\xi_3)^T\,|\,0 \le \xi_1 \le s_n+1,\ 0 \le \xi_2 \le l_n,\ 0 \le \xi_3 \le s_n+1\}$. In the RHS of (99), $l_n nh$ and $(nh)^3$ come from i_1 and $d\xi_1 d\xi_2 d\xi_3$, respectively . We can treat the other pairs almost in the same way and the details are omitted. Hence we have shown that

$$
\sum_{\mathcal{S}_1} |\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})| \le Cl_n(nh)^{4(d_1+d_2)}. \qquad (100)
$$

We proceed to $\mathcal{S}_2$. Since we can deal with $\mathcal{S}_3$ and $\mathcal{S}_4$ in the same way, the details about $\mathcal{S}_3$ and $\mathcal{S}_4$ are omitted. We still use $\mathcal{L}_p$ and $\mathcal{J}_q$ here. There is no $i_2 - i_1 = \lfloor \xi_1 nh \rfloor$ for $\mathcal{S}_2$.

We will find an upper bound of the summation of $|\mathrm{E}(\hat{\delta}_{2i_1}^2\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}^2\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})|$ over $\mathcal{S}_2$. We give expressions such as (91)-(93) only to

$$
\mathrm{E}(\hat{\delta}_{2i_1}^2\hat{\delta}_{2i_3}\hat{\delta}_{2i_4}) = \sum_{l_1,l_2,l_3,l_4} \tilde{a}_{i_1-l_1}\tilde{a}_{i_1-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4})
$$

since those of $\hat{\epsilon}_{2i}$ are similar to those of $\hat{\delta}_{2i}$. We put $\tilde{a}_{i_p-l_p} = 0$ when $i_p - l_p$ is not included in the definition of $\hat{\delta}_{2i_p}$.

We have for $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$,

$$
\sum_{\mathcal{L}_p} |\tilde{a}_{i_1-l_1}\tilde{a}_{i_1-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}|\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4}) \le \begin{cases} \mathrm{E}(\hat{\delta}_{2i_1}^2)\mathrm{E}(\hat{\delta}_{2i_3}\hat{\delta}_{2i_4}), & p=1 \\ \mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_3})\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_4}), & p=2 \\ \mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_3}), & p=3 \end{cases}. \qquad (101)
$$

In the case of $\mathcal{L}_4$, we have

$$
\sum_{l=-\infty}^{\infty} |\tilde{a}_{i_1-l}^2\tilde{a}_{i_3-l}\tilde{a}_{i_4-l}|\mathrm{E}(\zeta_1^4) \qquad (102)
$$

$$\leq \frac{C}{(nh)^{3-4d_1}} \int_{\bar{M}}^{\infty} \frac{d\xi_a}{\xi_a^{6-2d_1}(\xi_a+\xi_2)^{3-d_1}(\xi_a+\xi_2+\xi_3)^{3-d_1}}.$$

By exploiting (92), (101), (102), and those of $\hat{\epsilon}_{2i}$, we evaluate $\mathcal{T}(2,p,q)$ for every pair of $\{(p,q)\,|\,1 \leq p,q \leq 4\}$. We have for $(1,1)$,

$$\begin{aligned} &\mathcal{T}(2,1,1) \qquad (103)\\ \leq\ & Cl_n nh(nh)^{4(d_1+d_2)-4}(nh)^2 \int_{\Omega_{12}} \Big(\int_{\bar{M}}^{\infty} \frac{d\xi_a}{\xi_a^{6-2d_1}} \int_{\bar{M}}^{\infty} \frac{d\xi_a}{\xi_a^{3-d_1}(\xi_a+\xi_3)^{3-d_1}} \\ & \times \int_{\bar{M}}^{\infty} \frac{d\xi_b}{\xi_b^{6-2d_2}} \int_{\bar{M}}^{\infty} \frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_3)^{3-d_2}} \Big) d\xi_2 d\xi_3 \\ \leq\ & Cl_n^2(nh)^{4(d_1+d_2)-1}, \end{aligned}$$

where $\Omega_{12} = \{(\xi_2,\xi_3)^T\,|\,0 \leq \xi_2 \leq l_n,\ 0 \leq \xi_3 \leq s_n+1\}$. In the RHS of (103), $l_n nh$ and $(nh)^2$ come from i_1 and $d\xi_2 d\xi_3$, respectively .

We can treat the other pairs almost in the same way and the details are omitted. Recalling (98), we have

$$\sum_{\mathcal{S}_2} |\mathrm{E}(\hat{\delta}_{2i_1}\hat{\delta}_{2i_2}\hat{\delta}_{2i_3}\hat{\delta}_{2i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})| \leq Cl_n(nh)^{4(d_1+d_2)}. \qquad (104)$$

The proof of Lemma 5.6 is complete. □

Proof of Lemma 5.7. Lemma 5.7 can be established in the same way as Lemma 5.6 and we use the notation such as $\mathcal{S}_r$, $\mathcal{L}_p$, $\mathcal{J}_q$, and $\mathcal{T}(r,p,q)$. We represent $\hat{\delta}_{1i}$ and $\hat{\epsilon}_{1i}$ in ζ_l and η_j as

$$\hat{\delta}_{1i} = \sum_{j=-\lfloor nh \rfloor -1}^{\bar{M}\lfloor nh \rfloor} \tilde{a}_j \zeta_{i-j} = \sum_{l=i-\bar{M}\lfloor nh \rfloor}^{i+\lfloor nh \rfloor+1} \tilde{a}_{i-l}\zeta_l \qquad (105)$$

$$\hat{\epsilon}_{1i} = \sum_{j=-\lfloor nh \rfloor -1}^{\bar{M}\lfloor nh \rfloor} \tilde{b}_j \eta_{i-j} = \sum_{j=i-\bar{M}\lfloor nh \rfloor}^{i+\lfloor nh \rfloor+1} \tilde{b}_{i-j}\eta_j. \qquad (106)$$

When we calculate the autocovariances of $\hat{\delta}_{1i}$ and $\hat{\epsilon}_{1i}$, we use $|\tilde{a}_j|$ and $|\tilde{b}_j|$ instead of $\tilde{a}_j$ and $\tilde{b}_j$, respectively as in the proof of Lemma 5.6.

We have only to consider the summation of

$$|\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\epsilon}_{1i_1}\hat{\epsilon}_{1i_2}\hat{\epsilon}_{1i_3}\hat{\epsilon}_{1i_4})| \qquad (107)$$

over $\mathcal{S}_j$, $j = 1, \ldots, 8$.

We put

$$i_2 - i_1 = \lfloor \xi_1 nh \rfloor, \quad i_3 - i_2 = \lfloor \xi_2 nh \rfloor, \quad \text{and} \quad i_4 - i_3 = \lfloor \xi_3 nh \rfloor, \tag{108}$$

where $0 \le \xi_1 \le \bar{M} + 1$, $0 \le \xi_2 \le l_n$, and $0 \le \xi_3 \le \bar{M} + 1$ from (105) and (106).

We can deal with $\mathcal{S}_5 - \mathcal{S}_8$ as in the proof of Lemma 5.6 and the details are omitted.

We consider $\mathcal{S}_1$. Note that $\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})$ and $\mathrm{E}(\hat{\epsilon}_{1i_1}\hat{\epsilon}_{1i_2}\hat{\epsilon}_{1i_3}\hat{\epsilon}_{1i_4})$ have the same kind of expression as (89) and (90). As in (89) and (90), we put $\tilde{a}_{i_p - l_p} = 0$ and $\tilde{b}_{i_q - j_q} = 0$, when $i_p - l_p$ and $i_q - j_q$ are not included in the definitions of $\hat{\delta}_{1i_p}$ and $\hat{\epsilon}_{1i_q}$, respectively.

As in the proof of Lemma 5.6, we have for $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$,

$$\sum_{\mathcal{L}_p} |\tilde{a}_{i_1 - l_1}\tilde{a}_{i_2 - l_2}\tilde{a}_{i_3 - l_3}\tilde{a}_{i_4 - l_4}| \mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4}) \le \begin{cases} \mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2})\mathrm{E}(\hat{\delta}_{1i_3}\hat{\delta}_{1i_4}), & p = 1 \\ \mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_3})\mathrm{E}(\hat{\delta}_{1i_2}\hat{\delta}_{1i_4}), & p = 2 \\ \mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}), & p = 3 \end{cases}. \tag{109}$$

From Lemma 5.1, we have for $1 \le p \le q \le 4$,

$$\begin{aligned} & |\mathrm{E}(\hat{\delta}_{1i_p}\hat{\delta}_{1i_q})| \\ \le\ & C \sum_{l=-\lfloor nh \rfloor - 1}^{\bar{M}\lfloor nh \rfloor - (i_q - i_p)} |\tilde{a}_l \tilde{a}_{l + i_q - i_p}| \\ \le\ & C \sum_{l=-\lfloor nh \rfloor - 1}^{i_p - i_q - 1} \frac{1}{(nh)^{2-2d_1}} I\{i_p - i_q \ge -\lfloor nh \rfloor\} \\ & + C \sum_{l=(-\lfloor nh \rfloor - 1)\vee(i_p - i_q)}^{0} \frac{1}{(nh)^{1-d_1}(l + i_q - i_p + 1)^{1-d_1}} \\ & + C \sum_{l=1}^{\bar{M}\lfloor nh \rfloor - (i_q - i_p)} \frac{1}{(l+1)^{1-d_1}(l + i_q - i_p + 1)^{1-d_1}} \\ \le\ & C\Big\{\frac{1}{(nh)^{1-2d_1}} + \frac{1}{(i_q - i_p + 1)^{1-2d_1}}\Big\} \end{aligned} \tag{110}$$

In the case of $\mathcal{L}_4$, we have

$$\sum_{l=i_4 - \bar{M}\lfloor nh \rfloor}^{i_1 + \lfloor nh \rfloor + 1} |\tilde{a}_{i_1 - l}\tilde{a}_{i_2 - l}\tilde{a}_{i_3 - l}\tilde{a}_{i_4 - l}| \mathrm{E}(\zeta_1^4) \tag{111}$$

$$\leq \; C\Big\{\frac{1}{(nh)^{3-4d_1}} + \frac{1}{(nh)^{2-3d_1}(i_4 - i_3 + 1)^{1-2d_1}}$$
$$+\frac{1}{(nh)^{1-2d_1}(i_4 - i_2 + 1)^{1-2d_1}(i_3 - i_2 + 1)^{1-2d_1}}$$
$$+\frac{1}{(i_4 - i_1 + 1)^{1-2d_1}(i_3 - i_1 + 1)^{1-2d_1}(i_2 - i_1 + 1)^{1-2d_1}}\Big\}$$

(111) is rather complicated because $\tilde{a}_l$ is not equal to 0 for $-\lfloor nh \rfloor - 1 \leq l < 0$. We verify (111).

The LHS of (111) is rewritten as

$$\sum_{l=-\lfloor nh \rfloor - 1}^{\bar{M}\lfloor nh \rfloor + i_1 - i_4} |\tilde{a}_l \tilde{a}_{l+i_2-i_1} \tilde{a}_{l+i_3-i_1} \tilde{a}_{l+i_4-i_1}| \mathrm{E}(\zeta_1^4).$$

We define 5 subsets of the set of all integers, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, and $\mathcal{A}_5$, by

$$\mathcal{A}_1 = \{l \,|\, 0 \leq l \leq \bar{M}\lfloor nh \rfloor + i_1 - i_4\},$$
$$\mathcal{A}_2 = \{l \,|\, (i_1 - i_2) \vee (-\lfloor nh \rfloor - 1) \leq l < 0\},$$
$$\mathcal{A}_3 = \{l \,|\, (i_1 - i_3) \vee (-\lfloor nh \rfloor - 1) \leq l < (i_1 - i_2) \vee (-\lfloor nh \rfloor - 1)\},$$
$$\mathcal{A}_4 = \{l \,|\, (i_1 - i_4) \vee (-\lfloor nh \rfloor - 1) \leq l < (i_1 - i_3) \vee (-\lfloor nh \rfloor - 1)\},$$
$$\mathcal{A}_5 = \{l \,|\, -\lfloor nh \rfloor - 1 \leq l < (i_1 - i_4) \vee (-\lfloor nh \rfloor - 1)\}.$$

Then we have

$$\{l \,|\, -\lfloor nh \rfloor - 1 \leq l \leq \bar{M}\lfloor nh \rfloor + i_1 - i_4\} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \cup \mathcal{A}_5.$$

By using Lemma 5.1 and carrying out some calculations, we obtain

$$\sum_{\mathcal{A}_1} |\tilde{a}_l \tilde{a}_{l+i_2-i_1} \tilde{a}_{l+i_3-i_1} \tilde{a}_{l+i_4-i_1}|$$
$$\leq \frac{C}{(i_4 - i_1 + 1)^{1-2d_1}(i_3 - i_1 + 1)^{1-2d_1}(i_2 - i_1 + 1)^{1-2d_1}},$$
$$\sum_{\mathcal{A}_2} |\tilde{a}_l \tilde{a}_{l+i_2-i_1} \tilde{a}_{l+i_3-i_1} \tilde{a}_{l+i_4-i_1}|$$
$$\leq \frac{C}{(nh)^{1-2d_1}(i_4 - i_2 + 1)^{1-2d_1}(i_3 - i_2 + 1)^{1-2d_1}},$$
$$\sum_{\mathcal{A}_3} |\tilde{a}_l \tilde{a}_{l+i_2-i_1} \tilde{a}_{l+i_3-i_1} \tilde{a}_{l+i_4-i_1}|$$

$$\leq \frac{C}{(nh)^{2-3d_1}(i_4 - i_3 + 1)^{1-2d_1}},$$

$$\sum_{\mathcal{A}_4} |\tilde{a}_l \tilde{a}_{l+i_2-i_1} \tilde{a}_{l+i_3-i_1} \tilde{a}_{l+i_4-i_1}|$$

$$\leq \frac{C}{(nh)^{3-4d_1}},$$

$$\sum_{\mathcal{A}_5} |\tilde{a}_l \tilde{a}_{l+i_2-i_1} \tilde{a}_{l+i_3-i_1} \tilde{a}_{l+i_4-i_1}|$$

$$\leq \frac{C}{(nh)^{3-4d_1}},$$

where $\sum_{\mathcal{A}_s}$ means the summation over $\mathcal{A}_s$. Hence (111) follows from the above inequalities. In (111), some of $1-2d_1$ can replaced with $1-d_1$. However, $\xi_j^{1-2d_1}$ is more tractable in dealing with the integration over the regions containing 0.

We do not give the inequalities for $\hat{\epsilon}_{1i}$ such as (110) and (111) explicitly since they are trivial from (110) and (111).

Recalling (97) and (98), we consider $\mathcal{T}(1,p,q)$. By exploiting (109)–(111), we can bound $\mathcal{T}(1,p,q)$ by $Cl_n^2(nh)^{4(d_1+d_2)}$ for every pair of $\{(p,q)\,|\,1 \leq p,q \leq 4\}$. In fact we have for $(1,1)$,

$$\begin{aligned}
&\mathcal{T}(1,1,1) \qquad (112)\\
&\leq Cl_n nh(nh)^{4(d_1+d_2)-4}(nh)^3 \int_{\Omega_{21}} \Big(1+\frac{1}{\xi_1^{1-2d_1}}\Big)\Big(1+\frac{1}{\xi_3^{1-2d_1}}\Big)\\
&\qquad \times\Big(1+\frac{1}{\xi_1^{1-2d_2}}\Big)\Big(1+\frac{1}{\xi_3^{1-2d_2}}\Big)d\xi_1 d\xi_2 d\xi_3\\
&\leq Cl_n^2(nh)^{4(d_1+d_2)},
\end{aligned}$$

where $\Omega_{21} = \{(\xi_1,\xi_2,\xi_3)^T\,|\,0 \leq \xi_1 \leq \bar{M}+1,\ 0 \leq \xi_2 \leq l_n,\ 0 \leq \xi_3 \leq \bar{M}+1\}$. In the RHS of (112), $l_n nh$ and $(nh)^3$ come from i_1 and $d\xi_1 d\xi_2 d\xi_3$, respectively .

The other pairs can be treated almost in the same way and $\mathcal{T}(1,p,q)$ of those pairs are bounded by $Cl_n^2(nh)^{4(d_1+d_2)}$. Note that we have to be careful in dealing the pairs involving $\mathcal{L}_4$ and $\mathcal{J}_4$. The details are omitted. Hence we have shown that

$$\sum_{\mathcal{S}_1} |\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\epsilon}_{1i_1}\hat{\epsilon}_{1i_2}\hat{\epsilon}_{1i_3}\hat{\epsilon}_{1i_4})| \leq Cl_n^2(nh)^{4(d_1+d_2)}. \qquad (113)$$

We go on to $\mathcal{S}_2$. Since we can deal with $\mathcal{S}_3$ and $\mathcal{S}_4$ in the same way, the details about $\mathcal{S}_3$ and $\mathcal{S}_4$ are omitted. We give only necessary upper bounds in the case of $\mathcal{L}_4$ for $\mathcal{S}_3$ and $\mathcal{S}_4$ at the end of the proof.

We consider the summation of $|\mathrm{E}(\hat{\delta}^2_{1i_1}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\epsilon}^2_{1i_1}\hat{\epsilon}_{1i_3}\hat{\epsilon}_{1i_4})|$ over $\mathcal{S}_2$. We give expressions such as (109)-(111) only to

$$\mathrm{E}(\hat{\delta}^2_{1i_1}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4}) = \sum_{l_1,l_2,l_3,l_4} \tilde{a}_{i_1-l_1}\tilde{a}_{i_1-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4})$$

since those of $\hat{\epsilon}_{1i}$ are similar to those of $\hat{\delta}_{1i}$. We put $\tilde{a}_{i_p-l_p}=0$ when i_p-l_p is not included in the definition of $\hat{\delta}_{2i_p}$.

We have for $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$,

$$\sum_{\mathcal{L}_p} |\tilde{a}_{i_1-l_1}\tilde{a}_{i_1-l_2}\tilde{a}_{i_3-l_3}\tilde{a}_{i_4-l_4}|\mathrm{E}(\zeta_{l_1}\zeta_{l_2}\zeta_{l_3}\zeta_{l_4}) \le \begin{cases} \mathrm{E}(\hat{\delta}^2_{1i_1})\mathrm{E}(\hat{\delta}_{1i_3}\hat{\delta}_{1i_4}), & p=1 \\ \mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_3})\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_4}), & p=2 \\ \mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_3}), & p=3 \end{cases}. \quad (114)$$

In the case of $\mathcal{L}_4$, we have

$$\begin{aligned} &\sum_{l=-\infty}^{\infty} |\tilde{a}^2_{i_1-l}\tilde{a}_{i_3-l}\tilde{a}_{i_4-l}|\mathrm{E}(\zeta_1^4) \qquad (115)\\ &\le C \sum_{l=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor+i_1-i_4} |\tilde{a}^2_l\tilde{a}_{l+i_3-i_1}\tilde{a}_{l+i_4-i_1}| \\ &\le C\Big\{\frac{1}{(nh)^{3-4d_1}} + \frac{1}{(nh)^{2-3d_1}(i_4-i_3+1)^{1-2d_1}} \\ &\quad + \frac{1}{(i_3-i_1+1)^{1-2d_1}(i_4-i_1+1)^{1-2d_1}}\Big\}. \end{aligned}$$

By exploiting (110), (114), (115), and those of $\hat{\epsilon}_{1i}$, we evaluate $\mathcal{T}(2,p,q)$ for every pair of $\{(p,q)\,|\,1\le p,q\le 4\}$. We have for $(1,1)$,

$$\begin{aligned} &\mathcal{T}(2,1,1) \qquad (116)\\ &\le C l_n nh(nh)^{2(d_1+d_2)-2}(nh)^2\int_{\Omega_{22}}\Big(1+\frac{1}{\xi_3^{1-2d_1}}\Big)\Big(1+\frac{1}{\xi_3^{1-2d_2}}\Big)d\xi_2 d\xi_3 \\ &\le C l_n^2 (nh)^{4(d_1+d_2)}. \end{aligned}$$

where $\Omega_{22} = \{(\xi_2,\xi_3)^T\,|\,0\le\xi_2\le l_n,\ 0\le\xi_3\le\bar{M}+1\}$. In the RHS of (116), $l_n nh$ and $(nh)^2$ come from i_1 and $d\xi_2 d\xi_3$, respectively. Recall that $d_1+d_2>1/2$.

The other pairs can be treated almost in the same way and $\mathcal{T}(1,p,q)$ of those pairs are bounded by $Cl_n^2(nh)^{4(d_1+d_2)}$. We have to be careful in dealing the pairs involving $\mathcal{L}_4$ and $\mathcal{J}_4$. The details are omitted. Hence we have shown that

$$\sum_{\mathcal{S}_2} |\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\epsilon}_{1i_1}\hat{\epsilon}_{1i_2}\hat{\epsilon}_{1i_3}\hat{\epsilon}_{1i_4})| \le Cl_n^2(nh)^{4(d_1+d_2)}. \tag{117}$$

We omit the details about $\mathcal{S}_3$ and $\mathcal{S}_4$ by just giving upper bounds for $\mathcal{L}_4$.

We have for $\mathcal{S}_3$,

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty} |\tilde{a}_{i_1-l}\tilde{a}_{i_2-l}^2\tilde{a}_{i_4-l}|\mathrm{E}(\zeta_1^4) \qquad (118)\\
\le\ & C\sum_{l=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor+i_1-i_4} |\tilde{a}_l\tilde{a}_{l+i_2-i_1}^2\tilde{a}_{l+i_4-i_1}| \\
\le\ & C\Big\{\frac{1}{(nh)^{3-4d_1}} + \frac{1}{(nh)^{1-d_1}(i_4-i_2+1)^{1-2d_1}} \\
&+\frac{1}{(i_2-i_1+1)^{1-2d_1}(i_4-i_1+1)^{1-2d_1}}\Big\}.
\end{aligned}$$

We have for $\mathcal{S}_4$,

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty} |\tilde{a}_{i_1-l}\tilde{a}_{i_2-l}\tilde{a}_{i_3-l}^2|\mathrm{E}(\zeta_1^4) \qquad (119)\\
\le\ & C\sum_{l=-\lfloor nh\rfloor-1}^{\bar{M}\lfloor nh\rfloor+i_1-i_3} |\tilde{a}_l\tilde{a}_{l+i_2-i_1}\tilde{a}_{l+i_3-i_1}^2| \\
\le\ & C\Big\{\frac{1}{(nh)^{1-d_1}} + \frac{1}{(i_2-i_1+1)^{1-2d_1}(i_3-i_1+1)^{1-2d_1}}\Big\}.
\end{aligned}$$

Hence the proof of Lemma 5.7 is complete. □

Proof of Lemma 5.8. We only outline the proof of the former expression since the proofs of both of the expressions are similar to those of Lemmas 5.6 and 5.7. We use the notation such as $\mathcal{S}_r$, $\mathcal{L}_p$, $\mathcal{J}_q$, $\mathcal{T}(r,p,q)$, and Ω_{pq}.

We should consider the summation of

$$|\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})|$$

over $\mathcal{S}_j$, $j=1,\dots,8$. We define ξ_1, ξ_2, and ξ_3 as in (108) and use the upper bounds of $|\mathrm{E}(\hat{\delta}_{1i_1}\hat{\delta}_{1i_2}\hat{\delta}_{1i_3}\hat{\delta}_{1i_4})|$ and $|\mathrm{E}(\hat{\epsilon}_{2i_1}\hat{\epsilon}_{2i_2}\hat{\epsilon}_{2i_3}\hat{\epsilon}_{2i_4})|$ derived in the proofs of Lemmas 5.6 and 5.7.

We can treat the cases of $\mathcal{S}_5$–$\mathcal{S}_8$ in the same way as before and omit the details.

When we consider $\mathcal{S}_1$, we have

$$\begin{aligned} &\mathcal{T}(1,1,1) \qquad (120)\\ &\le Cl_n nh(nh)^{4(d_1+d_2)-4}(nh)^3\int_{\Omega_{21}}\Big\{\Big(1+\frac{1}{\xi_1^{1-2d_1}}\Big)\Big(1+\frac{1}{\xi_3^{1-2d_1}}\Big)\\ &\quad\times\int_{\bar M}^{\infty}\frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_1)^{3-d_2}}\int_{\bar M}^{\infty}\frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_3)^{3-d_2}}\Big\}d\xi_1 d\xi_2 d\xi_3\\ &\le Cl_n^2(nh)^{4(d_1+d_2)}, \end{aligned}$$

where $l_n nh$ and $(nh)^3$ come from i_1 and $d\xi_1 d\xi_2 d\xi_3$, respectively.

We can derive similar inequalities for the other pairs of $\{(p,q)\,|\,1\le p,q\le 4\}$ and each of $\mathcal{T}(1,p,q)$ is bounded by $Cl_n^2(nh)^{4(d_1+d_2)}$. Hence we have shown that

$$\sum_{\mathcal{S}_1}|\mathrm{E}(\hat\delta_{1i_1}\hat\delta_{1i_2}\hat\delta_{1i_3}\hat\delta_{1i_4})\mathrm{E}(\hat\epsilon_{2i_1}\hat\epsilon_{2i_2}\hat\epsilon_{2i_3}\hat\epsilon_{2i_4})|\le Cl_n^2(nh)^{4(d_1+d_2)}.$$

In the case of $\mathcal{S}_2$, we have

$$\begin{aligned} &\mathcal{T}(2,1,1) \qquad (121)\\ &\le Cl_n nh(nh)^{d_1+4d_2-3}(nh)^2\int_{\Omega_{22}}\Big\{\Big(1+\frac{1}{\xi_3^{1-2d_1}}\Big)\\ &\quad\times\int_{\bar M}^{\infty}\frac{d\xi_b}{\xi_b^{6-2d_2}}\int_{\bar M}^{\infty}\frac{d\xi_b}{\xi_b^{3-d_2}(\xi_b+\xi_3)^{3-d_2}}\Big\}d\xi_2 d\xi_3\\ &\le Cl_n^2(nh)^{4(d_1+d_2)}, \end{aligned}$$

where $l_n nh$ and $(nh)^2$ come from i_1 and $d\xi_2 d\xi_3$, respectively.

We can derive similar inequalities for the other pairs of $\{(p,q)\,|\,1\le p,q\le 4\}$ and each of $\mathcal{T}(2,p,q)$ is bounded by $Cl_n^2(nh)^{4(d_1+d_2)}$. Hence we have shown that

$$\sum_{\mathcal{S}_2}|\mathrm{E}(\hat\delta_{1i_1}\hat\delta_{1i_2}\hat\delta_{1i_3}\hat\delta_{1i_4})\mathrm{E}(\hat\epsilon_{2i_1}\hat\epsilon_{2i_2}\hat\epsilon_{2i_3}\hat\epsilon_{2i_4})|\le Cl_n^2(nh)^{4(d_1+d_2)}.$$

We can proceed in the same way in the cases of $\mathcal{S}_3$ and $\mathcal{S}_4$. Hence the proof of Lemmas 5.8 is complete.

□

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