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CONSENSUS BY PASSIONATE COMMUNICATION

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Abstract

This paper presents the communication process of passionate belief messages according to a protocol, and then to show that, by communication among players, the posteriors for a given event must be equal among them even if they have asymmetric information. In our setting, the players may send non-partitional messages based on their beliefs. We show that players can obtain the same posteriors under the communication by such noisy information.

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1 Introduction

People tend to believe their own considerations with extravagance and to take others' ones calmly. Indeed, consider a race or a contest where people anticipate who/which wins. Each person with asymmetric information conceives a various estimation or a different probability that every candidate wins. Each person will be confident in her/his own expectations, but take others' ones calmly. This paper shows that the communication of such confident expectations leads to a consensus: People with calm consideration for others' estimations obtain the identical probability for a fixed occurrence even through passionate communication.

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The literature of communication to lead players' information to symmetry was begun in order to address Aumann's agreement theorem [1]; if all players commonly know their posteriors of an event, then their posteriors must be identical. However, he did not address the issue of how common knowledge of players' posteriors is achieved. Geanakoplos and Polemarchakis [4] first address this issue by introducing a communication process between two players in which the players announce their posteriors to each other. In the process players can receive messages and revise their posteriors according to what they learn from the messages. As a result of this communication process, they come to a consensus on their posteriors. This implies that by the communication process their posteriors will eventually become common knowledge among them.

The model of Geanakoplos and Polemarchakis consists of only two players. In extending this framework to any finite number of players, Cave [3] considered communication among finitely many players. In his setting every player receives all messages that any one of them sends out. It is very similar to auctions in which a price announced by any one of the players is heard simultaneously by each and every player. It was shown that the posteriors of every player must become identical. However, a natural question suggests itself: Would it be possible for players to reach a consensus on posteriors even if communications among players do not simultaneously take place but only in pairwise manners? Parikh and Krasucki [10] first analyzed a case of pairwise communication among more than two players, and exhibited a communication protocol where players will eventually reach a consensus on posteriors. Our research in this paper is a development along the line of the work by Parikh and Krasucki.

In all of the models we referred above, both the knowledge of players and messages sent by them are represented by partitions of the state space. In the case of partitional messages, players always send the exact values of their posteriors for an occurrence in their messages each other. However, these messages mean that players are not emotional but are always rational. Nevertheless, we tend to be emotional, especially tend to believe ourselves as stated above. In our setting, players believe their own beliefs with extravagant while they have a kind of rationality for others; they are passionate for their beliefs and are calm for others'.

Ishikawa, Matsuhisa, and Akagawa (2001) first presented a communication process with belief messages. Players may send messages in non-partitional forms of the state space. They can revise their posteriors of an event as they receive messages from other players. Having revised their information after the receipt of a message from other player, they, in turn, send a new message to another player. The process is repeated infinitely many times. In this setting, since the messages sent by players may not be exact information, there is some difficulty to obtain same posteriors, that is, to reach a consensus. Hence, Ishikawa *et al.* required acyclic communication protocol as Krasucki [8]. The conclusion of this paper is that players will form a consensus on the posteriors of an event after an infinite round of communications according to a protocol independent of cyclicity even if messages sent out to players are in non-partitional forms.

We begin Section 2 by introducing the communication process with passionate messages. In Section 3 we prove Main Theorem and state some remarks.

2 Passionate Communication

2.1 Protocol

Let N be a set of finitely many *players* and i denote a player. We assume that players communicate with their opponents by sending *messages* over the time period $T = \{0, 1, \dots, \}$. A *protocol* defines a procedure of a communication process among players. For simplicity, we restrict our attention to protocols in which a recipient of a message at any time period immediately becomes a sender in the subsequent period. We impose one essential property on a protocol in our analysis of a communication process. This property will be referred as *repetitiveness.* It requires that every player must become a recipient (and a sender in the subsequent period) of a message repeatedly throughout the entire time periods. Let us state its definition formally.

Definition 1 *A protocol is a mapping, Pr of T into the product set $N \times N$, that assigns to each $t \in T$ an ordered pair of players $Pr(t) = (i(t), j(t))$ such that $i(t) = j(t-1)$. It is said to be repetitive if, for every player i and for any $k \in T$, there is $t_k \in T$ such that $i = j(t_k)$ and $t_k \geqq k$.*

In this definition the value of the protocol at time t, $\Pr(t) = (i(t), j(t))$, is interpreted as a pair consisting of the *sender* and the *recipient* of a message at time t. A repetitive protocol assures that every player receives and sends out her message to another player repeatedly throughout the entire time periods. It implies that, for every $i \in N$ and for every $k \in T$, there exists some $t_{ik} \in T$ such that $i = j(t_{ik})$, and for any $j \in N$, there exists some $t_j \in T$ with $k \leqq t_j \leqq t_{jk}$ such that $j = j(t_j)$.

A *state-space* Ω is a non-empty finite set, whose members are called *states*. An *event* is a subset of the state-space. We denote by 2^{Ω} the field of all subsets of it. An event E is said to *have occurred* in a state ω if $\omega \in E$.

To represent player' information structure, we define a class of mappings $(\Pi_i)_{i \in N}$ such that $\Pi_i : \Omega \to 2^{\Omega}$ satisfies the three properties, *Reflectivity*, *Transitivity* and *Symmetry*:

(Ref) $\omega \in \Pi_i(\omega)$;
(Trn) $\xi \in \Pi_i(\omega)$ implies $\Pi_i(\xi) \subseteqq \Pi_i(\omega)$;
(Sym) $\xi \in \Pi_i(\omega)$ implies $\omega \in \Pi_i(\xi)$.

These three properties imply that Π_i is a partition of Ω. A player i for whom $\Pi_i(\omega) \subseteqq E$ knows in the state ω that some state in the event E has occurred. This is interpreted as the set of states where i thinks are *possible* when ω occurs. We call Π_i i's *information structure* and $\Pi_i(\omega)$ i's *possibility set* at ω.

In a repetitive protocol Pr as defined above, at every time period there is a player i who sends a message informing about an event X that she believes to have occurred with some probability. Let us formally define this communication process by specifying messages sent by players. Given an event X, the *passionate communication process* $\langle \mathrm{Pr}, (Q_i^t, \Pi_i^t, M_{i(t)}^t)_{(i,t) \in N \times T} \rangle$ associated with an event X is defined as a "revision process" of the posteriors of the event X. Define $(Q_i^t, \Pi_i^t, q_i^t, M_{i(t)}^t)$ inductively as follows:

At time $\tau = 0$ Set, for all i, $Q_i^0 = \Pi_i$, $\Pi_i^0 = \Pi_i$, and $q_i^0(\omega) = \mu(X|\ \Pi_i^0(\omega))$ for each ω. Define the message $M_{i(0)}^0$ sent by $i(0)$ as a mapping from Ω into 2^{Ω} such that, for each $\omega \in \Omega$,

$$M_{i(0)}^0(\omega) = \{\xi|\ \mu(X|\ \Pi_{i(0)}^0(\xi)) \geq q_{i(0)}^0(\omega)\}.$$

At time $\tau \geqq 1$ Given that $(Q_i^\tau, \Pi_i^\tau, q_i^\tau, M_{i(\tau)}^\tau)$ are defined for all i up to $\tau = t-1$, we define for $\tau = t$ and all i,

$$Q_i^t(\omega) = \begin{cases} Q_i^{t-1}(\omega) \cap M_{i(t-1)}^{t-1}(\omega) & \text{if } i = j(t-1) \\ Q_i^{t-1}(\omega) & \text{if } i \neq j(t-1) \end{cases}$$

for every ω. Q_i^t represents the information learned by player i from the message $M_{i(t-1)}^{t-1}$ when i is its recipient at time t. After receiving a non-partitional information Q_i^t, player i deduces from it a consistent information [1] expressed

[1] Note that a non-partitional information Q_i^t contains inconsistency: Although player i can know the finer information at a state, his information does not re-

by a partition Π_i^t. For each i, Π_i^t is defined as a partition generated by Q_i^t:

$$\Pi_i^t(\omega) := \{\xi \in \Omega \mid Q_i^t(\xi) = Q_i^t(\omega)\}$$

for every ω. Then, define, for each i, q_i^t which represents the probabilistic assessment by player i of the relative likelihood of the occurrence of the event X conditional upon the player's information Π_i^t: For each ω,

$$q_i^t(\omega) = \mu(X \mid \Pi_i^t(\omega)).$$

$M_{i(t)}^t$, which represents a message sent by player $i(t)$, is defined by

$$M_{i(t)}^t(\omega) = \{\xi \mid \mu(X \mid \Pi_{i(t)}^t(\xi)) \geqq q_{i(t)}^t(\omega)\}$$

for every ω.

In the above process, player $i(t)$'s message $M_{i(t)}^t(\omega)$ at ω is specified as the set of states in which the sender $i(t)$ believes at ω that the event X must have occurred with probability at least equal to $q_{i(t)}^t(\omega)$. Though he obtains the exact value of his posterior, he believes that the event must have occurred with probability more than the exact value and sends the message. Since players' messages take a form of their excessive 'beliefs' on the occurrence of an event, we call the communication process defined above as a *passionate communication*. This belief message is much similar to p-belief Monderer and Samet (1989). However, while $p \in [0, 1]$ in their p-belief is always fixed, our beliefs in the messages can be revised according to players' information.

Furthermore players in our setting have a kind of rationality for others. This is because player i every time t generates a partition Π_i^t from Q_i^t. A non-partition information Q_i^t contains inconsistency as stated in Footnote 1. Nevertheless, players can refine messages sent from others by their own information: They are calm for the others' considerations.

2.3 Consensus

In a passionate communication process, $Q_i^t(\omega)$ are not empty for any time $t \in T$ as we have $\omega \in Q_i^t(\omega)$ for each i and $\omega \in \Omega$: Since $\omega \in \{\xi \mid \mu(X \mid \Pi_i^t(\xi)) = q_{i(t)}^t(\omega)\} \subseteq M_{i(t)}^t(\omega)$ for every t, and $Q_i^0(\omega) = \Pi_i(\omega) \ni \omega$, the intersection between $Q_i^t(\omega)$ and $M_{i(t)}^t(\omega)$ always has ω for every player i. Thus $\Pi_i^t(\omega)$ is also non-empty for any t, i, and ω. Therefore, $\{q_i^t\}_{t=0}^{\infty}$ is well-defined for any player i. If there exists some $\hat{t} \in T$ such that, for every player $i \in N$, and any state $\omega \in \Omega$, $q_i^t(\omega) = q_i^{\hat{t}}(\omega)$ at time $t \geqq \hat{t}$, then we denote $q_i^{\infty}(\omega) := q_i^{\hat{t}}(\omega)$. We

flect it another state. That is, when player i can obtain the information $Q_i(\xi)$ and $Q_i(\omega)$ such that $Q_i(\xi) \subsetneqq Q_i(\omega)$, $Q_i(\omega)$ does not reflect the finer information $Q_i(\xi)$.

say that a *consensus* on the assignment of posteriors of X is reached in the communication process if $q_i^\infty(\omega) = q_j^\infty(\omega)$ for any pair of players i, j and in any state ω.

3 Result and Remarks

3.1 Main Theorem

Our main theorem can formally be stated as follows.

Theorem 1 *Given any event, the passionate communication process associated with the event leads to a consensus among all the players on the assessment of the relative likelihood of its occurrence: There exists $t^* \in T$ and a probability $q^\infty : \Omega \to [0,1]$ such that, for each $\omega \in \Omega$, for any $t \geqq t^*$, and for every $i \in N$,*

$$q_i^\infty(\omega) = q^\infty(\omega).$$

In the passionate communication process, since players' messages are not partitions on the state space, each recipient cannot know the exact value at the state in the sent message. Ishikawa, *et al.* (2001) could not prove to reach the consensus with *cyclic* protocols. This is because the sent messages includes some states where a sender believes the probability of the occurrence without substance. In the following proof, we can show that such players' messages can be decomposed into each exact value of possible assignment.

Though we can prove the above theorem under the messages based on decision functions such as Parikh and Krasucki (1990) and Krasucki (1996), it is difficult to interpret it as human behaviors.

3.2 Proof of Main Theorem

Given an event X, let $\langle \Pr, (Q_i^t, \Pi_i^t, M_{i(t)}^t)_{(i,t)\in N\times T}\rangle$ be the passionate communication process associated with X. As defined in §2.2 (p. 5), for any state $\omega \in \Omega$, for every $t \in T$, and for each $i \in N$, we have

$$Q_i^t(\omega) = \begin{cases} Q_i^{t-1}(\omega) \cap M_{i(t)}^{t-1}(\omega) & \text{if } i = j(t-1) \\ Q_i^{t-1}(\omega) & \text{if } i \neq j(t-1). \end{cases}$$

It immediately follows that $Q_i^{t-1}(\omega) \supseteqq Q_i^t(\omega)$ for any state ω, for every t, and for each player i. We shall prove the following claim:

Claim 1: For any state ω and for each player i, there exists some t^* such that for any $t \geqq t^*$, we have

$$Q_i^t(\omega) = Q_i^{t^*}(\omega).$$

Suppose not; then, there exists a subsequence $\{Q_i^{t_k}(\omega)\}_{k=1}^{\infty}$ of the sequence $\{Q_i^t(\omega)\}_{t=0}^{\infty}$ for some state ω and a player i satisfying $Q_i^{t_k}(\omega) \neq Q_i^{t_{k+1}}(\omega)$ for all k. Now let $|E|$ be the cardinality of a set E. Since $\{Q_i^t(\omega)\}_{t=0}^{\infty}$ is a decreasing sequence, we have

$$|Q_i^{t_k}(\omega)| - 1 \geqq |Q_i^{t_{k+1}}(\omega)|$$

for all k. Set $k^* = |\Omega| + 2$; then,

$$\begin{aligned} 0 \leqq |Q_i^{t_k^*}(\omega)| &\leqq |Q_i^{t_{k^*-1}}(\omega)| - 1 \\ &\leqq |Q_i^{t_{k^*-2}}(\omega)| - 2 \\ &\vdots \\ &\leqq |Q_i^{t_1}(\omega)| - (\Omega + 1) < 0, \end{aligned}$$

a contradiction. This proves the above claim.

For the rest of our proof, set

$$t^* = \max\{t_i^* \in T |\ \forall \omega \in \Omega,\ \forall t \geqq t_i^*,\ Q_i^t(\omega) = Q_i^{t_i^*}(\omega)\}.$$

Since we have $Q_i^t(\omega) = Q_i^{t^*}(\omega)$ for any ω, for each player i and for every $t \geqq t^*$, we define $Q_i^{\infty}, \Pi_i^{\infty}$, and q_i^{∞} by $Q_i^{\infty}(\omega) = Q_i^{t^*}(\omega)$, $\Pi_i^{\infty}(\omega) = \Pi_i^{t^*}(\omega)$, and $q_i^{\infty}(\omega) = q_i^{t^*}(\omega)$, respectively, for every $\omega \in \Omega$ and for every $i \in N$. To prove the theorem, we would like to show that we have $q_i^{\infty}(\omega) = q_j^{\infty}(\omega)$ for any state ω and for every pair i, j of players.

Note that $\Pi_{j(t)}^{t+1}(\omega) \subseteqq M_{i(t)}^t(\omega)$ for any ω and for every t. Indeed, let $\zeta \in \Pi_{j(t)}^{t+1}(\omega)$. By the definition of $\Pi_{j(t)}^{t+1}$,

$$\begin{aligned} \zeta \in \Pi_{j(t)}^{t+1}(\omega) &= \{\xi \in \Omega |\ Q_{j(t)}^{t+1}(\xi) = Q_{j(t)}^{t+1}(\omega)\} \\ &= \{\xi \in \Omega |\ Q_{j(t)}^{t}(\xi) \cap M_{i(t)}^t(\xi) = Q_{j(t)}^{t}(\omega) \cap M_{i(t)}^t(\omega).\} \end{aligned}$$

It therefore follows that we have

$$\zeta \in Q_{j(t)}^t(\zeta) \cap M_{i(t)}^t(\zeta) = Q_{j(t)}^t(\omega) \cap M_{i(t)}^t(\omega) \subseteqq M_{i(t)}^t(\omega).$$

We will say that q is a possible assignment by player i if there is a state ω such that $q = q_i^{\infty}(\omega)$. Let the set of all possible assignments by player $i \in N$ be $\{q_{i1}, q_{i2}, \ldots, q_{iK_i}\}$ with $q_{i1} > q_{i2} > \cdots > q_{iK_i}$. Given a possible assignment q by i, define

$$E_i(q) := \{\omega \in \Omega |\ \mu(X |\ \Pi_i^{\infty}(\omega)) = q\}.$$

Then we shall prove the following:

Claim 2: For every $t \geqq t^*$,

$$E_{i(t)}(q_{i(t)1}) = \bigcup_{\xi \in E_{i(t)}(q_{i(t)1})} \Pi^{\infty}_{j(t)}(\xi). \tag{1}$$

Let $\zeta \in E_{i(t)}(q_{i(t)1})$. Then $\zeta \in \Pi^{\infty}_{j(t)}(\zeta) \subseteqq \cup_{\xi \in E_{i(t)}(q_{i(t)1})} \Pi^{\infty}_{j(t)}(\xi)$.

Conversely, let $\xi^* \in E_{i(t)}(q_{i(t)1})$ and $\zeta \in \Pi^{\infty}_{j(t)}(\xi^*)$. Then

$$\begin{aligned} \zeta \in \Pi^{\infty}_{j(t)}(\xi^*) = \Pi^{t}_{j(t)}(\xi^*) \subseteq M^{t}_{i(t)}(\xi^*) &= M^{\infty}_{i(t)}(\xi^*) \\ &= \{\omega \in \Omega |\ \mu(X|\ \Pi^{\infty}_{i(t)}(\omega)) \geqq q^{\infty}_{i(t)}(\xi^*)\} \\ &= \{\omega \in \Omega |\ \mu(X|\ \Pi^{\infty}_{i(t)}(\omega)) = q_{i(t)1}\} \\ &= E_{i(t)}(q_{i(t)1}). \end{aligned}$$

This proves the above claim.

Let us summarize three properties of conditional probabilities that will be used in our subsequent arguments.

Property (i) For any disjoint sets $A_1, \cdots, A_K$ such that $\mu(X|A_k) = q$ for $1 \leqq k \leqq K$,

$$\mu(X|\ \bigcup_{k=1}^{K} A_k) = q. \text{ [2]}$$

Property (ii) For any disjoint sets $A_1, \cdots, A_K$, there exists some real numbers $\lambda_k \in [0, 1]$ for $1 \leqq k \leqq K$ such that $\sum_{k=1}^{K} \lambda_k = 1$, and

$$\mu(X|\ \bigcup_{k=1}^{K} A_k) = \lambda_1 \mu(X|\ A_1) + \cdots + \lambda_K \mu(X|\ A_K) \text{ [3]}$$

with $\lambda_k > 0$ for $\mu(A_k) > 0$.

Property (iii) For any disjoint sets $A_1, \cdots, A_K$, if $\mu(X| \cup_k A_k) = \max_k \mu(X|\ A_k)$, then

$$\mu(X|\ \cup_k A_k) = \mu(X|\ A_k) \quad \text{for any } k \text{ with } \mu(A_k) > 0.$$

Property (i) is implied by Property (ii). Property (iii) is also an immediate consequence of Property (ii). Property (ii) is proved as follows: For any disjoint sets $A_1, \cdots, A_K$,

[2] This property is called *union consistency* in Cave [3], and *the sure thing principle* in Bacharach [2].

[3] This property is called *convexity* in Parikh and Krasucki [10].

$$
\begin{aligned}
\mu(X|\bigcup_{k=1}^{K} A_k) &= \frac{\mu(X\cap(\cup_{k=1}^{K}A_k))}{\mu(\cup_{k=1}^{K}A_k)} \\
&= \frac{\mu(X\cap A_1)+\cdots+\mu(X\cap A_K)}{\mu(\cup_k A_k)} \\
&= \frac{\mu(A_1)}{\mu(\cup_k A_k)}\frac{\mu(X\cap A_1)}{\mu(A_1)}+\cdots+\frac{\mu(A_K)}{\mu(\cup_k A_k)}\frac{\mu(X\cap A_K)}{\mu(A_K)} \\
&= \frac{\mu(A_1)}{\mu(\cup_k A_k)}\mu(X|\ A_1)+\cdots+\frac{\mu(A_K)}{\mu(\cup_k A_k)}\mu(X|\ A_K).
\end{aligned}
$$

Set $\lambda_k = \mu(A_k)/\mu(\cup_l A_l)$ for $k \in \{1, \cdots, K\}$, and we obtain Property (ii).

Recall $(i(t), j(t))$ is a pair of a sender and a recipient at time t. Since the protocol is repetitive, each i becomes a recipient at some $t_i \geqq t^*$, that is, $i = j(t_i)$. Consider the sender $i(t^*)$ at time t^*. Again by the repetitiveness of the protocol, there is $\bar{t} \geqq t'$ such that $j(\bar{t}) = i(t^*)$. The sender at t^* is the recipient at $\bar{t}$, and all the players become a recipient at least once during the time period between t^* and $\bar{t}$.

Hereafter let (i^*, j^*) be $(i(t^*), j(t^*))$, respectively. Let us prove the following:

Claim 3: For any player i, we have $q_{i1} = q_{i^*1}$.

Since $E_{i(t)}(q_{i(t)1})$ is a disjoint union of $j(t)$'s partition for any $t \geqq t^*$ by Claim 2, we denote $E_{i(t)}(q_{i(t)1}) = \cup_l \Pi^{\infty}_{j(t)}(\xi_l)$. Notice that $q_{i1} = \mu(X|\ E_i(q_{i1}))$ for any $i \in N$ by Property (i). Due to Property (ii), there is $\{\lambda_l\}_l$ with $\lambda_l \in [0, 1]$ and $\sum_l \lambda_l = 1$ such that

$$
q_{i(t)1} = \mu(X|\ E_{i(t)}(q_{i(t)1})) = \sum_l \lambda_l \mu(X|\ \Pi^{\infty}_{j(t)}(\xi_l))
$$

for any $t \geqq t^*$. Let us consider i^* and j^*. Then there is a state $\xi_1 \in E_{i^*}(q_{i^*1})$ with $q_{i^*1} \leqq \mu(X|\ \Pi^{\infty}_{j^*}(\xi_1)) = q^{\infty}_{j^*}(\xi_1) \leqq q_{j^*1}$. Similarly, there eixists $\xi_2 \in E_{j^*}(q_{j^*1})$ with $q_{j^*1} \leqq \mu(X|\ \Pi^{\infty}_{j(t^*+1)}(\xi_2)) = q^{\infty}_{j(t^*+1)1}$. Thus we obtain

$$
q_{i^*1} \leqq q^{\infty}_{j^*}(\xi_1) \leqq q_{j^*1} \leqq q^{\infty}_{j(t^*+1)}(\xi_2) \leqq q_{j(t^*+1)1} \leqq \cdots \leqq q_{j(\bar{t})1} = q_{i^*1}.
$$

The last equality follows from $j(\bar{t}) = i(t^*) = i^*$. We therefore conclude that we have $q_{i1} = q_{i^*1}$ for any player i because all the players become a recipient at least once during the time period between t^* and $\bar{t}$. This proves Claim 3.

Note that the largest possible assignments q_{i1} by all players are identical by Claim 3. By considering Claim 2, we obtain from Property (iii)

$$
\mu(X|\ \Pi^{\infty}_{j(t)}(\omega)) = q_{j(t)1} \tag{2}
$$

for any $\omega \in E_{i(t)}(q_{i(t)1})$ and for every $t \geqq t^*$. Furthermore, it implies $E_{i(t)}(q_{i(t)1}) \subseteq$

$E_{j(t)}(q_{j(t)1})$ for every $t \geqq t^*$. Hence, we have

$$E_{i^*}(q_{i^*1}) \subseteq E_{j^*}(q_{j^*1}) \subseteq \cdots \subseteq E_{j(\bar{t})}(q_{j(\bar{t})1}) = E_{i^*}(q_{i^*1})$$

because $j(\bar{t}) = i(t^*) = i^*$. Therefore, we obtain

$$E_i(q_{i1}) = E_{i^*}(q_{i^*1}) \text{ for any player } i. \tag{3}$$

Set $\underline{K} = \min\{K_i |\ i \in N\}$ where K_i is the index of the i's smallest possible assignment. We now prove the following:

Claim 4: For any player $i \in N$ and for every k with $1 \leqq k \leqq \underline{K}$,

$$E_i(q_{ik}) = E_{i^*}(q_{i^*k}), \quad \text{and} \quad q_{ik} = q_{i^*k}.$$

First we prove the following fact by an induction argument on k.

Fact 1: For any $t \geqq t^*$ and for each k with $1 \leqq k \leqq \underline{K}$,

$$E_{i(t)}(q_{i(t)k}) = \bigcup_{\xi \in E_{i(t)}(q_{i(t)k})} \Pi^{\infty}_{j(t)}(\xi).$$

For $k = 1$, the assertion of the fact follows from Claim 2.

Now suppose that the assertion holds up to $k-1$ for all $t \geqq t^*$. Given $t \geqq t^*$, let $\omega \in E_{i(t)}(q_{i(t)k})$. Then,

$$\omega \in \bigcup_{\xi \in E_{i(t)}(q_{i(t)k})} \Pi^{\infty}_{j(t)}(\xi)$$

as we have $\omega \in \Pi^{\infty}_{j(t)}(\omega)$.

Conversely, let $\omega \in \Pi^{\infty}_{j(t)}(\xi)$ for some $\xi \in E_{i(t)}(q_{i(t)k})$. Then,

$$\omega \in \Pi^{\infty}_{j(t)}(\xi) \subseteq M^{\infty}_{i(t)}(\xi) = \{\zeta |\ \mu(X |\ \Pi^{\infty}_{i(t)}(\zeta)) \geqq q^{\infty}_{i(t)}(\xi) = q_{i(t)k}\}.$$

The last set is equal to $E_{i(t)}(q_{i(t)1}) \cup \cdots \cup E_{i(t)}(q_{i(t)k})$. It, in turn, is equal to

$$\left(\bigcup_{\xi_1 \in E_{i(t)}(q_{i(t)1})} \Pi^{\infty}_{j(t)}(\xi_1)\right) \bigcup \cdots \bigcup \left(\bigcup_{\xi_{k-1} \in E_{i(t)}(q_{i(t)(k-1)})} \Pi^{\infty}_{j(t)}(\xi_{k-1})\right) \bigcup E_{i(t)}(q_{i(t)k}).$$

This is by the induction hypothesis because $E_{i(t)}(q_{i(t)n}) = \cup_{\xi \in E_{i(t)}(q_{i(t)n})} \Pi^{\infty}_{j(t)}(\xi)$ for any n with $1 \leqq n \leqq k-1$. Thus we must have $\omega \in E_{i(t)}(q_{i(t)k})$ as otherwise we would have $\omega \in \Pi^{\infty}_{j(t)}(\xi) \subseteq E_{i(t)}(q_{i(t)n})$ for some n with $1 \leqq n \leqq k-1$, a contradiction.

Next we prove the following:

Fact 2: For any $t \geqq t^*$ and for any k with $1 \leqq k \leqq \underline{K}$, we have

$$E_{i(t)}(q_{i(t)k}) \subseteq E_{j(t)}(q_{j(t)k}).$$

We use an induction argument on k. For $k = 1$, we have already proved $E_{i(t)}(q_{i(t)1}) = E_{i^*}(q_{i^*1})$ for any player i by (3). Thus, in particular, we have $E_{i(t)}(q_{i(t)1}) = E_{j(t)}(q_{j(t)1})$ for any $t \geqq t^*$.

Now suppose that the assertion holds up to $k-1$ for any $t \geqq t^*$. If the assertion does not hold for k, then there is some $\hat{t} \geqq t^*$ such that

$$E_{i(\hat{t})}(q_{i(\hat{t})k}) \neq E_{j(\hat{t})}(q_{j(\hat{t})k}). \tag{4}$$

It implies that we have $q^{\infty}_{j(\hat{t})}(\hat{\omega}) \neq q_{j(\hat{t})k}$ for some $\hat{\omega} \in E_{i(\hat{t})}(q_{i(\hat{t})k})$. We have two cases:

(I) For some n with $1 \leqq n \leqq k-1$, $q^{\infty}_{j(\hat{t})}(\hat{\omega}) = q_{j(\hat{t})n}$.
(II) For some n with $k+1 \leqq n \leqq K_{j(\hat{t})}$, $q^{\infty}_{j(\hat{t})}(\hat{\omega}) = q_{j(\hat{t})n}$.

In case (I), by the induction hypothesis, we have $\hat{\omega} \in E_{j(\hat{t})}(q_{j(\hat{t})n}) = E_{i(\hat{t})}(q_{i(\hat{t})n})$ in contradiction to $\hat{\omega} \in E_{i(\hat{t})}(q_{i(\hat{t})k})$.

In case (II), let us note

$$q^{\infty}_{j(t)}(\omega) \leqq q_{j(t)k} \ \text{ for any } t \geqq t^* \text{ and for any } \omega \in E_{i(t)}(q_{i(t)k}). \tag{5}$$

Indeed, if $q^{\infty}_{j(t)}(\xi) > q_{j(t)k}$ for some $t \geqq t^*$ and for some $\xi \in E_{i(t)}(q_{i(t)k})$, we have $q^{\infty}_{j(t)}(\xi) = q_{j(t)n}$ for some n with $1 \leqq n \leqq k-1$. Then, by the induction hypothesis, $\xi \in E_{j(t)}(q^{\infty}_{j(t)}(\xi)) = E_{j(t)}(q_{j(t)n}) = E_{i(t)}(q_{i(t)n})$ in contradiction to $\xi \in E_{i(t)}(q_{i(t)k})$. Hence, we have $q^{\infty}_{j(t)}(\omega) \leqq q_{j(t)k}$ for any $\omega \in E_{i(t)}(q_{i(t)k})$ and for any $t \geqq t^*$.

By Fact 1 and Property (ii), there is $\{\lambda_l\}_l$ with $\lambda_l \in [0,1]$ and $\sum_l \lambda_l = 1$ such that

$$q_{i(t)k} = \mu(X|\ E_{i(t)}(q_{i(t)k})) = \sum_l \lambda_l \mu(X|\ \Pi^{\infty}_{j(t)}(\xi_l)) \tag{6}$$

for any $t \geqq t^*$. Let us consider i^* and j^*. Thus there is a state $\xi_1 \in E_{i^*}(q_{i^*k})$ with $q_{i^*k} \leqq \mu(X|\ \Pi^{\infty}_{j^*}(\xi_1)) = q^{\infty}_{j^*}(\xi_1) \leqq q_{j^*k}$. The last inequality is due to (5). Similarly, there is $\xi_2 \in E_{j^*}(q_{j^*k})$ with $q_{j^*k} \leqq \mu(X|\ \Pi^{\infty}_{j(t^*+1)}(\xi_2)) \leqq q^{\infty}_{j(t^*+1)k}$. By continuing the same arguments, we obtain

$$q_{i^*k} \leqq q^{\infty}_{j^*}(\xi_1) \leqq q_{j^*k} \leqq q^{\infty}_{j(t^*+1)}(\xi_2) \leqq q_{j(t^*+1)k} \leqq \cdots \leqq q_{j(\bar{t})k} = q_{i^*k}.$$

The last equality follows from $j(\bar{t}) = i(t^*) = i^*$. We therefore conclude that we have

$$q_{ik} = q_{i^*k} \text{ for any player } i \tag{7}$$

because all the players become a recipient at least once during the time period between t^* and $\bar{t}$.

It follows from (5), (7), and Property (iii) that we have

$$q_{j(t)k} = q^{\infty}_{j(t)}(\omega) \text{ for any } \omega \in E_{i(t)}(q_{i(t)k}) \text{ and for any } t \geqq t^*.$$

Thus, we have $E_{i(t)}(q_{i(t)k}) \subseteq E_{j(t)}(q_{j(t)k})$ for all $t \geqq t^*$. This proves Fact 2.

It now follows from Fact 2 by the repetitiveness of the protocol that we have:

$$E_{i^*}(q_{i^*k}) \subseteq E_{j^*}(q_{j^*k}) \subseteq \cdots \subseteq E_{j(\bar{t})}(q_{j(\bar{t})k}) = E_{i^*}(q_{i^*k})$$

as $j(\bar{t}) = i(t^*) = i^*$. Thus, we obtain $E_i(q_{ik}) = E_{i^*}(q_{i^*k})$ for any $i \in N$ which proves the first half of Claim 4. To complete its proof, we show the following:

Fact 3: For any player i and for each k such that $1 \leqq k \leqq \underline{K}$, we have

$$q_{ik} = q_{i^*k}.$$

We first show that, for any $t \geqq t^*$, and for each k with $1 \leqq k \leqq \underline{K}$, we have $q_{i(t)k} = q_{j(t)k}$. Suppose there are some $\hat{t} \geqq t^*$ and some k with $1 \leqq k \leqq \underline{K}$ such that $q_{i(\hat{t})k} \neq q_{j(\hat{t})k}$. Then, since $E_{i(\hat{t})}(q_{i(\hat{t})k}) = E_{j(\hat{t})}(q_{j(\hat{t})k})$ by the first half of Claim 4, we have $q_{i(\hat{t})k} \neq q^{\infty}_j(\omega)$ for any $\omega \in E_{i(\hat{t})}(q_{i(\hat{t})k})$. As Property (ii) and Fact 1 imply $q_{i(t)} = \sum_l \lambda_l \mu(X|\ \Pi_{j(t)}(\xi_l))$ for some $\{\lambda_l\}_l$, it follows that there are $\xi_{l'}, \xi_{l''} \in E_{i(\hat{t})}(q_{i(\hat{t})k})$ such that $q^{\infty}_{j(\hat{t})}(\xi_{l'}) < q_{i(\hat{t})k} < q^{\infty}_{j(\hat{t})}(\xi_{l''})$. This is a contradiction because $\xi_{l'}, \xi_{l''} \in E_{j(\hat{t})}(q_{j(\hat{t})k})$. Thus, the proof of Claim 4 is complete.

Recall $\underline{K} = \min\{K_i|\ i \in N\}$. As the last step, we prove the following:

Claim 5: $\underline{K} = \max\{K_i|\ i \in N\}$.

Let $j = \operatorname{argmin}\{K_i|\ i \in N\}$. We have $\Omega = \bigcup_{l=1}^{K_j} E_j(q_{jl})$. Thus, we must have $\Omega = \bigcup_{l=1}^{\underline{K}} E_i(q_{il})$ for each player $i \in N$ as $E_i(q_{ik}) = E_j(q_{jk})$ for any k with $1 \leqq k \leqq K_j = \underline{K}$ by Claim 4. Therefore, $E_i(q_{il})$ must be empty for all $l > \underline{K}$ because $\{E_i(q_{il})\}_l^{K_i}$ is a partition of Ω. Thus, it would induce a contradiction unless we have $K_i = K_j = \underline{K}$ for all $i \in N$. This proves Claim 5.

Finally, define $q^{\infty} : \Omega \to [0,1]$ by $q^{\infty}(\omega) = q^{\infty}_{i^*}(\omega)$ for all $\omega \in \Omega$. Then, given $i \in N$ and $\omega \in \Omega$, there is some k with $1 \leqq k \leqq \underline{K}$ such that $\omega \in E_i(q_{ik})$ as $\Omega = \bigcup_{k=1}^{\underline{K}} E_i(q_{ik})$. Since $E_i(q_{ik}) = E_{i^*}(q_{i^*k})$ by Claim 4, we obtain $q^{\infty}_i(\omega) = q_{i^*k} = q^{\infty}_{i^*}(\omega) = q^{\infty}(\omega)$. This completes the proof of the theorem.

4 Example

In this section we illustrate the communication process discussed in the above. Consider the communication among three players A, B, and C. The state space Ω is $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. They have their own initial information structure at time $t = 0$ as Figure 3. Let μ be the common prior on Ω as follows:

$$\mu(\omega_1) = \mu(\omega_2) = \frac{1}{8}, \quad \mu(\omega_3) = \frac{1}{4}, \quad \mu(\omega_4) = \frac{1}{2}.$$

Let the event X be $\{\omega_1\}$.

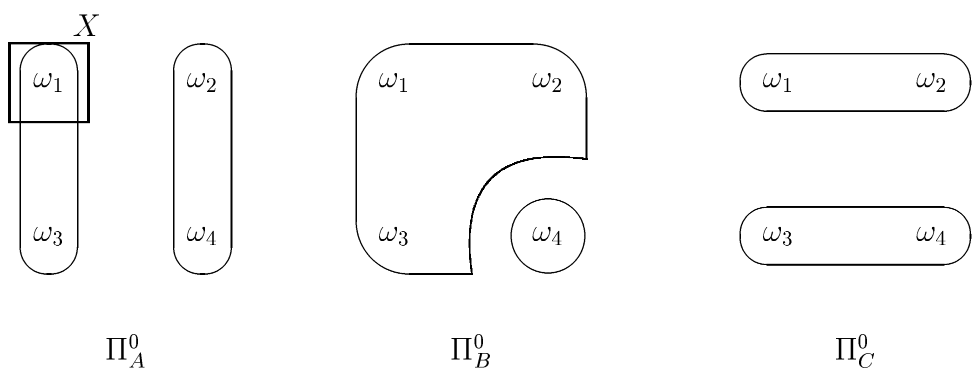

Fig. 3. Players' Information Structures.

The players communicate by sending the messages each other. Consider the communication protocol as shown in Figure 4. We assume that they make use of this repetitive protocol.

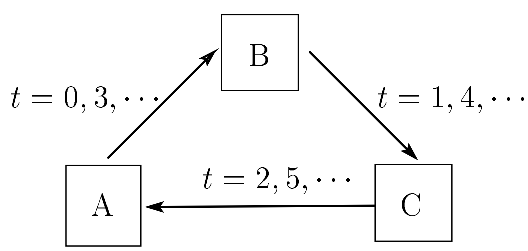

Fig. 4. The Repetitive Protocol for Communication.

First, A sends his message to B at time $t = 0$ as Table 4. B receives it and consists of his own partition as Figure 5. Since the other players A and C do not receive any information this time, their information structures do not change at all.

Table 4
A's message at time $t = 0$.

	ω_1	ω_2	ω_3	ω_4
q_A^0	$1/3$	0	$1/3$	0
M_A^0	$\{\omega_1, \omega_3\}$	Ω	$\{\omega_1, \omega_3\}$	Ω

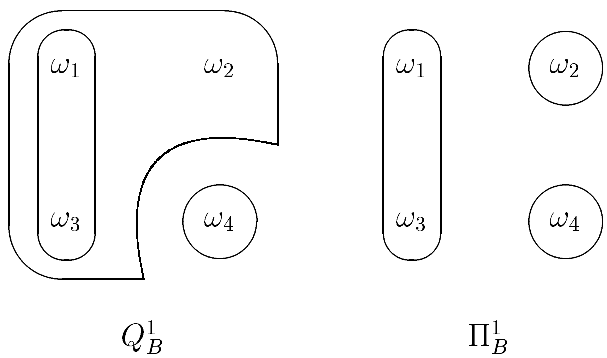

Fig. 5. B's information structure at time $t = 1$.

In the similar way, one player sends information and another player receives it each time while following the shown protocol. Table 5–9 and Figures 6–9 respectively show the sender's message and players' information structures each time after time 2.

Table 5
B's message at time $t = 1$.

	ω_1	ω_2	ω_3	ω_4
q_B^1	$1/3$	0	$1/3$	0
M_B^1	$\{\omega_1, \omega_3\}$	Ω	$\{\omega_1, \omega_3\}$	Ω

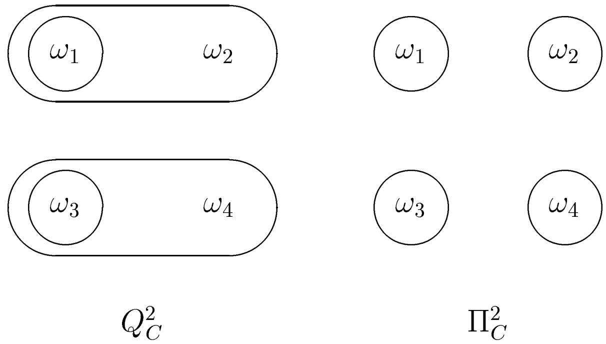

Fig. 6. C's information structure at time $t = 2$.

Table 6
C's message at time $t = 2$.

	ω_1	ω_2	ω_3	ω_4
q_C^2	1	0	0	0
M_C^2	$\{\omega_1\}$	Ω	Ω	Ω

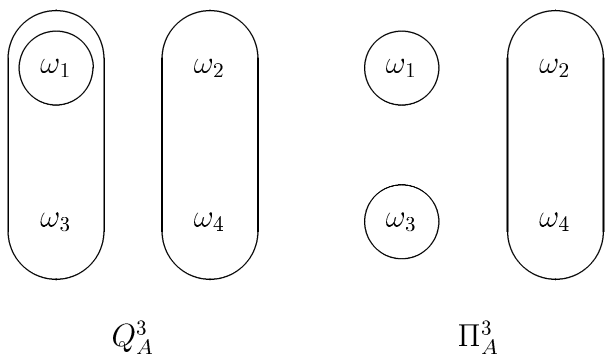

Fig. 7. A's information structure at time $t = 3$.

Table 7
A's message at time $t = 3$.

	ω_1	ω_2	ω_3	ω_4
q_A^3	1	0	0	0
M_A^3	$\{\omega_1\}$	Ω	Ω	Ω

See Table 9, and we can observe that consensus on the limiting values of the posteriors about X is guaranteed in the example. Figure 9 shows the stable

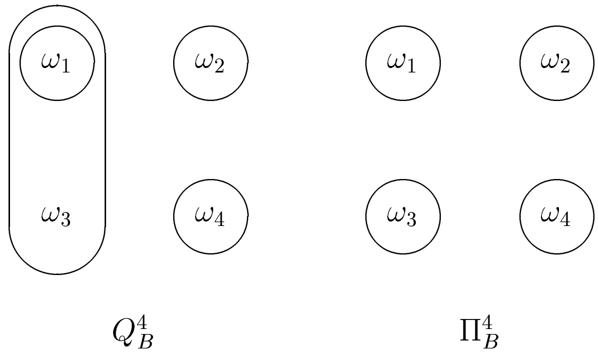

Fig. 8. B's information structure at time $t = 4$.

Table 8
B's message at time $t = 4$.

	ω_1	ω_2	ω_3	ω_4
q_B^4	1	0	0	0
M_B^4	$\{\omega_1\}$	Ω	Ω	Ω

Table 9
Players' messages at time $t \geqq 4$.

	ω_1	ω_2	ω_3	ω_4
q_A^∞	1	0	0	0
q_B^∞	1	0	0	0
q_C^∞	1	0	0	0
M_A^∞	$\{\omega_1\}$	Ω	Ω	Ω
W_B^∞	$\{\omega_1\}$	Ω	Ω	Ω
W_C^∞	$\{\omega_1\}$	Ω	Ω	Ω

information structures in reaching consensus.

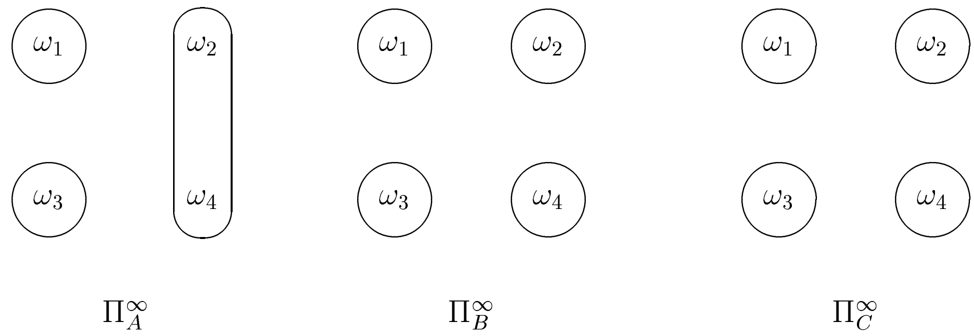

Fig. 9. Players' Information structures at time $t \geqq 4$.

5 Concluding Remarks

This paper investigates how players reach consensus in the p-belief communication. Though we consider communication pairwise, it can be also shown in the context of the communication among a part of players if only players make use of the round protocol. If we consider the case of the communication among all players, it is the same model of Cave [3].

In our model the following two are essential; (i) players evaluate their own beliefs of an event by the conditional probabilities, and (ii) they can generate partitions from their information. The first mentioned often in the above means that their decision functions, intended to be conditional probabilities, satisfy convexity and union consistency. Krasucki [8] pursued how players reach consensus without convexity. The second implies that their messages are disjoint unions of the element of their own generated partitions even if the messages are not partitions. Therefore we can apply the convexity.

Finally we remark on Heifetz [5]. He paid attention to Parikh and Krasucki's informal claim: Even when the players reach consensus, the consensus need not become common knowledge among them. Heifetz said that this claim is somewhat puzzling because the communication value is *formally* common knowledge in reaching consensus. To solve this puzzle, he introduced a broader space incorporating the time into the basic state space. He distinguished between the cases that players know who communicates whom and that they do not. One possibility in his model is that consensus is reached by mutual knowledge of the communication protocol[4]. Krasucki [8] introduced the *acyclic* protocol because the decision functions in his model do not satisfy (weakly) convex. We may prove that mutual knowledge of the protocol enables to reach consensus even on the cyclic one.

[4] This possibility was suggested by Professor Takashi Shimizu

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