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Abstract

We propose the concept of a *universal social ordering,* defined on the set of pairs of an allocation and a preference profile of any finite population. It is meant to unify evaluations and comparisons of social states with populations of possibly different sizes with various characteristics. The universal social ordering not only evaluates policy options for a given population but also compares social welfare across populations, as in international or intertemporal comparisons of living standards. It also makes it possible to evaluate policy options which affect the size of the population or the preferences of its members. We study how to extend the theory of social choice in order to select such orderings on a rigorous axiomatic basis. Key ingredients in this analysis are attitudes with respect to population size and the bases of interpersonal comparisons.

Keywords: social choice, universal social orderings, maximin principle, interpersonal comparisons.

JEL Classification Numbers: D63, D71.

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1 Introduction

Welfare economics and the theory of social choice, since Samuelson's (1947) and Arrow's (1951) seminal contributions, have mostly focused on the issue of evaluating social states and the impact of public policies for a given population with given preferences, over a domain of possible profiles of preferences. There are, however, other kinds of social evaluation that are often needed. For instance, the measurement of growth is often criticized for focusing on the volume of production and failing to accurately reflect the evolution of welfare, but measuring the evolution of welfare, especially over a long period of time, would require making comparisons of social welfare across populations with different size and different preferences. Similarly, international comparisons involve comparing the situations of countries with different populations. Moreover, policies which may affect the size of the population or the preferences of its members cannot be assessed with the standard tools of social choice.

In this paper, we propose an extension of welfare economics and social choice theory meant to cover these important needs for ethical evaluation. A *universal social ordering* evaluates and ranks states that are described by pairs of a distribution of resources and characteristics of the corresponding population of any size. Such an ordering makes it possible not only to answer standard questions of social choice — "Is an allocation better than another, for a given population?" — but also any question of the following sort: "Is the situation of a certain population at a certain time or location better than that of another population at another time or location?"

Moreover, as the size of the populations involved in such questions can be any from a single individual to billions, a universal social ordering also encompasses interpersonal comparisons — "Is an individual consuming a certain bundle with certain preferences better-off than another individual with another bundle and different preferences?". Thus, our present study may be considered an attempt to unify various kinds of social or individual comparisons commonly accomplished in welfare economics and social choice theory, and we focus on "consistency" between these comparisons of different types. In this unified framework, we will see that some analytical separation is possible between the question of interpersonal comparisons and the question of social aggregation because for any given ordering that compares individual situations, the considerations relevant to extending this ordering into a universal social ordering are basically the same.

The questions addressed here were already raised by Sen (1976, 1979) when he examined how to make international comparisons of living standards. In particular, Sen examined the question of comparing situations of populations with different preferences. He was confronted with the difficulty that it may happen that population A is better-off, in its own eyes, than population B, while population B deems itself better-off than population A. As a result, the criterion proposed by Sen was incomplete and could not rank all possible situations. We propose a way to solve this difficulty and the orderings studied in this paper are complete even when different populations have different preferences. In addition, Sen examined how to compare situations of populations of different sizes, and he noticed that in the context of international comparisons it is quite natural to require the social ordering to be indifferent to the size of the population. Indeed, it would be strange to consider the population of Luxembourg less well-off than the Chinese population just because of size. He deduced that the social criterion could focus on the statistical distribution of individual situations for a normalized population size.

Although indifference to size appears very reasonable in the context of international comparisons, there are other contexts, studied in particular by the theory of population ethics,[1] in which a definite preference about the size of the population is defensible. For instance, assessing the evolution of the world population, or even the evolution of a particular nation over time, may involve considerations on the optimal size of the population. We thus think that different universal social orderings which reflect different ethical attitudes toward population size should be called for, depending on the context, e.g., depending on whether one wants to determine the optimal size of the world population or to compare the situation of two different countries. While neutrality with respect to population size seems reasonable for the latter exercise, a more positive attitude toward size (when welfare is sufficiently high) may be adopted for the former. In this respect,

[1]Important recent contributions to this theory include Blackorby, Bossert and Donaldson (2005) and Broome (2004).

the theory of universal social orderings proposed in this paper is quite general and may be useful to address a broad set of issues, with different universal social orderings being devised for different contexts.

The basic ethical principles for comparisons of social states in this paper are taken from a recent literature at the intersection of the theory of social choice and the theory of fair allocation. For fixed populations, this literature proposes social orderings that incorporate fairness principles.[2] We extend this approach to the evaluation of allocations for variable populations.

The paper is organized as follows. The formal framework and the notion of universal social ordering are introduced in Section 2. Basic ethical requirements about the social aggregation part of the problem are described in Section 3. The main results are stated in Section 4 and proved in Section 5. They consist of axiomatic characterizations of two families of universal social orderings, which differ in their attitude toward population size and may therefore be applicable to different contexts of social evaluation, or correspond to genuinely different ethical views of population ethics. These are two *families* of orderings, not just two orderings, and for each family a particular member is defined by the way in which interpersonal comparisons are performed. In Section 6 we show how simple fairness conditions may impose specific metrics for interpersonal comparisons and thereby guide the choice of a particular member for each family. Section 7 concludes.

2 The model

The model describes situations of finite populations with ordinal preferences over consumption bundles.

The set of real numbers (resp., natural numbers) is denoted $\mathbb{R}$ (resp., $\mathbb{N}$). Let N be the countably infinite set of potential individuals. Let ℓ be the finite number of commodities. We assume that the set N and the number ℓ are fixed. Let $\mathcal{S}$ be the set of all non-empty finite subsets of N, i.e., the set of possible populations. For every $S \in \mathcal{S}$, $|S|$ denotes the

[2] The theory of fair allocation rules is surveyed in Moulin and Thomson (1997) and in Thomson (2004). Surveys on fair social orderings can be found in Fleurbaey (2006) and Maniquet (2007).

cardinality of S, i.e., the size of the population.

A reflexive, transitive, and complete binary relation is called an *ordering*. In every particular social state to be evaluated, involving a population $S \in \mathcal{S}$, each individual $i \in S$ is endowed with a preference ordering R_i on a consumption set $X \subseteq \mathbb{R}^\ell$. To fix ideas, we assume throughout the paper that $X = \mathbb{R}^\ell_+$, but the theorems in Section 4 hold for any X that is convex, bounded from below (i.e., there is $q \in \mathbb{R}^\ell$ such that $q \leq x$ for all $x \in X$)[3] and upper-comprehensive (i.e., if $x \in X$ and $y \geq x$, then $y \in X$). Let $\mathcal{R}$ be the set of all continuous, convex, and weakly monotonic (i.e., $x_i \geq y_i$ implies $x_i \, R_i \, y_i$ and $x_i \gg y_i$ implies $x_i \, P_i \, y_i$) preference orderings on X.[4] For all $R_i \in \mathcal{R}$ and all $x_i \in X$, the *indifference set at x_i for R_i* is defined as $I(x_i, R_i) := \{y_i \in X \mid y_i \, I_i \, x_i\}$. Let $S \in \mathcal{S}$ be given. A *preference profile for S* is a list of preference orderings of the members of S: $R_S := (R_i)_{i \in S} \in \mathcal{R}^{|S|}$. An *allocation for S* is a vector $x_S := (x_i)_{i \in S} \in X^{|S|}$.

A *universal social ordering* is an ordering $\succsim$ defined on $\bigcup_{S \in \mathcal{S}} [X^{|S|} \times \mathcal{R}^{|S|}]$. For all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, and all $(y_T, R'_T) \in X^{|T|} \times \mathcal{R}^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ can be interpreted as follows: the state in which the members of group S with the preferences R_S consume x_S is at least as good as the state in which the members of group T with the preferences R'_T consume y_T. For convenience, we let (x_S, R_S) also denote the vector $(x_i, R_i)_{i \in S} \in (X \times \mathcal{R})^{|S|}$, and we identify $X^{|S|} \times \mathcal{R}^{|S|}$ with $(X \times \mathcal{R})^{|S|}$.

Throughout the paper, every universal social ordering is assumed to be anonymous, i.e., for all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, and all $(y_T, R'_T) \in X^{|T|} \times \mathcal{R}^{|T|}$, $(x_S, R_S) \sim (y_T, R'_T)$ if there is a bijection $\mu : S \to T$ such that for all $i \in S$, $x_i = y_{\mu(i)}$ and $R_i = R'_{\mu(i)}$.

A universal social ordering can be used for various kinds of evaluations, such as:

1. Comparisons of allocations for a given population (a region, a country, the world). For group S with preferences R_S, which allocation is socially better, x_S or y_S?

2. International comparisons of allocations. Which social state is better, country S with the allocation x_S and population preferences R_S or country T with the alloca-

[3]Vector inequalities are as usual: $\geq, >$, and $\gg$.

[4]Continuity and weak monotonicity of preferences are indispensable for our results, but convexity is not and is introduced only to make it clear that no result depends on non-standard preferences.

tion y_T and preferences R'_T?

3. Intertemporal comparisons of allocations. At which time are the people better-off, the present time when the people with preferences R_S consume x_S, or one hundred years ago when the people with preferences R'_T consumed y_T?

4. Interpersonal comparisons of individual states. Which individual state is better, individual i with preferences R_i consuming x_i or individual j with preferences R_j consuming y_j?

Most of the literature on social choice theory addresses issues like (1) in this list and only compares allocations for a given population. The concept of universal ordering enlarges the scope of evaluations and provides a unified framework to also make comparisons as in the other items of the list.

3 Axioms

In order to find reasonable universal social orderings, we first formulate properties of such orderings. A list of these properties, usually called *axioms* in the theory of social choice, is proposed in this section. The properties are classified into three groups: The first group is about the informational basis of comparisons of individual states, the second about the fairness of social states with a fixed population, and the third about the consistency or relationship between comparisons of states with variable populations.

3.1 Informational basis of comparisons of individual states

The first property expresses a basic principle of consumer sovereignty: In the evaluation of a given individual's states, the universal social ordering should espouse this individual's preferences over consumption bundles.

Consumer Sovereignty. For all $i \in N$, all $R_i \in \mathcal{R}$, and all $x_i, y_i \in X$, $(x_i, R_i) \succsim (y_i, R_i)$ if and only if $x_i \, R_i \, y_i$.

The second axiom requires that, in order to evaluate and compare states for a given individual, it should be sufficient to look at the indifference sets of the individual at the consumption bundles under consideration.

Individual Hansson Independence. For all $i \in N$, all $(x_i, R_i), (y_i, R'_i) \in \mathbb{R}^\ell_+ \times \mathcal{R}$, and all $R''_i, R'''_i \in \mathcal{R}$, if $I(x_i, R_i) = I(x_i, R''_i)$ and $I(y_i, R'_i) = I(y_i, R'''_i)$, then $(x_i, R_i) \succsim (y_i, R'_i)$ if and only if $(x_i, R''_i) \succsim (y_i, R'''_i)$.

The third axiom requires the evaluation of individual states not to be sensitive to infinitesimal changes in the bundle consumed by the individual.[5]

Individual Continuity. For all $i \in N$, all $(x_0, R_0) \in \mathbb{R}^\ell_+ \times \mathcal{R}$, and all $R_i \in \mathcal{R}$, the sets $\left\{x_i \in \mathbb{R}^\ell_+ \mid (x_i, R_i) \succsim (x_0, R_0)\right\}$ and $\left\{x_i \in \mathbb{R}^\ell_+ \mid (x_0, R_0) \succsim (x_i, R_i)\right\}$ are closed.

3.2 Fairness of social states with a fixed population

The next axiom is a fairness requirement which is inspired by the Pigou-Dalton transfer principle and is adapted here to our multidimensional framework. This axiom has been playing a central role in the theory of fair social orderings. It recommends transfers from an agent to another when the latter has less of every good in his bundle, provided that these two agents have the same preferences. Note that the post-transfer allocation is only required to be at least as good as the pre-transfer allocation, although all the orderings studied in the next section will actually strictly prefer the post-transfer allocation.

Pigou-Dalton for Equal Preferences. For all $S \in \mathcal{S}$, all $R_S \in \mathcal{R}^{|S|}$, all $x_S, y_S \in X^{|S|}$, and all $i, j \in S$, if $R_i = R_j$, and there exists $i, j \in S$ and $\delta \in \mathbb{R}^\ell_{++}$ such that

$$y_i \gg x_i = y_i - \delta \gg x_j = y_j + \delta \gg y_j,$$

and $x_k = y_k$ for all $k \neq i, j$, then $(x_S, R_S) \succsim (y_S, R_S)$.

[5] Although this may sound like a merely technical condition, it is shown in the appendix that in its absence one cannot exclude social orderings which give absolute priority to the best-off in some cases.

3.3 Relationship between comparisons of states with variable populations

We now turn to axioms dealing directly or indirectly with the issue of population size. The first one is separability, requiring an agent who has the same bundle in two allocations to play no role in the evaluation of these two allocations, so that removing him from the population would not affect the evaluation.

Separability. For all $S \in \mathcal{S}$ with $|S| \geq 2$, all $(x_S, R_S), (y_S, R'_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, and all $i \in S$, if $x_i = y_i$ and $R_i = R'_i$, then $(x_S, R_S) \succsim (y_S, R'_S)$ if and only if $(x_{S\setminus\{i\}}, R_{S\setminus\{i\}}) \succsim (y_{S\setminus\{i\}}, R'_{S\setminus\{i\}})$.

The next axiom is similar but it extends separability to the case in which the two allocations involve different populations both of which contain the same "unconcerned" individual.

Strong Separability. For all $S, T \in \mathcal{S}$ with $S \cap T \neq \emptyset$ and $|S|, |T| \geq 2$, all $i \in S \cap T$, and all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$ and $(y_T, R'_T) \in X^{|T|} \times \mathcal{R}^{|T|}$, if $x_i = y_i$ and $R_i = R'_i$, then $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{S\setminus\{i\}}, R_{S\setminus\{i\}}) \succsim (y_{T\setminus\{i\}}, R'_{T\setminus\{i\}})$.

A more radical indifference to population size is introduced in the next axiom which says that only the distribution of individual situations matters, not the size of the population. This requirement seems particularly suitable for international comparisons of standards of living or for evaluations of welfare growth over time. Here we need to introduce the replication operator. For any positive integer k, let $x_{kS} := (\underbrace{x_S, ..., x_S}_{k \text{ times}})$ and $R_{kS} := (\underbrace{R_S, ..., R_S}_{k \text{ times}})$. As we assume that every universal social ordering is anonymous, the pair (x_{kS}, R_{kS}) can be evaluated by every ordering even though kS is not, strictly speaking, an element of $\mathcal{S}$.

Replication Indifference. For all $S \in \mathcal{S}$, for all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, all $k \in \mathbb{N}$, $(x_S, R_S) \sim (x_{kS}, R_{kS})$.

This axiom was introduced by Sen (1976). He used the property to extend an index of real national income to the cases of different sizes of population.

The next axiom is borrowed from the theory of population ethics[6] and requires that it should be possible to add a new individual to the population without changing the social value of the state.

Indifferent Addition. For all $S \in \mathcal{S}$, and all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, there exist $x_i \in X$ and $R_i \in \mathcal{R}$ such that $(x_{S\cup\{i\}}, R_{S\cup\{i\}}) \sim (x_S, R_S)$.

4 Solutions

In this section we introduce and characterize two different families of universal social orderings on the basis of the axioms defined in the previous section. Each family contains a variety of specific orderings which may differ in particular about how to perform interpersonal comparisons. The specification of interpersonal comparisons will be the topic of Section 6.

A new piece of notation is necessary. Let a universal social ordering $\succsim$ be given. For all $S \in \mathcal{S}$ and all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, let $\theta(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$ be a vector of the pairs (x_i, R_i) arranged by increasing order, i.e., such that for all $k \in \{1, \ldots, |S| - 1\}$, $\theta_{k+1}(x_S, R_S) \succsim \theta_k(x_S, R_S)$. The following property of universal social orderings involves the lexicographic extension of the maximin criterion, applied to populations of the same size.

Leximin. A universal social ordering $\succsim$ is a *leximin* ordering if for all $S, T \in \mathcal{S}$ with $|S| = |T|$, all $x_S, y_T \in X^{|S|}$ and all $R_S, R'_T \in \mathcal{R}^{|S|}$,

(i) $(x_S, R_S) \succ (y_T, R'_T)$ if and only if there exists $m \in \mathbb{N}$, $m \leq |S|$, such that $\theta_k(x_S, R_S) \sim \theta_k(y_T, R'_T)$ for all $k < m$, and $\theta_m(x_S, R_S) \succ \theta_m(y_T, R'_T)$; and

(ii) $(x_S, R_S) \sim (y_T, R'_T)$ if and only if $\theta_k(x_S, R_S) \sim \theta_k(y_T, R'_T)$ for all $k \leq |S|$.

The two families of orderings defined below are subfamilies of the family of leximin orderings and diverge on the attitude toward population size. The first family, called

[6] Blackorby, Bossert and Donaldson (2005), in particular, make use of a similar axiom.

relative leximin, is completely neutral about population size and is only concerned about the distribution of individual well-being.

Relative leximin. A universal social ordering $\succsim$ is a *relative leximin* ordering if (i) it is a *leximin* ordering, and (ii) for all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{|T|S}, R_{|T|S}) \succsim (y_{|S|T}, R'_{|S|T})$.

To define the second family of orderings, we consider functions C such that for all $S \in \mathcal{S}$, and all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, $C(x_S, R_S) \in X \times \mathcal{R}$. Such a function is called a *critical level function* because it is used in the following definition in such a way that the addition of a new individual i to (x_S, R_S) is neutral if his situation is the pair $(x_i, R_i) = C(x_S, R_S)$.

Critical level leximin. A universal social ordering $\succsim$ is a *critical level leximin* ordering if (i) it is a *leximin* ordering, and (ii) there exists a critical level function C such that for all $S, T \in \mathcal{S}$ with $|S| < |T|$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if there exist $Q \subseteq N \setminus S$ with $|Q| = |T| - |S|$ and $(x_Q, R_Q) = ((x^1, R^1), \ldots, (x^{|Q|}, R^{|Q|})) \in (X \times \mathcal{R})^{|Q|}$ such that $(x^1, R^1) = C(x_S, R_S)$ and $(x^k, R^k) = C((x_S, R_S), (x^1, R^1), \ldots, (x^{k-1}, R^{k-1}))$ for all $k \in \{2, \ldots, |Q|\}$, and $(x_{S\cup Q}, R_{S\cup Q}) \succsim (y_T, R'_T)$.

5 Characterizations

We are now ready to characterize these two families of solutions on the basis of the axioms introduced in section 3.

Theorem 1 *Assme that a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*, and* Individual Continuity*. Then,* $\succsim$ *satisfies* Pigou-Dalton for Equal Preferences*,* Separability*, and* Replication Indifference *if and only if it is a* relative leximin *ordering.*

A leximin ordering is not continuous, although the orderings characterized here rely on a continuous ordering of individual states. There is no paradox in this configuration because the two notions of continuity apply at different levels. The discontinuity of a leximin ordering occurs only in prioritizing individuals when several individuals have conflicting interests. This is fully compatible with having a continuous measure of individual welfare. Adding full continuity of $\succsim$ to the list of axioms in our theorems would entail an impossibility.

Theorem 2 *Assume that a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*, and* Individual Continuity*. Then,* $\succsim$ *satisfies* Pigou-Dalton for Equal Preferences*,* Separability*, and* Indifferent Addition *if and only if it is a* critical level leximin *ordering.*

Although Replication Indifference and Indifferent Addition are generally compatible, they become incompatible in the presence of the other axioms and the two families singled out in these theorems are disjoint. In order to see this, consider $(x, R) \prec (y, R')$. By Indifferent Addition, there must exist (z, R'') such that $((x, R), (z, R''), (y, R')) \sim ((x, R), (y, R'))$. By Replication Indifference, this implies that

$$\begin{aligned} &((x, R), (x, R), (z, R''), (z, R''), (y, R'), (y, R')) \\ \sim \quad &((x, R), (x, R), (x, R), (y, R'), (y, R'), (y, R')) \, . \end{aligned}$$

The latter is impossible, because the left-hand allocation is preferred by a Leximin ordering if $(z, R'') \succ (x, R)$ and the right-hand allocation is preferred if $(z, R'') \precsim (x, R)$.

Whereas a relative leximin ordering is fully specified once a leximin ordering for fixed populations is given, a critical level leximin ordering involves an additional free parameter, namely, the critical level function C. The next result, which is similar to results from the theory of population ethics (Blackorby, Bossert and Donaldson, 2005), provides some precision about this function: it can be chosen to be constant if the universal social ordering satisfies Strong Separability.

Constant critical level leximin. A universal social ordering $\succsim$ is a *constant critical level leximin* ordering if it satisfies the following properties:

(i) $\succsim$ is a leximin ordering, and

(ii) there exists $(x_0, R_0) \in X \times \mathcal{R}$ such that for all $S, T \in \mathcal{S}$ with $|S| < |T|$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{S\cup Q}, R_{S\cup Q}) \succsim (y_T, R'_T)$ where $Q \subseteq N \setminus S$, $|Q| = |T| - |S|$, and $(x_i, R_i) = (x_0, R_0)$ for all $i \in Q$.

Theorem 3 *Assume that a universal social ordering $\succsim$ satisfies* Consumer Sovereignty*,* Individual Hansson Independence*, and* Individual Continuity*. Then, $\succsim$ satisfies* Pigou-Dalton for Equal Preferences*,* Strong Separability*, and* Indifferent Addition *if and only if it is a* constant critical level leximin *ordering.*

6 Proofs

The proofs of these theorems involve several lemmas. In the appendix, we check that all axioms are needed for the necessity parts of the theorems.

Let $x_i, y_i \in X$ and $R_i, R'_i \in \mathcal{R}$. We say that $I(x_i, R_i)$ *is above* $I(y_i, R'_i)$ if for every $z'_i \in I(y_i, R'_i)$, there exists $z_i \in I(x_i, R_i)$ such that $z_i \gg z'_i$. Note that if $I(x_i, R_i)$ is above $I(y_i, R'_i)$, then by weak monotonicity of preferences $I(x_i, R_i) \cap I(y_i, R'_i) = \varnothing$.

Lemma 1 *If a universal social ordering $\succsim$ satisfies* Consumer Sovereignty *and* Individual Hansson Independence*, then for all $R, R' \in \mathcal{R}$, and all $x, y \in X$, if $I(x, R) = I(y, R')$ then $(x, R) \sim (y, R')$; and if $I(x, R)$ is above $I(y, R')$, then $(x, R) \succ (y, R')$.*

Proof. Let $R, R' \in \mathcal{R}$, and $x, y \in X$ be such that $I(x, R) = I(y, R')$. Suppose that $(x, R) \prec (y, R')$. As $I(x, R) = I(y, R')$, one has $y \in I(x, R)$ and Consumer Sovereignty implies $(y, R) \sim (x, R)$. By transitivity, $(y, R) \prec (y, R')$. Since $I(y, R) = I(y, R')$, a direct application of Individual Hansson Independence implies $(y, R') \prec (y, R)$, a contradiction.

If $I(x, R)$ is above $I(y, R')$, then there exists R_0 such that $I(x, R) = I(x, R_0)$ and $I(y, R') = I(y, R_0)$. By weak monotonicity of preferences, xP_0y. By Consumer Sovereignty, $(x, R_0) \succ (y, R_0)$. Therefore, by Individual Hansson Independence, $(x, R) \succ (y, R')$. ■

Lemma 2 *Assume that a universal social ordering* $\succsim$ *satisfies* Separability*. Then, for all* $S, T \in \mathcal{S}$ *with* $|S| = |T|$*, all* $x_S, y_T \in X^{|S|}$*, and all* $R_S, R'_T \in \mathcal{R}^{|S|}$*, if there exists a bijection* $\mu : S \to T$ *such that* $(x_i, R_i) \succsim (y_{\mu(i)}, R'_{\mu(i)})$ *for all* $i \in S$*, then* $(x_S, R_S) \succsim (y_T, R'_T)$*; and if in addition,* $(x_i, R_i) \succ (y_{\mu(i)}, R'_{\mu(i)})$ *for some* $i \in S$*, then* $(x_S, R_S) \succ (y_T, R'_T)$*.*

Proof. Let $S, T \in \mathcal{S}$ with $|S| = |T|$, $x_S, y_T \in X^{|S|}$, and $R_S, R'_T \in \mathcal{R}^{|S|}$. Define $R''_S \in \mathcal{R}^{|S|}$ and $z_S \in X^{|S|}$ as $R''_i = R'_{\mu(i)}$ and $z_i = y_{\mu(i)}$ for all $i \in S$. Since $\succsim$ is anonymous, we have $(z_S, R''_S) \sim (y_T, R'_T)$.

Suppose that $(x_i, R_i) \succsim (y_{\mu(i)}, R'_{\mu(i)})$ for all $i \in S$. As $(z_i, R''_i) \precsim (x_i, R_i)$ for all $i \in S$, it follows from Separability that

$$(z_S, R''_S) \precsim ((x_1, z_{S\setminus\{1\}}), (R_1, R''_{S\setminus\{1\}})) \precsim ((x_{\{1,2\}}, z_{S\setminus\{1,2\}}), (R_{\{1,2\}}, R''_{S\setminus\{1,2\}})) \precsim \cdots$$
$$\cdots \precsim (x_S, R_S).$$

By transitivity, $(x_S, R_S) \succsim (y_T, R'_T)$.

If $(z_i, R''_i) \prec (x_i, R_i)$ for some $i \in S$, strict preference occurs in one of these chains and by transitivity, $(x_S, R_S) \succ (y_T, R'_T)$. ■

It follows from Lemma 2 that Consumer Sovereignty and Separability imply Strong Pareto: For all $S \in \mathcal{S}$, all $R_S \in \mathcal{R}^{|S|}$, and all $x_S, y_S \in X^{|S|}$, if $x_i \, R_i \, y_i$ for all $i \in S$, then $(x_S, R_S) \succsim (y_S, R_S)$, and if in addition $x_i \, P_i \, y_i$ for some $i \in S$, then $(x_S, R_S) \succ (y_S, R_S)$.

We now introduce a stronger version of Hansson Independence.

Hansson Independence For all $S, T \in \mathcal{S}$, for all $(x_S, R_S) \in \mathbb{R}_+^{|S|\ell} \times \mathcal{R}^{|S|}$ and $(y_T, R'_T) \in \mathbb{R}_+^{|T|\ell} \times \mathcal{R}^{|T|}$, all $R''_S \in \mathcal{R}^{|S|}$ and $R'''_T \in \mathcal{R}^{|T|}$, if $I(x_i, R_i) = I(x_i, R''_i)$ for all $i \in S$ and $I(y_i, R'_i) = I(y_i, R'''_i)$ for all $i \in T$, then $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_S, R''_S) \succsim (y_T, R'''_T)$.

Lemma 3 *If a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*, and* Separability*, then it satisfies* Hansson Independence*.*

Proof. Assume that a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, and Separability. Let $S, T \in \mathcal{S}$, $(x_S, R_S) \in \mathbb{R}_+^{|S|\ell} \times \mathcal{R}^{|S|}$,

$(y_T, R'_T) \in \mathbb{R}_+^{|T|\ell} \times \mathcal{R}^{|T|}$, $R''_S \in \mathcal{R}^{|S|}$, and $R'''_T \in \mathcal{R}^{|T|}$. Suppose that $I(x_i, R_i) = I(x_i, R''_i)$ for all $i \in S$ and $I(y_i, R'_i) = I(y_i, R'''_i)$ for all $i \in T$. By Lemma 1, $(x_i, R_i) \sim (x_i, R''_i)$ for all $i \in S$ and $(y_i, R'_i) \sim (y_i, R'''_i)$ for all $i \in T$. It follows from Lemma 2 that $(x_S, R_S) \sim (x_S, R''_S)$ and $(y_T, R'_T) \sim (y_T, R'''_T)$. Hence, by transitivity, $(x_S, R_S) \succsim (y_T, R'_T) \Leftrightarrow (x_S, R''_S) \succsim (y_T, R'''_T)$. ■

Lemma 4 *Assume that a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*,* Pigou-Dalton for Equal Preferences*, and* Separability*. Then, for all* $S \in \mathcal{S}$*, all* $R_S \in \mathcal{R}^{|S|}$*, all* $x_S, y_S \in X^{|S|}$*, and all* $i, j \in S$*, if*

$$(y_i, R_i) \succ (x_i, R_i) \succ (x_j, R_j) \succ (y_j, R_j),$$

and $(x_k, R_k) \succ (y_k, R_k)$ *for all* $k \neq i, j$*, then* $(x_S, R_S) \succ (y_S, R_S)$*.*

Proof. By Lemmas 2 and 3, $\succsim$ satisfies Strong Pareto and Hansson Independence. From Fleurbaey (2007b, Lemma 1), one deduces that for $X = \mathbb{R}_+^\ell$, if a universal social ordering $\succsim$ satisfies Strong Pareto, Hansson Independence, and Pigou-Dalton for Equal Preferences, then it satisfies the following property, which we call *Property P*: For all $S \in \mathcal{S}$, all $R_S \in \mathcal{R}^{|S|}$, all $x_S, y_S \in X^{|S|}$, and all $i, j \in S$, if $R_i = R_j$ and $y_i \, P_i \, x_i \, P_i \, x_j \, P_i \, y_j$, and $x_k \, P_k \, y_k$ for all $k \neq i, j$, then $(x_S, R_S) \succ (y_S, R_S)$. (This result extends to any set X that is convex, bounded from below and upper-comprehensive.)

Let $S \in \mathcal{S}$, $R_S \in \mathcal{R}^{|S|}$, $x_S, y_S \in X^{|S|}$, and $i, j \in S$. Assume that $(y_i, R_i) \succ (x_i, R_i) \succ (x_j, R_j) \succ (y_j, R_j)$ and $(x_k, R_k) \succ (y_k, R_k)$ for all $k \neq i, j$. By Consumer Sovereignty and continuity of R_j, there is $z_j \in X$ such that $(x_j, R_j) \succ (z_j, R_j) \succ (y_j, R_j)$. Let $q \in X$ and $R_0 \in \mathcal{R}$ be such that $I(q, R_0)$ is above $I(y_i, R_i)$ and $I(x_j, R_j)$, $I(x_j, R_0) = I(x_j, R_j)$, $I(y_j, R_0) = I(y_j, R_j)$, and $I(z_j, R_0) = I(z_j, R_j)$. By Lemma 1, $(q, R_0) \succ (y_i, R_i)$, $(x_j, R_0) \sim (x_j, R_j)$, $(y_j, R_0) \sim (y_j, R_j)$, and $(z_j, R_0) \sim (z_j, R_j)$. By Lemma 2,

$$\big((q, y_j, y_{S\setminus\{i,j\}}), \big(R_0, R_0, R_{S\setminus\{i,j\}}\big)\big) \succ (y_S, R_S).$$

By Property P,

$$\big((x_j, z_j, x_{S\setminus\{i,j\}}), \big(R_0, R_0, R_{S\setminus\{i,j\}}\big)\big) \succ \big((q, y_j, y_{S\setminus\{i,j\}}), \big(R_0, R_0, R_{S\setminus\{i,j\}}\big)\big).$$

By Lemma 2 and the fact that $(x_i, R_i) \succ (x_j, R_j) \sim (x_j, R_0)$ and $(x_j, R_j) \succ (z_j, R_j) \sim (z_j, R_0)$,

$$(x_S, R_S) \succ \left((x_j, z_j, x_{S\setminus\{i,j\}}), \left(R_0, R_0, R_{S\setminus\{i,j\}}\right)\right).$$

By transitivity, $(x_S, R_S) \succ (y_S, R_S)$. ■

Lemma 5 *If a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty, Individual Hansson Independence *and* Individual Continuity, *then for all* $t \in \mathbb{N}$, $t \geq 2$, *for all* $i \in N$, *all* $R_1, \ldots, R_t \in \mathcal{R}$, *and all* $x_1, \ldots, x_t \in X$ *such that* $(x_1, R_1) \precsim \cdots \precsim (x_t, R_t)$ *there exists* $R' \in \mathcal{R}$ *and* $z_1, \ldots, z_t \in X$ *such that* $(x_k, R_k) \sim (z_k, R')$ *for all* $k = 1, \ldots, t$.

Proof. Consider $R_1, \ldots, R_t \in \mathcal{R}$ and $x_1, \ldots, x_t \in X$ such that $(x_1, R_1) \precsim \cdots \precsim (x_t, R_t)$. Let $q \in X$ and $R_0 \in \mathcal{R}$ be such that $I(q, R_0)$ is above $I(x_1, R_1)$ and $I(x_t, R_t)$. There exists $R' \in \mathcal{R}$ such that $I(q, R') = I(q, R_0)$ and $I(x_1, R') = I(x_1, R_1)$, and there exists $R'' \in \mathcal{R}$ such that $I(q, R'') = I(q, R_0)$ and $I(x_t, R'') = I(x_t, R_t)$.

By Consumer Sovereignty and weak monotonicity of preferences, $(q, R'') \succ (x_t, R'')$ and therefore, by Lemma 1, $(q, R_0) \succ (x_t, R_t)$. As $(q, R_0) \sim (q, R')$, one has $(q, R') \succ (x_t, R_t)$. Let $z_1 = x_1$. One has $(z_1, R') \sim (x_1, R_1)$. Take any $k = 2, \ldots, t$. One has $(x_1, R_1) \precsim (x_k, R_k) \precsim (x_t, R_t) \sim (x_t, R'') \prec (q, R'') \sim (q, R')$, implying $(x_1, R') \precsim (x_k, R_k) \prec (q, R')$. By Individual Continuity, there is $z_k \in X$ such that $(x_k, R_k) \sim (z_k, R')$.

■

Lemma 6 *Assume that a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty, Individual Continuity, Individual Hansson Independence, Pigou-Dalton for Equal Preferences, *and* Separability. *Then, for all* $S \in \mathcal{S}$, *all* $x_S, y_S \in X^{|S|}$, *and all* $R_S \in \mathcal{R}^{|S|}$, *if* $\theta_1(x_S, R_S) \succ \theta_1(y_S, R_S)$, *then* $(x_S, R_S) \succ (y_S, R_S)$.

Proof. By Lemma 5, there exist $R'_S \in \mathcal{R}$ and $x'_S, y'_S \in X^{|S|}$ such that for all $i, j \in S$, $R'_i = R'_j$ and for all $i \in S$, $(x'_i, R'_i) \sim (x_i, R_i)$ and $(y'_i, R'_i) \sim (y_i, R_i)$. By Lemma 2, $(x'_S, R'_S) \sim (x_S, R_S)$ and $(y'_S, R'_S) \sim (y_S, R_S)$.

By a repeated application of Lemma 4 (or, simply, Property P from the proof of that lemma) and Lemma 2, $\theta_1(x'_S, R'_S) \succ \theta_1(y'_S, R'_S)$ implies $(x'_S, R'_S) \succ (y'_S, R'_S)$. This step is standard and is just sketched here. Start from (y'_S, R'_S), raise all individuals except a worst-off i_0 above $\theta_{|S|}(x'_S, R'_S)$ —an improvement by Lemma 2. Then, for each $i \neq i_0$, pull i down to a situation equivalent to $\theta_1(x'_S, R'_S)$ while i_0 is moved up but remains below $\theta_1(x'_S, R'_S)$ —an improvement by Lemma 4. The resulting allocation is worse than (x'_S, R'_S) by Lemma 2. By transitivity, $(x'_S, R'_S) \succ (y'_S, R'_S)$.

By transitivity, one has $(x_S, R_S) \succ (y_S, R_S)$. ■

Lemma 7 *If a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*,* Individual Continuity*,* Pigou-Dalton for Equal Preferences*, and* Separability*, then it is a* leximin *ordering.*

Proof. Let $S, T \in \mathcal{S}$ such that $|S| = |T|$, $R_S, R'_T \in \mathcal{R}^{|S|}$, and $x_S, y_T \in X^{|S|}$. If $\theta_k(x_S, R_S) \sim \theta_k(y_T, R'_T)$, then by Lemma 2, $(x_S, R_S) \sim (y_T, R'_T)$.

Assume that there exists $m \in \mathbb{N}$, $m \leq |S|$, such that $\theta_k(x_S, R_S) \sim \theta_k(y_T, R'_T)$ for all $k \in \mathbb{N}$ with $k < m$, and $\theta_m(x_S, R_S) \succ \theta_m(y_T, R'_T)$. Take $V \in \mathcal{S}$ such that $|V| = |S|$. By Lemma 5, there exist $x'_V, y'_V \in X^{|S|}$ and $R''_V \in \mathcal{R}^{|S|}$ such that

(i) $R''_i = R''_j$ for all $i, j \in V$,

(ii) for some bijections $\mu_S : V \to S$ and $\mu_T : V \to T$, one has $(x'_i, R''_i) \sim (x_{\mu_S(i)}, R_{\mu_S(i)})$ and $(y'_i, R''_i) \sim (y_{\mu_T(i)}, R'_{\mu_T(i)})$ for all $i \in V$, and

(iii) $x'_i = y'_i$ for all $i \in V$ such that for some $k < m$, $(x'_i, R''_i) \sim \theta_k(x_S, R_S)$.

Let $M \subset V$ denote the subgroup of the $m-1$ agents in V satisfying condition (iii) above. Let z_M be such that for all $i \in M$, $(z_i, R''_i) \sim \theta_m(x'_V, R''_V) \sim \theta_m(x_S, R_S)$. By Lemma 6, one has $\left((z_M, x'_{V\setminus M}), R''_V\right) \succ \left((z_M, y'_{V\setminus M}), R''_V\right)$. By Separability, $\left((z_M, x'_{V\setminus M}), R''_V\right) \succsim \left((z_M, y'_{V\setminus M}), R''_V\right)$ if and only if $(x'_V, R''_V) \succsim (y'_V, R''_V)$. Therefore, $(x'_V, R''_V) \succ (y'_V, R''_V)$. By Lemma 2, $(x'_V, R''_V) \sim (x_S, R_S)$ and $(y'_V, R''_V) \sim (y_T, R'_T)$. By transitivity, $(x_S, R_S) \succ (y_T, R_T)$. ■

Remark 1 In Lemma 6, Separability could be replaced by the following property requiring a monotonic relation of the evaluation of social states to the evaluations of individual situations:

Monotonicity. For all $S \in \mathcal{S}$, for all $(x_S, R_S), (y_S, R'_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, if $(x_i, R_i) \succsim (y_i, R'_i)$ for all $i \in S$, then $(x_S, R_S) \succsim (y_S, R'_S)$; if, in addition, $(x_i, R_i) \succ (y_i, R'_i)$ for some $i \in S$, then $(x_S, R_S) \succ (y_S, R'_S)$.

In Lemma 7, the proof only uses the following weak version of Separability, in which the unconcerned agent is not removed from the population:

Weak Separability. For all $S \in \mathcal{S}$ such that $|S| \geq 2$, all $(x_S, R_S), (y_S, R'_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, all $i \in S$, and all $x'_i \in X$, if $x_i = y_i$ and $R_i = R'_i$, then $(x_S, R_S) \succsim (y_S, R'_S) \Leftrightarrow \left(\left(x'_i, x_{S\setminus\{i\}}\right), R_S\right) \succsim \left(\left(x'_i, y_{S\setminus\{i\}}\right), R'_S\right)$.

As a consequence, Theorems 1 and 2 admit variants in which Monotonicity is added to the list of axioms and Weak Separability is substituted for Separability. (End of Remark)

Lemma 8 *If a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*,* Individual Continuity*,* Pigou-Dalton for Equal Preferences*,* Separability*, and* Replication Indifference*, then it is a* relative leximin *ordering.*

Proof. Assume that a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, Separability, and Replication Indifference. By Lemma 7, $\succsim$ is a leximin ordering. Let $S, T \in \mathcal{S}$, $x_S \in X^{|S|}$, $y_T \in X^{|T|}$, $R_S \in \mathcal{R}^{|S|}$, and $R_T \in \mathcal{R}^{|T|}$. By Replication Invariance, $(x_S, R_S) \sim (x_{|T|S}, R_{|T|S})$ and $(y_T, R_T) \sim (y_{|S|T}, R_{|S|T})$. Therefore, $(x_S, R_S) \succsim (y_T, R_T) \Leftrightarrow (x_{|T|S}, R_{|T|S}) \succsim (y_{|S|T}, R_{|S|T})$. ■

Lemma 9 *If a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*,* Individual Continuity*,* Pigou-Dalton for Equal Preferences*,* Separability*, and* Indifferent Addition*, then it is a* critical leval leximin *ordering.*

Proof. Assume that a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, Separability, and Indifferent Addition. By Lemma 7, $\succsim$ is a leximin ordering.

For all $S \in \mathcal{S}$, and all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, define $C(x_S, R_S) \in X \times \mathcal{R}$ as a pair $(x_i, R_i) \in X \times \mathcal{R}$ such that $((x_S, R_S), (x_i, R_i)) \sim (x_S, R_S)$. By Indifferent Addition, such a pair (x_i, R_i) exists.

Let $S, T \in \mathcal{S}$ with $|S| < |T|$, $x_S \in X^{|S|}$, $y_T \in X^{|T|}$, $R_S \in \mathcal{R}^{|S|}$, and $R_T \in \mathcal{R}^{|T|}$. Let $Q \subseteq N \setminus S$ and $(x_Q, R_Q) \in X^{|Q|} \times \mathcal{R}^{|Q|}$ be such that $|Q| = |T| - |S|$.and $(x_Q, R_Q) = ((x^1, R^1), \ldots, (x^{|Q|}, R^{|Q|}))$ with $(x^1, R^1) = C(x_S, R_S)$ and $(x^k, R^k) = C((x_S, R_S), (x^1, R^1), \ldots, (x^{k-1}, R^{k-1}))$ for all $k \in \{2, \ldots, |Q|\}$. Then, by construction, we have $(x_{S \cup Q}, R_{S \cup Q}) \sim (x_S, R_S)$. By transitivity, $(x_S, R_S) \succsim (y_T, R_T) \Leftrightarrow (x_{S \cup Q}, R_{S \cup Q}) \succsim (y_T, R_T)$. Thus, $\succsim$ is a critical level leximin ordering. ■

Lemma 10 *If a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*,* Individual Continuity*,* Pigou-Dalton for Equal Preferences*,* Strong Separability*, and* Indifferent Addition*, then it is a* constant critical level leximin *ordering.*

Proof. Assume that a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, Strong Separability, and Indifferent Addition. By Theorem 2, $\succsim$ is a critical level leximin ordering.

Let $S, T \in \mathcal{S}$, $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, $(y_T, R'_T) \in X^{|T|} \times \mathcal{R}^{|T|}$. Let $x_0 \in X$, $R_0 \in \mathcal{R}$ be such that $(x_S, R_S) \sim ((x_S, x_0), (R_S, R_0))$. Take some arbitrary $x_1 \in X$, $R_1 \in \mathcal{R}$.

By Strong Separability $((x_S, x_1), (R_S, R_1)) \sim ((x_S, x_0, x_1), (R_S, R_0, R_1))$. By Strong Separability again, $(x_1, R_1) \sim ((x_0, x_1), (R_0, R_1))$, implying $((y_T, x_1), (R'_T, R_1)) \sim ((y_T, x_0, x_1), (R'_T, R_0, R_1))$ and finally $(y_T, R'_T) \sim ((y_T, x_0), (R'_T, R_0))$. This shows that the constant function $C(x_S, R_S) = (x_0, R_0)$ for all $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$ is a critical level function for $\succsim$. ■

Proof of Theorem 1. Assume that a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, and Individual Continuity. If it satisfies Pigou-Dalton for Equal Preferences, Separability, and Replication Indifference, then by Lemma 8, it is a relative leximin ordering. Conversely, if it is a relative leximin ordering, then, as can be easily checked, it satisfies Separability, Pigou-Dalton for Equal Preferences, and Replication Indifference. ■

Proof of Theorem 2. Assume that a a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, and Individual Continuity. If it satisfies Pigou-Dalton for Equal Preferences, Separability, and Indifferent Addition, then by Lemma 9, it is a *critical level leximin* ordering. Conversely, if it is a critical level leximin ordering, then it satisfies Separability, Pigou-Dalton for Equal Preferences, and Indifferent Addition. ■

Lemma 11 *If a universal social ordering $\succsim$ is a* constant critical level leximin *ordering, then it satisfies* Strong Separability*.*

Proof. Assume that $\succsim$ is a constant critical level leximin ordering with the constant critical level $(x_0, R_0) \in X \times \mathcal{R}$. Let $S, T \in \mathcal{S}$ be such that $|T| > |S| \geq 2$ and $S \cap T \neq \emptyset$. Let $(x_S, R_S) \in X^{|S|} \times \mathcal{R}^{|S|}$, and $(y_T, R'_T) \in X^{|T|} \times \mathcal{R}^{|T|}$. Assume that for some $i \in S \cap T$, $x_i = y_i$ and $R_i = R'_i$. Let $Q \subseteq N \backslash S$ be such that $|Q| = |T| - |S|$, and define $(x_Q, R_Q) \in X^{|Q|} \times \mathcal{R}^{|Q|}$ by $(x_j, R_j) = (x_0, R_0)$ for all $j \in Q$. Then, by the definition of a constant critical level leximin ordering,

(i) $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{S \cup Q}, R_{S \cup Q}) \succsim (y_T, R'_T)$, and

(ii) $(x_{S \backslash \{i\}}, R_{S \backslash \{i\}}) \succsim (y_{T \backslash \{i\}}, R'_{T \backslash \{i\}})$ if and only if $(x_{(S \backslash \{i\}) \cup Q}, R_{(S \backslash \{i\}) \cup Q}) \succsim (y_{T \backslash \{i\}}, R'_{T \backslash \{i\}})$.

By Separability,

(iii) $(x_{S \cup Q}, R_{S \cup Q}) \succsim (y_T, R'_T)$ if and only if $(x_{(S \backslash \{i\}) \cup Q}, R_{(S \backslash \{i\}) \cup Q}) \succsim (y_{T \backslash \{i\}}, R'_{T \backslash \{i\}})$.

It follows from (i), (ii), and (iii) that $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{S \backslash \{i\}}, R_{S \backslash \{i\}}) \succsim (y_{T \backslash \{i\}}, R'_{T \backslash \{i\}})$. ■

Proof of Theorem 3. Assume that a a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence and Individual Continuity. If it satisfies Pigou-Dalton for Equal Preferences, Strong Separability, and Indifferent Addition, then by Lemma 10, it is a constant critical leval leximin ordering. Conversely, if it is a constant critical level leximin ordering, then it satisfies Pigou-Dalton for Equal Preferences, Indifferent Addition, and, by Lemma 11, Strong Separability. ■

7 Interpersonal comparisons

The families of universal social orderings characterized in the previous section are left imprecise on an important issue. They do not specify how interpersonal comparisons, i.e., relations of the sort $(x_i, R_i) \succsim (x_j, R_j)$, should be made. Any specification of this comparison that is exclusively based on the indifference sets $I(x_i, R_i)$ and $I(x_j, R_j)$ is compatible with the axioms of the theorems. Interpersonal comparisons of this sort are commonplace in welfare economics (in particular, in Bergson-Samuelson welfare economics, in cost-benefit analysis, and in the theory of fair allocation), and one can argue that recent philosophical theories of justice formulated in terms of resources have added support to the economic tradition of rejecting non-ordinal utility information in interpersonal comparisons.[7]

As illustrations of such interpersonal comparisons, consider the following two examples, defined for the case $X = \mathbb{R}^\ell_+$ that is studied in this paper:

(1) The Pazner-Schmeidler interpersonal comparisons.

For each individual i, each bundle x_i and each preference relation R_i, consider the fraction of a given reference bundle $\omega \in \mathbb{R}^\ell_{++}$ that individual i considers as equally desirable as x_i :

$$\lambda_\omega(x_i, R_i) := \min\{\lambda \in \mathbb{R}_+ \mid \lambda\omega \; R_i \; x_i\}.$$

Then, compare the situations (x_i, R_i) and (y_j, R_j) by the index λ_ω:

$$(x_i, R_i) \succsim (y_j, R_j) \Leftrightarrow \lambda_\omega(x_i, R_i) \geq \lambda_\omega(y_j, R_j).$$

(2) The money-metric interpersonal comparisons.

For each individual i, each bundle x_i and each preference relation R_i, calculate the minimum amount of expenditure needed to obtain the same satisfaction as with x_i when the price vector is a certain reference $p^* \in \mathbb{R}^\ell_{++}$:

$$e_{p^*}(x_i, R_i) := \min\left\{e \in \mathbb{R}_+ \mid \exists y_i \in \mathbb{R}^\ell_+,\; p^* y_i \leq e \text{ and } y_i \, R_i \, x_i\right\}.$$

[7] For such an argument, see, e.g., Fleurbaey (2007a).

Then, compare the situations (x_i, R_i) and (y_j, R_j) by the expenditure function e_{p^*}:

$$(x_i, R_i) \succsim (y_j, R_j) \Leftrightarrow e_{p^*}(x_i, R_i) \geq e_{p^*}(y_j, R_j).$$

It is not difficult to provide axiomatic justifications of these interpersonal rankings. Consider the following axioms, which involve the Pigou-Dalton transfer principle restricted to some specific situations. The first axiom restricts the application of the principle to allocations in which all bundles are proportional to the reference bundle.

Pigou-Dalton for ω-Proportional Bundles For all $S \in \mathcal{S}$, all $R_S \in \mathcal{R}^{|S|}$, and all $x_S, y_S \in \mathbb{R}_+^{|S|\ell}$, if x_i and y_i are proportional to ω for all $i \in S$, and there exist $i, j \in S$ and $\delta \in \mathbb{R}_{++}^{\ell}$ such that

$$y_i \gg x_i = y_i - \delta \gg x_j = y_j + \delta \gg y_j,$$

and $x_k = y_k$ for all $k \neq i, j$, then $(x_S, R_S) \succsim (y_T, R_T)$.

The second axiom restricts the application of the Pigou-Dalton principle to allocations in which all bundles are chosen by the agents in budgets defined with the price vector p^*. Let us say that, for a given R_i, a bundle x_i is "best for its p^*-value" if for all $q \in \mathbb{R}_+^{\ell}$ such that $p^* q \leq p^* x_i$, one has $x_i \, R_i \, q$.

Pigou-Dalton for p^*-Budgets For all $S \in \mathcal{S}$, all $R_S \in \mathcal{R}^{|S|}$, and all $x_S, y_S \in \mathbb{R}_+^{|S|\ell}$, if x_i and y_i are best for their p^*-value for all $i \in S$, and there exist $i, j \in S$ and $\delta \in \mathbb{R}_{++}^{\ell}$ such that

$$y_i \gg x_i = y_i - \delta \gg x_j = y_j + \delta \gg y_j,$$

and $x_k = y_k$ for all $k \neq i, j$, then $(x_S, R_S) \succsim (y_T, R_T)$.

If either of these two axioms is added to the list of axioms of Theorems 1, 2 or 3, then one obtains the Pazner-Schmeidler or the minimum expenditure comparison in the interpersonal comparison part of the corresponding ordering. We only state one of these results and leave it to the reader to formulate the other similar theorems.

Theorem 4 *Assume that a universal social ordering* $\succsim$ *satisfies* Consumer Sovereignty*,* Individual Hansson Independence*, and* Individual Continuity*. Then,* $\succsim$ *satisfies* Pigou-Dalton for Equal Preferences*,* Separability*,* Replication Indifference*, and* Pigou-Dalton for ω-Proportional Bundles *if and only if it is the* relative leximin ordering with the Pazner-Schmeidler interpersonal comparisons*: For all* $i, j \in N$*, and all* $(x_i, R_i), (y_j, R_j) \in X \times \mathcal{R}$*,* $(x_i, R_i) \succsim (y_j, R_j)$ *if and only if* $\lambda_\omega(x_i, R_i) \geq \lambda_\omega(y_j, R_j)$*.*

Proof. Assume that a universal social ordering $\succsim$ satisfies Consumer Sovereignty, Hansson Independence, Pigou-Dalton for Equal Preferences, Separability, Replication Indifference, and Pigou-Dalton for ω-Proportional Bundles. By Theorem 1, it is a (relative) leximin ordering.

Suppose, on the contrary, that it does not always rely on Pazner-Schmeidler comparisons. Then there are two individual states (y_i, R_i) and (y_j, R_j) such that either $(y_i, R_i) \precsim (y_j, R_j)$ although $\lambda_\omega(y_i, R_i) > \lambda_\omega(y_j, R_j)$, or $(y_i, R_i) \prec (y_j, R_j)$ although $\lambda_\omega(y_i, R_i) \geq \lambda_\omega(y_j, R_j)$ By Consumer Sovereignty, there is no loss of generality in assuming that y_i and y_j are proportional to ω, implying that $\lambda_\omega(y_i, R_i)\omega = y_i$ and $\lambda_\omega(y_j, R_j)\omega = y_j$.

Consider the first case. Let $\delta \in \mathbb{R}^\ell_{++}$ be proportional to ω and be such that

$$y_i \gg x_i = y_i - \delta \gg x_j = y_j + \delta \gg y_j.$$

By Pigou-Dalton for ω-Proportional Bundles, $((x_i, x_j), (R_i, R_j)) \succsim ((y_i, y_j), (R_i, R_j))$. On the other hand, by Consumer Sovereignty,

$$(x_i, R_i) \prec (y_i, R_i) \precsim (y_j, R_j) \prec (x_j, R_j),$$

which implies, as $\succsim$ is a leximin ordering, that $((x_i, x_j), (R_i, R_j)) \prec ((y_i, y_j), (R_i, R_j))$, a contradiction.

Consider the second case. By Individual Continuity, there is $z_i \in X$ (proportional to ω) such that $(y_i, R_i) \prec (z_i, R_i) \prec (y_j, R_j)$ and $\lambda_\omega(z_i, R_i) > \lambda_\omega(y_j, R_j)$. This is impossible because it is an instance of the first case.

Therefore $(y_i, R_i) \precsim (y_j, R_j)$ if and only if $\lambda_\omega(y_i, R_i) \leq \lambda_\omega(y_j, R_j)$.

Conversely, if a universal social ordering $\succsim$ is a relative leximin ordering with Pazner-Schmeidler interpersonal comparisons, then obviously it satisfies Pigou-Dalton for ω-Proportional Bundles. ■

8 Conclusion

This paper has introduced the notion of universal social orderings and proposed two types of solutions derived from an axiomatic analysis. The relative leximin orderings are suitable in contexts where population size is a matter of indifference, such as international comparisons of living standards. In contrast, the critical level leximin orderings are not indifferent about population size and appears relevant for the evaluation of global populations or economic growth.

The proofs of the results reveal that three ingredients of a universal social ordering have been analyzed separately here: 1) the aggregation criterion defines the degree of inequality aversion in the trade-off between conflicting individual interests (Lemma 7); 2) the comparison of situations with different population sizes involves specific axioms like Replication Indifference or Indifferent Addition (Theorems 1–3); 3) interpersonal comparisons are specified with the help of other axioms which have been introduced after the others (Theorem 4). This separation may be specific to our list of axioms, as the literature on social orderings contains results in which parts (1) and (3) are intertwined.[8]

With the specification of interpersonal comparisons as exemplified in the previous section, the relative leximin orderings are fully specified. But for the critical level leximin, the critical level remains to be determined, even if one accepts the conclusion of Theorem 3 that it should be a constant. We suspect that our framework, although more concrete than the standard model of the welfare economics of population, is still too abstract to help determine what the critical level should be. A theory of the critical level would require a richer description of lives, enabling the analyst to decipher the conditions deciding whether a life is worth living for an individual, or worth adding to a given society. We

[8]See in particular Maniquet and Sprumont (2004).

leave this issue for future research.

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Appendix

The appendix checks that each axiom is needed for the necessity parts of the theorems. That is, for each theorem in Section 4, we show that removing an axiom creates new possible universal social orderings. Let $\geq_{\text{lex}}$ denote the leximin ordering on real vectors (i.e., it lexicographically compares the smallest component, then the second smallest component, and so on).

Theorem 1

(1) Consumer Sovereignty

Define $\succsim$ by reference to an arbitrary price vector $p \in \mathbb{R}^{\ell}_{++}$ as follows: for all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if

$$\frac{1}{|S|}\sum_{i \in S} px_i \geq \frac{1}{|T|}\sum_{i \in T} px_i$$

This ordering satisfies Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, Separability, and Replication Indifference, but violates Consumer Sovereignty and is not a leximin ordering.

(2) Individual Hansson Independence.

Let $\mathcal{R}^* \subset \mathcal{R}$ be the set of preference orderings R for which there exists a continuous utility function U_R representing R such that $U_R(0) = 0$ and $U_R(x_i) + U_R(x_j) \geq U_R(y_i) +$

$U_R(y_j)$ whenever there exists $\delta \in \mathbb{R}^{\ell}_{++}$ such that

$$y_i \gg x_i = y_i - \delta \gg x_j = y_j + \delta \gg y_j.$$

Let $\mathcal{R}^{**} = \mathcal{R} \setminus \mathcal{R}^{*}$. The set $\mathcal{R}^{**}$ is not empty, as can be shown by the following example. Let $X = \mathbb{R}^2_+$, and let R be defined by the following utility function v : (i) $v(x_1, x_2) = x_1 + 1 - 1/x_2$ for $x_2 > 0$; (ii) $v(x_1, x_2) = -\infty$ for $x_2 = 0$. The ordering R is continuous, convex, and weakly monotonic. Suppose that R has another representation U_R satisfying the above properties. As $U_R(0) = 0$, there is $q \in (0, 1)$ such that $0 < U_R(q, q) < U_R(1, 1)/2$. Let (t, s) be such that $ts = 1$ and $t > \max\{3, -2v(q, q)\}$. The fact that $t > -2v(q, q)$ implies that $v(t-1, 2s/3) < v(q, q)$ because $v(t-1, 2s/3) = t - 3/(2s) = -t/2$. Since $v(t-1, 2s/3) < v(q, q)$, one also has $U_R(t-1, 2s/3) < U_R(q, q)$ and, a fortiori, $U_R(t-2, s/3) < U_R(q, q)$. Therefore,

$$U_R(t-2, s/3) + U_R(t-1, 2s/3) < 2U_R(q, q) < U_R(1, 1).$$

On the other hand,

$$(t, s) \gg (t-1, 2s/3) = (t, s) - (1, s/3) \gg \\ (t-2, s/3) = (t-3, 0) + (1, s/3) \gg (t-3, 0),$$

which implies, by the second property of U_R,

$$U_R(t-2, s/3) + U_R(t-1, 2s/3) \geq U_R(t-3, 0) + U_R(t, s).$$

As $U_R(t-3, 0) \geq 0$, one has $U_R(t-3, 0) + U_R(t, s) \geq U_R(1, 1)$, implying

$$U_R(t-2, s/3) + U_R(t-1, 2s/3) \geq U_R(1, 1).$$

This yields a contradiction, therefore R has no representation U_R satisfying the required properties.

Let $\succsim_{PS}$ be the relative leximin ordering with the Pazner-Schmeidler interpersonal comparisons by the reference bundle $\omega \in X$.

Define $\succsim$ as follows. For each $R \in \mathcal{R}^{*}$, choose a utility function U_R representing R that satisfies the above conditions. For all $S, T \in \mathcal{S}$ such that $|S| = |T|$, all $(x_S, R_S) \in$

$(\mathbb{R}_+^\ell \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if either

$$\sum_{i \in \{j \in S | R_j \in \mathcal{R}^*\}} U_{R_i}(x_i) > \sum_{i \in \{j \in T | R'_j \in \mathcal{R}^*\}} U_{R'_i}(y_i)$$

or

$$\sum_{i \in \{j \in S | R_j \in \mathcal{R}^*\}} U_{R_i}(x_i) = \sum_{i \in \{j \in T | R'_j \in \mathcal{R}^*\}} U_{R'_i}(y_i) \text{ and}$$
$$(x_i, R_i)_{i \in \{j \in S | R_j \notin \mathcal{R}^*\}} \succsim_{PS} (y_i, R'_i)_{i \in \{j \in T | R'_j \notin \mathcal{R}^*\}},$$

with the convention that $\sum_{i \in \varnothing} U_{R_i}(x_i) = 0$ and for all $(x_S, R_S) \in (\mathbb{R}_+^\ell \times \mathcal{R})^{|S|}$, $(x_S, R_S) \succ_{PS} (y_i, R'_i)_{i \in \varnothing}$. When $|S| \neq |T|$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{|T|S}, R_{|T|S}) \succsim (y_{|S|T}, R'_{|S|T})$ as defined above.

This ordering satisfies Consumer Sovereignty, Individual Continuity, Pigou-Dalton for Equal Preferences, Separability, and Replication Indifference, but violates Individual Hansson Independence and is not a leximin ordering.

(3) Individual Continuity.

Define $\succsim$ as follows. For all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (\mathbb{R}_+^\ell \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if either
(i) $(x_S, R_S) \succ_{PS} (y_T, R'_T)$, or
(ii) $(x_S, R_S) \sim_{PS} (y_T, R'_T)$ and $|T|$ times the number of agents i from S for which $x_i P_i q$ for all $q \in \mathbb{R}_+^\ell \setminus \mathbb{R}_{++}^\ell$ is at least as great as $|S|$ times the number of agents i from T for which $y_i P'_i q$ for all $q \in \mathbb{R}_+^\ell \setminus \mathbb{R}_{++}^\ell$.

This ordering satisfies Consumer Sovereignty, Individual Hansson Independence, Pigou-Dalton for Equal Preferences, Separability, and Replication Indifference, but violates Individual Continuity and is not a leximin ordering, as can be seen by the following example. Let $x_1 = y_1 = \omega$, $x_2 = x_3 = y_2 = y_3 = 2\omega$. Let R_1 and $R'_2 = R'_3$ be Leontief preferences (with cusp on the ray of ω), and R'_1, $R_2 = R_3$ be linear preferences. If $\succsim$ were a leximin ordering, one should have $\left(x_{\{1,2,3\}}, R_{\{1,2,3\}}\right) \succ \left(y_{\{1,2,3\}}, R'_{\{1,2,3\}}\right)$ because

$$(y_1, R'_1) \prec (x_1, R_1) \prec (x_2, R_2) = (x_3, R_3) \prec (y_2, R'_2) = (y_3, R'_3)$$

but the reverse preference $\left(x_{\{1,2,3\}}, R_{\{1,2,3\}}\right) \prec \left(y_{\{1,2,3\}}, R'_{\{1,2,3\}}\right)$ actually holds.

(4) Pigou-Dalton for Equal Preferences.

Define $\succsim$ as follows. For all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$,

$$(x_S, R_S) \succsim (y_T, R'_T) \Leftrightarrow \frac{1}{|S|} \sum_{i \in S} \lambda_\omega(x_i, R_i) \geq \frac{1}{|T|} \sum_{i \in T} \lambda_\omega(y_i, R'_i).$$

This $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Separability, and Replication Indifference, but violates Pigou-Dalton for Equal Preferences and is not a leximin ordering.

(5) Separability.

Let $\omega \in X$ be given. Define $\succsim$ as follows. For all $i, j \in N$, all $R_i, R_j \in \mathcal{R}$, $(x_i, R_i) \succsim (y_j, R_j)$ if and only if $\lambda_\omega(x_i, R_i) \geq \lambda_\omega(y_j, R_j)$. For all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $\theta_1(x_S, R_S) \succsim \theta_1(y_T, R'_T)$.

This ordering satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, and Replication Indifference, but violates Separability and is not a leximin ordering.

(6) Replication Indifference

The critical level leximin ordering with the Pazner-Schmeidler interpersonal comparisons satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, and Separability, but violates Replication Indifference and is not a relative leximin ordering.

Theorem 2

(1) Consumer Sovereignty

Let $p \in \mathbb{R}^\ell_{++}$ be given. Define $\succsim$ as follows: for all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if

$$\sum_{i \in S} px_i \geq \sum_{i \in T} px_i$$

This ordering satisfies Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, (Strong) Separability, and Indifferent Addition, but violates Consumer Sovereignty and is not a leximin ordering.

(2) Individual Hansson Independence

Let $\succsim_{PS_0}$ be the critical level leximin ordering with the Pazner-Schmeidler interpersonal comparisons by the reference bundle $\omega \in X$ and the critical level being (x_0, R_0) for some fixed $x_0 \in X$ and some fixed $R_0 \in \mathcal{R}^{**}$.

Define $\succsim$ as follows. For all $S, T \in \mathcal{S}$ such that $|S| = |T|$, all $(x_S, R_S) \in (\mathbb{R}^\ell_+ \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if either

$$\sum_{i \in \{j \in S | R_j \in \mathcal{R}^*\}} U_{R_i}(x_i) > \sum_{i \in \{j \in T | R'_j \in \mathcal{R}^*\}} U_{R'_i}(y_i)$$

or

$$\sum_{i \in \{j \in S | R_j \in \mathcal{R}^*\}} U_{R_i}(x_i) = \sum_{i \in \{j \in T | R'_j \in \mathcal{R}^*\}} U_{R'_i}(y_i) \text{ and}$$
$$(x_i, R_i)_{i \in \{j \in S | R_j \notin \mathcal{R}^*\}} \succsim_{PS_0} (y_i, R'_i)_{i \in \{j \in T | R'_j \notin \mathcal{R}^*\}}.$$

For all $S, T \in \mathcal{S}$ with $|S| < |T|$, all $(x_S, R_S) \in (\mathbb{R}^\ell_+ \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (\mathbb{R}^\ell_+ \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{S\cup Q}, R_{S\cup Q}) \succsim (y_T, R'_T)$ as defined above, where $|S \cup Q| = |T|$ and $(x_Q, R_Q) = ((x_0, R_0), \ldots, (x_0, R_0))$; and $(y_T, R'_T) \succsim (x_S, R_S)$ if and only if $(y_T, R'_T) \succsim (x_{S\cup Q}, R_{S\cup Q})$.

This ordering satisfies Consumer Sovereignty, Individual Continuity, Pigou-Dalton for Equal Preferences, (Strong) Separability, and Indifferent Addition, but violates Individual Hansson Independence and is not a leximin ordering.

(3) Individual Continuity.

Let $\succsim^*$ denote the ordering defined in (3) for Theorem 1, and let $(x_0, R_0) \in X \times \mathcal{R}$ be given. Define $\succsim$ as follows. For all $S, T \in \mathcal{S}$ with $|S| = |T|$, $\succsim$ coincides with $\succsim^*$. For all $S, T \in \mathcal{S}$ with $|S| < |T|$, all $(x_S, R_S) \in (\mathbb{R}^\ell_+ \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (\mathbb{R}^\ell_+ \times \mathcal{R})^{|T|}$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_{S\cup Q}, R_{S\cup Q}) \succsim^* (y_T, R'_T)$ where $(x_Q, R_Q) = ((x_0, R_0), \ldots, (x_0, R_0))$; and $(y_T, R'_T) \succsim (x_S, R_S)$ if and only if $(y_T, R'_T) \succsim^* (x_{S\cup Q}, R_{S\cup Q})$.

This ordering satisfies Consumer Sovereignty, Individual Hansson Independence, Pigou-Dalton for Equal Preferences, (Strong) Separability, and Indifferent Addition, but violates Individual Continuity, and is not a leximin ordering.

(4) Pigou-Dalton for Equal Preferences

Define $\succsim$ as follows. For all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$,

$$(x_S, R_S) \succsim (y_T, R'_T) \Leftrightarrow \sum_{i \in S} \lambda_\omega(x_i, R_i) \geq \sum_{i \in T} \lambda_\omega(y_i, R'_i).$$

This $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, (Strong) Separability, and Indifferent Addition, but violates Pigou-Dalton for Equal Preferences and is not a leximin ordering.

(5) Separability

Define $\succsim$ as in (5) for Theorem 1. Then, $\succsim$ satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, and Indifferent Addition, but violates Separability and is not a leximin ordering.

(6) Indifferent Addition

Define $\succsim$ as follows. For all $S, T \in \mathcal{S}$, all $(x_S, R_S) \in (X \times \mathcal{R})^{|S|}$, and all $(y_T, R'_T) \in (X \times \mathcal{R})^{|T|}$,

(i) if $|S| > |T|$, then $(x_S, R_S) \succ (y_T, R'_T)$, and

(ii) if $|S| = |T|$, $(x_S, R_S) \succsim (y_T, R'_T)$ if and only if $(x_S, R_S) \succsim_{PS} (y_T, R'_T)$.

This ordering satisfies Consumer Sovereignty, Individual Hansson Independence, Individual Continuity, Pigou-Dalton for Equal Preferences, and (Strong) Separability, but violates Indifferent Addition and is not a critical level leximin ordering.

Theorem 3

The examples for Theorem 2 also show that each axiom in Theorem 3 is necessary for its necessity part.