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REVISED PROOF OF SKOLEM'S THEOREM*

TAKASHI NAGASHIMA

In [N 91], the author intended to present an easy finitary proof of Skolem's Theorem but unfortunately it turned out to contain some serious errors. This corrected version is self-contained and readers' knowledge of [N 91] is not assumed. Skolem's Theorem is the following statement: *In the classical predicate logic, let f be a k-ary function symbol not contained in a formula* $\forall x_1 \dots \forall x_k \exists y A(x_1, \dots, x_k, y) \supset B$. *Then*

$$\forall x_1 \dots \forall x_k \exists y A(x_1, \dots, x_k, y) \supset B$$

is valid if and only if

$$\forall x_1 \dots \forall x_k A(x_1, \dots, x_k, f(x_1, \dots, x_k)) \supset B$$

is valid.

We use the logical symbols $\neg$, $\wedge$, $\vee$, $\supset$, $\forall$ and $\exists$. Different letters are used for free variables and for bound variables. Any set consisting of function symbols and predicate symbols is called a *language*. The language obtained by adding function symbols f, g, ... and predicate symbols P, Q, ... to any language $\mathscr{L}$ is written $\mathscr{L} \cup \{f, g, \dots, P, Q, \dots\}$. *Terms* and *formulae* are constructed according to the usual syntactic rules. Any formula of the form $(A \supset B) \wedge (B \supset A)$ is abbreviated as $A \equiv B$. For any formula $A(a)$, the formula

$$\exists x A(x) \wedge \forall x \forall y (A(x) \wedge A(y) \supset x = y)$$

is abbreviated as $\exists! x A(x)$. If two formal expressions A and B differ only in their bound variables, A and B are *congruent* [K 52, §33] or A is an *alphabetical variant* of B [T 75, §3].

A *cedent* is a sequence of zero or more formulae separated by commas. A *sequent* is an expression of the form

$$\Gamma \longrightarrow \Theta$$

where Γ and Θ are any decents. *Partition of cedent* is defined as follows:

(1) If Γ is the empty cedent then $[\Gamma; \Gamma]$ is the only partition of Γ.

(2) If $[\Gamma_1; \Gamma_2]$ is a partition of Γ then $[\Gamma_1, A; \Gamma_2]$ and $[\Gamma_1; \Gamma_2, A]$ are partitions of Γ, A.

A *partition* of a sequent $\Gamma \rightarrow \Theta$ is an ordered pair (of sequents)

$$[\Gamma_1 \longrightarrow \Theta_1; \Gamma_2 \longrightarrow \Theta_2]$$

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where $[\Gamma_1; \Gamma_2]$ is any partition of Γ and $[\Theta_1; \Theta_2]$ is any partition of Θ.

For any formula or a cedent or a sequent S, let

$$\mathrm{FV}(S)=\text{the set of free variables occurring in } S,$$

$$\mathrm{BV}(S)=\text{the set of bound variables occurring in } S,$$

$$\mathrm{Func}(S)=\text{the set of function symbols occurring in } S.$$

For any formula or a cedent or a sequent S, $\mathrm{Pred}^{+}(S)$ (resp. $\mathrm{Pred}^{-}(S)$) denotes the set of predicate symbols occurring positively (resp. negatively) in S and $\mathrm{Pred}(S)$ denotes the set of all predicate symbols occurring in S.

Let $\mathscr{L}$ be a language. A term t is an *$\mathscr{L}$-term* if $\mathrm{Func}(t)\subset\mathscr{L}$. If $\mathrm{Func}(A)\cup\mathrm{Pred}(A)\subset\mathscr{L}$ then A is an *$\mathscr{L}$-formula.* The language $\mathrm{Func}(A)\cup\mathrm{Pred}(A)$ is called *the language of A* and denoted as $\mathscr{L}(A)$. Similar notations and terminologies are used for cedents and sequents.

For any formal expression S and for any (free or bound) variable v, $S[v:=t]$ denotes the result of replacing all occurrences of variable v in S with t. When a formula is denoted as $A(a)$, the expression $A(a)[a:=t]$ is abbreviated as $A(t)$.

The system LK_e (*LK with equality*) is an extension of LK obtained by adding the following schemata for initial sequents:

$$\longrightarrow t=t$$

$$s=t, A(s)\longrightarrow A(t)$$

where a is a free variable, s and t are terms and $A(a)$ is an atomic formula. This system is equivalent to LK_e in [T 75, §7]. LK_e is also equivalent to the system LKG [N 66], which is an extension of LK obtained with additional inference schemata

$$\frac{t=t,\ \Gamma\longrightarrow\Theta}{\Gamma\longrightarrow\Theta} \quad \text{and} \quad \frac{\Gamma\longrightarrow\Theta,\ A(s)\quad A(t),\ \Delta\longrightarrow\Lambda}{s=t,\ \Gamma,\ \Delta\longrightarrow\Theta,\ \Lambda}$$

where a is a free variable, s and t are terms and $A(a)$ is an atomic formula.

A *derivation* (or a *proof figure*) is defined as usual. A sequent S is *LK-provable* and denoted as $\vdash S$ is there exists an LK-derivation of S. If there exists an LK-derivation $\mathscr{H}$ of a sequent S and if all sequents in $\mathscr{H}$ are $\mathscr{L}$-sequents, then S is *LK-provable in $\mathscr{L}$* and denoted as $\mathscr{L}\vdash S$. Corresponding terminologies and notations are used aldo for LK_e. If the equality symbol $=$ is not contained in $\mathscr{L}$, then LK_e-provability in $\mathscr{L}$ is clearly equivalent to LK-provability in $\mathscr{L}$.

If $A(a_1,\ldots,a_n)$ is a formula and $x_1,\ldots,x_n$ are distinct bound variables not occurring in this formula, then the formal expression

$$\lambda x_1\ldots x_n.A(x_1\ldots,x_n)$$

is an n-ary *abstract* [T 75, §20]. If V is $\lambda x_1\ldots x_n.A(x_1,\ldots,x_n)$ and if $t_1,\ldots,t_n$ are terms then $V(t_1,\ldots t_n)$ denotes the formula $A(t_1,\ldots,t_n)$. For any n-ary abstract V, define

$$\mathrm{Func}(V)=\mathrm{Func}(V(a_1,\ldots,a_n)),$$

$$\mathrm{Pred}(V)=\mathrm{Pred}(V(a_1,\ldots,a_n))$$

where $a_1,\ldots,a_n$ are free variables not occurring in V. An abstract V is an *$\mathscr{L}$-absrtact* if $\mathrm{Func}(V)\cup\mathrm{Pred}(V)\subset\mathscr{L}$.

For any k-ary predicate symbol P and any k-ary abstract V, the result of substituting V for P in a formula or a cedent or a sequent S is denoted as $S[P:=V]$.

The key idea of our proof is replacing a function symbol by a predicate symbol [K 52, §74]. Let $\mathscr{L}$ be a language, let f be a k-ary function symbol not contained in $\mathscr{L}$ and let F be a $(k+1)$-ary predicate symbol not contained in $\mathscr{L}$. The *$(f;F)$-transformation* applies to $\mathscr{L}\cup\{f\}$-terms and to $\mathscr{L}\cup\{f\}$-formulae. Any $\mathscr{L}\cup\{f\}$-term is transformed into a unary $\mathscr{L}\cup\{=,F\}$-abstract and any $\mathscr{L}\cup\{f\}$-formula is transformed into an $\mathscr{L}\cup\{=,F\}$-formula. Definition is by the following induction.

(1) a^* is $\lambda u.(u=a)$.
(2) $f(t_1,\ldots,t_k)^*$ is
$\lambda u.\exists x_1\ldots\exists x_k(t_1^*(x_1)\wedge\ldots\wedge t_k^*(x_k)\wedge F(x_1,\ldots,x_k,u))$
where $u, x_1,\ldots,x_k\notin\mathrm{BV}(t_1^*)\cup\ldots\cup\mathrm{BV}(t_k^*)$.
(3) $g(t_1,\ldots,t_n)^*$ is
$\lambda u.\exists x_1\ldots\exists x_n(t_1^*(x_1)\wedge\ldots\wedge t_n^*(x_n)\wedge u=g(x_1,\ldots,x_n))$
where $u, x_1,\ldots,x_n\notin\mathrm{BV}(t_1^*)\cup\ldots\cup\mathrm{BV}(t_n^*)$.
(4) $P(t_1,\ldots,t_n)^*$ is
$\exists x_1\ldots\exists x_n(t_1^*(x_1)\wedge\ldots\wedge t_n^*(x_n)\wedge P(x_1,\ldots,x_n))$
where $x_1,\ldots,x_n\notin\mathrm{BV}(t_1^*)\cup\ldots\cup\mathrm{BV}(t_n^*)$.
(5) $(A\wedge B)^*$ is $A^*\wedge B^*$, $(A\vee B)^*$ is $A^*\vee B^*$, $(A\supset B)^*$ is $A^*\supset B^*$ and $(\neg A)^*$ is $\neg A^*$.
(6) $(\forall xA(x))^*$ is $\forall yA^*(y)$ and $(\exists xA(x))^*$ is $\exists yA^*(y)$ where y is any bound variable such that $y\notin\mathrm{BV}(A(a)^*)$.

Example. In case of $k=1$, $(f(a)=b)^*$ is (any alphabetical variant of)

$$\exists x\exists y(\exists z(z=a\wedge F(z,x))\wedge y=b\wedge x=y).$$

For any $(k+1)$-ary predicate symbol F, the *existence condition* [Mo 82] $\mathrm{Ex}(F)$ is the formula

$$\forall x_1\ldots\forall x_k\exists yF(x_1,\ldots,x_k,y)$$

and the *uniqueness condition* [Mo 82] $\mathrm{Un}(F)$ is the formula

$$\forall x_1\ldots\forall x_k\forall y\forall z(F(x_1,\ldots,z_k,y)\wedge F(x_1,\ldots,x_k,z)\supset y=z).$$

Lemma 1. *An $\mathscr{L}$-sequent is LK-provable if and only if it is LK-provable in $\mathscr{L}$. An $\mathscr{L}$-sequent is LK_e-provable if and only if it is LK_e-provable in $\mathscr{L}\cup\{=\}$.* □

Proof. The first part of Lemma is a direct consequence of Gentzen's cut-elimination theorem [G 35]. The latter part follows from cut-elimination theorem of LK_e [T 75, §7] or cut-elimination theorem of *LKG* [N 66]. □

Lemma 2. *Let P be a k-ary predicate symbol, V be a k-ary $\mathscr{L}$-abstract and S be any $\mathscr{L}$-sequent. If $\mathscr{L}\vdash S$ then $\mathscr{L}\vdash S[P:=V]$. If $\mathscr{L}\vdash_e S$ then $\mathscr{L}\vdash_e S[P:=V]$.* □

Proof. Case LK: By cut-elimination theorem, there exists an cut-free LK-derivation $\mathscr{H}$ of S. Applying *redesignation of free variables* [G 35, III 3.10], $\mathscr{H}$ can be converted into a cut-free LK-derivation $\mathscr{H}'$ of S such that no eigenvariable of $\mathscr{H}'$ occurs in V. Substitute V for P in every sequent of $\mathscr{H}'$. The result of substitution is easily verified to be an LK-derivation of $S[P:=V]$. Similarly for Case LK_e. □

Now let a k-ary function symbol f and a $(k+1)$-ary predicate symbol F be fixed. We state some Lemmas concerning the $(f;F)$-transformation.

Lemma 3. *For any* $\mathscr{L}$*-term* t,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F)\longrightarrow \exists!x t^*(x). \quad \square$$

Proof. By induction on the structure of t. □

Lemma 4. *For any free variable* a, *any* $\mathscr{L}$*-term* t *and any* $\mathscr{L}$*-formula* A,

$$\mathscr{L}\cup\{=\}\vdash_e \longrightarrow t^*(a)\equiv a=t,$$

$$\mathscr{L}\cup\{=\}\vdash_e \longrightarrow A^*\equiv A. \quad \square$$

Lemma 5. *For any free variable* a *and any* $\mathscr{L}$*-terms* s *and* t,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),t^*(a)\longrightarrow s^*(b)\equiv s[a:=t]^*(b). \quad \square$$

Proof. By induction on the structure of s. □

Lemma 6. *If* $=\notin\mathscr{L}$, $F\notin\mathscr{L}$, a *is a free variable,* t *is a* $\mathscr{L}$*-term and* A *is a* $\mathscr{L}$*-formula, then*

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),t^*(a)\longrightarrow A^*\equiv A[a:=t]^*. \quad \square$$

Proof. By induction on the structure of A. □

Lemma 7. *If* $=\notin\mathscr{L}$, $F\notin\mathscr{L}$, a *is a free variable,* t *is an* $\mathscr{L}$*-term and* $A(a)$ *is an* $\mathscr{L}$*-formula, then*

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\forall xA(x)^*\longrightarrow A(t)^*,$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),A(t)^*\longrightarrow \exists xA(x)^*. \quad \square$$

Proof. Let $a\notin\mathrm{FV}(\forall xA(x))\cup\mathrm{FV}(t)$. By Lemma 6,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),t^*(a)\longrightarrow A(a)^*\equiv A(t)^*.$$

Hence

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),t^*(a),A(a)^*\longrightarrow A(t)^*,$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),t^*(a),\forall xA(x)^*\longrightarrow A(t)^*,$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\exists xt^*(x),\forall xA(x)^*\longrightarrow A(t)^*.$$

By Lemma 3,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\forall xA(x)^*\longrightarrow A(t)^*.$$

The latter half is proved similarly. □

Lemma 8. *For any $\mathscr{L}\cup\{f\}$-sequent $\Gamma\to\Theta$, if $\mathscr{L}\cup\{f\}\vdash\Gamma\to\Theta$* then

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\Gamma^*\longrightarrow\Theta^*. \quad \square$$

Proof. By cut-elimination theorem, there exists a cut-free $\mathscr{L}\cup\{f\}$-*LK*-derivation $\mathscr{H}$ of $\Gamma\to\Theta$. It suffices to prove by induction that

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\Phi^*\longrightarrow\Psi^*$$

for any sequent $\Phi\to\Psi$ occurring in $\mathscr{H}$. The statement is evident for initial sequents. We divide cases according to the inference whose lower sequent is $\Phi\to\Psi$.
Case $(\to\exists)$. Let the inference be

$$\frac{\Delta\longrightarrow\Lambda, A(t)}{\Delta\longrightarrow\Lambda, \exists xA(x)}\ (\to\exists).$$

Since $\mathscr{L}\cup\{f\}\vdash\Delta\to\Lambda,A(t)$,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\Delta^*\longrightarrow\Lambda^*,A(t)^*$$

by the inductive hypothesis. By Lemma 7,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),A(t)^*\longrightarrow\exists xA(x)^*,$$

hence

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\Delta^*\longrightarrow\Lambda^*,\exists xA(x)^*.$$

Case $(\forall\to)$. Similarly by Lemma 7.
All the other cases are straightforward. □

Theorem 9 (Lyndon). *For any partition $[\Gamma_1\to\Theta_1;\Gamma_2\to\Theta_2]$ of an LK-provable sequent $\Gamma\to\Theta$, if*

$$\mathrm{Pred}^-(\Gamma_1\longrightarrow\Theta_1)\cap\mathrm{Pred}^+(\Gamma_2\longrightarrow\Theta_2)\neq\phi$$

or

$$\mathrm{Pred}^+(\Gamma_1\longrightarrow\Theta_1)\cap\mathrm{Pred}^-(\Gamma_2\longrightarrow\Theta_2)\neq\phi$$

then there exists a formula C satisfying the following properties:

(1) $\vdash\Gamma_1\to\Theta_1$, C and $\vdash C,\Gamma_2\to\Theta_2$.
(2) $\mathrm{FV}(C)\subset\mathrm{FV}(\Gamma_1\to\Theta_1)\cap\mathrm{FV}(\Gamma_2\to\Theta_2)$.
(3) $\mathrm{Pred}^+(C)\subset\mathrm{Pred}^-(\Gamma_1\to\Theta_1)\cap\mathrm{Pred}^+(\Gamma_2\to\Theta_2)$.
(4) $\mathrm{Pred}^-(C)\subset\mathrm{Pred}^+(\Gamma_1\to\Theta_1)\cap\mathrm{Pred}^-(\Gamma_2\to\Theta_2)$. □

Any formula C satisfying (1)–(4) is called a *Lyndon interpolant of* the partition $[\Gamma_1\to\Theta_1;\Gamma_2\to\Theta_2]$.

Lyndon's proof is not finitary but a finitary proof can be carried out with Maehara's method [Ma 73, §8.3], [T 75, §6].

Theorem 10. *If a sequent S contains no equality symbol and if* $\vdash_e S$ *then* $\vdash S$. □

Proof. This is an easy application of cut-elimination theorem of *LK* with equality [Ma 73, §6.6], [T 75, §7], [N 66]. □

Theorem 11 (Skolem). *Let f be a k-ary function symbol not occurring in*

$$\forall x_1 \ldots \forall x_k \exists y A(x_1,\ldots,x_k,y),\Delta,\Lambda.$$

If

$$\vdash \forall x_1 \ldots \forall x_k A(x_1,\ldots,x_k,f(x_1,\ldots,x_k)),\Delta \longrightarrow \Lambda$$

then

$$\vdash \forall x_1 \ldots \forall x_k \exists y A(x_1,\ldots,x_k,y),\Delta \longrightarrow \Lambda. \quad \square$$

Proof. For the sake of simplicity in notation, we assume $k=1$. Let $\mathscr{L}=\mathscr{L}(\forall x \exists y A(x,y),\Delta,\Lambda)$ and let F be a 2-ary predicate symbol not contained in $\mathscr{L}$. Assume

$$\vdash \forall x A(x,f(x)),\Delta \longrightarrow \Lambda.$$

Whence follows

$$\mathscr{L}\cup\{f\}\vdash \forall x A(x,f(x)),\Delta \longrightarrow \Lambda \tag{1}$$

by Lemma 1. By Lemma 8,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\forall x A(x,f(x))^*,\Delta^* \longrightarrow \Lambda^*. \tag{2}$$

Because $f(a)^*(b)$ is $\exists x(x=a \wedge F(x,b))$,

$$\{=,F\}\vdash_e f(a)^*(b)\equiv F(a,b). \tag{3}$$

$$\mathscr{L}\cup\{=\}\vdash_e \longrightarrow A(a,b)^*\equiv A(a,b) \tag{4}$$

and

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),f(a)^*(b) \longrightarrow A(a,b)^*\equiv A(a,f(a)) \tag{5}$$

follows immediately from Lemmas 4 and 6 respectively. From (3), (4) and (5) we obtain successively

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),F(a,b) \longrightarrow A(a,f(a))^*\equiv A(a,b),$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),F(a,b) \longrightarrow A(a,f(a))^*\equiv F(a,b)\wedge A(a,b),$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),F(a,b) \longrightarrow A(a,f(a))^*\equiv \exists y(F(a,y)\wedge A(a,y)),$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\mathrm{Ex}(F) \longrightarrow A(a,f(a))^*\equiv \exists y(F(a,y)\wedge A(a,y)),$$

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F) \longrightarrow \forall x A(x,f(x))^*\equiv \forall x \exists y(F(x,y)\wedge A(x,y)). \tag{6}$$

By Lemma 4,

$$\mathscr{L}\cup\{=\}\vdash_e \longrightarrow B^*\equiv B$$

for any member B of Δ,Λ. Therefore

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Ex}(F),\mathrm{Un}(F),\forall xA(x,f(x))^*,\Delta\longrightarrow\Lambda \tag{7}$$

follows from (2). From (6) and (7) follows

$$\mathscr{L}\cup\{=,F\}\vdash_e \forall x\exists y(F(x,y)\wedge A(x,y)),\mathrm{Ex}(F),\mathrm{Un}(F),\Delta\longrightarrow\Lambda. \tag{8}$$

Consider the partition

$$[\mathrm{Un}(F)\longrightarrow\quad;\forall x\exists y(F(x,u)\wedge A(x,y)),\mathrm{Ex}(F),\Delta\longrightarrow\Lambda]$$

of this sequent. Then

$$\mathrm{Pred}^-(\mathrm{Un}(F)\longrightarrow)=\{=\},\quad \mathrm{Pred}^+(\mathrm{Un}(F)\longrightarrow)=\{F\}$$

and

$$F\notin\mathrm{Pred}^+(\forall x\exists y(F(x,y)\wedge A(x,y)),\mathrm{Ex}(F),\Delta\longrightarrow\Lambda),$$

Let C be a Lyndon interpolant of this partition. Then C satisfies $\mathrm{Pred}(C)\subset\{=\}$,

$$\mathscr{L}\cup\{=,F\}\vdash_e \mathrm{Un}(F)\longrightarrow C \tag{9}$$

and

$$\mathscr{L}\cup\{=,F\}\vdash_e C,\forall x\exists y(F(x,y)\wedge A(x,y)),\mathrm{Ex}(F),\Delta\longrightarrow\Lambda. \tag{10}$$

Substitute $\lambda uv.(u=v)$ for F in (9) and apply Lemma 2. Then

$$\mathscr{L}\cup\{=\}\vdash_e \forall x\forall y\forall z(x=y\wedge x=z\supset y=z)\longrightarrow C,$$

hence

$$\mathscr{L}\cup\{=\}\vdash_e \longrightarrow C. \tag{11}$$

From (10), (11) and

$$\mathscr{L}\cup\{F\}\vdash_e \forall x\exists y(F(x,y)\wedge A(x,y))\longrightarrow\mathrm{Ex}(F),$$

it follows

$$\mathscr{L}\cup\{=,F\}\vdash_e \forall x\exists y(F(x,y)\wedge A(x,y)),\Delta\longrightarrow\Lambda.$$

By substitution of $\lambda uv.A(u,v)$ for F, we obtain

$$\mathscr{L}\cup\{=\}\vdash_e \forall x\exists y(A(x,y)\wedge A(x,y)),\Delta\longrightarrow\Lambda.$$

Hence

$$\mathscr{L}\cup\{=\}\vdash_e \forall x\exists yA(x,y),\Delta\longrightarrow\Lambda. \tag{12}$$

By Lemma 10, we conclude

$$\mathscr{L}\vdash\forall x\exists yA(x,y),\Delta\longrightarrow\Lambda.\quad\square$$

Remark. Another proof is sketched in [Mo 82], which can be stated as follows. From (8) we have

$$\mathscr{L}\cup\{=,F\}\vdash_e\forall x\forall y(F(x,y)\supset A(x,y)),\mathrm{Ex}(F),\mathrm{Un}(F),\Delta\longrightarrow\Lambda. \tag{13}$$

Since $F\notin\mathrm{Pred}^-(\forall x\forall y(F(x,y)\supset A(x,y)),\Delta\rightarrow\Lambda)$,

$$\mathscr{L}\cap\{=,F\}\vdash_e\forall x\forall y(F(x,y)\supset A(x,y)),\mathrm{Ex}(F),\Delta\longrightarrow\Lambda \tag{14}$$

by [Mo 82, Theorem 1]. We obtain (12) by substituting $\lambda uv.A(u,v)$ for F.

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