Hitotsubashi Journal of Economics 51 (2010), pp.115-128.

COMPARING COURNOT AND STACKELBERG DUOPOLY*

Eiichi Chuman

School of Science and Engineering, University of Tsukuba
Tsukuba, Ibaraki 305-8577, Japan
eichuman@hotmail.com

Received December 2009; Accepted July 2010

Abstract

Equilibrium properties of Cournot and Stackelberg duopoly are compared with respect to outputs, profits, and welfare, the results of which depend on some conditions expressed in terms of the elasticities of the cost and demand functions. In particular, the case where the condition owing to Fisher (1961), Hahn (1962), and Okuguchi (1964, 1976, 1999) is not satisfied exactly corresponds to the case where the marginal revenue curve is steeper than the demand curve, the marginal cost curve is decreasing, and the demand curve is elastic to some extent, the case of which gives rise to different results from Okuguchi's (1999).

Key words: Cournot and Stackelberg duopoly, the elasticities of the cost and demand functions, Fisher-Hahn-Okuguchi condition
JEL Classification Codes: D43, L13

I. *Introduction*

In a sequential game with two identical players, Gal-Or (1985) explored the conditions on which a first mover (a leader) or a second mover (a follower) gains a higher profit than the other and showed that the first-mover or second-mover advantage over the rival arises if the reaction function of the follower is downward or upward sloping, respectively.

Okuguchi (1999) compared Cournot duopoly (a simultaneous-mover game) and Stackelberg duopoly (a sequential-mover game) with respect to outputs and profits and showed the results as follows. First, if the follower's reaction function is downward sloping, then the outputs and the profits are the largest in the (Stackelberg) leader case, the second largest in the Cournot case, and the smallest in the (Stackelberg) follower case. Second, if the follower's reaction function is upward sloping, then the outputs are the largest in the Cournot case, the second largest in the follower case, and the smallest in the leader case, whereas the profits are the largest in the follower case, the second largest in the leader case, and the smallest in the Cournot case.

Fisher (1961), Hahn (1962), and Okuguchi (1964, 1976, 1999) assumed that (1) for each firm, the marginal cost should rise more rapidly than the marginal revenue, with other firms

* I am grateful to an anonymous referee for helpful suggestions and comments.

expanding their 'collective' output, and that (2) for each firm, the marginal cost should not fall more rapidly than the demand for total industry output. The first assumption was given as a sufficient condition for each firm to maximize the profit with respect to its output, which is known as the second-order condition for the optimum. The second assumption was given as a condition for the Cournot equilibrium to be stable, which we call *Fisher-Hahn-Okuguchi condition*.

In this paper we compare equilibrium properties of Cournot and Stackelberg duopoly with respect to outputs, profits, and *welfare*, also examining the case where F-H-O condition is *not* satisfied whereas the second-order condition for the optimum is still satisfied, the case of which Okuguchi (1999) did not examine. In order to deal simultaneously with conditions on whether the follower's reaction function is downward or upward sloping and those on whether F-H-O condition is satisfied or not, given the second-order condition for the optimum, we express these conditions in terms of the elasticities of the cost and demand functions.

We show that the results of comparison with respect to outputs and profits are different from Okuguchi's (1999) in the case where F-H-O condition is not satisfied whereas the second-order condition for the optimum is still satisfied. That is, the output is the largest in the follower case, the second largest in the Cournot case, and the smallest in the leader case, even if the follower's reaction function is downward sloping, whereas the profit is the largest in the Cournot case.

As for consumer surplus, the results of comparison are summarized as follows. Consumer surplus is *larger* in the Cournot equilibrium than in the Stackelberg equilibrium if the follower's reaction function is upward sloping and not so steep. On the contrary, consumer surplus is *smaller* in the Cournot equilibrium if the follower's reaction function is downward sloping, or if the follower's reaction function is upward sloping and so steep.

As for producer surplus, the results of comparison are summarized as follows. Producer surplus is *larger* in the Cournot equilibrium than in the Stackelberg equilibrium if (1) the marginal revenue function is steeper than the demand function, the marginal cost functions are non-decreasing, and the demand function is weakly convex and less elastic, or if (2) the marginal revenue function is steeper than the demand function, the marginal cost functions are decreasing, and the demand function is weakly convex and elastic to some extent, with some additional condition for the outputs. On the contrary, producer surplus is *smaller* in the Cournot equilibrium if (3) the demand function is steeper than the marginal revenue function, the marginal cost functions are non-decreasing, and the demand function is convex to some extent and less elastic, if (4) the demand function is steeper than the marginal revenue function, the marginal cost functions are decreasing, and the demand function is convex to some extent and elastic to some extent, or if (5) the marginal cost functions are steeper than the marginal revenue function, the marginal cost functions are non-decreasing, and the demand function is strongly convex and more elastic.

The remainder of this paper is organized as follows. In Section II we formulate basic models of Cournot and Stackelberg duopoly. In Section III we compare equilibrium properties of Cournot and Stackelberg duopoly with respect to outputs, profits, and welfare, i.e., consumer surplus, producer surplus, and total surplus (defined as the sum of consumer surplus and producer surplus). We attempt to classify the results of comparison by some conditions on the slope of the follower's reaction function expressed in terms of the elasticities of the cost and demand functions. In Section IV we provide concluding remarks.

II. *Basic Models*

We assume that there are two firms, labeled 1, 2 respectively, in an industry, each of which produces x_i ($i=1, 2$) units of homogenous commodities to be sold in a single market; all the outputs are non-negative, i.e., $x_i \geq 0$, $i=1, 2$. Denote by $X=x_1+x_2$ the total output in the industry.

Let $p=P(X)$ be the (inverse) demand, where p is the market price of the good. We assume that the function $P(X)$ is continuous, twice differentiable, and monotonically decreasing. We also assume that $0 \leq x_i \leq M$, $i=1, 2$, such that $P(X)=0$ if $X=x_1+x_2 \geq M$ for some $M>0$.

Let $C_i(x_i)$ be the (total) cost function of firm i that produces x_i units of the output. We assume that the function $C_i(x_i)$ is continuous, twice differentiable, and monotonically non-decreasing.

The profit of firm i is written in the form $\pi_i(x_i, x_j)=x_iP(X)-C_i(x_i)$, $i, j=1, 2, j \neq i$. In order to guarantee the non-negative profit, we may assume that $C_i(0)=0$ for all i.[1] Note that since $P(X)$ and $C_i(x_i)$ are continuous and twice differentiable by assumption, so is $\pi_i(x_i, x_j)$.

Firm i maximizes the profit $\pi_i(x_i, x_j)=x_iP(X)-C_i(x_i)$ with respect to the output x_i. The first-order condition for the optimum for firm i is given by

$$P(\varphi_i(x_j)+x_j)+\varphi_i(x_j)P'(\varphi_i(x_j)+x_j)-C_i'(\varphi_i(x_j))=0, \tag{1}$$

where $\varphi_i(x_j)=x_i^C$ is the reaction function of firm i. One could show the following properties of the function φ_i, which we use hereafter:[2]

(i) φ_i *is continuous.*
(ii) φ_i *is differentiable in an open neighborhood of* x_j^C.

A point (x_1^C, x_2^C) at which the curves φ_1 and φ_2 cross with each other defines the Cournot duopoly equilibrium. In order to guarantee that the solution x_i^C of equation (1) uniquely exists, we assume that π_i is strictly concave, namely:

Condition 1. $\partial^2\pi_i/\partial x_i^2=(P'+x_iP'')+(P'-C_i'')<0$ at x_i^C for $i=1, 2$.

Given Condition 1, the second-order condition for the optimum is satisfied. *Throughout the paper we assume that Condition 1 is always satisfied.*

Differentiating (1) with respect to x_j and arranging terms, we have

$$\varphi_i'(x_j)=-(\partial^2\pi_i/\partial x_j\partial x_i)/(\partial^2\pi_i/\partial x_i^2)=-(P'+x_iP'')/\{(P'+x_iP'')+(P'-C_i'')\}, \tag{2}$$

which gives the slope of the reaction curve of firm i. From (2) we see that, given Condition 1, $\varphi_i'<0$ (resp. >0) as $\partial^2\pi_i/\partial x_j\partial x_i=P'+x_iP''<0$ (resp. >0), that is, quantities of the outputs are

[1] As for this point, see Okuguchi (1976).

[2] Let U be open in $\mathbf{R}^2$ and let $\partial\pi_i/\partial x_i$: $U \rightarrow \mathbf{R}$ be continuous and differentiable. Let $(x_i^C, x_j^C) \in U$, that is, U is an open neighborhood of a point (x_i^C, x_j^C), and assume that $\partial\pi_i(x_i^C, x_j^C)/\partial x_i=0$ but $\partial\pi_j(x_i^C, x_j^C)/\partial x_j \neq 0$. By the Implicit Function Theorem we observe that there exist an open neighborhood $U'$of x_j^C and a continuous and differentiable function φ_i: $U' \rightarrow \mathbf{R}$ such that $\varphi_i(x_j^C)=x_i^C$ and $\partial\pi_i(\varphi_i(x_j), x_j)/\partial x_i=0$ for all $x_j \in U'$. Okuguchi (1976) proved that the follower's reaction function is continuous, focusing on the property that a real-valued and strictly concave function on a set in $\mathbf{R}^2$ has a unique (global) maximum. As for the proof, see Takayama (1986).

strategic substitutes (resp. complements).

The following is what we call *Fisher-Hahn-Okuguchi condition*:

Condition 2. $P'-C_i''<0$ at x_i^C for $i=1, 2$.

Note that Condition 2 is equivalent to saying that $|\partial^2\pi_i/\partial x_i^2|>|\partial^2\pi_i/\partial x_j\partial x_i|$ at x_i^C for $i, j=1, 2, j\neq i$. F-H-O condition means that for each firm, the marginal cost should not fall more rapidly than the demand for total industry output.[3]

It follows that at x_i^C for $i=1, 2$,

$$\begin{aligned}&P'+x_iP''=(P/X\varepsilon)(1+\sigma_i/\eta)<0 \text{ (resp. } >0) \text{ as } 1+\sigma_i/\eta>0 \text{ (resp. } <0),\\&P'-C_i''=(P/X\varepsilon)\{1-\mu_i(1+\varepsilon)/\sigma_i\}<0 \text{ (resp. } >0) \text{ as } -\mu_i+\sigma_i>\text{(resp. } <) \ \mu_i\varepsilon,\\&(P'+x_iP'')+(P'-C_i'')=(P/X\varepsilon)\{2+\sigma_i/\eta-\mu_i(1+\varepsilon)/\sigma_i\}<0 \text{ as } -\mu_i+\sigma_i(2+\sigma_i/\eta)>\mu_i\varepsilon,\end{aligned} \tag{3}$$

where $\sigma_i=x_i/X$ with $0<\sigma_i<1$ represents the ratio of the size of firm i's production to the total production of the good, i.e., the market share of firm i's products; $\varepsilon=P/XP'$ with $\varepsilon<-1$ by (1), $\eta=P'/XP''$ with $\eta\geq 0$ (resp. <0) as $P''\leq 0$ (resp. >0), and $\mu_i=x_iC_i''/C_i'$ with $\mu_i\geq$ (resp. $<$) 0 as $C_i''\geq 0$ (resp. <0) represent the elasticity of the demand function, the elasticity of the slope of the demand function, and the elasticities of the marginal cost functions, respectively.[4] Using the terminology in Brander and Spencer (1984), we may say that η and μ_i measure the "relative curvature" of P and that of C_i at x_i^C, respectively. Since

$$\begin{aligned}&-1+\sigma_i/\mu_i\geq\text{(resp. } <) -1 \text{ as } \mu_i\geq 0 \text{ (resp. } <0),\\&-1+(\sigma_i/\mu_i)(2+\sigma_i/\eta)\geq\text{(resp. } <) -1+\sigma_i/\mu_i \text{ as } \mu_i(1+\sigma_i/\eta)\geq\text{(resp. } <) \ 0,\\&-1+(\sigma_i/\mu_i)(2+\sigma_i/\eta)\geq\text{(resp. } <) -1 \text{ as } \mu_i(2+\sigma_i/\eta)\geq\text{(resp. } <) \ 0,\end{aligned} \tag{4}$$

we have the following lemma:

Lemma 1. *In any of the following cases, Condition 1 is satisfied*:

(*a*) *We have* $P'+x_iP''<0$ *and* $P'-C_i''<0$ *if and only if*
(i) $\sigma_i/\eta>-1$, $\mu_i\geq 0$, *and* $\varepsilon<-1$, *or* (ii) $\sigma_i/\eta>-1$, $\mu_i<0$, *and* $-1+\sigma_i/\mu_i<\varepsilon<-1$.

(*b*) *We have* $P'+x_iP''>0$ *and* $P'-C_i''<0$ *if and only if*
(i) $-2<\sigma_i/\eta<-1$, $\mu_i\geq 0$, *and* $\varepsilon<-1$, (ii) $-2<\sigma_i/\eta<-1$, $\mu_i<0$, *and* $-1+(\sigma_i/\mu_i)(2+\sigma_i/\eta)<\varepsilon<-1$, *or* (iii) $\sigma_i/\eta<-2$, $\mu_i\geq 0$, *and* $\varepsilon<-1+(\sigma_i/\mu_i)(2+\sigma_i/\eta)$.

(*c*) *We have* $P'+x_iP''<0$ *and* $P'-C_i''>0$ *if and only if* $\sigma_i/\eta>-1$, $\mu_i<0$, *and* $-1+(\sigma_i/\mu_i)(2+\sigma_i/\eta)<\varepsilon<-1+\sigma_i/\mu_i$.

Note that Case (*a*) and Case (*b*) in Lemma 1 correspond to the case where Condition 1 and Condition 2 are satisfied and Case (*c*) corresponds to the case where Condition 2 is not satisfied whereas Condition 1 is still satisfied. Also note that $-1<\varphi_i'<0$ in Case (*a*), $\varphi_i'>0$ in Case (*b*), and $\varphi_i'<-1$ in Case (*c*). Thus the slope of firm i's reaction function has been expressed by the elasticities of the demand function and the market share of firm i's products. In other words, strategic substitution/complementarity is regarded as a consequence, not as an

[3] The Cournot (oligopoly or duopoly) equilibrium may be *unstable* even if F-H-O condition is satisfied. As for this point, see Seade (1980), Al-Nowaihi and Levine (1985), and Chuman (2008, 2009).

[4] In particular, the equality of the second of the three equations in (3) can be derived as follows: Since $P(1+1/\varepsilon)=C_i'$ by (1), expressing that for firm i the marginal revenue is equal to the marginal cost at x_i^C, we have $P'-C_i''=(P/X)\{(XP'/P)-(X/x_i)(x_iC_i''/C_i')(C_i'/P)\}=(P/X\varepsilon)\{1-\mu_i(1+\varepsilon)/\sigma_i\}$, using $C_i'/P=1+1/\varepsilon$.

assumption often appeared in the industrial organization literature.[5]

Remark. Let $\sigma_1=\sigma_2=1/2$ and $\mu_1=\mu_2=\mu$ in Lemma 1. Then by (3) we can check that:

(*a*) $-1<\varphi_2'<0$ if and only if (i) $\eta\geq 0$ or $\leq -1/2$, $\mu\geq 0$, and $\varepsilon<-1$, or (ii) $\eta\geq 0$ or $\leq -1/2$, $\mu<0$, and $-1+1/2\mu<\varepsilon<-1$;

(*b*) $0<\varphi_2'<1$ if and only if (i) $\mu\geq 0$, $-1/2<\eta<-1/3$, and $\varepsilon<-1$, (ii) $\mu\geq 0$, $-1/3<\eta<-1/4$, and $\varepsilon<-1+(1/\mu)(3/2+1/2\eta)$, (iii) $\mu\geq 0$, $-1/4<\eta<0$, and $\varepsilon<-1+(1/\mu)(3/2+1/2\eta)$, or (iv) $\mu<0$, $-1/2<\eta<-1/3$, and $-1+(1/\mu)(3/2+1/2\eta)<\varepsilon<-1$; $\varphi_2'>1$ if and only if (v) $\mu\geq 0$, $-1/3<\eta<-1/4$, and $-1+(1/\mu)(3/2+1/2\eta)<\varepsilon<-1$, (vi) $\mu\geq 0$, $-1/4<\eta<0$, and $-1+(1/\mu)(3/2+1/2\eta)<\varepsilon<-1+(1/\mu)(1+1/4\eta)$, (vii) $\mu<0$, $-1/2<\eta<-1/3$, and $-1+(1/\mu)(1+1/4\eta)<\varepsilon<-1+(1/\mu)(3/2+1/2\eta)$, or (viii) $\mu<0$, $-1/3<\eta<-1/4$, and $-1+(1/\mu)(1+1/4\eta)<\varepsilon<-1$;

(*c*) $\varphi_2'<-1$ if and only if $\eta\geq 0$ or $\leq -1/2$, $\mu<0$, and $-1+(1/\mu)(1+1/4\eta)<\varepsilon<-1+1/2\mu$.

In addition, we can check that conditions (*b*)(i)(ii)(v) correspond to Case (*b*)(i) in Lemma 1, conditions (*b*)(iv)(vii)(viii) correspond to Case (*b*)(ii) in Lemma 1, and conditions (*b*)(iii)(vi) correspond to Case (*b*)(iii) in Lemma 1.

Further note that $\sigma_i/\eta>-1$, corresponding to the case where the demand function is not too convex, is equivalent to that $|2P'+x_iP''|>|P'|$, that is, the marginal revenue function is steeper than the demand function; $-2<\sigma_i/\eta<-1$ is equivalent to that $|2P'+x_iP''|<|P'|$, that is, the demand function is steeper than the marginal revenue function; $\sigma_i/\eta<-2$, corresponding to the case where the demand function is too convex, is equivalent to that $0<2P'+x_iP''<C_i''$, that is, the marginal cost functions are strictly increasing and steeper than the marginal revenue function.

In summary, Case (*a*) in Lemma 1 occurs if (i) the marginal revenue function is steeper than the demand function, the marginal cost functions are non-decreasing, and the demand function is less elastic, or if (ii) the marginal revenue function is steeper than the demand function, the marginal cost functions are decreasing, and the demand function is elastic to some extent; Case (*b*) occurs if (i) the demand function is steeper than the marginal revenue function, the marginal cost functions are non-decreasing, and the demand function is less elastic, if (ii) the demand function is steeper than the marginal revenue function, the marginal cost functions are decreasing, and the demand function is elastic to some extent, or if (iii) the marginal cost functions are steeper than the marginal revenue function, the marginal cost functions are non-decreasing, and the demand function is more elastic; Case (*c*) occurs if the marginal revenue function is steeper than the demand function, the marginal cost functions are decreasing, and the demand function is elastic to some extent.

Hereafter we focus on the case where all firms are identical, i.e., $C_i(x_i)=C(x_i)$, $i=1, 2$. Denote the total output and firm *i*'s profit in the Cournot duopoly equilibrium by X^C, where $X^C=x_1^C+x_2^C$, and π_i^C, respectively. Since all firms compete in the same product market and produce with the identical cost functions by hypothesis, we have $x_1^C=x_2^C$ and $\pi_1^C=\pi_2^C$. Immediately, $\sigma_1=\sigma_2=1/2$ and $\mu_1=\mu_2=\mu$ (constant).

Let us explore the Stackelberg duopoly. Let firm 1 be the leader and firm 2 the follower. Firm 2, taking firm 1's output x_1 as given, maximizes the profit $\pi_2(x_1, x_2)=x_2P(x_1+x_2)-C(x_2)$ with respect to the output x_2. Firm 1, knowing firm 2's reaction function $\varphi_2(x_1)$, maximizes the

[5] As for this point, see Brander (1995) and Shapiro (1989), for example.

profit $\pi_1(x_1, \varphi_2(x_1)) = x_1P(x_1 + \varphi_2(x_1)) - C(x_1)$ with respect to the output x_1.[6] Denote by x_2^F and x_1^L the equilibrium output of the follower and that of the leader, respectively, where $x_2^F = \varphi_2(x_1^L)$. Denote by X^S the total output in the Stackelberg duopoly equilibrium, where $X^S = x_1^L + x_2^F$.

The first-order condition for firm 2 to maximize $\pi_2(x_1, x_2) = x_2P(x_1 + x_2) - C(x_2)$ with respect to x_2, taking x_1 as given, is given by

$$P(X^S) + x_2^F P'(X^S) - C'(x_2^F) = 0. \tag{5}$$

The first-order condition for firm 1 to maximize $\pi_1(x_1, \varphi_2(x_1)) = x_1P(x_1 + \varphi_2(x_1)) - C(x_1)$ with respect to x_1 is given by

$$P(X^S) + x_1^L\{1 + \varphi_2'(x_1^L)\}P'(X^S) - C'(x_1^L) = 0. \tag{6}$$

Denote by π_i^L and π_i^F, $i = 1, 2$, the profit of the leader and that of the follower in the Stackelberg duopoly equilibrium, respectively. Since all firms are identical and compete in the same product market by hypothesis, we have $x_1^L = x_2^L$, $x_1^F = x_2^F$, $\pi_1^L = \pi_2^L$, and $\pi_1^F = \pi_2^F$. So it will suffice to explore firm 1's side alone.

III. *Comparison of the Two Duopoly Equilibria*

Let us now compare equilibrium properties of the Cournot and Stackelberg duopoly with respect to outputs, profits, and welfare.

First, as for the results of comparison with respect to the equilibrium outputs and profits, we have the following proposition:[7]

Proposition 1.

(a) *In the case where* $-1 < \varphi_2' < 0$ *(that is, the follower's reaction function is downward sloping and not so steep),* $x_1^L > x_1^C > x_1^F$ *and* $\pi_1^L > \pi_1^C > \pi_1^F$.

(b) *In the case where* $0 < \varphi_2' < 1$ *(that is, the follower's reaction function is upward sloping and not so steep),* $\pi_1^F > \pi_1^L > \pi_1^C$ *and* $x_1^C > x_1^F > x_1^L$*; in the case where* $\varphi_2' > 1$ *(that is, the follower's reaction function is upward sloping and so steep),* $\pi_1^F > \pi_1^L > \pi_1^C$ *and* $x_1^F > x_1^L > x_1^C$.

(c) *In the case where* $\varphi_2' < -1$ *(that is, the follower's reaction function is downward sloping and so steep),* $x_1^F > x_1^C > x_1^L$*;* $\pi_1^C > \pi_1^F > \pi_1^L$ *if* $P(X^S) - C'(x'') > 0$ *and* $\pi_1^C > \pi_1^L > \pi_1^F$ *otherwise, where* x'' *is a number between* x_1^L *and* x_1^F *such that* $C(x_1^L) - C(x_1^F) = (x_1^L - x_1^F)C'(x'')$.

Proof. See Appendix A1. ■

Intuition for the results in Proposition 1 can be explained as follows. As for Part (a), which corresponds to the strategic-substitutes case (i.e., $\partial^2\pi_2/\partial x_1\partial x_2 < 0$), firm 1, by producing

---

[6] We can also consider the situation in which there exist one leader and $n-1$ ($n \geq 3$) followers in the industry. The followers cooperatively maximize the profit π_2 with respect to their 'joint' output $x_2 = \sum_{i=1}^{n-1} x_{2i}$, taking the leader's output x_1 as given. Immediately, $\pi_2(x_1, x_2) = x_2P(x_1 + x_2) - C(x_2) > \sum_{i=1}^{n-1} \pi_{2i}(x_1, x_{2i}) = \sum_{i=1}^{n-1}\{x_{2i}P(x_1 + x_2) - C(x_{2i})\}$ if and only if $C(\sum_{i=1}^{n-1} x_{2i}) < \sum_{i=1}^{n-1} C(x_{2i})$, that is, the 'joint' cost of production is smaller under cooperation than under non-cooperation. The followers' reaction function $\varphi_2(x_1)$ will be defined as usual. The leader, knowing $\varphi_2(x_1)$, non-cooperatively maximizes the profit $\pi_1(x_1, \varphi_2(x_1)) = x_1P(x_1 + \varphi_2(x_1)) - C(x_1)$ with respect to x_1.

[7] Okuguchi (1999) had the same results as those in Part (a) and the first case of Part (b) in Proposition 1, focusing not on the elasticities ε, η, μ but on the slope of firm i's reaction function, i.e., the sign of φ_2'.

the output x_1 more, forces firm 2 to gain the marginal profit $\partial\pi_2/\partial x_2$ less and to produce the output x_2 less, which leads to $x_1^L > x_1^C$ and $x_2^C > x_2^F$. Assuming gross substitutes (i.e., $\partial\pi_i/\partial x_j = x_i P' < 0$ for $i, j = 1, 2, j \neq i$), we have $\pi_1^L > \pi_1^C$ and $\pi_2^C > \pi_2^F$. As for Part (*b*), which corresponds to the strategic-complements case (i.e., $\partial^2\pi_2/\partial x_1 \partial x_2 > 0$), in case $0 < \varphi_2{}' < 1$, firm 1, by producing x_1 less, forces firm 2 to gain the marginal profit $\partial\pi_2/\partial x_2$ less and to produce the output x_2 less, which leads to $x_1^L < x_1^C$ and $x_2^C > x_2^F$. Assuming gross substitutes, we have $\pi_1^L > \pi_1^C$ and $\pi_2^C > \pi_2^F$.

Note that the results of comparison in Part (*c*) are much different from those in Part (*a*), both of which correspond to the strategic-substitutes cases, in that whereas the output of the leader is the *largest* in Part (*a*), it is the *smallest* in Part (*c*). This is because whereas the output of the leader is larger than that of the follower if F-H-O condition is satisfied, this is not the case if F-H-O condition is not satisfied. Likewise, as for Part (*b*), the results of comparison in the second case are much different from those in the first case, both of which correspond to the strategic-complements cases, in that whereas the output of the Cournot player is the *largest* in the first case, it is the *smallest* in the second case. This is because whereas the output of the Cournot player is larger than that of the Stackelberg leader if the follower's reaction function is not so steep, this is not the case if the follower's reaction function is so steep.

Define consumer surplus by $\Gamma(X) = \int_0^X P(Y)dY - P(X)X$. Write $\Gamma^S = \Gamma(X^S)$ and $\Gamma^C = \Gamma(X^C)$. Then we have the following proposition:

Proposition 2.

(*a*) *In the case where* $-1 < \varphi_2{}' < 0$, *consumer surplus is smaller in the Cournot equilibrium than in the Stackelberg equilibrium.*

(*b*) *In the case where* $0 < \varphi_2{}' < 1$, *consumer surplus is larger in the Cournot equilibrium than in the Stackelberg equilibrium; in the case where* $\varphi_2{}' > 1$, *consumer surplus is smaller in the Cournot equilibrium than in the Stackelberg equilibrium.*

(*c*) *In the case where* $\varphi_2{}' < -1$, *consumer surplus is smaller in the Cournot equilibrium than in the Stackelberg equilibrium.*

Proof. See Appendix A2. ■

Intuition for the results in Proposition 2 can be explained as follows. $X^S >$ (resp. $<$) X^C corresponds to $P(X^S) <$ (resp. $>$) $P(X^C)$, which corresponds to $\Gamma^S >$ (resp. $<$) Γ^C. In other words, the larger the total output in the industry, the lower the price of the good, and hence consumer surplus is the larger.

Proposition 2 means that consumer surplus is *smaller* in the Cournot equilibrium than in the Stackelberg equilibrium if the follower's reaction function is downward sloping or if the follower's reaction function is upward sloping and so steep. On the contrary, consumer surplus is *larger* in the Cournot equilibrium if the follower's reaction function is upward sloping and not so steep.

Define producer surplus by $\Pi = \pi_1 + \pi_2$. Write $\Pi^S = \pi_1^L + \pi_2^F = P(X^S)X^S - \{C(x_1^L) + C(x_2^F)\}$ and $\Pi^C = \pi_1^C + \pi_2^C = P(X^C)X^C - \{C(x_1^C) + C(x_2^C)\}$. Then we have the following proposition:

Proposition 3.

(*a*) *In the case where* $-1 < \varphi_2{}' < 0$ *with* $\eta \leq -1/2$ *and* $\mu \geq 0$, *producer surplus is larger in the Cournot equilibrium than in the Stackelberg equilibrium.*

(*b*) *In the case where* $\varphi_2'>0$, *producer surplus is smaller in the Cournot equilibrium than in the Stackelberg equilibrium.*

(*c*) *In the case where* $\varphi_2'<-1$ *with* $\eta\leq-1/2$, *producer surplus is larger in the Cournot equilibrium than in the Stackelberg equilibrium, provided that* $x^{+}>x^{++}$, *where* x^{+} *is a number between* x_1^C *and* x_1^L *such that* $C(x_1^L)-C(x_1^C)=(x_1^L-x_1^C)C'(x^{+})$ *and* x^{++} *is a number between* x_2^C *and* x_2^F *such that* $C(x_2^C)-C(x_2^F)=(x_2^C-x_2^F)C'(x^{++})$.

Proof. See Appendix A3. ■

Proposition 3 means that producer surplus is larger in the Cournot equilibrium than in the Stackelberg equilibrium if (1) the marginal revenue function is steeper than the demand function, the marginal cost functions are non-decreasing, and the demand function is weakly convex and less elastic. On the contrary, producer surplus is *smaller* in the Cournot equilibrium if (2) the demand function is steeper than the marginal revenue function, the marginal cost functions are non-decreasing, and the demand function is convex to some extent and less elastic, if (3) the demand function is steeper than the marginal revenue function, the marginal cost functions are decreasing, and the demand function is convex to some extent and elastic to some extent, if (4) the marginal cost functions are steeper than the marginal revenue function, the marginal cost functions are non-decreasing, and the demand function is strongly convex and more elastic, or if (5) the marginal revenue function is steeper than the demand function, the marginal cost functions are decreasing, and the demand function is weakly convex and elastic to some extent, together with $x^{+}>x^{++}$.

Define total surplus by $\Omega=\Gamma+\Pi$. Write $\Omega^S=\Gamma^S+\Pi^S$ and $\Omega^C=\Gamma^C+\Pi^C$. Then we have the following proposition:

Proposition 4.

(*a*) *In the case where* $-1<\varphi_2'<0$, *total surplus is smaller in the Cournot equilibrium than in the Stackelberg equilibrium if* $C''\geq0$, $\varphi_2(x^*)>x^*$, *and* $-1\leq-P'/(P'-C'')<\varphi_2'$, *if* $C''<0$, $\varphi_2(x^*)<x^*$, *and* $-P'/(P'-C'')<-1<\varphi_2'$, *or if* $C''\geq0$, $\varphi_2(x^*)>x^*$, *and* $-1<\varphi_2'<-P'/(P'-C'')$, *where* x^* *is a number between* x_1^L *and* x_1^C *such that* $\Omega(x_1^L)-\Omega(x_1^C)=(x_1^L-x_1^C)\Omega'(x^*)$.

(*b*) *In the case where* $0<\varphi_2'<1$, *total surplus is larger in the Cournot equilibrium than in the Stackelberg equilibrium if* $C''\geq0$ *and* $\varphi_2(x^*)>x^*$, *or if* $C''<0$ *and* $\varphi_2(x^*)<x^*$; *in the case where* $\varphi_2'>1$, *total surplus is smaller in the Cournot equilibrium than in the Stackelberg equilibrium if* $C''\geq0$ *and* $\varphi_2(x^*)>x^*$, *or if* $C''<0$ *and* $\varphi_2(x^*)<x^*$.

(*c*) *In the case where* $\varphi_2'<-1$, *total surplus is larger in the Cournot equilibrium than in the Stackelberg equilibrium if* $\varphi_2(x^*)<x^*$.

Proof. See Appendix A4. ■

Proposition 4 implies that, as long as the follower's reaction function is not so steep (i.e., $|\varphi_2'|<1$), the policy such that welfare is improved in behalf of *consumers* will be preferable. That is, the policy such that the Stackelberg leader should be advantageous in the industry will be preferable in case $-1<\varphi_2'<0$, whereas the policy such that the Cournot player should be advantageous in the industry will be preferable in case $0<\varphi_2'<1$. On the contrary, as long as the follower's reaction function is so steep (i.e., $|\varphi_2'|>1$), the policy such that welfare is improved in behalf of *producers* will be preferable. That is, the policy such that the Stackelberg follower should be advantageous in the industry will be preferable in case $\varphi_2' > 1$ and in case $\varphi_2' < -1$.

IV. *Concluding Remarks*

In this paper we have compared equilibrium properties of Cournot and Stackelberg duopoly with respect to outputs, profits, and welfare, focusing on whether F-H-O condition is satisfied or not, provided that the second-order condition for the optimum is still satisfied. The analysis can be extended to a case with symmetric more-than-two firms. We have pointed out that the results of comparison depend on some conditions expressed in terms of the elasticities of the cost and demand functions. In particular, we have shown that the case where F-H-O condition is not satisfied exactly corresponds to the case where the marginal revenue function is steeper than the demand function, the marginal cost functions are decreasing, and the demand function is elastic to some extent, the case of which gives rise to different results from Okuguchi's (1999).

We have derived conditions for a first mover, a second mover, or a simultaneous mover to be more advantageous than the others, those of which are expressed in terms of the elasticities of the cost and demand functions. Conversely, given the Cournot and Stackelberg duopoly, will the elasticities be estimated as those derived above? It is left to future studies to investigate this problem.

Appendix

In this Appendix, we first introduce several lemmas, which we use in turn to prove the propositions.

It follows from (2) that

$$1+\varphi_2'(x_1)=(P'-C'')/\{(P'+x_2P'')+(P'-C'')\}>0 \text{ (resp. } <0) \text{ as } P'-C''<0 \text{ (resp. } >0). \quad \text{(A1)}$$

It follows from (A1) and (6) that

$$\begin{gathered} P(X^S)-C'(x_1^L)=-x_1^L\{1+\varphi_2'(x_1^L)\}P'(X^S)>0 \text{ (resp. } <0) \\ \text{as } 1+\varphi_2'(x_1^L)>0 \text{ (resp. } <0\text{), i.e., } P'-C''<0 \text{ (resp. } >0). \end{gathered} \quad \text{(A2)}$$

It follows from (5) that

$$P(X^S)-C'(x_1^F)=-x_1^F P'(X^S)>0. \quad \text{(A3)}$$

It follows from (A2) and (A3) that

$$C'(x_1^L)-C'(x_2^F)=\{(x_1^L-x_2^F)+x_1^L\varphi_2'(x_1^L)\}P'(X^S). \quad \text{(A4)}$$

As for the left-hand side of (A4), since by the Mean Value Theorem there is some number x' between x_1^L and x_2^F such that $C'(x_1^L)-C'(x_2^F)=(x_1^L-x_2^F)C''(x')$, we obtain

$$(x_1^L-x_2^F)\{P'(X^S)-C''(x')\}=-x_1^L\varphi_2'(x_1^L)P'(X^S). \quad \text{(A5)}$$

Putting $x_1^F = x_2^F$ into (A5), we obtain

$$\mathrm{sign}(x_1^L - x_1^F)\{P'(X^S) - C''(x')\} = \mathrm{sign}\{\varphi_2'(x_1^L)\}. \tag{A6}$$

In light of (A6), we have

$$\begin{aligned}&\text{(i) If } P' - C'' < 0, \text{ then } x_1^L > x_1^F \text{ (resp. } x_1^L < x_1^F\text{) as } \varphi_2' < 0 \text{ (resp. } > 0\text{).}\\ &\text{(ii) If } P' - C'' > 0, \text{ then } x_1^L < x_1^F \text{ as } \varphi_2' < 0.\end{aligned} \tag{A7}$$

By the Mean Value Theorem, we have $x_2^F - x_2^C = \varphi_2(x_1^L) - \varphi_2(x_1^C) = (x_1^L - x_1^C)\varphi_2'(x^0)$, where x^0 is a number between x_1^L and x_1^C. Hence

$$X^S - X^C = (x_1^L - x_1^C) + (x_2^F - x_2^C) = (x_1^L - x_1^C)\{1 + \varphi_2'(x^0)\}. \tag{A8}$$

It follows that $\pi_1^L - \pi_1^F = (x_1^L - x_1^F)P(X^S) - \{C(x_1^L) - C(x_1^F)\}$ and hence

$$\pi_1^L - \pi_1^F = (x_1^L - x_1^F)\{P(X^S) - C'(x'')\}, \tag{A9}$$

where x'' is a number between x_1^L and x_1^F such that $C(x_1^L) - C(x_1^F) = (x_1^L - x_1^F)C'(x'')$.

Consider Case (a) in Lemma 1. By the first part of (A7), since $-1 < \varphi_2' < 0$, we see that

$$x_1^L - x_1^C > x_1^F - x_1^C = x_2^F - x_2^C = \varphi_2(x_1^L) - \varphi_2(x_1^C) = (x_1^L - x_1^C)\varphi_2'(x^0), \tag{A10}$$

whence $(x_1^L - x_1^C)\{1 - \varphi_2'(x^0)\} > 0$, which implies that $x_1^L > x_1^C$. Hence $x_1^F < x_1^C$ by (A10). We see that

$$\begin{aligned}\pi_1^F &= x_1^F P(x_1^L + x_2^F) - C(x_1^F)\\ &< x_1^F P(x_1^C + x_2^F) - C(x_1^F) \text{ since } x_1^L > x_1^C\\ &< x_1^F P(x_1^F + x_2^C) - C(x_1^C) \text{ since } x_1^C = x_2^C \text{ and } x_1^F = x_2^F\\ &< x_1^C P(x_1^C + x_2^C) - C(x_1^C) = \pi_1^C,\end{aligned} \tag{A11}$$

where the last inequality holds since $x_1 P(x_1 + x_2^C) - C(x_1)$ has a maximum at $x_1 = x_1^C$. In like manner,

$$\begin{aligned}\pi_1^C &= x_1^C P(x_1^C + \varphi_2(x_1^C)) - C(x_1^C)\\ &< x_1^C P(x_1^C + \varphi_2(x_1^L)) - C(x_1^C) \text{ since } x_2^F = \varphi_2\,(x_1^L) < \varphi_2(x_1^C) \text{ as } x_1^L > x_1^C\\ &< x_1^L P(x_1^L + x_2^F) - C(x_1^L) = \pi_1^L,\end{aligned} \tag{A12}$$

where the last inequality holds since $x_1 P(x_1 + x_2^F) - C(x_1)$ has a maximum at $x_1 = x_1^L$. If $C'' \geq 0$, then, since $x_1^F < x'' < x_1^L$ by the first part of (A7), $C'(x_1^F) \leq C'(x'') \leq C'(x_1^L)$. Taking into account (A2), we obtain $P(X^S) - C'(x'') \geq P(X^S) - C'(x_1^L) > 0$. Hence $\pi_1^L > \pi_1^F$ by (A9). On the other hand, if $C'' < 0$, then, since $x_1^F < x'' < x_1^L$ by the first part of (A7), $C'(x_1^L) < C'(x'') < C'(x_1^F)$. Taking into account (A3), we obtain $P(X^S) - C'(x'') > P(X^S) - C'(x_1^F) > 0$. Hence we have $\pi_1^L > \pi_1^F$ by (A9).

Consider Case (b) in Lemma 1. By the first part of (A7), since $\varphi_2' > 0$, we see that

$$x_1^L - x_1^C < x_1^F - x_1^C = x_2^F - x_2^C = \varphi_2(x_1^L) - \varphi_2(x_1^C) = (x_1^L - x_1^C)\varphi_2'(x^0), \tag{A13}$$

whence $(x_1^L - x_1^C)\{1 - \varphi_2'(x^0)\} < 0$, which implies that $x_1^L < x_1^C$ if $0 < \varphi_2' < 1$ and $x_1^L > x_1^C$ if $\varphi_2' > 1$. Hence by (A13), $x_1^F < x_1^C$ if $0 < \varphi_2' < 1$ and $x_1^F > x_1^C$ if $\varphi_2' > 1$. First, if $0 < \varphi_2' < 1$, then we have

$$\begin{aligned}\pi_1^C &= x_1^C P(x_1^C + x_2^C) - C(x_1^C)\\ &< x_1^C P(x_1^L + x_2^F) - C(x_1^C) \text{ since } x_1^L < x_1^C \text{ and } x_1^L + x_2^F = X^S < X^C = x_1^C + x_2^C \text{ by (A8)}\\ &< x_1^F P(x_1^L + x_2^F) - C(x_1^F) = \pi_1^F,\end{aligned} \tag{A14}$$

where the last inequality holds since $x_1P(x_1+x_2^{\mathrm{F}})-C(x_1)$ has a maximum at $x_1=x_1^{\mathrm{F}}$, and

$$\begin{aligned}\pi_1^{\mathrm{C}}&=x_1^{\mathrm{C}}P(x_1^{\mathrm{C}}+\varphi_2(x_1^{\mathrm{C}}))-C(x_1^{\mathrm{C}})\\ &<x_1^{\mathrm{C}}P(x_1^{\mathrm{C}}+\varphi_2(x_1^{\mathrm{L}}))-C(x_1^{\mathrm{C}}) \text{ since } x_2^{\mathrm{F}}=\varphi_2(x_1^{\mathrm{L}})<\varphi_2(x_1^{\mathrm{C}}) \text{ as } x_1^{\mathrm{L}}<x_1^{\mathrm{C}}\\ &<x_1^{\mathrm{L}}P(x_1^{\mathrm{L}}+x_2^{\mathrm{F}})-C(x_1^{\mathrm{L}})=\pi_1^{\mathrm{L}},\end{aligned}\tag{A15}$$

where the last inequality holds since $x_1P(x_1+x_2^{\mathrm{F}})-C(x_1)$ has a maximum at $x_1=x_1^{\mathrm{L}}$. If $C''\geq 0$, then, since $x_1^{\mathrm{F}}>x''>x_1^{\mathrm{L}}$ by the first part of (A7), $C'(x_1^{\mathrm{F}})\geq C'(x'')\geq C'(x_1^{\mathrm{L}})$. Taking into account (A3), we obtain $P(X^{\mathrm{S}})-C'(x'')\geq P(X^{\mathrm{S}})-C'(x_1^{\mathrm{F}})>0$. Hence $\pi_1^{\mathrm{L}}<\pi_1^{\mathrm{F}}$ by (A9). If $C''<0$, then, since $x_1^{\mathrm{F}}>x''>x_1^{\mathrm{L}}$ by the first part of (A7), $C'(x_1^{\mathrm{F}})<C'(x'')<C'(x_1^{\mathrm{L}})$. Taking into account (A2), we obtain $P(X^{\mathrm{S}})-C'(x'')>P(X^{\mathrm{S}})-C'(x_1^{\mathrm{L}})>0$. Hence we have $\pi_1^{\mathrm{L}}<\pi_1^{\mathrm{F}}$ by (A9). Second, if $\varphi_2'>1$, then we have $\pi_1^{\mathrm{C}}<\pi_1^{\mathrm{F}}$, $\pi_1^{\mathrm{L}}<\pi_1^{\mathrm{F}}$, and $\pi_1^{\mathrm{C}}<\pi_1^{\mathrm{L}}$ in like manner.

Consider Case (*c*) in Lemma 1. By the second part of (A7), since $\varphi_2'<-1$, we see that $x_1^{\mathrm{L}}-x_1^{\mathrm{C}}<x_1^{\mathrm{F}}-x_1^{\mathrm{C}}=x_2^{\mathrm{F}}-x_2^{\mathrm{C}}=\varphi_2(x_1^{\mathrm{L}})-\varphi_2(x_1^{\mathrm{C}})=(x_1^{\mathrm{L}}-x_1^{\mathrm{C}})\varphi_2'(x^0)$, and hence $(x_1^{\mathrm{L}}-x_1^{\mathrm{C}})\{1-\varphi_2'(x^0)\}<0$, which implies that $x_1^{\mathrm{L}}<x_1^{\mathrm{C}}$. Immediately, we have $X^{\mathrm{S}}-X^{\mathrm{C}}=(x_1^{\mathrm{L}}-x_1^{\mathrm{C}})\{1+\varphi_2'(x^0)\}>0$ by (A8) and $x_1^{\mathrm{F}}-x_1^{\mathrm{C}}=(x_1^{\mathrm{L}}-x_1^{\mathrm{C}})\varphi_2'(x^0)>0$. Then

$$\begin{aligned}\pi_1^{\mathrm{F}}&=x_1^{\mathrm{F}}P(x_1^{\mathrm{L}}+x_2^{\mathrm{F}})-C(x_1^{\mathrm{F}})\\ &<x_1^{\mathrm{F}}P(x_1^{\mathrm{C}}+x_2^{\mathrm{C}})-C(x_1^{\mathrm{F}}) \text{ since } x_1^{\mathrm{L}}+x_2^{\mathrm{F}}=X^{\mathrm{S}}>X^{\mathrm{C}}=x_1^{\mathrm{C}}+x_2^{\mathrm{C}} \text{ by (A8)}\\ &<x_1^{\mathrm{C}}P(x_1^{\mathrm{C}}+x_2^{\mathrm{C}})-C(x_1^{\mathrm{C}})=\pi_1^{\mathrm{C}},\end{aligned}\tag{A16}$$

where the last inequality holds since $x_1P(x_1+x_2^{\mathrm{C}})-C(x_1)$ has a maximum at $x_1=x_1^{\mathrm{C}}$, and

$$\begin{aligned}\pi_1^{\mathrm{L}}&=x_1^{\mathrm{L}}P(x_1^{\mathrm{L}}+\varphi_2(x_1^{\mathrm{L}}))-C(x_1^{\mathrm{L}})\\ &<x_1^{\mathrm{L}}P(x_1^{\mathrm{L}}+\varphi_2(x_1^{\mathrm{C}}))-C(x_1^{\mathrm{L}}) \text{ since } \varphi_2(x_1^{\mathrm{L}})>\varphi_2(x_1^{\mathrm{C}})=x_2^{\mathrm{C}} \text{ as } x_1^{\mathrm{L}}<x_1^{\mathrm{C}}\\ &<x_1^{\mathrm{C}}P(x_1^{\mathrm{C}}+x_2^{\mathrm{C}})-C(x_1^{\mathrm{C}})=\pi_1^{\mathrm{C}},\end{aligned}\tag{A17}$$

where the last inequality holds since $x_1P(x_1+x_2^{\mathrm{C}})-C(x_1)$ has a maximum at $x_1=x_1^{\mathrm{C}}$. Since $C''<0$ and $x_1^{\mathrm{L}}<x''<x_1^{\mathrm{F}}$ by the second part of (A7), $C'(x_1^{\mathrm{L}})>C'(x'')>C'(x_1^{\mathrm{F}})$. Taking into account (A2) and (A3), we obtain $P(X^{\mathrm{S}})-C'(x_1^{\mathrm{F}})>P(X^{\mathrm{S}})-C'(x'')>P(X^{\mathrm{S}})-C'(x_1^{\mathrm{L}})$, where $P(X^{\mathrm{S}})-C'(x_1^{\mathrm{F}})>0$ and $P(X^{\mathrm{S}})-C'(x_1^{\mathrm{L}})<0$. Hence by (A9), we have $\pi_1^{\mathrm{L}}<\pi_1^{\mathrm{F}}$ if $P(X^{\mathrm{S}})-C'(x'')>0$ and $\pi_1^{\mathrm{L}}>\pi_1^{\mathrm{F}}$ if $P(X^{\mathrm{S}})-C'(x'')<0$.

In summary, we have the following result:

Lemma 2.
(*a*) *In Case (a) in Lemma 1*, $x_1^{\mathrm{L}}>x_1^{\mathrm{C}}>x_1^{\mathrm{F}}$ *and* $\pi_1^{\mathrm{L}}>\pi_1^{\mathrm{C}}>\pi_1^{\mathrm{F}}$.
(*b*) *In Case (b) in Lemma 1*, $\pi_1^{\mathrm{F}}>\pi_1^{\mathrm{L}}>\pi_1^{\mathrm{C}}$; $x_1^{\mathrm{C}}>x_1^{\mathrm{F}}>x_1^{\mathrm{L}}$ *if* $0<\varphi_2'<1$ *and* $x_1^{\mathrm{F}}>x_1^{\mathrm{L}}>x_1^{\mathrm{C}}$ *if* $\varphi_2'>1$.
(*c*) *In Case (c) in Lemma 1*, $x_1^{\mathrm{F}}>x_1^{\mathrm{C}}>x_1^{\mathrm{L}}$; $\pi_1^{\mathrm{C}}>\pi_1^{\mathrm{F}}>\pi_1^{\mathrm{L}}$ *if* $P(X^{\mathrm{S}})-C'(x'')>0$ *and* $\pi_1^{\mathrm{C}}>\pi_1^{\mathrm{L}}>\pi_1^{\mathrm{F}}$ *otherwise, where* x'' *is a number between* x_1^{L} *and* x_1^{F} *such that* $C(x_1^{\mathrm{L}})-C(x_1^{\mathrm{F}})=(x_1^{\mathrm{L}}-x_1^{\mathrm{F}})C'(x'')$.

We have the following lemma:

Lemma 3.
(*a*) *In Case (a) in Lemma 1*, $X^{\mathrm{S}}>X^{\mathrm{C}}$.
(*b*) *In Case (b) in Lemma 1*, $X^{\mathrm{S}}<X^{\mathrm{C}}$ *if* $0<\varphi_2'<1$ *and* $X^{\mathrm{S}}>X^{\mathrm{C}}$ *if* $\varphi_2'>1$.
(*c*) *In Case (c) in Lemma 1*, $X^{\mathrm{S}}>X^{\mathrm{C}}$.

Proof. Remember that $-1<\varphi_2'<0$, $\varphi_2'>0$, and $\varphi_2'<-1$ in Case (*a*), Case (*b*), and Case (*c*) in Lemma 1, respectively. Then the proof is immediate by (A8) and Lemma 2. ■

A1. *Proof of Proposition 1* The proof is immediate from Lemma 2 since $-1<\varphi_2'<0$, $\varphi_2'>0$, and $\varphi_2'<-1$ in Case (*a*), Case (*b*), and Case (*c*) in Lemma 1, respectively. ■

A2. *Proof of Proposition 2* By the integral by parts we have $\Gamma(X)=\int_0^X\{-YP'(Y)\}\,dY>0$. Note that $\{\Gamma(X)\}'=-XP'>0$. As for Part (*a*), since $X^S>X^C$ by Lemma 3, $-X^SP'(X^S)>-X^CP'(X^C)$. Hence $\Gamma^S=\Gamma(X^S)>\Gamma(X^C)=\Gamma^C$. As for Part (*b*), $X^S<X^C$ if $0<\varphi_2'<1$ and $X^S>X^C$ if $\varphi_2'>1$ by Lemma 3. Hence $\Gamma^S<\Gamma^C$ if $0<\varphi_2'<1$ and $\Gamma^S>\Gamma^C$ if $\varphi_2'>1$. As for Part (*c*), since $X^S>X^C$ by Lemma 3, $-X^SP'(X^S)>-X^CP'(X^C)$. Hence $\Gamma^S>\Gamma^C$, as desired. ■

A3. *Proof of Proposition 3* It follows that $\Pi^S-\Pi^C=(\pi_1^L-\pi_1^C)+(\pi_2^F-\pi_2^C)$, where, by way of the Mean Value Theorem,

$$\begin{aligned}\pi_1^L-\pi_1^C&=P(X^S)x_1^L-P(X^C)x_1^C-\{C(x_1^L)-C(x_1^C)\}\\&=(x_1^L-x_1^C)\{P(X^S)-C'(x^+)\}+x_1^C\{P(X^S)-P(X^C)\},\end{aligned}$$

where x^+ is a number between x_1^L and x_1^C such that $C(x_1^L)-C(x_1^C)=(x_1^L-x_1^C)C'(x^+)$, and

$$\begin{aligned}\pi_2^F-\pi_2^C&=P(X^S)x_2^F-P(X^C)x_2^C-\{C(x_2^F)-C(x_2^C)\}\\&=(x_2^F-x_2^C)\{P(X^S)-C'(x^{++})\}+x_2^C\{P(X^S)-P(X^C)\}\\&=(x_1^L-x_1^C)\varphi_2'(x^0)\{P(X^S)-C'(x^{++})\}+x_2^C\{P(X^S)-P(X^C)\},\end{aligned}$$

where x^{++} is a number between x_2^F and x_2^C such that $C(x_2^F)-C(x_2^C)=(x_2^F-x_2^C)C'(x^{++})$. By using $P(X^S)-P(X^C)=(X^S-X^C)P'(X^0)$, where X^0 is a number between X^S and X^C, together with $X^S-X^C=(x_1^L-x_1^C)\{1+\varphi_2'(x^0)\}$, we have

$$\Pi^S-\Pi^C=(x_1^L-x_1^C)[P(X^S)-C'(x^+)+X^CP'(X^0)+\varphi_2'(x^0)\{P(X^S)-C'(x^{++})+X^CP'(X^0)\}]. \quad \text{(A18)}$$

Remind that concerning the slope of the follower's reaction function, $-1<\varphi_2'<0$ in Part (*a*), $\varphi_2'>0$ in Part (*b*), and $\varphi_2'<-1$ in Part (*c*) in Proposition 1.

As for Part (*a*) in Proposition 3, first note that $x_1^L>x_1^C$, and hence $X^S>X^0$ $(>X^C)$ by Lemma 3. If $C''\geq 0$ (i.e., $\mu\geq 0$) then, since $x_1^F<x^{++}$ and hence $C'(x_1^F)\leq C'(x^{++})$, we obtain

$$\begin{aligned}&P(X^S)-C'(x^{++})+X^CP'(X^0)\\&\leq P(X^S)-C'(x_1^F)+X^CP'(X^0)=-x_1^FP'(X^S)+X^CP'(X^0)\ \text{(by (A3))}\\&<-x_1^FP'(X^S)+X^CP'(X^S)\ \text{if } P''>0\ \text{(i.e., } \eta\leq -1/2)\\&=\{(x_1^C-x_1^F)+x_2^C\}P'(X^S)<0.\end{aligned}$$

Therefore, since $C''\geq 0$, $\varphi_2'>-1$, and $x^+>x^{++}$, we obtain

$$\begin{aligned}&P(X^S)-C'(x^+)+X^CP'(X^0)+\varphi_2'(x^0)\{P(X^S)-C'(x^{++})+X^CP'(X^0)\}\\&<P(X^S)-C'(x^+)+X^CP'(X^0)-\{P(X^S)-C'(x^{++})+X^CP'(X^0)\}=C'(x^{++})-C'(x^+)<0.\end{aligned}$$

Hence $\Pi^S<\Pi^C$ by (A18).

As for Part (*b*), the proof is immediate from Proposition 1: $\Pi^S=\pi_1^L+\pi_2^F>\Pi^C=\pi_1^C+\pi_2^C$.

As for Part (*c*), first note that $x_1^L<x_1^C$, and hence $X^S>X^0$ $(>X^C)$. Since $C''<0$ (i.e., $\mu<0$), $x_1^L<x^{++}$, and $C'(x_1^L)>C'(x^{++})$, we obtain

$$\begin{aligned}&P(X^S)-C'(x^{++})+X^CP'(X^0)\\&<P(X^S)-C'(x_1^L)+X^CP'(X^0)=-x_1^L\{1+\varphi_2'(x_1^L)\}P'(X^S)+X^CP'(X^0)\ \text{(by (A2))}\end{aligned}$$

$$< -x_1^L\{1+\varphi_2'(x_1^L)\}P'(X^S)+X^C P'(X^S) \text{ if } P''>0 \text{ (i.e., } \eta\leq -1/2)$$
$$=\{(x_1^C-x_1^L)+x_2^C-x_1^L\varphi_2'(x_1^L)\}P'(X^S)<0.$$

Therefore, if $x^+ > x^{++}$ then, since $C''<0$ and $\varphi_2'<-1$, we obtain

$$P(X^S)-C'(x^+)+X^C P'(X^0)+\varphi_2'(x^0)\{P(X^S)-C'(x^{++})+X^C P'(X^0)\}$$
$$>P(X^S)-C'(x^+)+X^C P'(X^0)-\{P(X^S)-C'(x^{++})+X^C P'(X^0)\}=C'(x^{++})-C'(x^+)>0.$$

Hence $\Pi^S<\Pi^C$ by (A18). This completes the proof of the proposition. ■

A4. *Proof of Proposition 4* We can write total surplus as $\Omega(x_1)=\int_0^X P(Y)dY-C(x_1)-C(\varphi_2(x_1))$, where $X=x_1+\varphi_2(x_1)$. It follows by the Mean Value Theorem that

$$\Omega(x_1^L)-\Omega(x_1^C)=(x_1^L-x_1^C)\Omega'(x^*), \tag{A19}$$

where x^* is a number between x_1^L and x_1^C and

$$\Omega'(x_1)=\varphi_2'(x_1)\{P(X)-C'(\varphi_2(x_1))\}+\{P(X)-C'(x_1)\}.$$

Note that

$$\{P(x_1+\varphi_2(x_1))-C'(\varphi_2(x_1))\}'=\varphi_2'(P'-C'')+P'>0 \text{ (resp. } <0)$$
$$\text{as (i) } \varphi_2'<\text{(resp. } >)\ -P'/(P'-C'') \text{ if } P'-C''<0,$$
$$\text{or (ii) } \varphi_2'>\text{(resp. } <)\ -P'/(P'-C'') \text{ if } P'-C''>0,$$
$$-P'/(P'-C'')\geq \text{(resp. } <)\ -1 \text{ as } C''\geq 0 \text{ (resp. } <0) \text{ if } P'-C''<0,$$
$$-P'/(P'-C'')>0 \text{ if } P'-C''>0.$$

As for Part (a), first note that $P'-C''<0$ and $x_1^C<x^*<x_1^L$. If $C''\geq 0$ and $-1\leq -P'/(P'-C'')<\varphi_2'$, then $\{P(.)-C'(.)\}'<0$, whence $P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))>P(x_1^L+\varphi_2(x_1^L))-C'(\varphi_2(x_1^L))>0$ by (A3). Then we see that

$$\Omega'(x^*)=\varphi_2'(x^*)\{P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))\}+\{P(x^*+\varphi_2(x^*))-C'(x^*)\}$$
$$>-\{P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))\}+\{P(x^*+\varphi_2(x^*))-C'(x^*)\}=C'(\varphi_2(x^*))-C'(x^*),$$

the last inequality of which is >0 if $\varphi_2(x^*)>x^*$. On the other hand, if $C''<0$ and $-P'/(P'-C'')<-1<\varphi_2'$, then $\{P(.)-C'(.)\}'<0$, whence $P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))>P(x_1^L+\varphi_2(x_1^L))-C'(\varphi_2(x_1^L))>0$ by (A3). Then we see that

$$\Omega'(x^*)=\varphi_2'(x^*)\{P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))\}+\{P(x^*+\varphi_2(x^*))-C'(x^*)\}$$
$$>-\{P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))\}+\{P(x^*+\varphi_2(x^*))-C'(x^*)\}=C'(\varphi_2(x^*))-C'(x^*),$$

the last inequality of which is >0 if $\varphi_2(x^*)<x^*$. If $C''\geq 0$ and $-1<\varphi_2'<-P'/(P'-C'')$, then $\{P(.)-C'(.)\}'>0$, whence $P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))>P(x_1^C+\varphi_2(x_1^C))-C'(\varphi_2(x_1^C))>0$ by (1). Then we see that

$$\Omega'(x^*)=\varphi_2'(x^*)\{P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))\}+\{P(x^*+\varphi_2(x^*))-C'(x^*)\}$$
$$>-\{P(x^*+\varphi_2(x^*))-C'(\varphi_2(x^*))\}+\{P(x^*+\varphi_2(x^*))-C'(x^*)\}=C'(\varphi_2(x^*))-C'(x^*),$$

the last inequality of which is >0 if $\varphi_2(x^*)>x^*$. In those three cases, we have $\Omega^S=\Omega(x_1^L)>\Omega(x_1^C)=\Omega^C$ by (A19).

In like manner, as for Part (b), we have $\Omega^S<\Omega^C$ in case $0<\varphi_2'<1$ and $\Omega^S>\Omega^C$ in case $\varphi_2'>1$ by (A19) if $C''\geq 0$, $P'-C''<0$, and $\varphi_2(x^*)>x^*$, or if $C''<0$, $P'-C''<0$, and $\varphi_2(x^*)<x^*$; as for Part (c), we

have $\Omega^S < \Omega^C$ by (A19) if $\varphi_2(x^*) < x^*$. This completes the proof of the proposition. ■

Remark. It follows by the Mean Value Theorem that $\varphi_2(x) - x = \varphi_2(0) + x\{\varphi_2'(x^\#) - 1\}$, where $0 < x^\# < x$. Hence $\varphi_2(x^*) >$ (resp. $<$) x^* as $\varphi_2'(x^\#) >$ (resp. $<$) $1 + \{P(\varphi_2(0)) - C'(\varphi_2(0))\}/x^* P'(\varphi_2(0))$. For example, we have $\varphi_2(x^*) > x^*$ if $\varphi_2' > 1$ and $\varphi_2(x^*) < x^*$ if $\varphi_2' < 1$ with x^* sufficiently large.

References

Brander, J. A. (1995), "Strategic Trade Policy," in *Handbook of International Economics*, vol. 3, North-Holland.

Brander, J. A., and B. J. Spencer (1984), "Trade Warfare: Tariffs and Cartels," *Journal of International Economics*, vol. 16, pp. 227-242.

Burger, E. (1963), *Introduction to the Theory of Games*, Prentice-Hall.

Chuman, E. (2008), "Stability and Instability of the Cournot Equilibrium," *Australian Economic Papers*, vol. 47, pp. 259-263.

Chuman, E. (2009), "Stability and Instability of the Cournot Equilibrium: Erratum," *Australian Economic Papers*, vol. 48, p. 103.

Fisher, F. M. (1961), "The Stability of the Cournot Oligopoly Solution: The Effects of Speeds Adjustment and Increasing Marginal Costs," *Review of Economic Studies*, vol. 28, pp. 125-135.

Frank Jr., C. R., and R. E. Quandt (1963), "On the Existence of Cournot Equilibrium," *International Economic Review*, vol. 4, 92-96.

Friedman, J. W. (1977), *Oligopoly and the Theory of Games*, North-Holland.

Gal-Or, E. (1985), "First Mover and Second Mover Advantages," *International Economic Review*, vol. 26, 649-653.

Hahn, F. H. (1962), "The Stability of Cournot Oligopoly Solution," *Review of Economic Studies*, vol. 29, pp. 329-331.

McManus, M., and R. E. Quandt (1961), "Comments on the Stability of the Cournot Oligopoly Model," *Review of Economic Studies*, vol. 28, pp. 136-139.

Al-Nowaihi, A., and P. L. Levine (1985), "The Stability of the Cournot Oligopoly Model: A Reassessment," *Journal of Economic Theory*, vol. 35, pp. 307-321.

Okuguchi, K. (1964), "The Stability of the Cournot Oligopoly Solution: A Further Generalization," *Review of Economic Studies*, vol. 31, pp. 143-146.

Okuguchi, K. (1976), *Expectations and Stability in Oligopoly Models*, Springer-Verlag.

Okuguchi, K. (1999), "Cournot and Stackelberg Duopolies Revised," *Japanese Economic Review*, vol. 50, pp. 363-367.

Seade, J. (1980), "The Stability of Cournot Revisited," *Journal of Economic Theory*, vol. 23, pp. 15-27.

Shapiro, C. (1989), "Theories of Oligopoly Behavior," in *Handbook of Industrial Organization*, vol. 1, North-Holland.

Takayama, A. (1986), *Mathematical Economics*, 2nd edition, Princeton University Press.

Theocharis, R. D. (1960), "On the Stability of the Cournot Solution on the Oligopoly Problem," *Review of Economic Studies*, vol. 27, pp. 133-134.