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Abstract

Economic systems generate various distributions of opportunity sets for individuals to choose consumption bundles. This paper presents an axiomatic analysis on distributions of opportunity sets. We introduce several reasonable properties of distributions of opportunity sets, and characterize the distributions of opportunity sets in the market economy by these properties.

Keywords: distribution of opportunity sets, market economy, Walrasian distribution, axiomatic characterization.
JEL Classification Numbers: D50, D60, D70, P00

1 Introduction

Economic and social systems are assessed on various criteria. For example, the market system has been evaluated on such criteria as efficiency, equity, and freedom. On the efficiency ground, modern economic theory has established the fundamental theorems of welfare economics (Arrow (1951) and Debreu (1951, 1954)), which state that the market system, if perfectly operated, generates a socially efficient outcome. Though the efficiency aspect of the market system is very appealing, many writers have also argued that a most compelling advantage of the market system lies in its promotion of individual freedom in terms of opportunities to achieve as well as its guarantee of individual liberty in terms of autonomy of decisions.[1] Sen (1988) stresses the importance of the opportunity aspect of freedom:

> It seems reasonable to argue that if we really do attach importance to the actual opportunity that each person has, subject to feasibility, to lead the life that he or she would choose, then the opportunity aspect of freedom must be quite central to social evaluation (Sen, 1988, p.527).

Following Sen (1993), we consider that the degree of freedom of choice enjoyed by an individual in a social and economic system is reflected in the set of opportunities available to the individual. For example, in the market system, a consumer's opportunities can be described by her budget set, namely, the set of consumption bundles that are available and affordable for her. Indeed, standard textbooks of microeconomics start with postulating that each consumer chooses her most preferred bundle in her budget set. In contrast, the central planning system assigns to each individual a specific bundle, giving her no freedom of choice. Somewhere in between, distortions and/or regulations in the market will change opportunities available to individuals. For instance, if the government subsidizes low-income individuals to purchase necessities, their opportunities are different from those of high-income individuals; if, on the other hand, the government sets a regulation on the quantity of consumption of gasoline for each individual, then each individual's budget set is truncated by the limit. Such examples are abundant and all illustrate the following: each social and economic system gives

[1] We refer to, for instance, Hayek (1976), Nozick (1974), Buchanan (1986), Friedman (1962), Kornai (1988), and Lindbeck (1988) for further discussions on these and related issues.

various degrees of freedom for individuals; these various degrees of freedom for individuals are reflected by their opportunity sets.

In order to better understand and compare the opportunity aspect of individual freedom in various systems, it is essential to study properties of distributions of opportunity sets generated by the systems. To the best of our knowledge, however, there exist few studies on general properties of distributions of opportunity sets.[2] In this paper, we formulate several reasonable properties of distributions of opportunity sets, and then, by using these properties, we characterize the distribution of opportunity sets generated by the most prominent economic system in the modern age, the market system.

We will show that, a distribution of opportunity sets satisfies three properties, *Feasibility*, *No Coordination Surplus*, and *Minimum Attainability* (see below for informal descriptions of these properties) if and only if it is a distribution of opportunity sets generated by the market system, which we call a *Walrasian* distribution. In what follows, let us explain more about these properties that characterize Walrasian distributions.

Concerning our first property, *Feasibility*, we note first that, for most economic systems, opportunity sets of individuals are *inter-dependent*. This point has been observed by Basu (1987) and Pattanaik (1994). In fact, on a moment's reflection of an Edgeworth box diagram, one will see that whether a consumption bundle in a budget set is attainable or not for an individual depends on the choices of the other individuals. For example, suppose that there are two individuals, i and j, each of whom is endowed with 5 units of bread and 3 units of meat, and that the price of each good is unity. Then, individual i's choice of 6 units of bread and 2 units of meat is not compatible with individual j's choice of 7 units of bread and 1 unit of meat, while it is compatible with j's choice of 4 units of bread and 4 units of meat. Hence, to formulate a very basic notion of feasibility of a distribution of opportunity sets, a careful consideration is necessary. Certainly, it is too strong to require that *any* combination of choices of individuals from their respective opportunity sets constitute a feasible allocation. In fact, under a very mild assumption, the only distribution of opportunity sets satisfying this requirement is the central planning system which assigns to each individual a specific bundle or less.

[2]There is a related literature on ranking opportunity sets. But its focus is on ranking opportunity sets by a *single individual*, and do not consider *distributions* of opportunity sets in a social setting. See, for instance, Barbera, Bossert and Pattanaik (2004), and Pattanaik and Xu (1990, 2000).

To consider various types of distributions of opportunity sets, therefore, we need a weaker notion of feasibility. Here we propose to define a feasible distribution of opportunity sets as one in which each individual's choice from the boundary of her opportunity set constitutes a feasible allocation with *some* choices of the other individuals from the boundaries of their respective opportunity sets. Our axiom of Feasibility requires that a distribution of opportunity sets be feasible in this weak sense.

The second property, *No Coordination Surplus*, says that no "coordination" of choices by individuals over their respective opportunity sets gives these individuals a larger total amount of commodities. More formally, it requires that, for any group M of individuals, and any vector of consumption bundles $(x_i)_{i \in M}$ of the members of M such that each x_i belongs to the boundary of i's opportunity set, there exists no vector of consumption bundles $(y_i)_{i \in M}$ in their respective opportunity sets such that $\sum_{i \in M} y_i > \sum_{i \in M} x_i$. This property is essential for a decentralized decision by each individual to be incentive compatible. If it were violated, a group of individuals would want to increase each member's consumption by coordinating its decisions.

The third property, *Minimum Attainability*, states that, for any feasible allocation in the economy, at least one consumption bundle in the allocation should be contained in the opportunity set of the corresponding individual. This condition excludes the case in which no individual can attain his consumption bundle at some allocation in the given economic system even though it is a feasible allocation. Such a case arises, for instance, in the central planning system.

Each of our three properties is a fairly weak and natural requirement, and yet these properties together are enough to characterize the class of distributions of opportunity sets generated by the market system. We show that a distribution of opportunity sets satisfies Feasibility, No Coordination Surplus, and Minimum Attainability if and only if it is a Walrasian distribution with positive prices.

The rest of the paper is organized as follows. In Section 2, we present notation and basic definitions. We also formally define a Walrasian distribution of opportunity sets. Section 3 introduces and discusses properties of distributions of opportunity sets. Section 4 presents our characterization of the Walrasian distributions of opportunity sets, and Section 5 concludes. The proof of the main result is contained in the appendix.

2 Basic Definitions and Notation

There are n individuals and k goods. Let $N := \{1, \dots, n\}$ be the set of individuals, and $K := \{1, \dots, k\}$ the set of goods. An *allocation* is a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^{nk}_+$ where for each $i \in N$, $x_i = (x_{i1}, \dots, x_{ik}) \in \mathbb{R}^k_+$ is a *consumption bundle* of individual i.[3] There exists some fixed amount of social endowments of goods, which are represented by the vector $\omega \in \mathbb{R}^k_{++}$. An allocation $x \in \mathbb{R}^{nk}_+$ is *feasible* if $\sum_{i=1}^n x_i \leq \omega$.[4] Let Z be the set of all feasible allocations. For each $x \in \mathbb{R}^{nk}_+$, and each $i \in N$, we denote $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

For each $i \in N$, an *opportunity set* for individual i is a set in $\mathbb{R}^k_+$ that is non-empty, compact, and comprehensive. Recollect that a set $S \in \mathbb{R}^k_+$ is *comprehensive* if for all $x, y \in \mathbb{R}^k_+$, $x \in S$ and $y \leq x$ imply $y \in S$. Comprehensiveness is a reasonable assumption on opportunity sets under free disposal of goods. Let $\mathcal{S} := \{S \subset \mathbb{R}^k_+ \mid S \text{ is non-empty, compact, and comprehensive}\}$. A *distribution of opportunity sets* is an n-tuple $(S_1, \dots, S_n) \in \mathcal{S}^n$.

Given a *price vector* $p \in \mathbb{R}^k_+$ and an *income* $m_i \in \mathbb{R}_+$, define $B(p, m_i) := \{x_i \in \mathbb{R}^k_+ \mid p \cdot x_i \leq m_i\}$ and $B^*(p, m_i) := \{x_i \in \mathbb{R}^k_+ \mid p \cdot x_i = m_i\}$. Let $X(\omega) := \{x_i \in \mathbb{R}^k_+ \mid x_i \leq \omega\}$.

Walrasian distribution: A distribution of opportunity sets $(S_1, \dots, S_n) \in \mathcal{S}^n$ is *Walrasian* if there exist $p \in \mathbb{R}^k_+$ and $(m_1, \dots, m_n) \in \mathbb{R}^n_+$ such that $\sum_{i \in N} m_i = p \cdot \omega$ and $S_i = B(p, m_i) \cap X(\omega)$ for all $i \in N$. If, in addition, $p \gg 0$, then we call the distribution of opportunity sets a *Walrasian distribution with positive prices.*

Notice that our definition of a Walrasian distribution of opportunity sets slightly departs from the standard definition of a distribution of budget sets because each budget set is bounded by the resource constraint. This notion of *constrained budget sets* was introduced by Hurwicz, Maskin and Postlewaite (1982, 1995), and plays an important role in implementation theory. In particular, Hurwicz, Maskin and Postlewaite (1982, 1995) showed that the unconstrained Walrasian correspondence is not implementable in Nash equilibria, but the constrained Walrasian correspondence is.

For all $(S_1, \dots, S_n) \in \mathcal{S}^n$, and all $M \subseteq N$, define $\sum_{j \in M} S_j := \{x \in \mathbb{R}^k_+ \mid$

[3]As usual, $\mathbb{R}_+$ is the set of all non-negative real numbers, and $\mathbb{R}_{++}$ is the set of all positive real numbers.

[4]Vector inequalities are defined as follows: For all $x, y \in \mathbb{R}^k_+$, $x \geq y \Leftrightarrow (x - y) \in \mathbb{R}^k_+$; $x > y \Leftrightarrow [x \geq y \text{ and } x \neq y]$; and $x \gg y \Leftrightarrow (x - y) \in \mathbb{R}^k_{++}$.

$\exists (x_j)_{j \in M} \in \Pi_{j \in M} S_j$ such that $x = \sum_{j \in M} x_j\}$. For all $S \subseteq \mathbb{R}^k_+$, define

$$\partial^+ S \;\; := \;\; \{x \in S \mid \nexists y \in S \text{ such that } y > x\}.$$

That is, $\partial^+ S$ is the "undominated boundary of S". For all $x \in \mathbb{R}^k_+$, and all $\varepsilon \in \mathbb{R}_{++}$, let $D(x, \varepsilon) := \{y \in \mathbb{R}^k_+ \mid ||y - x|| < \varepsilon\}$. For all $S \subseteq \mathbb{R}^k_+$, define $\text{int} S := \{x \in S \mid \exists \varepsilon \in \mathbb{R}_{++}, D(x, \varepsilon) \subseteq S\}$.

3 Properties of Distributions of Opportunity Sets

An opportunity set of an individual prescribes the range of alternatives from which she can choose. In social situations, however, choices made by individuals involved are often interdependent. Thus, it is too strong to require that, for *all* $x = (x_1, \ldots, x_n) \in \mathbb{R}^{nk}_+$, if $x_i \in S_i$ for all $i \in N$, then $x \in Z$, i.e., $\sum_{i \in N} x_i \leq \omega$. This requirement says that every individual can attain any consumption bundle in her opportunity set *regardless of the choices of the other individuals*. A distribution of opportunity sets $(S_1, \ldots, S_n)$ satisfies this condition if and only if, there exists $\bar{x} \in \mathbb{R}^{nk}_+$ such that $\bar{x} \in Z$, and, for all $i \in N$, $S_i \subseteq \{x \in \mathbb{R}^k_+ \mid x \leq \bar{x}\}$. Hence, we need to weaken the requirement in order to consider and compare reasonable distributions of opportunity sets. Our proposal is the following:

Feasibility: For every $i \in N$ and every $x_i \in \partial^+ S_i$, there exists $x_{-i} \in \Pi_{j \neq i} \partial^+ S_j$ such that $(x_1, \ldots, x_n) \in Z$.

Feasibility states that, for each individual, any choice from the boundary of her opportunity set is attainable for *some* choices of the other individuals from the boundaries of their respective opportunity sets. It is a very weak, reasonable requirement. Were it violated, some alternative a_i of individual i from the boundary of her opportunity set is compatible with *no* combination of choices of the other individuals from the boundaries of their respective opportunity sets. Since we consider the case where objects of individual choice are *goods* (not *bads*), each individual would choose from the boundary of her opportunity set, given freedom of choice. Then, individual i can never attain the alternative a_i. Our feasibility condition excludes such cases.

As we will show, Walrasian distributions of opportunity sets satisfy this condition, and so do many other kinds of distributions. Figure 1 depicts

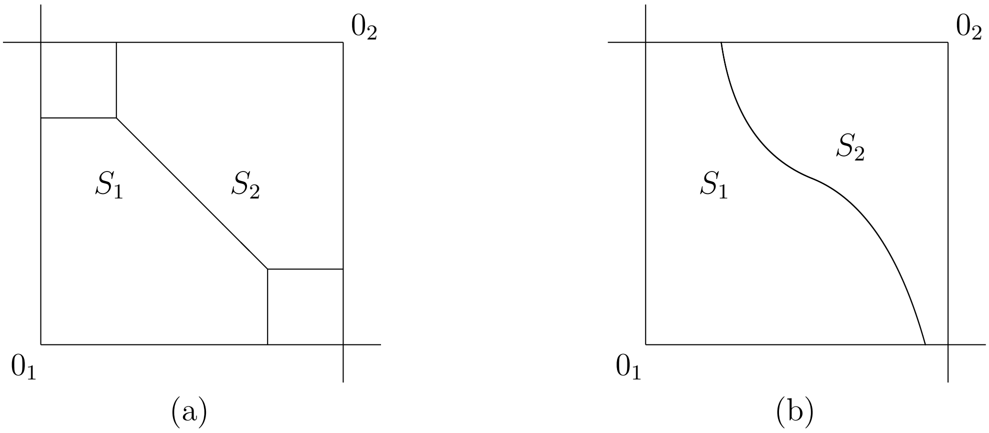

Figure 1: Examples of feasible distributions of opportunity sets

several examples where each corresponding distribution of opportunity sets satisfies Feasibility.

There are distributions of opportunity sets that do not satisfy Feasibility. Consider the following example. Let $k = 2$, $n = 2$ and $\omega = (2, 2)$. Let $S_1 = \{z \in \mathbb{R}^2_+ \mid z \leq (1, 2)\} = S_2$. Then, (S_1, S_2) does not satisfy Feasibility because $\partial^+ S_1 = \{(1, 2)\}$ and $\partial^+ S_2 = \{(1, 2)\}$, and $(1, 2) + (1, 2) = (2, 4) > \omega$.

Given a feasible distribution, it may still leave rooms for improving opportunities of all individuals. Consider the following example. Let $k = 2$, $n = 2$ and $\omega = (4, 4)$. Let $S_1 = \{z \in \mathbb{R}^2_+ \mid z \leq (2, 2)\} = S_2$. Then, (S_1, S_2) satisfies Feasibility. However, we can improve both individuals' opportunities. For example, consider $S'_1 = S_1 \cup \{z \in \mathbb{R}^2_+ \mid z \leq (1, 3)\}$ and $S'_2 = S_2 \cup \{z \in \mathbb{R}^2_+ \mid z \leq (3, 1)\}$. Because $\partial^+ S'_1 = \{(2, 2), (1, 3)\}$ and $\partial^+ S'_2 = \{(2, 2), (3, 1)\}$, the distribution (S'_1, S'_2) is feasible, and it may be argued that S'_1 offers more opportunities to individual 1 than S_1, and S'_2 offers more opportunities to individual 2 than S_2.

To understand our next property, let $x_i \in S_i$ and $x_j \in S_j$ be the chosen consumption bundles of individuals i and j in their opportunity sets, respectively. Suppose that both x_i and x_j are on the boundaries of their opportunity sets respectively, and hence each individual cannot obtain more goods by herself. Suppose, however, that there exist other alternatives $y_i \in S_i$ and $y_j \in S_j$ such that $y_i + y_j > x_i + x_j$. Then, the two individuals will coordinate their actions, and do not choose x_i and x_j but choose y_i and y_j in order to obtain a larger total amount of goods. By appropriately redistributing the total among them, each individual can receive a greater amount of goods than x_i or x_j. Thus, coordination among individuals is beneficial in this case. Our next property says that such cases never arise, and hence coordination of choices among any individuals generates no surplus for them.

No Coordination Surplus: For every $M \subseteq N$, if $x_i \in \partial^+ S_i$ for all $i \in M$, then there exists no $(y_i)_{i \in M} \in \Pi_{i \in M} S_i$ such that $\sum_{i \in M} y_i > \sum_{i \in M} x_i$.

We will show that Walrasian distributions of opportunity sets satisfy No Coordination Surplus later. There are many other distributions of opportunity sets that satisfy this property. For examle, let $\bar{z} \in \mathbb{R}^2_+$ be given, and define $S_i = \{z \in \mathbb{R}^2_+ \mid z \leq \bar{z}\}$ for every $i \in N$. Then, the distribution $(S_1, \ldots, S_n)$ satisfies No Coordination Surplus.

There are also distributions of opportunity sets that do not satisfy this property. Let $M = \{1, 2\} \subseteq N$, and define $S_1 = \{z \in \mathbb{R}^2_+ \mid z_1^2 + z_2^2 \leq 4\} = S_2$. Let $x_1 = (0, 2)$ and $x_2 = (2, 0)$. Then, $x_i \in \partial^+ S_i$ for every $i \in M$, and $x_1 + x_2 = (2, 2)$. Let $y_1 = (1.4, 1.4) \in S_1$ and $y_2 = (1.4, 1.4) \in S_2$. Then, $y_1 + y_2 = (2.8, 2.8) \gg x_1 + x_2$.

The last property requires that given any feasible allocation, among the n consumption bundles specified in the allocation, there must be at least one bundle belonging to the opportunity set of the corresponding individual. In other words, any feasible allocation must be "attainable" for some individual by his choice from his opportunity set.

Minimum Attainability: For every $x \in Z$, there exists $i \in N$ such that $x_i \in S_i$.

Walrasian distributions of opportunity sets satisfy Minimum Attainability as we show later. There are many other distributions that satisfy this property as well as those that violate it. In Figure 1, case (b) satisfies Minimum Attainability whereas case (a) viloates it.

4 Characterizations of the Walrasian Distribution of Opportunity Sets

We have introduced three reasonable properties of distributions of opportunity sets. Each of these properties is fairly weak, and there are many kinds of distributions of opportunity sets that satisfy it. However, requiring the three properties together singles out one particular type of distributions of opportunity sets, namely, Walrasian distributions with positive prices.

First, we show that Walrasian distributions of opportunity sets with positive prices satisfy all these properties.

Proposition 1 *Every Walrasian distribution of opportunity set with positive prices satisfies* Feasibility*,* No Coordination Surplus*, and* Minimum Attainability*.*

Proof. Let $(S_1, \ldots, S_n) \in \mathcal{S}^n$ be a Walrasian distribution of opportunity sets with positive prices. Let $p \in \mathbb{R}^k_{++}$ and $(m_1, \ldots, m_n) \in \mathbb{R}^n_+$ be such that $S_i = \{x_i \in \mathbb{R}^k_+ \mid p \cdot x_i \leq m_i \text{ and } x_i \leq \omega\}$ for every $i \in N$, and $\sum_{i \in N} m_i = p \cdot \omega$.

Feasibility: Let $i \in N$ and $x_i \in \partial^+ S_i$ be given. Then, $p \cdot x_i = m_i$. Define $v := \omega - x_i$. Notice that $v \geq 0$. If $v = 0$, then $S_j = \{0\}$ for all $j \neq i$, and $S_i = X(\omega)$. Clearly, this distribution of opportunity sets is feasible. Consider $v > 0$ next. For every $j \in N$ with $j \neq i$, let $\alpha_j \in \mathbb{R}_+$ be such that $p \cdot (\alpha_j v) = m_j$, and define $x_j := \alpha_j v$. Then, $x_j \in \partial^+ S_j$ for every $j \in N$. On the other hand, we have $p \cdot v = p \cdot (\omega - x_i) = \sum_{h \in N} m_h - m_i = \sum_{j \neq i} m_j = \sum_{j \neq i} p \cdot (\alpha_j v)$. Hence, $(1 - \sum_{j \neq i} \alpha_j) p \cdot v = 0$. Since $p \cdot v > 0$, it follows that $\sum_{j \neq i} \alpha_j = 1$. Therefore, $\sum_{h \in N} x_h = x_i + \sum_{j \neq i} x_j = x_i + \sum_{j \neq i} \alpha_j v = x_i + v = \omega$. Thus, we have proven that $(S_1, \ldots, S_n)$ is feasible.

No Coordination Surplus: Let $M \subseteq N$ be given. For every $y \in \sum_{j \in M} S_j$, there exists $(x_j)_{j \in M} \in \Pi_{j \in M} S_j$ such that $y = \sum_{j \in M} x_j$, and hence $p \cdot y = \sum_{j \in M} p \cdot x_j \leq \sum_{j \in M} m_j$. Let $z \in \sum_{j \in M} \partial^+ S_j$. Obviously, $z \in \sum_{j \in M} S_j$. There exists $(w_j)_{j \in M} \in \Pi_{j \in M} \partial^+ S_j$ such that $z = \sum_{j \in M} w_j$, and thus $p \cdot z = \sum_{j \in M} p \cdot w_j = \sum_{j \in M} m_j$. Since $p \gg 0$, there cannot exist $y \in \sum_{j \in M} S_j$ with $y > z$. Therefore, $z \in \partial^+ (\sum_{j \in M} S_j)$. We have shown that $\sum_{j \in M} \partial^+ S_j \subseteq \partial^+ (\sum_{j \in M} S_j)$.

Minimum Attainability: Suppose, on the contrary, that there exists a feasible allocation $x \in Z$ such that for every $i \in N$, $x_i \notin S_i$. Then, for every $i \in N$, $p \cdot x_i > m_i$. Hence, $\sum_{i \in N} p \cdot x_i > \sum_{i \in N} m_i = p \cdot \omega$. However, since $\sum_{i \in N} x_i \leq \omega$ and $p \gg 0$, we must have $\sum_{i \in N} p \cdot x_i \leq p \cdot \omega$, which is a contradiction. Thus, for every $x \in Z$, there exists $i \in N$ such that $x_i \in S_i$. ■

Our main theorem is a characterization of the class of Walrasian distributions of opportunity sets with positive prices.

Theorem 1 *A distribution of opportunity sets* $(S_1, \ldots, S_n) \in \mathcal{S}^n$ *satisfies* Feasibility*,* No Coordination Surplus*, and* Minimum Attainability *if and only if it is Walrasian distribution of opportunity sets with positive prices.*

The proof of Theorem 1 is long, and is relegated in the appendix. It should be noted that the properties in the above theorem are logically independent:

(i) case (a) of Figure 1 satisfies Feasibility and No Coordination Surplus, but violates Minimum Attainability;

(ii) case (b) of Figure 1 satisfies Feasibility and Minimum Attainability, but violates No Coordination Surplus;

(iii) the distribution of the opportunity sets defined by setting $S_i = X(\omega)$ for all $i \in N$ satisfies No Coordination Surplus and Minimum Attainability, but violates Feasibility.

5 Conclusion

In this paper, we have provided a new approach to economic systems by focusing on distributions of opportunity-freedom for individuals generated in the systems. In particular, we have axiomatically characterized the distributions of opportunity sets in the market system with three properties: Feasibility, No Coordination Surplus, and Minimum Attainability. Each property captures some characteristic of the market system. Feasibility reflects and captures the inter-dependence of opportunity sets of different individuals in a social setting. No Coordination Surplus is a necessary condition for decentralized decisions to be incentive compatible. Were it violated, a group of individuals would be able to "improve" each member's situation by coordinating its decisions. Minimum Attainability requires the non-existence of a feasible allocation that is "unreachable" for all individuals in the following sense: for every feasible allocation, at least one individual has his consumption bundle specified by the feasible allocation belonging to his opportunity set.

For each of these three properties, there are many other distributions of opportunity sets that meet the other two properties. However, the only distributions that satisfy all the three properties are the distributions of opportunity sets generated by the market system.

Two final remarks are in order. First, one may think that *convexity* of opportunity sets is a desirable and natural property. Convexity of opportunity sets is essential for the optimal choice correspondence to be continuous with respect to changes in opportunity sets. It is also intuitively natural if opportunity sets are regarded as the set of available consumption bundles during some period of time such as a week or a month: if two consumption plans are attainable in a month, then the average of the two plans should also be attainable. It is then interesting to note that this important and natural property, convexity, of opportunity sets is not imposed *a priori*, but is *implied by* the three properties in our theorem. (See Lemma 3 in the appendix.) Thanks to this result, we can fully utilize the separating hyperplane theorem to establish our main theorem.

Second, there is no equity-type property (in any sense) imposed on a distribution of opportunity-freedom in our characterization theorem. In fact, the extreme distribution such that $S_1 = X(\omega)$ and $S_i = \{0\}$ for all $i \neq 1$ satisfies all the three properties and it is Walrasian. This aspect reflects the very nature of the market system. No mechanism to realize an equitable distribution of opportunity sets is embedded in the market system itself. In order to consider equity in distributions of opportunity-freedom, we need to introduce a measure of equity. As a further and future study, it would be interesting to formally define notions of equity in distributions of opportunity sets and investigate their implications.

Our approach to analyze economic systems based on distributions of opportunity sets has opened a new avenue for investigating and making comparisons of various economic systems. The present study has axiomatically characterized the distribution of opportunity sets generated by the market system. It would be interesting to investigate other economic systems.

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Appendix

To prove Theorem 1, it is useful to introduce the following property:

Tightness: A distribution of opportunity sets $(S_1, \ldots, S_n) \in \mathcal{S}^n$ is *tight* if for every $i \in N$, and every $x_i \in \partial S_i$, there exists $x_{-i} \in \Pi_{j \neq i} \partial S_j$ such that $\sum_{h \in N} x_h = \omega$.

Notice that Tightness implies Feasibility.

The proof of Theorems 1 relies on the following sequence of lemmas.

Lemma 1 *If* $S := (S_1, \ldots, S_n) \in \mathcal{S}^n$ *satisfies* Feasibility *and* Minimum Attainability*, then it satisfies* Tightness*.*

Proof. Suppose that $S = (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies Feasibility and Minimum Attainability, but violates Tightness. Then, there exist $i \in N$ and $x_i \in \partial^+ S_i$ such that for all $x_{-i} \in \Pi_{j \neq i} \partial^+ S_j$, $x_i + \sum_{j \neq i} x_j \neq \omega$. By Feasibility, however, there exists $y_{-i} \in \Pi_{j \neq i} \partial^+ S_j$ with $x_i + \sum_{j \neq i} y_j < \omega$. Define $z \in \mathbb{R}_+^{nk}$ by $z_i := x_i + \frac{1}{n}(\omega - x_i - \sum_{j \neq i} y_j) > x_i$ and $z_j := y_j + \frac{1}{n}(\omega - x_i - \sum_{j \neq i} y_j) > y_j$. Then, z is a feasible allocation, but for all $h \in N$, $z_h \notin S_h$. This contradicts the supposition that S satisfies Minimum Attainability. ■

For each $x_i \in \mathbb{R}_+^k$ and each $\varepsilon > 0$, define $V(x_i, \varepsilon) := \{y_i \in \mathbb{R}_+^k \mid |y_{i\ell} - x_{i\ell}| < \varepsilon$ for every $\ell \in K\}$. Recollect that $X(\omega) = \{a \in \mathbb{R}_+^k \mid a \leq \omega\}$.

Lemma 2 *Suppose that* $S := (S_1, \ldots, S_n) \in \mathcal{S}^n$ *satisfies* Feasibility *and* Minimum Attainability*. For every* $i \in N$ *and every* $x_i \in S_i$*, if*

$$V(x_i, \varepsilon) \cap (\mathbb{R}_+^k \setminus S_i) \cap X(\omega) \neq \emptyset$$

for every $\varepsilon > 0$*, then* $x_i \in \partial^+ S_i$*.*

Proof. Suppose that $S = (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies Feasibility and Minimum Attainability. Since S is feasible, $S_i \subseteq X(\omega)$ for every $i \in N$. By Lemma 1, S satisfies Tightness. Suppose, on the contrary, that there exist $i \in N$ and $x_i \in S_i$ such that $V(x_i, \varepsilon) \cap (\mathbb{R}_+^k \setminus S_i) \cap X(\omega) \neq \emptyset$, for every $\varepsilon > 0$, and yet $x_i \notin \partial^+ S_i$. Then, there exists $z_i \in \partial^+ S_i$ with $z_i > x_i$. By Tightness, there is $z_{-i} \in \Pi_{j \neq i} \partial^+ S_j$ such that $z_i + \sum_{j \neq i} z_j = \omega$.

Define the allocation $t \in \mathbb{R}_+^{nk}$ by $t_i := x_i$ and $t_j := z_j + \frac{1}{n-1}(z_i - x_i) > z_j$ for every $j \neq i$. Then, t is a feasible allocation and $t_j \notin S_j$ for every $j \neq i$.

Let $K(x_i) := \{\ell \in K \mid \omega_\ell > x_{i\ell}\}$. If $K(x_i) = \emptyset$, then $x_{i\ell} = \omega_\ell$ for every $\ell \in K$, and comprehensiveness of S_i implies $S_i = X(\omega)$. Hence, we have $(\mathbb{R}_+^k \setminus S_i) \cap X(\omega) = \emptyset$, which is a contradiction. Thus, $K(x_i) \neq \emptyset$. For each $\varepsilon > 0$, define the consumption bundle $x_i(\varepsilon) \in \mathbb{R}_+^k$ by

$$x_{i\ell}(\varepsilon) = \begin{cases} x_{i\ell} + \varepsilon & \text{if } \ell \in K(x_i) \\ x_{i\ell} & \text{otherwise.} \end{cases}$$

Claim: $x_i(\varepsilon) \notin S_i$ for every $\varepsilon > 0$.

Suppose, on the contrary, that $x_i(\varepsilon) \in S_i$ for some $\varepsilon > 0$. Let $y_i \in V(x_i, \varepsilon)$. If $y_{i\ell} > \omega_\ell$ for some $\ell \in K$, then $y_i \notin X(\omega)$. Otherwise, $y_{i\ell} \leq \omega_\ell$ for all $\ell \in K$. Then, $y_i \leq x_i(\varepsilon) \in S_i$. It follows from comprehensiveness of S_i that $y_i \in S_i$ and hence $y_i \notin (\mathbb{R}^k_+ \setminus S_i)$. Therefore, we have $V(x_i, \varepsilon) \cap (\mathbb{R}^k_+ \setminus S_i) \cap X(\omega) = \emptyset$, which is a contradiction. Thus, the claim has been proven.

For each $\ell \in K(x_i)$, define $J(\ell) := \{j \in N \setminus \{i\} \mid t_{j\ell} > 0\}$. Since $\ell \in K(x_i)$, we have $t_{i\ell} = x_{i\ell} < \omega_\ell$. Together with $\sum_{h \in N} t_{h\ell} = \omega_\ell$, it follows that $J(\ell) \neq \emptyset$. Let $\varepsilon_\ell > 0$ be such that $t_{j\ell} - \varepsilon_\ell \geq 0$ for every $j \in J(\ell)$.

For every $j \in N \setminus \{i\}$, since S_j is closed and $t_j \notin S_j$, it follows that there exists $\varepsilon_j > 0$ such that $V(t_j, \varepsilon_j) \cap S_j = \emptyset$.

Let

$$\varepsilon^* := \min\{\min_{\ell \in K(x_i)} \varepsilon_\ell, \min_{j \in N \setminus \{i\}} \varepsilon_j\}.$$

Define the allocation $v \in \mathbb{R}^{nk}_+$ by

$$v_{i\ell} = \begin{cases} x_{i\ell} + |J(\ell)| \frac{\varepsilon^*}{2} & \text{if } \ell \in K(x_i) \\ x_{i\ell} & \text{otherwise.} \end{cases}$$

and for every $j \neq i$,

$$v_{j\ell} = \begin{cases} t_{j\ell} - \frac{\varepsilon^*}{2} & \text{if } \ell \in K(x_i) \text{ and } j \in J(\ell) \\ t_{j\ell} & \text{otherwise.} \end{cases}$$

Then, v is a feasible allocation. By the above claim, we have $v_i \notin S_i$. For every $j \in N \setminus \{i\}$, because $\frac{\varepsilon^*}{2} < \varepsilon^* \leq \varepsilon_j$ and $V(t_j, \varepsilon_j) \cap S_j = \emptyset$, it follows that $v_j \notin S_j$. This contradicts the supposition that S satisfies Minimum Attainability. ■

Lemma 3 *If $S := (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies* Feasibility*,* No Coordination Surplus *and* Minimum Attainability*, then S_i is convex for all $i \in N$.*

Proof. Suppose that $S = (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies Feasibility, No Coordination Surplus and Minimum Attainability. By Lemma 1, S satisfies Tightness. Suppose, on the contrary, that S_i is not convex for some $i \in N$. Then, there exist $x_i, y_i \in \partial^+ S_i$ such that $z_i := \frac{1}{2}(x_i + y_i) \notin S_i$. Since $x_i, y_i \in S_i$ and S is feasible, it follows that $x_i, y_i \in X(\omega)$ and hence $z_i \in X(\omega)$ as well. Let $\alpha := \max\{\alpha' \in \mathbb{R} \mid \alpha' z_i \in S_i\}$. Because S_i is compact and comprehensive, such α exists and $0 \leq \alpha < 1$. Let $v_i := \alpha z_i$. It is clear that for every $\varepsilon > 0$, $V(v_i, \varepsilon) \cap (\mathbb{R}^k_+ \setminus S_i) \cap X(\omega) \neq \emptyset$. By Lemma 2, $v_i \in \partial^+ S_i$. It follows from Tightness that there exist $x_{-i}, y_{-i}, v_{-i} \in \Pi_{j \neq i} \partial^+ S_j$ such that

$$x_i + \sum_{j \neq i} x_j = \omega$$

$$
\begin{aligned}
y_i + \sum_{j \neq i} y_j &= \omega \\
v_i + \sum_{j \neq i} v_j &= \omega.
\end{aligned}
$$

Since $v_i < \frac{1}{2}(x_i + y_i)$, we have $\sum_{j \neq i} v_j = \omega - v_i > \omega - \frac{1}{2}(x_i + y_i)$. Then, we have

$$
\begin{aligned}
v_i + \sum_{j \neq i} x_j &< \frac{1}{2}(x_i + y_i) + \sum_{j \neq i} x_j \\
&= x_i + \sum_{j \neq i} x_j - \frac{1}{2}x_i + \frac{1}{2}y_i \\
&= \omega - \frac{1}{2}x_i + \frac{1}{2}y_i \\
y_i + \sum_{j \neq i} v_j &> y_i + \omega - \frac{1}{2}(x_i + y_i) \\
&= \omega - \frac{1}{2}x_i + \frac{1}{2}y_i
\end{aligned}
$$

Hence,

$$
y_i + \sum_{j \neq i} v_j > v_i + \sum_{j \neq i} x_j.
$$

Therefore, $(v_i, x_{-i}) \notin \partial^+(\sum_{h \in N} S_h)$. But this means that S does not satisfy No Coordination Surplus, which is a contradiction. ■

Lemma 4 *If $S := (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies* Feasibility*,* No Coordination Surplus *and* Minimum Attainability*, then for all $i \in N$, there exist $p^i \in \mathbb{R}^k_+, p^i \neq 0$ and $m_i \in \mathbb{R}_+$ such that $\partial^+ S_i \subseteq B^*(p^i, m_i) \cap X(\omega)$.*

Proof. Assume that $S = (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies Feasibility, No Coordination Surplus and Minimum Attainability. By Lemma 1, S satisfies Tightness. It follows from Lemma 3 that S_i is convex for every $i \in N$. Let $i \in N$. By Feasibility, $\partial^+ S_i \subset X(\omega)$. Define $T_i := \sum_{j \in N \setminus \{i\}} S_j$ and $V_i := \{\omega\} - T_i := \{x_i \in \mathbb{R}^k \mid \exists z_i \in T_i, x_i = \omega - z_i\}$. Since S_j is convex for all $j \in N \setminus \{i\}$, T_i is convex, and so is $(-T_i)$. Hence, V_i is convex.

By No Coordination Surplus,

$$
\sum_{j \in N \setminus \{i\}} \partial^+ S_j \subseteq \partial^+ T_i. \tag{1}
$$

It follows from Tightness and (1) that

(A) for all $y_i \in \partial^+ S_i$, there exists $z_i \in \partial^+ T_i$ such that $y_i + z_i = \omega$.

In order to show that

$$V_i \cap \mathrm{int} S_i = \emptyset, \tag{2}$$

suppose, to the contrary, that there exists $x_i \in V_i \cap \mathrm{int} S_i$. Since $x_i \in \mathrm{int} S_i$, there exists $x_i' \in \partial^+ S_i$ with $x_i' > x_i$. From (A), there exists $z_i \in \partial^+ T_i$ such that $x_i' + z_i = \omega$. On the other hand, since $x_i \in V_i$, there exists $z_i' \in T_i$ with $x_i = \omega - z_i'$. Together we have $z_i' = \omega - x_i > \omega - x_i' = z_i$, which contradicts $z_i \in \partial^+ T_i$. Thus, (2) must hold true.

By the separating hyperplane theorem, there exist $p^i \in \mathbb{R}^k, p^i \neq 0$ and $m_i \in \mathbb{R}$ such that $S_i \subseteq \{x_i \in \mathbb{R}^k_+ \mid p^i \cdot x_i \leq m_i\}$ and $V_i \subseteq \{x_i \in \mathbb{R}^k_+ \mid p^i \cdot x_i \geq m_i\}$. Since S_i is comprehensive, it follows that $p^i > 0$. To complete the proof, we need to show that

$$\partial^+ S_i \subseteq \{x_i \in \mathbb{R}^k_+ \mid p^i \cdot x_i = m_i\} \tag{3}$$

Suppose, to the contrary, that there exists $s_i \in \partial^+ S_i$ such that $p^i \cdot s_i < m_i$. From (A) above, there exists $t_i \in \partial^+ T_i$ with $s_i + t_i = \omega$. Then, $p \cdot (\omega - t_i) = p \cdot s_i < m_i$. But since $\omega - t_i \in V_i$, we have $p \cdot (\omega - t_i) \geq m_i$. This is a contradiction. Thus, the relation (3) must hold true. ■

Lemma 5 *If $S := (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies* Feasibility, No Coordination Surplus *and* Minimum Attainability*, then, for all $i \in N$, there exist $p^i \in \mathbb{R}^k_{++}$ and $m_i \in \mathbb{R}_+$ such that $S_i = B(p^i, m_i) \cap X(\omega)$.*

Proof. Suppose that $S = (S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies Feasibility, No Coordination Surplus and Minimum Attainability. By Lemma 1, S satisfies Tightness. Let $N^0 := \{i \in N \mid S_i = \{0\}\}$ and $N^* := N \setminus N^0$. By Tightness, $N^* \neq \emptyset$. If $i \in N^0$, then it is obvious that $S_i = B(p^i, 0) \cap X(\omega)$ for any $p^i \in \mathbb{R}^k_{++}$. If $N^* = \{i\}$, then Tightness implies that $S_i = X(\omega)$. Then, it is clear that for any $p^i \in \mathbb{R}^k_{++}$, $S_i = B(p^i, m_i) \cap X(\omega)$ with $m_i = p^i \cdot \omega$.

In the rest of the proof, we assume that $|N^*| \geq 2$. Let $i \in N^*$. By Lemma 4, there exists $p^i \in \mathbb{R}^k_+$ and $m_i \in \mathbb{R}_+$ such that $\partial^+ S_i \subseteq B^*(p^i, m_i) \cap X(\omega)$.

Step 1: $m_i > 0$.

Suppose, to the contrary, that $m_i = 0$. Let $M := N^* \setminus \{i\} \neq \emptyset$. We will show that $\sum_{j \in M} S_j \supseteq X(\omega)$. Suppose, to the contrary, that there exists

$y_0 \in \mathbb{R}^k_+$ such that $y_0 \leq \omega$, but $y_0 \notin \sum_{j\in M} S_j$. Since $\sum_{j\in M} S_j$ is closed, there exists $z_0 \in \mathbb{R}^k_+$ with $z_0 \ll \omega$ and $z_0 \notin \sum_{j\in M} S_j$. For each $j \in M$, define $\lambda_j \in \mathbb{R}_+$ by

$$\lambda_j := \max\{\lambda \in \mathbb{R}_+ \mid \lambda z_0 \in S_j\}.$$

By definition, $\sum_{j\in M} \lambda_j z_0 = (\sum_{j\in M} \lambda_j) z_0 \in \sum_{j\in M} S_j$. Since $z_0 \notin \sum_{j\in M} S_j$, we must have $\sum_{j\in M} \lambda_j < 1$. Define

$$\varepsilon := \frac{1 - \sum_{j\in M} \lambda_j}{|N^*| - 1} > 0.$$

Define an allocation $z \in \mathbb{R}^{nk}_+$ by $z_j := (\lambda_j + \varepsilon) z_0$ for each $j \in M$, and $z_i := \omega - z_0 \gg 0$. Then, $\sum_{j\in N} z_j = \omega$. However, from the definition of λ_j, we have $z_j \notin S_j$ for all $j \in M$. Because $S_i \subseteq B(p, 0)$ and $p^i > 0$, it follows that $z_i \notin S_i$. This is a contradiction with Minimum Attainability of S. Thus, we have shown that $\sum_{j\in M} S_j \supseteq X(\omega)$. By No Coordination Surplus, $\partial^+ \sum_{j\in M} S_j \supseteq \sum_{j\in M} \partial^+ S_j$. Therefore, there exists no $(y_j)_{j\in M} \in \Pi_{j\in M} \partial^+ S_j$ such that $\sum_{j\in M} y_j < \omega$. On the other hand, since $i \notin N^0$, there exists $x_i \in \partial^+ S_i$ with $x_i > 0$. By Tightness, there exists $(y_j)_{j\in M} \in \Pi_{j\in M} \partial^+ S_j$ such that $x_i + \sum_{j\in M} y_j = \omega$, and hence $\sum_{j\in M} y_j = \omega - x_i < \omega$. This is a contradiction.

Step 2: $S_i = B(p^i, m_i) \cap X(\omega)$.

In view of Lemma 4, it is enough to show that $S_i \supseteq B(p^i, m_i) \cap \{x_i \in \mathbb{R}^k_+ \mid x_i \leq \omega\}$. Suppose, to the contrary, that there exists $x_i \in \mathbb{R}^k_+$ such that $x_i \in B(p^i, m_i) \cap X(\omega)$ but $x_i \notin S_i$. Since S_i is comprehensive, for all $x'_i \in \mathbb{R}^k_+$ with $x'_i \geq x_i$, $x'_i \notin S_i$. From Step 1, $m_i > 0$. Hence one can find $x'_i \in B(p^i, m_i) \cap X(\omega)$ such that $x'_i \notin S_i$ and $p^i \cdot x'_i > 0$. Because S_i is closed in $\mathbb{R}^k_+$, there exists $x^*_i \in \mathbb{R}^k_+$ such that $x^*_i \notin S_i$ and

$$p^i \cdot x^*_i < p^i \cdot x'_i \leq m_i. \tag{4}$$

Next we will show that $(\omega - x^*_i) \notin \sum_{j\in M} S_j$. Suppose, to the contrary, that $(\omega - x^*_i) \in \sum_{j\in M} S_j$. Since $\{\omega\} - \sum_{j\in M} S_j \subseteq \{x_0 \in \mathbb{R}^k_+ \mid p^i \cdot x_0 \geq m_i\}$ from the proof of Lemma 4, we have $p^i \cdot [\omega - (\omega - x^*_i)] \geq m_i$. Hence, $p^i \cdot x^*_i \geq m_i$, which is in contradiction with (4). Thus, it must be true that $(\omega - x^*_i) \notin \sum_{j\in M} S_j$. Then, by a similar argument to the proof of Step 1, we can find a feasible allocation $z \in \mathbb{R}^{nk}_+$ such that $z_h \notin S_h$ for all $h \in N$. This is a contradiction with Minimum Attainability of S.

Step 3: $p^i \gg 0$.

Suppose that for some good $\ell \in \{1, \dots, k\}$, $p^i_\ell = 0$. Then, for all $x_i \in \partial^+ S_i$, $x_{i\ell} = \omega_\ell$. By Tightness,

(B) for all $j \in N^*$ with $j \neq i$, and all $x_j \in \partial^+ S_j$, $x_{j\ell} = 0$.

On the other hand, by applying Steps 1 and 2 to each $j \in N^*$ with $j \neq i$, we obtain that for all $j \in N^*$ with $j \neq i$, there exist $p^j \in \mathbb{R}^k_+$ and $m_j > 0$ such that $S_j = B(p^j, m_j) \cap X(\omega)$. Since $m_j > 0$, there exists $x_j \gg 0$ such that $p^j \cdot x_j \leq m_j$, and hence $x_j \in S_j$. But then, there exists $x'_j \in \partial^+ S_j$ with $x'_j \geq x_j \gg 0$. This contradicts (B). Thus, it must be true that $p^i \gg 0$. This completes the proof. ■

Lemma 6 *If $S := (S_1, \dots, S_n) \in \mathcal{S}^n$ satisfies* Feasibility, No Coordination Surplus *and* Minimum Attainability, *then there exist $p \in \mathbb{R}^k_{++}$ and $(m_1, \dots, m_n) \in \mathbb{R}^n_+$ such that $S_i = B(p, m_i) \cap X(\omega)$ for all $i \in N$, and $\sum_{i \in N} m_i = p \cdot \omega$.*

Proof. Assume that $S = (S_1, \dots, S_n) \in \mathcal{S}^n$ satisfies Feasibility, No Coordination Surplus and Minimum Attainability. By Lemma 1, S satisfies Tightness. Let $N^0 := \{i \in N \mid S_i = \{0\}\}$ and $N^* := N \setminus N^0$. By Tightness, $N^* \neq \emptyset$. It follows from Lemma 5 that for each $i \in N^*$, there exist $p^i \in \mathbb{R}^k_{++}$ and $m_i \in \mathbb{R}_{++}$ such that $S_i = B(p^i, m_i) \cap X(\omega)$.

Claim: for all $i, j \in N^*$, and all $s, t \in K$,

$$\frac{p^i_s}{p^i_t} = \frac{p^j_s}{p^j_t}.$$

Suppose, on the contrary, that $\frac{p^i_s}{p^i_t} \neq \frac{p^j_s}{p^j_t}$ for some $i, j \in N^*$ and some $s, t \in K$. Without loss of generality, let $\frac{p^i_s}{p^i_t} > \frac{p^j_s}{p^j_t}$. Since $p^i, p^j \gg 0$, $m_i, m_j > 0$, and $p^i \cdot \omega > m_i$, there exist $\varepsilon > 0$, $a_i \in \partial^+ S_i$ and $b_j \in \partial^+ S_j$ such that $a'_i := (a_{i1}, \dots, a_{is} - \varepsilon, \dots, a_{it} + \frac{p^i_s}{p^i_t}\varepsilon, \dots, a_{in}) \in \partial^+ S_i$ and $b'_j := (b_{j1}, \dots, b_{js} + \varepsilon, \dots, b_{jt} - \frac{p^j_s}{p^j_t}\varepsilon, \dots, b_{jn}) \in \partial^+ S_j$. Since S satisfies No Coordination Surplus, it follows that $a_i + b_j \in \partial^+(S_i + S_j)$. On the other hand, because $a'_i \in S_i$ and $b'_j \in S_j$, we have $a'_i + b'_j \in (S_i + S_j)$. Observe that $a'_{ir} + b'_{jr} = a_{ir} + b_{jr}$ for all $r \neq t$, and $a'_{it} + b'_{jt} = a_{it} + b_{jt} + [\frac{p^i_s}{p^i_t} - \frac{p^j_s}{p^j_t})]\varepsilon > a_{it} + b_{jt}$. Therefore,

$a_i + b_j \notin \partial^+(S_i + S_j)$, which contradicts $a_i + b_j \in \partial^+(S_i + S_j)$. Thus, it must be true that $\frac{p^i_s}{p^i_t} = \frac{p^j_s}{p^j_t}$ for all $i, j \in N^*$, and all $s, t \in K$.

Choose $i \in N^*$, and define $p := (1, \frac{p^i_2}{p^i_1}, \ldots, \frac{p^i_n}{p^i_1}) \in \mathbb{R}^k_{++}$. Then, it follows from the above claim that $p^j = p^j_1 p$ for all $j \in N^*$. For each $j \in N^*$, define $m'_i := m_i / p^j_1$. Then, we have $S_j = B(p, m'_i) \cap X(\omega)$ for all $j \in N^*$. For all $j \in N^0$, let $m'_j = 0$. Then, it is clear that for all $j \in N^0$, $S_j = \{0\} = B(p, m'_j) \cap X(\omega)$.

Let $i \in N^*$, and let $x_i \in \partial^+ S_j$. By Lemma 1, there exist $(x_j)_{j \in N \setminus \{i\}} \in \Pi_{j \in N \setminus \{i\}} \partial S_j$ such that $\sum_{h \in N} x_h = \omega$. For all $h \in N$, because $\partial^+ S_h = B^*(p, m'_h) \cap X(\omega)$, it follows that $p \cdot x_h = m'_h$. Then,

$$\begin{aligned} \sum_{h \in N} m'_h &= \sum_{h \in N} p \cdot x_h \\ &= p \cdot \sum_{h \in N} x_h \\ &= p \cdot \omega. \end{aligned}$$

This completes the proof. ■

Proof of Theorem 1:
By Proposition 1, every Walrasian distribution of opportunity sets with positive prices satisfies Feasibility, No Coordination Surplus and Minimum Attainability. It follows from Lemma 6 that if a distribution of opportunity sets $(S_1, \ldots, S_n) \in \mathcal{S}^n$ satisfies Feasibility, No Coordination Surplus and Minimum Attainability, then it is Walrasian with positive prices. ■