On the condition of no unbounded profit with bounded risk

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Abstract

As a simple corollary to Delbaen and Schachermayer's fundamental theorem of asset pricing [5] [6] [7], we prove, in a general finite-dimensional semimartingale setting, that the no unbounded profit with bounded risk (NUPBR) condition is equivalent to the existence of a strict martingale density for the price process. This extends the result of Choulli and Stricker [2] to the càdlàg cases, and refines partially the second main result of Karatzas and Kardaras [15] concerning the existence of an equivalent supermartingale deflator. The proof uses the technique of numéraire change.

1 Introduction

Harrison and Kreps [12] and Harrison and Pliska [13] first discovered, for discrete time finite-state stochastic models of financial markets, that the absence of arbitrage opportunity (*no arbitrage* or NA for short) is equivalent to the existence of a martingale measure for the price process. Their theorem and its various generalizations are called (versions of) the *fundamental theorem of asset pricing.* Each version of the theorem gives precise definitions of "NA" and "martingale measure" in various settings, and uses functional analysis to prove the equivalence. For discrete time models, a definitive extension to the finite-horizon infinite-state cases was proved by Dalang, Morton, and Willinger [3], and an infinite-horizon extension was given by Schachermayer [20]. A paper by Kabanov and Stricker [14] also gives a deep insight into mathematics concerning discrete-time models.

Most of the continuous time versions of the theorem involve stochastic calculus as well, since it is natural to model the return of a continuous trading by an Itô integral. Here we must exclude doubling-type strategies even for

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finite-horizon settings, and several definitions of *admissible* strategies have been proposed in the literature. It has also turned out that the NA condition alone is not sufficient to be equivalent to the existence of an equivalent martingale measure: we need a property stronger than NA. Various continuous time assertions have been proved by many authors, Kreps [16], Duffie and Huang [10], Stricker [21], Lakner [18], Delbaen [4], and Kusuoka [17] to name just a few, but a definitive measure-free version is given by Delbaen and Schachermayer, first in [5] for locally bounded process and then in [7] for general semimartingales (see also their monograph [8]). In Delbaen and Schachermayer's papers, a strategy is called admissible if its Itô integral with respect to the price process never goes below some predetermined level. With the definition, they proved that the existence of an equivalent sigma-martingale measure is equivalent to the *no free lunch with vanishing risk (NFLVR)* condition that is somewhat stronger than NA. Corollary 3.8 of their paper [5] also shows how much stronger NFLVR is compared with NA: the price process satisfies NFLVR if and only if it satisfies NA as well as what is called today the *no unbounded profit with bounded risk (NUPBR)* condition, i.e.,

$$\text{NFLVR} \iff \text{NA} + \text{NUPBR}.$$

The precise definitions of the NA and the NUPBR conditions are presented in Section 2 of the present paper.

NUPBR is not just an auxiliary condition. For continuous semimartingales, Choulli and Stricker [2] proved the equivalence of the three properties: NUPBR, the structure condition, and the existence of a strict martingale density. The precise definition of strict martingale density is presented in Section 2 of the present paper. Surprisingly, their proof uses only stochastic calculus. Also, the second main result (Theorem 3.12) of the recent paper of Karatzas and Kardaras [15] proves the equivalence of the three properties: NUPBR, the existence of a numéraire portfolio, and the existence of an equivalent supermartingale deflator (see Remarks 7 and 8 of the present paper) for the price process. They do not assume path continuity but assume that each component of the price process is strictly positive. Stricker and Yan [22] generalized the optimal decomposition theorem of Kramkov and Schachermayer [14], originally assuming the existence of an equivalent local martingale measure, to the case where the price process has a strict martingale density. Goll and Kallsen [11] shows that the existence of an equivalent supermartingale deflator is sufficient for considering log-optimal portfolio problems.

The aim of the present paper is to prove, in a general finite-dimensional semimartingale setting, that the NUPBR condition is equivalent to the existence of a strict martingale density for the price process. This generalizes the above cited result of Choulli and Stricker [2] to the càdlàg cases. For the proof we change numéraires, a technique introduced by Delbaen and Shirakawa [9] and Delbaen and Schachermayer [6], and reduce the problem to Delbaen and Schachermayer's fundamental theorem of asset pricing [5] [7].

This paper is organized as follows. In Section 2, we give some definitions and state our results, i.e., Theorem 5 and Proposition 6. The proposition is an intermediate result needed for the proof of the main theorem. Two remarks are added concerning the relationship of our result with Theorem 3.12 of Karatzas and Kardaras [15]. Proposition 6 and Theorem 5 are proved in Sections 3 and 4, respectively.

2 Definitions and the Results

Let $S = \{ S_t \}_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued (càdlàg) semimartingale on a stochastic basis $\big(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t\in[0,T]}, P\big)$ with the usual assumptions, where $T \in (0,\infty)$ is a fixed finite time horizon. The process S is interpreted as the discounted price process of d risky assets. We do not assume the path continuity nor the positivity of S.

Throughout this paper, we assume càdlàg versions for all Itô integrals. Moreover, our conventions about time 0 follow those of the book by Rogers and Williams [19].

As in the literature, we also define the following four notions.

Definition 1. An $\mathbb{R}^d$-valued predictable process $H = \{ H_t \}_{t\in(0,T]}$ is called a *1-admissible strategy* for S if H is S-integrable and if

$$\int_{0+}^{t} H\,dS \geq -1, \qquad t \in [0,T], \qquad \text{a.s.}$$

For such an H, we define the $\mathbb{R}$-valued semimartingale $W^H = \{ W^H_t \}_{t\in[0,T]}$ by

$$W^H_t \;:=\; 1 + \int_{0+}^{t} H\,dS$$

and call it the *wealth process* associated with the strategy H.

Definition 2. The semimartingale S is said to satisfy the *no unbounded profit with bounded risk (NUPBR) condition* if the set

$$\mathcal{K}_1 \;:=\; \left\{ \int_{0+}^{T} H\,dS \;\middle|\; H \text{ is a 1-admissible strategy for } S \right\}$$

is bounded in L^0.

Definition 3. The semimartingale S is said to satisfy the *no arbitrage (NA) condition* if 0 is a maximal element in the set $\mathcal{K}_1$, i.e., if there does not exist any 1-admissible strategy H such that

$$\int_{0+}^{T} H\,dS \;\geq\; 0 \quad \text{a.s. and} \quad P\Big(\int_{0+}^{T} H\,dS > 0 \Big) \;>\; 0.$$

Definition 4. An $\mathbb{R}$-valued càdlàg local martingale $Z = \{ Z_t \}_{t\in[0,T]}$ is called a *strict martingale density* for S if the following three properties hold:

- Z is strictly positive;
- $E[\, Z_0 \,] \, < \, \infty$;
- ZS is an $\mathbb{R}^d$-valued sigma-martingale.

Our main result is the following

Theorem 5. *The semimartingale S satisfies the NUPBR condition if and only if there exists a strict martingale density for S.*

As an intermediate result for the proof of Theorem 5, we also prove the following proposition concerning the numéraire-free property of NUPBR.

Proposition 6. *Suppose the semimartingale S satisfies the NUPBR condition. Then, for each 1-admissible strategy H satisfying $W_T^H > 0$ a.s., we have the following two properties:*

(i) $W_t^H > 0$ *and* $W_{t-}^H > 0$ *for* $\forall t \in (0, T]$, *a.s.;*

(ii) *the $\mathbb{R}^{d+1}$-valued semimartingale*

$$\tilde{S}_t \; := \; \Big(\frac{S_t}{W_t^H}, \, \frac{1}{W_t^H} \Big)$$

also satisfies the NUPBR condition.

We give two remarks about the relationship of our result with Theorem 3.12 of Karatzas and Kardaras [15]. Remark 7 shows that our theorem refines partially their result, and Remark 8 shows that our refinement is only partial.

Remark 7. A strict martingale density Z is an equivalent supermartingale deflator, i.e., for each 1-admissible strategy H the process $\{\, Z_t \, W_t^H \,\}_{t \in [0,T]}$ is a supermartingale. Indeed, it holds by the Itô formula that

$$\begin{aligned} d\,(ZW^H)_t \;\; &= \;\; W_{t-}^H \, dZ_t \; + \; Z_{t-} \, H_t \, dS_t \; + \; H_t \, d[Z, S]_t \\ &= \;\; W_{t-}^H \, dZ_t \; + \; \Big\{ H_t \, d(ZS)_t \; - \; H_t \, S_{t-} \, dZ_t \Big\}, \end{aligned}$$

and thus ZW^H is a sigma-martingale. Since it is nonnegative and its initial value Z_0 has a finite expectation, we see that

$$E\Big[\sup_{t \in [0,T]} \big(Z_t W_t^H - Z_0 W_0^H \big)^- \Big] \; \leq \; E[\, Z_0 \,] \; < \; \infty.$$

It then follows from Proposition 3.3 of Ansel and Stricker [1] that the process ZW^H is a local martingale and hence it is a supermartingale.

Remark 8. We give an essentially one-period example where the price process has some strict martingale densities but none of them can be expressed as the reciprocal of the wealth process associated with a 1-admissible strategy. This is a sharp contrast to equivalent supermartingale deflators: Karatzas and Kardaras [15] shows the equivalence between the NUPBR property and the existence of a numéraire portfolio. Note that the reciprocal of the wealth process associated with a numéraire portfolio is, by definition, an equivalent supermartingale deflator. It should also be noted that Choulli and Stricker [2] proves that, if S is a *continuous* semimartingale and if it has some strict martingale densities, then one of them can be expressed as the reciprocal of the wealth process associated with a 1-admissible strategy.

Our example is as follows. Suppose that X is an $\mathbb{R}$-valued random variable on a probability space that satisfies:

- $||X||_1 \;<\; \infty$;
- $||X^+||_\infty \;=\; \infty$;
- $0 \;<\; c := ||X^-||_\infty \;<\; \infty$;
- $P(X = -c) \;>\; 0$.

Define the $\mathbb{R}$-valued process $S = \{\, S_t \,\}_{t\in[0,T]}$ by

$$S_t \;:=\; \begin{cases} 0 & \text{if } \; 0 \le t < T, \\ X & \text{if } \; t = T. \end{cases}$$

The filtration is assumed to be (the usual augmentation of) the natural filtration of S. Clearly, S has an equivalent martingale measure so it has a strict martingale density. In this setting, every wealth process associated with a 1-admissible strategy for S is of the form

$$W^h_t \;:=\; \begin{cases} 1 & \text{if } \; 0 \le t < T, \\ 1 + hX & \text{if } \; t = T \end{cases}$$

for some $h \in [0, \frac{1}{c}]$. We consider three cases.

Case 1: $E[\,X\,] \;<\; 0$. For this case, none of the strict martingale densities can be expressed as the reciprocal of W^h. Indeed, it is easy to see that $\frac{1}{W^0} \equiv 1$ is not a strict martingale density for S. If $h = \frac{1}{c}$, the corresponding wealth process becomes zero with positive probability, so we cannot consider its reciprocal. Also, if $h \in (0, \frac{1}{c})$, it follows from Jensen's inequality that

$$E\Big[\,\frac{1}{W^h_T}\,\Big] \;=\; E\Big[\,\frac{1}{1+hX}\,\Big] \;\ge\; \frac{1}{1\,+\,h\,E[X]} \;>\; 1 \;=\; E\Big[\,\frac{1}{W^h_0}\,\Big],$$

so the process $\frac{1}{W^h}$ cannot be a local martingale. Note that $\frac{1}{W^0} \;\equiv\; 1$ is an equivalent supermartingale density for this case.

Case 2: $E[\,X\,] = 0$. For this case, S is already a martingale and $\frac{1}{W^0} \equiv 1$ is a strict martingale density.

Case 3: $E[\,X\,] > 0$. For this case, there exists some $h^* \in (0, \frac{1}{c})$ such that $\frac{1}{W^{h^*}}$ is a strict martingale density. Indeed, since the random variable $\frac{1}{W_T^h}$ is bounded for each fixed $h \in [0, \frac{1}{c})$, Lebesgue's dominated convergence theorem gives the following two properties:

- The function $[0, \frac{1}{c}) \ni h \mapsto E\big[\, \frac{1}{W_T^h} \,\big] \in (0, \infty)$ is C^1;
- $\frac{d}{dh} E\big[\, \frac{1}{W_T^h} \,\big] \Big|_{h=0+} = E[-X] < 0.$

It also follows from the assumption $P(X = -c) > 0$ that $\lim_{h \nearrow \frac{1}{c}} E\big[\, \frac{1}{W_T^h} \,\big] = \infty$. Thus $\exists h^* \in (0, \frac{1}{c})$ such that $E\big[\, \frac{1}{W_T^{h^*}} \,\big] = 1 = \frac{1}{W_0^{h^*}}$, i.e., $\frac{1}{W^{h^*}}$ is a martingale. The process $\frac{S}{W^{h^*}}$ is also a martingale, since

$$E\Big[\, \frac{S_T}{W_T^{h^*}} \,\Big] = \frac{1}{h^*} E\Big[1 - \frac{1}{W_T^{h^*}} \Big] = 0 = \frac{S_0}{W_0^{h^*}}.$$

3 Proof of Proposition 6

3.1 Proof of (i)

We will prove that the event

$$A^H := \Big\{ \exists t \in (0, T] \quad \text{such that} \quad W_t^H = 0 \quad \text{or} \quad W_{t-}^H = 0 \Big\}$$

has probability zero. It suffices to show that the event

$$A_n^H := A^H \cap \Big\{ W_T^H > \frac{1}{n} \Big\}$$

has probability zero for each fixed $n \in \mathbb{N}$, since $\cup_{n=1}^{\infty} A_n^H = A^H$. For each integer $k \geq n$ we define the stopping time

$$\tau_k^H := \inf \Big\{ t \in [0, T] \;\Big|\; W_t^H \leq \frac{1}{k} \Big\},$$

with $\inf \emptyset := T$ by convention. Then we have

$$\big\{ \tau_k^H < T \big\} \supset A_n^H.$$

We consider $k\, H\, \mathbf{1}_{(\tau_k^H, T]}$: it is easy to show that this is a 1-admissible strategy and

$$\int_{0+}^{T} k\, H\, \mathbf{1}_{(\tau_k^H, T]}\, dS \;\geq\; k \left(W_T^H - \frac{1}{k} \right) = k\, W_T^H - 1$$

on the event A_n^H. Therefore, if $P(A_n^H) > 0$ we could derive a contradiction to the NUPBR condition for S. $\square$

3.2 Proof of (ii)

Note first that $\tilde{S}$ is a semimartingale by **(i)** and by Itô's formula. For each $\mathbb{R}^{d+1}$-valued $\tilde{S}$-integrable process $\tilde{K}$, denote its first d component by K^S and the last component by K^1. Define the $\mathbb{R}$-valued predictable process $K^W = \{ K^W_t \}_{t\in(0,T]}$ by

$$
\begin{aligned}
K^W_t \quad &:= \int_{0+}^{t} \tilde{K}\, d\tilde{S} \;-\; \tilde{K}_t \cdot \tilde{S}_t \\
&= \int_{0+}^{t-} \tilde{K}\, d\tilde{S} \;-\; \tilde{K}_t \cdot \tilde{S}_{t-},
\end{aligned}
$$

where the dot $\cdot$ denotes the scalar product of $d{+}1$-dimensional vectors. Then the $\mathbb{R}^{d+2}$-valued predictable process $\big(K^W,\, K^S,\, K^1 \big)$ is a self-financing strategy, with zero initial wealth, for the $\mathbb{R}^{d+2}$-valued semimartingale

$$
\left\{ \Big(1,\, \frac{S_t}{W^H_t},\, \frac{1}{W^H_t} \Big) \right\}_{t\in[0,T]}.
$$

We see from Itô's formula that the self-financing property remains invariant by discounting and

$$
\begin{aligned}
\int_{0+}^{t} \tilde{K}\, d\tilde{S} \quad &= \int_{0+}^{t} (K^W, K^S, K^1)\, d\Big(1,\, \frac{S}{W^H},\, \frac{1}{W^H} \Big) \\
&= \frac{1}{W^H_t} \int_{0+}^{t} (K^W, K^S, K^1)\, d\big(W^H,\, S,\, 1 \big) \\
&= \frac{1}{W^H_t} \int_{0+}^{t} (\, K^W H \,+\, K^S \,)\, dS, \qquad t\in[0,T], \quad \text{a.s.} \qquad (1)
\end{aligned}
$$

Consequently, it holds that

$$
\begin{aligned}
&\left\{ \int_{0+}^{T} \tilde{K}\, d\tilde{S} \;\Big|\; \tilde{K} \text{ is a 1-admissible strategy for } \tilde{S} \right\} \\
= \quad &\left\{ \frac{1}{W^H_T} \int_{0+}^{T} (\, K^W H \,+\, K^S \,)\, dS \;\Big|\; \frac{1}{W^H_t} \int_{0+}^{t} (\, K^W H \,+\, K^S \,)\, dS \ge -1,\ t\in[0,T],\ \text{ a.s.} \right\} \\
&\quad \text{by (1)} \\
= \quad &\left\{ \frac{1}{W^H_T} \int_{0+}^{T} (\, K^W H \,+\, K^S \,)\, dS \;\Big|\; \int_{0+}^{t} (\, K^W H \,+\, K^S \,)\, dS \ge -W^H_t,\ t\in[0,T],\ \text{ a.s.} \right\} \\
\subset \quad &\left\{ \frac{1}{W^H_T} \int_{0+}^{T} J\, dS \;\Big|\; J \text{ is } S\text{-integrable and } \int_{0+}^{t} J\, dS \ge -W^H_t,\ t\in[0,T],\ \text{ a.s.} \right\}.
\end{aligned}
$$

Since the L^0-boundedness property remains invariant by the multiplication of an $\mathbb{R}$-valued random variable, it suffices to show that the set

$$
\left\{ \int_{0+}^{T} J\, dS \;\Big|\; J \text{ is } S\text{-integrable and } \int_{0+}^{t} J\, dS \ge -W^H_t,\ t\in[0,T],\ \text{ a.s.} \right\}
$$

is bounded in L^0. The set equals

$$\left\{ \int_{0+}^{T} J\, dS \;\middle|\; J + H \text{ is a 1-admissible strategy for } S \right\}$$

and further equals

$$\left\{ \int_{0+}^{T} I\, dS \;-\; \int_{0+}^{T} H\, dS \;\middle|\; I \text{ is a 1-admissible strategy for } S \right\},$$

where $I := J + H$. Since the L^0-boundedness property remains invariant by the addition of an $\mathbb{R}$-valued random variable, the proof is complete. □

4 Proof of Theorem 5

4.1 Proof of the 'if' part

Choulli and Stricker [2] and Karatzas and Kardaras [15] already gave a proof of this part. For sake of completeness, we present a proof. Suppose that Z is a strict martingale density for S. For each 1-admissible strategy H, we have

$$\begin{aligned} \left| Z_T \int_{0+}^{T} H\, dS \right| &= Z_T \left| \int_{0+}^{T} H\, dS \right| \\ &\leq Z_T \left(2 + \int_{0+}^{T} H\, dS \right), \end{aligned}$$

and thus

$$E\left[\left| Z_T \int_{0+}^{T} H\, dS \right| \right] \leq E\left[Z_T \left(2 + \int_{0+}^{T} H\, dS \right) \right]. \tag{2}$$

The same argument as in the above Remark 4 yields that the process

$$\left\{ Z_t \left(2 + \int_{0+}^{t} H\, dS \right) \right\}_{t \in [0,T]}$$

is a supermartingale with initial value $2Z_0$, and thus the right-hand side of (2) does not exceed $E[\, 2Z_0 \,]$. It then follows that the set

$$\left\{ Z_T \int_{0+}^{T} H\, dS \;\middle|\; H \text{ is a 1-admissible strategy for } S \right\}$$

is bounded in L^1 and consequently bounded in the L^0 sense as well. Since the L^0-boundedness property remains invariant by the multiplication of an $\mathbb{R}$-valued random variable, the process S satisfies the NUPBR condition. □

4.2 Proof of the 'only if' part

Assume the NUPBR condition for S. Define the set

$$\mathcal{D} \;:=\; \Big\{ X \;\Big|\; X \ge 0 \text{ a.s. and } \exists \{ H^{(n)} \}_n : \text{ a sequence of 1-admissible strategies for } S \text{ such that } \int_{0+}^{T} H^{(n)}\, dS \to X \text{ a.s. } \Big\},$$

which is non-empty since $0 \in \mathcal{D}$. Then the same argument as in §4 of Delbaen and Schachermayer [5] shows that $\mathcal{D}$ has a maximal element X^* and furthermore that there exists a 1-admissible strategy H^* such that $X^* = \int_{0+}^{T} H^*\, dS$. Their argument proceeds with the assumption of the NFLVR condition for the price process, but it should be noted that the proof of this property in their paper uses only the NUPBR condition.

Since $W_T^{H^*} \ge 1$ a.s. by the definition of $\mathcal{D}$, it follows from Proposition 6 that the $\mathbb{R}^{d+1}$-valued process

$$\tilde{S}_t \;:=\; \Big(\frac{S_t}{W_t^{H^*}}, \frac{1}{W_t^{H^*}} \Big)$$

is a semimartingale and satisfies the NUPBR condition. For the rest of the proof, we will prove that $\tilde{S}$ satisfies the NFLVR condition as well: Delbaen and Schachermayer [7] then shows the existence of a probability measure $Q \sim P$ under which $\tilde{S}$ is a sigma-martingale, and thus our proof will be complete with the strict martingale density

$$Z_t \;:=\; \frac{1}{W_t^{H^*}} \frac{dQ}{dP}\Big|_{\mathcal{F}_t}.$$

Since Corollary 3.8 of Delbaen and Schachermayer [5] shows the equivalence NFLVR $\iff$ NA + NUPBR, it remains to prove that $\tilde{S}$ satisfies the NA condition.

We assume that NA did not hold, i.e., $\exists \tilde{K}$ such that

(a) $\displaystyle\int_{0+}^{t} \tilde{K}\, d\tilde{S} \ge -1, \quad t \in [0,T], \quad \text{a.s.}$

(b) $\displaystyle\int_{0+}^{T} \tilde{K}\, d\tilde{S} \;\ge\; 0 \quad \text{a.s.}$

(c) $\displaystyle P\Big(\int_{0+}^{T} \tilde{K}\, d\tilde{S} > 0 \Big) \;>\; 0,$

and we will derive a contradiction. Assuming the same notations as in the proof of Proposition 5, we use Equation (1) to rewrite the above three conditions as

(a′) $\displaystyle\int_{0+}^{t} (\, K^W H^* \,+\, K^S \,)\, dS \ge -W_t^{H^*}, \quad t \in [0,T], \quad \text{a.s.},$

(b′) $\displaystyle\int_{0+}^{T} (\, K^W H^* \;+\; K^S \,)\, dS \;\geq\; 0 \quad$ a.s.,

(c′) $\displaystyle P\Big(\int_{0+}^{T} (\, K^W H^* \;+\; K^S \,)\, dS \;>\; 0 \Big) \;>\; 0,$

respectively. We define the $\mathbb{R}^d$-valued predictable process $J = \{\, J_t \,\}_{t \in (0,T]}$ by

$$J_t \;:=\; \big(K_t^W H_t^* \;+\; K_t^S \big) \;+\; H_t^*$$

and rewrite the three conditions further as

(a″) $\displaystyle\int_{0+}^{t} J\, dS \geq -1, \quad t \in [0,T], \quad$ a.s.,

(b″) $\displaystyle\int_{0+}^{T} J\, dS \;\geq\; \int_{0+}^{T} H^*\, dS \quad$ a.s.,

(c″) $\displaystyle P\Big(\int_{0+}^{T} J\, dS > \int_{0+}^{T} H^*\, dS \Big) \;>\; 0,$

respectively. This contradicts the maximality of the random variable $\int_{0+}^{T} H^*\, dS$ in the set $\mathcal{D}$. $\square$

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