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Why are Trend Cycle Decompositions of Alternative Models So Different?

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Abstract

When a certain procedure is applied to extract two component processes from a single observed process, it is necessary to impose a set of restrictions that defines two components. One popular restriction is the assumption that the shocks to the trend and cycle are orthogonal. Another is the assumption that the trend is a pure random walk process. The unobserved components (UC) model (Harvey, 1985) assumes both of the above, whereas the BN decomposition (Beveridge and Nelson, 1981) assumes only the latter. Quah (1992) investigates a broad class of decompositions by making the former assumption only.

This paper provides a general framework in which alternative trend-cycle decompositions are regarded as special cases, and examines alternative decomposition schemes from the perspective of the frequency domain. We find that as long as the US GDP is concerned, the conventional UC model is inappropriate for the trend-cycle decomposition. We agree with Morley et al (2003) that the UC model is simply misspecified. However, this does not imply that the UC model that allows for the correlated shocks is a better model specification. The correlated UC model would lose many attractive features of the conventional UC model.

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1 Introduction

It has been a common practice in empirical macroeconomic analysis to treat a time-series process such as an output level as the sum of its long-run trend and the short run fluctuation from the trend. These two components are sometimes called the trend and cycle or the permanent and transitory. This type of decomposition is clearly motivated by the modern macroeconomic theory, which is accustomed to separate analytically the long-run equilibrium of an economy from the short-run adjustment of the economy to an occasional shock. In the case of output, the long-run equilibrium process is treated by the economic growth theory, while the theory of business cycles investigates the short-run fluctuations of output from the trend and the impacts of government policies on its short-run behavior.

When a certain procedure is applied to extract two processes from a single observed process, however, it is necessary to impose a set of restrictions that defines two component processes. When there are overidentified restrictions, they can be tested, which might be interpreted as a specification test. One popular restriction is the assumption that the shocks to the trend and cycle are orthogonal. Another is the assumption that the trend is a pure random walk process. The unobserved components (UC) model (Harvey, 1985) assumes both of the above, whereas the BN decomposition (Beveridge and Nelson, 1981) assumes only the latter. Quah (1992) investigates a broad class of decompositions by making the former assumption only.

The orthoganlity assumption has several desirable features to model the output process. First, when two shocks are orthogonal, the total variability is the sum of variabilities of two shocks, and hence the relative importance of each shock is easily caculated. Second, the dynamic response of y_t to each shock can be computed and interpreted nicely. If two shocks were correlated, distinguishing the impact of one shock from the other would be difficult. Hence, some people think that the orthogonality assumption is indispensable for the decomposition to be a useful analytical tool.

The random walk assumption for the trend component has also some attractive features. First, such a trend is well defined and always identifiable without any additional assumptions. Second, it can be interpreted as the long-run forecast of y_t. A drawback for this assumption is that it makes the trend quite volatile.

This paper has two contributions to the literature. First, it provides a general framework in which alternative trend-cycle decompositions are regarded as special cases. In this way, we can find a link between what are otherwise viewed as different models with no apparent connection. Second, the paper examines alternative decomposition schemes from the perspective of the frequency domain. Several authors have included the spectral approach in their analysis of the trend-cycle decomposition, either merely superficially or quite substantially (Watson 1986, Quah 1992, Lippi & Reichlin 1992, Proietti 2006, and others). This paper attempts to provide a broad view of alternative decomposition schemes in terms of the spectral representation. The latter is quite natural because the aim of decomposing a macroeconomic process into the long-run and short-run process is inherently connected to the notion of frequecies of the original process.

The organization of this paper is as follows. In the next section, we develop the general form of the UC model and present the reduced form ARIMA model, the conventional (uncorrelated) UC model and the correlated UC model as special cases. We show the regression of the cycle shock on the trend shock as a link to connect the above three models. In section 3, we turn our attention to the US GDP process. Through this empirical examination of the alternative decomposition schemes, we ask two important questions: (i) Is the UC model misspecified? and (ii) Does the correlated UC model make sense? In section 4, we provide a brief conclusion.

2 UC and ARIMA Models

The trend-cycle decomposition of y_t in the conventional form of the unobserved component (UC) model (Harvey 1985) is given by

$$
\begin{aligned}
y_t &= \tau_t + c_t && (1)\\
\tau_t &= \mu + \tau_{t-1} + \eta_t && (2)\\
c_t &= \tilde{\psi}(L)\varepsilon_t && (3)
\end{aligned}
$$

where τ_t is the trend component, c_t is the cycle component of y_t, and $\tilde{\psi}(z) = \frac{\tilde{\theta}(z)}{\tilde{\phi}(z)}$, $\tilde{\theta}(z)$ and $\tilde{\phi}(z)$ are the polynomials of z. Both $\tilde{\theta}(z)$ and $\tilde{\phi}(z)$ have all roots outside of the unit circle. Let

$$\begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} \sim N(0, \mathbf{\Sigma}) \tag{4}$$

be possibly correlated bivariate normal random variables, where

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_\eta^2 & \sigma_{\eta\varepsilon} \\ \sigma_{\eta\varepsilon} & \sigma_\varepsilon^2 \end{bmatrix}. \tag{5}$$

Now consider the regression of ε_t on η_t

$$\varepsilon_t = \beta\eta_t + e_t \tag{6}$$

where $\beta = \frac{\sigma_{\eta\varepsilon}}{\sigma_\eta^2}$. Write the UC model as (1),(2),(3) with

$$\begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} \begin{bmatrix} \eta_t \\ e_t \end{bmatrix} \tag{4'}$$

where

$$\begin{bmatrix} \eta_t \\ e_t \end{bmatrix} \sim N(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_\eta^2 \begin{bmatrix} 1 & 0 \\ 0 & \gamma^2 \end{bmatrix}) \tag{5'}$$

where $\gamma^2 = \sigma_e^2/\sigma_\eta^2$ so that $\sigma_\varepsilon^2 = (\beta^2 + \gamma^2)\sigma_\eta^2$.

The model (1),(2),(3),(4′),(5′) is a general UC model for which the uncorrelated UC model as well as the ARIMA model as a special case. If β is set equal to zero, we have the uncorrelated UC model of Harvey (1985). If γ is set equal to zero, the model becomes the single source of error (SSOE) UC model (Anderson et. al., 2006), equivalent to the ARIMA model. To see the point above, take the first difference of (1) and use (2) and (3) to obtain

$$\begin{aligned} \Delta y_t &= \mu + \eta_t + \psi^*(L)\varepsilon_t \\ &= \mu + [1 + \beta\psi^*(L)]\eta_t + \psi^*(L)e_t \\ &= \mu + \zeta(L)\eta_t^* + \psi^*(L)e_t \end{aligned} \tag{7}$$

where $\psi^*(z) = (1-z)\tilde{\psi}(z) = 1 + \psi_1^* z + \psi_2^* z^2 + \ldots$ and $\zeta(z) = 1 + \zeta_1 z + \zeta_2 z^2 + \ldots$ Note that $1 + \beta\psi^*(z) = (1+\beta) + \beta\psi_1^* z + \beta\psi_2^* z^2 + \ldots$ Therefore, we find

$$\eta_t^* = (1+\beta)\eta_t \sim N(0, (1+\beta)^2\sigma_\eta^2) \tag{8}$$

and

$$\zeta_j = (\frac{\beta}{1+\beta})\psi_j^* \quad j = 1, 2, 3 \ldots \tag{9}$$

Recall the Wold representation of Δy_t as

$$\Delta y_t = \mu + \psi(L) u_t \tag{10}$$

where $u_t \sim WN(0, \sigma_u^2)$. The ARIMA model admits $\psi(z)$ be expressed as $\psi(z) = \frac{\theta(z)}{\phi(z)}$ with finite order polynomials of $\theta(\cdot)$ and $\phi(\cdot)$.

2.1 Single source of error UC model

Suppose $\gamma = 0$ now. Then $e_t = 0$ with probability 1, which in turn implies from (6) that $\varepsilon_t = \beta\eta_t$. Therefore (7) reduces to

$$\Delta y_t = \mu + \zeta(L)\eta_t^* \tag{11}$$

Comparing (10) and (11), the uniqueness of the Wold representation implies

$$\begin{aligned} u_t &= \eta_t^* = (1+\beta)\eta_t & (12)\\ \sigma_u^2 &= (1+\beta)^2\sigma_\eta^2 \\ \psi(z) &= \zeta(z) = \frac{1}{1+\beta}\left[1 + \beta\psi^*(z)\right] & (13) \end{aligned}$$

Setting $z = 1$ in (13) implies

$$\psi(1) = \frac{1}{1+\beta} \tag{14}$$

or

$$\beta = \frac{1-\psi(1)}{\psi(1)}$$

Given $\psi(z)$ and β, solving (13) for $\tilde{\psi}(z)$ yields

$$\begin{aligned} \tilde{\psi}(z) &= \frac{1+\beta}{\beta}\left[\psi(z) - \frac{1}{1+\beta}\right]/(1-z) \\ &= \frac{\psi(z) - \psi(1)}{[1-\psi(1)](1-z)} = \frac{\psi^+(z)}{1-\psi(1)} & (15) \end{aligned}$$

where $\psi(z) - \psi(1) = (1-z)\psi^+(z)$.

Unless $\psi(1) = 1$, there is a one to one mapping between an ARIMA model and the corresponding SSOE UC model. To see this, suppose $y_t \sim ARIMA(p, 1, q)$ and the corresponding UC model has $c_t \sim ARMA(p, q^*)$ where $q^* = \max(p, q-1)$. Then in addition to the common parameters μ and $\phi_1, \cdots, \phi_p$, we need to have a mapping from $\left(\theta_1, \cdots, \theta_q, \sigma_u^2\right)$ to $\left(\tilde{\theta}_1, \cdots, \tilde{\theta}_{q*}, \sigma_\eta^2, \beta\right)$. (14) yields β from the former set of parameter values. (13) gives σ_η^2 from β and σ_u^2. Then (15) leads to the mapping from $(\theta_1, \cdots, \theta_q)$ to $\left(\tilde{\theta}_1, \cdots, \tilde{\theta}_{q*}\right)$. It is easy to see that the resulting trend and cycle are, respectively, identical to those for the BN decomposition. That is,

$$\tau_t = \mu t + \sum_{s=0}^{t} \eta_s = \mu t + \frac{1}{1+\beta} \sum u_s = \mu t + \psi(1) \sum_{s=0}^{t} u_s$$

and

$$c_t = \tilde{\psi}(L)\varepsilon_t = \beta\tilde{\psi}(L)\eta_t = \frac{\beta}{1+\beta}\tilde{\psi}(L)u_t = [1-\psi(1)]\tilde{\psi}(L)u_t.$$

We find in this case

$$\begin{aligned} cov(\eta_t, \varepsilon_t) &= cov(\psi(1)u_t, [1-\psi(1)]\, u_t) \\ &= \psi(1)\,[1-\psi(1)]\,\sigma_u^2 \end{aligned}$$

and

$$corr(\eta_t, \varepsilon_t) = \left\{ \begin{array}{cl} 1 & \text{if } \psi(1) < 1 \\ -1 & \text{if } \psi(1) > 1 \end{array} \right. .$$

2.2 Correlated UC model

We now consider a more general UC model given in (1)-(5). Let $f_v(w)$ be the spectral density matrix of $\boldsymbol{v}_t = [\eta_t, \varepsilon_t]'$. Then $f_v = \frac{1}{2\pi}\boldsymbol{\Sigma}$ and the spectrum of Δy is given by

$$f_{\Delta y}(\omega) = \frac{1}{2\pi}\left[1, \psi^*(e^{-i\omega})\right] \boldsymbol{\Sigma} \left[\begin{array}{c} 1 \\ \psi^*(e^{i\omega}) \end{array} \right] \tag{16}$$

Now (6) implies that $\boldsymbol{\Sigma}$ can be written as

$$\boldsymbol{\Sigma} = \sigma_\eta^2 \left(\left[\begin{array}{c} 1 \\ \beta \end{array} \right] \left[\begin{array}{cc} 1 & \beta \end{array} \right] + \left[\begin{array}{cc} 0 & 0 \\ 0 & \gamma^2 \end{array} \right] \right)$$

Substituting the above expression into (16), we obtain

$$
\begin{aligned}
f_{\Delta y}(\omega) &= \frac{\sigma_\eta^2}{2\pi}\left|1+\beta\psi^*(e^{-i\omega})\right|^2 + \frac{\gamma^2\sigma_\eta^2}{2\pi}\left|\psi^*(e^{-i\omega})\right|^2 \\
&= \frac{\sigma_{\eta^*}^2}{2\pi}\left|\zeta(e^{-i\omega})\right|^2 + \frac{\gamma^2\sigma_\eta^2}{2\pi}\left|\psi^*(e^{-i\omega})\right|^2 \qquad (17)
\end{aligned}
$$

where $\sigma_{\eta^*}^2 = (1+\beta)^2\sigma_\eta^2$ and $\zeta(z)$ is given in (13).

The equality (17) reveals an important point. Suppose, for a while, that y_t is generated from the ARIMA model given in (10) with $\psi(z) = \frac{\theta(z)}{\phi(z)}$. Then $f_{\Delta y}(\omega) = \frac{\sigma_u^2}{2\pi}\left|\psi(e^{-i\omega})\right|^2$ and we can always set σ_e^2 equal to zero to obtain $\sigma_{\eta*}^2\left|\zeta(e^{-i\omega})\right|^2 = \sigma_u^2\left|\psi(e^{-i\omega})\right|^2$ by (14) and (15). In other words, the likelihood of a correlated UC model is always maximized by choosing the SSOE model.

The above point might sound a little odd because Morley, Nelson and Zivot (2003, MNZ hereafter) show that the correlated UC model fitted to the US GDP has typically a negative correlation between η_t and ε_t close to but not exactly equal to -1. This confusing phenomenon is due to the issue of model approximation and the identification problem, which is discussed later.

Now applying the Kolmogorov's Lemma to both sides of inequality $f_{\Delta y}(\omega) \geq \frac{\sigma_{\eta*}^2}{2\pi}\left|\zeta(e^{-i\omega})\right|^2$ and noticing $\sigma_u^2 = \exp[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log 2\pi f_{\Delta y}(\omega)d\omega]$, we obtain $\sigma_u^2 \geq \sigma_{\eta^*}^2 = (1+\beta)^2\sigma_\eta^2$. Since $\psi(1)^2\sigma_u^2 = \sigma_\eta^2$, it follows that $\psi(1)^2 \leq \frac{1}{(1+\beta)^2}$ so $\psi(1) > 1$ implies $\beta < 0$, which is the result of Nagakura & Zivot(2006).

This result is important, since $\psi(1) > 1$ for many macroeconomic series such as the US GDP process. It tells that the negative correlation between the trend and cycle shocks is simply the consequence of the persistence of the original process. And it would be incorrect and misleading to try to find any structural interpretation from it.

2.3 Uncorrelated UC model

The conventional UC models developed by Harvey and others consists of (1)-(5) with $\sigma_{\eta\varepsilon} = 0$. Setting $\beta = 0$ and noticing $\sigma_\varepsilon^2 = \gamma^2\sigma_\eta^2 = \sigma_e^2$ in (17), we obtain the spectrum of Δy_t as

$$f_{\Delta y}(\omega) = \frac{\sigma_\eta^2}{2\pi} + \frac{\sigma_\varepsilon^2}{2\pi}\left|\psi^*(e^{-i\omega})\right|^2 \geq \frac{\sigma_\eta^2}{2\pi}$$

which implies that the UC representation with uncorrelated trend and cycle shocks is 'feasible' only when the spectrum of Δy_t has the global minimum at frequency zero, which is shown by Lippi and Reichlin(1994). This fact appears to render the conventional UC models (proposed by Harvey and others) inappropriate for describing the important macroeconomic series such as the US GDP. Actually it is the main conclusion of MNZ, which has a quite important implication on empirical macroeconomic model building. A question is, however, as Harvey point out, whether BN decomposition would also lose its attractiveness in such a case.

It is easy to see that if we apply the BN decomposition to the ARIMA model implied by any uncorrelated UC model, the resulting trend and cycle are identical to those in the original UC model.

3 US GDP

In this section we examine a variety of trend-cycle models when applied to the US GDP process. Our special focus is on the comparison between the BN decomposition and the UC models. Data used in our analysis are the quarterly data on the US real output level from 1947Q1 through 2009Q1.

3.1 Cycles of the UC GDP implied by models

The left hand column of Figure 1 displays the plots of the cycle component implied by the models popular among macroeconomists: (a) the BN, (b) the UC(0), (c) the Perron-Wada (2009) Trend-break model, (d) the HP filter, and (e) the Bandpass filter (Christiano & Fitzgerald 2003). In fact, (d) and (e) are not originally proposed as procedures for decomposition. The shaded areas indicate the NBER recessions. The right hand column shows the corresponding spectral density of the cycle component of each model. The shaded area of each figure in this column indicates the range of business cycle frequencies (defined by the period of 6 to 24 quarters).

A striking finding here is that, as seen in the right column of Figure 1(a), the BN cycle almost totally fails to capture the business cycle frequencies of the US output. The left column shows that the BN cycle exhibits too much fluctuations and does not fit the NBER recessions in any sense. This fact has been pointed out by many authors, which discredits the BN approach as a way to separate the cyclical components of the US output from the trend in a conventional sense.

The cycles in the UC(0) model (Figure 1(b)) and the Trend-break model (Figure 1(c)) are similar except that the trend after 1972 is adjusted down in the latter model (as emphasized in Perron & Wada 2009). Also the cycles in the HP filter and the Bandpass filter are similar. Note that the spectral densities of these cycles are equal to zero at frequency zero, implying that they are I(-1) processes. In other words, both filters work when the GDP process is not only I(1) but also I(2). Not surprisingly the cycle process obtained by the Bandpass filter captures the business cycle frequencies very well. So is the HP filter.

The cycles implied by UC(0), Trend-break models and the HP filter look more consistent with the NBER business cycle chronology than other models. This might imply that inclusion of the frequency range a little lower than what is implied by 6 years period would help make cycles more consistent with what we regard conventionally as the business cycle. The UC(0) model does at least reasonable job in this respect.

3.2 Is the UC model misspecified?

We now turn our attention more closely to the comparison between the BN decomposition and the UC models applied to the US GDP. Table 1 reports the estimates of the model parameters for the BN decomposition of the ARIMA(2,1,2) model of the US GDP as well as its corresponding UC models.

The estimates are obtained for the general UC model given in (1) , (2) , (3)

$(4'), (5')$ with restrictions imposed on parameters corresponding to each case, described in the second row of the table. The second column of the table reports the SSOE-UC model (or the BN decomposition), the third and fourth columns the UC(0) models, and the last two columns the correlated UC models.

As is widely known, the size of the variance of the trend shock (σ_η^2) is quite different between the BN case and the UC(0) case. The former is more than twice as large as the latter. This problem greatly confuses many empirical macroeconomists when the size of the random walk part of the GDP is such a crucial issue. Watson (1986) points out that the UC(0) model assumes zero correlation between the trend and cycle shocks, whereas the BN decomposition assumes the two are perfectly correlated. MNZ go a step further and argue that the correlated UC model can be identified when y_t is assumed to be generated from ARIMA(2,1,2) process and the cycle component in the UC model is ARMA(2,0) process. They find that the ML estimate of the correlation of the two shocks is close to negative one, and that the zero correlation is rejected by the likelihood ratio test. The fifth column of Table 1 reports our estimates for this model, which is consistent with MNZ. Our estimated correlation is -0.94, which is quite close to MNZ's -0.98.

The top panel of Figure 2 displays the spectral density functions of Δy_t based on the BN model (or ARIMA) and the UC(0) model (with the shaded area showing the Business cycle frequencies). The two functions look quite different except for a high frequency region. The spectrum of Δy_t based on the ARIMA model starts with about 0.22 at frequency zero, peaks at frequency 0.2π, then declines quickly, and overlaps with the spectrum implied by the UC(0) model after frequency 0.5π. In contrast, the spectrum for the UC(0) model start with 0.04 at frequency zero, peaks at frequency 0.06π and then gradually declines. The panels (b) and (c) display the 95% confidence bands of Δy_t for the ARIMA model and the UC(0) model, respectively. To construct the confidence bands we use bootstrapped error terms to generate artificial data[1]. The spectrum in the figure corresponds to the median value of simulated spectral ordinates.

The entire empirical distributions for the two spectra at frequency zero

[1]The method used for bootstrapping UC0 and ARMA model is a revised procedure in RATS based on Stoffer and Wall (1991). We thank Tom Doan for his coding help.

are displayed in Figure 3. The horizontal axis measures the long-run variance of Δy_t divided by 2π, namely $\hat{\sigma}_\eta^2/2\pi$. The modes of the distributions are reached at 0.03 and 0.22, quite close but not necessarily equal to our estimated values (0.04 and 0.22). The estimated value is clearly smaller than the lower 2.5% quantile of the empirical distribution of the trend shock variance based on the ARIMA(2,1,2) model.

A natural question is what causes these two estimates to be so different. The answer lies in misspecification of the UC model. As we saw in section 2.3, the spectrum of Δy_t in the UC form is the sum of the spectrum of a white noise ($\Delta\tau_t$) and I(-1) process (Δc_t). The former is a positive constant $\left(\sigma_\eta^2/2\pi\right)$ and the latter is a hump-shaped curve starting from the origin at frequency zero. It implies that the ordinate of the spectrum for Δy_t is always higher than that at frequency zero, which contradicts the shape of Δy_t in the ARIMA representation (seen in Figure 2(a) and (b)). Lippi and Reichlin (1992) show that $\psi(1) < 1$ is a necessary condition for Δy_t to have the conventional UC representation with uncorrelated shocks. The estimates of $\psi(1)$ for the US GDP under ARIMA(2,1,2) is 1.28. The estimated parameters for the uncorrelated UC model is the maximizer of the misspecified likelihood function. The trend shock variance $\left(\sigma_\eta^2\right)$ has to be significantly underestimated in order to give a room to let the spectrum of Δc_t have a positive ordinate. But then c_t has to have a near unit root to compensate the underestimation at frequency zero. As is seen in Figure 2(a), distortion of the spectrum shape is substantial.

3.3 Does the correlated UC model make sense?

MNZ relax the zero correlation assumption between the trend and cycle shocks in the conventional UC model with AR(2) cycle and find the estimated correlation close to negative one, and observe that, with this nonzero correlation, all other parameter estimates get indistinguishable from those in the ARIMA model (or the BN model). Our estimates in the fifth column of Table 1 (labeled with correlated UC model with $\theta = 0$) reproduce their result. Its entries are almost identical to those in the second column (labeled with BN decomposition), including the likelihood values. Based on this result, MNZ reject the zero correlation between and the trend and cycle.

If we assume that y_t is generated from ARIMA(2,1,2) process, it has six parameters $(\mu, \sigma_\eta^2, \phi_1, \phi_2, \theta_1, \theta_2)$. The correlated UC model with ARMA(2,1) cycle (denoted UC-AR(2,1)), on the other hand, has 7 parameters (μ, σ_η^2, ϕ_1, ϕ_2, θ, β, γ) in our notation and $\left(\mu, \sigma_\eta^2, \phi_1, \phi_2, \theta, \rho_{\eta\varepsilon}, \sigma_\varepsilon^2\right)$ in MNZ notation. Therefore, the latter is not identified. Proietti (2006) argues that, for the process to be identified we need to impose either $\beta = 0$ or $\theta = 0$ (equivalently $\rho_{\eta\varepsilon} = 0$ or $\theta = 0$). The reduced form of UC-AR(2,0) is also ARIMA(2,1,2), so this process must be more restrictive than UC-AR(2,1). However, the estimated values of other parameters are not different (as seen in the 3rd and 4th columns in Table 1 under the labels '$\beta = 0$ & $\theta = 0$' and '$\beta = 0$ & $\theta \neq 0$'). When θ is set at zero, we can estimate the correlation between the trend and cycle shocks. MNZ follow this argument and use correlated UC-AR(2,0) as the reference model. They find the shock correlation is estimated as negative, close to -1 and very significant. At the same time the likelihood value and the parameter estimates are almost same as the reduced form ARIMA model.

This interesting finding motivates them to propose the correlated UC model as a correct model specification, as opposed to the conventional uncorrelated UC model. To say that the correlated UC model is a correct model specification is far beyond saying that the uncorrelated UC model is misspecified. The former argument is not as persuasive as the latter for two reasons. One is related to the issue of approximation and identification. The other concerns the problem of interpretation.

To see these two issues, let us get back to the estimated model. When we set $\theta = 0$, we can estimate β and γ. Our estimate for this model is reported in the 5th column of Table 1 under the label UC-UR with $\theta = 0$. The estimates imply the shock correlation is -0.940 (see Table 1), which is close to the estimate of Oh et al (2008) of -0.9487. However, θ is not necessarily equal to zero for the model to be identified. For instance, $\theta = 0.082$, the model is fully identified and we get the estimated correlation equal to -1.0 (see the last column of Table 1). The likelihood value is the same. This appears reasonable since this value of θ is the MLE for the SSOE-UC model, where the correlation is set at -1.0. In fact, when θ shifts from 0.082 to 0.0, the shock correlation moves from -1.00 to -0.940 and the likelihood stays the same. MNZ's correlated UC model is simply one point and no special status within this continuous range of (θ, ρ) pairs.

The second issue is the interpretation problem. In the conventional UC model, the two shocks are uncorrelated. In macroeconomic analysis, the fundamental structural shocks are often assumed to be uncorrelated, and the movements of macroeconomic variables such as output or inflation are considered as the outcome of those shocks that hit the economy simultaneously. Those orthogonal shocks can be analyzed in the variance decomposition analysis and the impulse response function. When the shocks are correlated, however, those powerful analytical devices cannot be used.

4 Conclusion

This paper provides a convenient general framework of the UC model in which the standard ARIMA model, the single source of error (SSOE) UC model, the conventional uncorrelated UC model, and the correlated UC model are special cases of the general model. It helps understand the link between different models with no apparent connection. A frequency domain perspective applied to this general form of the UC model provides important insight into understanding the difference between the BN decomposition and the decomposition based on the UC model.

We find that as long as the US GDP is concerned, the conventional UC model is inappropriate for the trend-cycle decomposition. As pointed out by MNZ, the conventional UC model is simply misspecified. However, this does not imply that the UC model that allows for the correlated shocks is a better model specification. When macroeconomic variables are best captured with ARIMA models, the SSOE model with perfectly correlated shocks is a best model since it is simply another representation of the ARIMA model itself. Moreover, the correlated UC model would lose many attractive features of the conventional UC model.

Table 1: Parameters Estimates of Trend-Cycle Models

Models	BN decomposition SSOE-UC Model (ARIMA)	Uncorrelated UC Model (UC(0))		Correlated UC Model (UC-UR)	
Restrictions on the General UC Model	$\gamma = 0$	$\beta = 0$		$\theta = 0$	$\theta = 0.082$
		$\theta = 0$	$\theta \neq 0$		
μ	0.794*	0.809*	0.807*	0.794*	0.794*
	(0.067)	(0.041)	(0.047)	(0.075)	(0.075)
ϕ_1	1.333*	1.522*	1.491*	1.333*	1.333*
	(0.146)	(0.103)	(0.102)	(0.152)	(0.127)
ϕ_2	-0.734*	-0.582*	-0.562*	-0.734*	-0.734*
	(0.173)	(0.110)	(0.106)	(0.170)	(0.129)
θ	0.082	[0.0]	1.000	[0.0]	[0.082]
	(0.196)		(3.054)		
σ_η	1.17*	0.603*	0.720*	1.17*	1.170*
	(0.141)	(0.103)	(0.051)	(0.140)	(0.145)
β	-0.563*	[0.0]	[0.0]	-0.535*	-0.563*
	(0.202)			(0.150)	(0.128)
γ	[0.0]	1.060*	0.446	0.194	0.006
		(0.324)	(0.690)	(0.264)	(5.484)
Likelihood	-329.580	-331.883	-331.883	-329.580	-329.580
Implied rho	-1.000	0.000	0.000	-0.940	-1.000

Notes:

1. Standard errors are in parentheses.
2. '*' indicates the significance level at 5% level.
3. Figures in [ ] indicate the values imposed rather than estimated.
4. 'Implied rho' stands for the correlation of trend and cycle shocks calculated as $\frac{\hat{\beta}}{\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}$.

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