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Power Laws in Firm Productivity

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We estimate firm productivity for about 3.2 million firms from 30 countries. We find that the distribution of firm productivity in each country, which is measured by total factor productivity (TFP), has a power law upper tail. However, the power law exponent of a TFP distribution in a country tends to be greater than that of a sales distribution in that country, indicating that the upper tail of a TFP distribution is less heavy compared to that of a sales distribution. We also find that the power law exponent of a TFP distribution tends to be greater than the power law exponents associated with the number of workers or tangible fixed assets. Given the idea that the sales of a firm is determined by the amount of various inputs employed by the firm (i.e., "production function" in the terminology of economics), these results suggest that the heavy tail of a sales distribution in a country comes not from the tail of a TFP distribution, but from the tail of the distribution of the number of workers or tangible fixed assets.

§1. Introduction

Ever since Pareto reported 120 years ago that the distribution of personal wealth follows a power law, both physicists and economists have investigated the distributions of a wide variety of size variables.[1)] Some economists found in the 1950s and 1960s that firm size distributions have fatter tails than log-normal distributions,[2)–4)] while physicists showed in the 1990s and later that the distributions of firm size variables, including firm sales, the number of workers, and tangible fixed assets, are characterized by power law tails.[5)–9)] The mechanism behind firm growth was also investigated so as to learn more about where the power law tails of firm size distributions come from.[10)–13)]

More recently, researchers have shifted their focus to the relationship between firm sales, the number of workers, and tangible fixed assets.[14)–16)] It was shown that this relationship is not linear but nonlinear,[5),14)] and that there exists a scaling law for the relationship.[14)] On the other hand, some economists studied this relationship based on the idea of "production function",[17)–20)] in which the output of a firm is determined by the amount of inputs employed by the firm, including the number of workers, tangible fixed assets, and productivity. Among several alternative functional

forms for production function proposed by economists, the most widely used one is a Cobb-Douglas production function,[21] which is given by $Y = AK^{\alpha}L^{\beta}$, where Y is output, L is the number of workers, K is tangible fixed assets, and A is productivity (more precisely, total factor productivity, or TFP). The exponents α and β are constant parameters, which are usually estimated empirically.

Note that Y, L, and K are reported by firms in their financial statements, so that they are all observable. However, A is not observable, so that we need to estimate it. Recently, it was found using firm level data from major countries, including the U.S. and Japan, that the relationship between Y, L, and K is well described by a Cobb-Douglas production function at least for mega firms (i.e. firms that belong to the tail part of firm size distributions).[14),17)] This suggests a method to estimate A from the data; namely, we first estimate the exponents α and β of a Cobb-Douglas production function using the data of Y, L, and K, and then compute A by $A = Y/(K^{\alpha}L^{\beta})$. In the present paper, we will estimate TFP in this way, and then look at the tail part of its distribution. Specifically, we are interested in whether the distribution of TFP is characterized by power law tails, as in the case of the distributions for Y, L, and K, and whether the tail of a TFP distribution is thicker or thinner than the tails associated with Y, L, and K.

The rest of the paper is organized as follows. In Sections 2 to 4, we estimate TFP for about 3.2 million firms from 30 countries, with the sample period of 2004 to 2008. The data we will use is from "Orbis," a database compiled by Bureau van Dajk Electronic Publishing. In Section 2, we give a brief review of the method proposed by Ishikawa et al. for estimating a Cobb- Douglas production function. In Section 3, we estimate two key parameters, α and β, of a Cobb-Douglas function and compare them across countries, as well as across industries. In Section 4, we show that the distribution of TFP is characterized by a power law tail, and that its power law exponent is greater than those for Y, L, and K for most countries, as well as for most industries. In Section 5, we use value added instead of sales as a measure of output in estimating TFP. This is closer to the traditional treatment adopted by economists. We repeat the same exercise as in Sections 3 and 4 to confirm that the basic results remain unchanged. Section 6 concludes the paper.

§2. Estimation of Cobb-Douglas production functions

It is shown by previous studies that various firm size variables, including firm sales (S), the number of workers (L), and tangible fixed assets (K), follow power law distributions in many countries. That is,

$$P_{>}(S) \propto S^{-\mu_S} \quad \text{for} \quad S > S_0 \tag{2·1}$$

$$P_{>}(L) \propto L^{-\mu_L} \quad \text{for} \quad L > L_0 \tag{2·2}$$

$$P_{>}(K) \propto K^{-\mu_K} \quad \text{for} \quad K > K_0 \tag{2·3}$$

where S_0, L_0, and K_0 represent the size thresholds, and μ_S, μ_L, and μ_K are the power law exponents associated with S, L, and K. Ishikawa et al. found two statistical laws regarding the relationships between S, L, and K.[14] The first law is that the

joint probability distributions of S and L, S and K, L and K are characterized by nonlinear functions of the form

$$P_J(L,S) = P_J\left(\left(\frac{S}{A_{LS}}\right)^{\frac{1}{\nu_{LS}}}, A_{LS}L^{\nu_{LS}}\right) \quad \text{for} \quad L > L_0, \tag{2·4}$$

$$P_J(S,K) = P_J\left(\left(\frac{K}{A_{SK}}\right)^{\frac{1}{\nu_{SK}}}, A_{SK}S^{\nu_{SK}}\right) \quad \text{for} \quad S > S_0, \tag{2·5}$$

$$P_J(L,K) = P_J\left(\left(\frac{K}{A_{LK}}\right)^{\frac{1}{\nu_{LK}}}, A_{LK}L^{\nu_{LK}}\right) \quad \text{for} \quad L > L_0, \tag{2·6}$$

where the coefficients A_{LS}, A_{SK}, A_{LK} and ν_{LS}, ν_{SK}, ν_{LK} are fixed numbers. The second law is that S, L, and K satisfy Gibrat's law as

$$P\left(\frac{S}{A_{LS}L^{\nu_{LS}}} \mid L\right) = P\left(\frac{S}{A_{LS}L^{\nu_{LS}}}\right) \quad \text{for} \quad L > L_0, \tag{2·7}$$

$$P\left(\frac{K}{A_{SK}S^{\nu_{SK}}} \mid S\right) = P\left(\frac{K}{A_{SK}S^{\nu_{SK}}}\right) \quad \text{for} \quad S > S_0, \tag{2·8}$$

$$P\left(\frac{K}{A_{LK}L^{\nu_{LK}}} \mid L\right) = P\left(\frac{K}{A_{LK}L^{\nu_{LK}}}\right) \quad \text{for} \quad L > L_0. \tag{2·9}$$

From these two laws, Ishikawa et al. derives the following relational expression.[14)]

$$S = AK^{\alpha}L^{\beta} \quad \text{for} \quad L > L_0 \quad \text{and} \quad K > K_0, \tag{2·10}$$

where the coefficients α, β are fixed numbers, and A is a stochastic variable which is independent of K and L, so that A satisfies

$$P(A \mid K, L) = P(A). \tag{2·11}$$

In economics, Eq. (2.10) is known as a Cobb-Douglas production function, while the variable A is referred to as total factor productivity, or TFP.

In estimating the coefficients α and β in Eq. (2.10), we need pay attention to the issue of multicollinearity; namely, K and L are highly correlated so that a simple application of ordinary least squares does not work. To overcome this issue, Ishikawa et al. proposes to introduce new variables, Z_1 and Z_2, which are defined as

$$\log Z_1 = \frac{\log L}{\sigma_{\log L}} + \frac{\log K}{\sigma_{\log K}}, \tag{2·12}$$

$$\log Z_2 = \frac{\log L}{\sigma_{\log L}} - \frac{\log K}{\sigma_{\log K}}, \tag{2·13}$$

where $\sigma_{\log L}$ and $\sigma_{\log K}$ are the standard deviations of $\log K$ and $\log L$. Then Eq. (2.10) is transformed into

$$S = AZ_1^{\theta_1}Z_2^{\theta_2} \quad \text{for} \quad L > L_0 \ \text{ and } \ K > K_0, \tag{2·14}$$

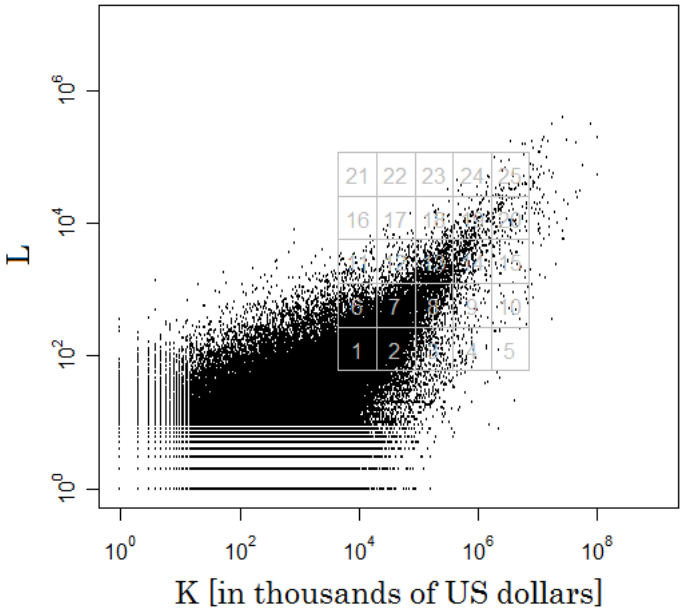

Fig. 1. Scatter plots of tangible fixed assets and the number of workers for Japanese firms in 2008. We evenly divide the entire (K, L) plane (i.e. $4.5 \times 10^3 \leq K \leq 7.2 \times 10^6$ and $60 \leq L \leq 1.2 \times 10^5$) into 5×5 sub-areas.

where

$$\alpha = \frac{\theta_1 - \theta_2}{\sigma_{\log K}}; \ \beta = \frac{\theta_1 + \theta_2}{\sigma_{\log L}}. \tag{2·15}$$

Note that, by construction, Z_1 and Z_2 are not correlated to each other in the regions $L > L_0$ and $K > K_0$. Therefore, we can safely apply an ordinary least square to Eq. (2.14) so as to obtain estimates for α and β.

We run this regression using Japanese data in 2008 to find that α and β are equal to 0.18 and 0.85, respectively. Based on these estimates, we conduct the following exercise. In Figure 1, we plot K and L, and divide the entire (K, L) space (i.e. $4.5 \times 10^3 \leq K \leq 7.2 \times 10^6$ and $60 \leq L \leq 1.2 \times 10^5$) evenly into 5×5 sub-areas. Then, we produce a probability density function of the estimated TFP for each of the sub-areas. The results are presented in Figure 2. It is confirmed that the probability density functions of A are almost identical irrespective of the values of K and L, indicating that the estimated A is indeed independent of K and L.

§3. Estimates of α and β

Figure 3 presents the estimates of α and β for 30 countries, whose list is available in Table II in the appendix. Figure 3 shows that β is greater than α for all countries except Serbia and the Netherlands, suggesting that firm sales depend more closely on the number of workers than on the amount of tangible fixed assets. Figure 4 presents the estimates of α and β for different industries. The horizontal axis of Figure 4 represents the SIC (Standard Industrial Classification) code, which is given

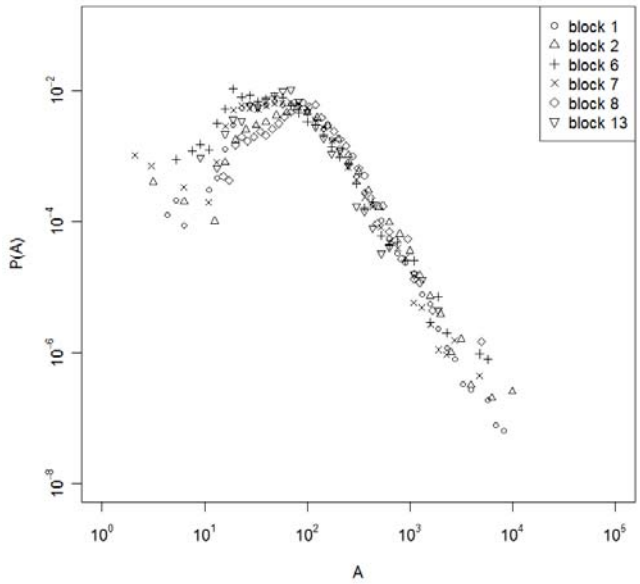

Fig. 2. PDF of total factor productivity. The block number refers to the corresponding sub-area in Figure 1.

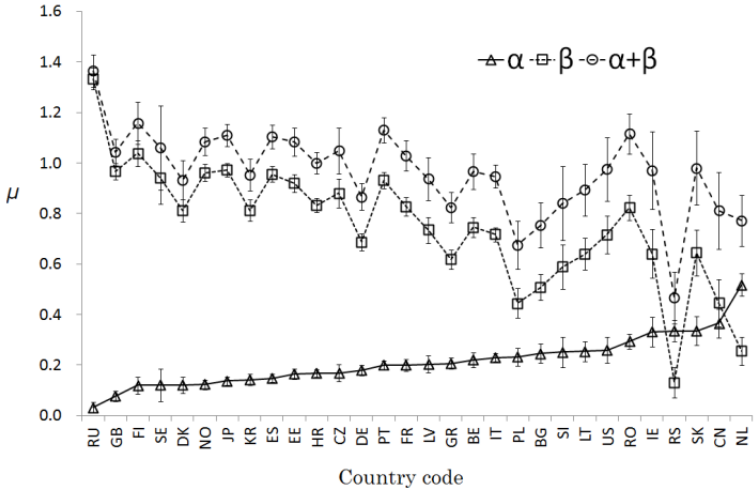

Fig. 3. The estimates of α, β, and $\alpha + \beta$ for different countries. The horizontal axis represents the country code, which is given in Table II in the appendix.

in Table III in the appendix.[22)] We see that the estimate of α tends to be somewhere around 0.4, and β tends to be around 0.6, with the exception of "insurance agents, brokers, and services" (SIC code no.64) where β is very close to unity.

Turning to the sum of α and β, we see in Figure 3 that the sum of α and β is close to unity for most countries, although there are several countries, like Serbia, in

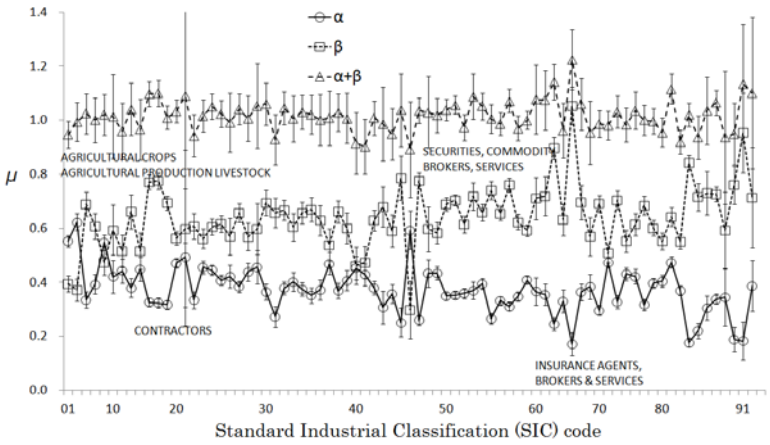

Fig. 4. The estimates of α, β, and $\alpha+\beta$ for different industries. The horizontal axis represents the SIC code, which is given in Table III in the appendix.

which the sum of α and β is significantly smaller than unity. Somewhat interestingly, there is a tendency that the sum of α and β is smaller for countries with larger α. Also, we see in Figure 4 that the sum of α and β is very close to unity for almost all industries. In economics, $\alpha+\beta=1$ implies a constant return to scale; namely, if one increases each of K and L by, say, ten percent, then S also increases by ten percent. The above empirical results suggest that this property holds in most countries, as well as in most industries.

§4. Total factor productivity

Given the estimates of α and β in hand, we now proceed to the estimation of total factor productivity. Using Eq. (2.10), we calculate total factor productivity as $A = S/(K^{\alpha}L^{\beta})$. Figure 5 shows the cumulative distribution functions of total factor productivity obtained in this way for Japanese firms in the years of 2004 to 2008. We see that the tail part of the distribution for each year satisfies:

$$P_{>}(A) \propto A^{-\mu_A} \quad \text{for} \quad A > A_0. \tag{4·1}$$

indicating that total factor productivity follows a power law distribution with an exponent of μ_A. Note that the power law exponent is about 1.65, and it remains unchanged over time.

Figure 6 shows the power law exponents of total factor productivity for 27 countries, together with the power law exponents of sales (μ_S), the number of workers (μ_L), and tangible fixed assets (μ_K). These power law exponents are estimated using a new method proposed by recent studies.[9),23)] That is, through a set of statistical tests, we identify a range of a variable in which the variable follows a power law distribution. A power law exponent is estimated using observations only in that range. We see that

$$\mu_K < \mu_S \leq \mu_L < \mu_A. \tag{4·2}$$

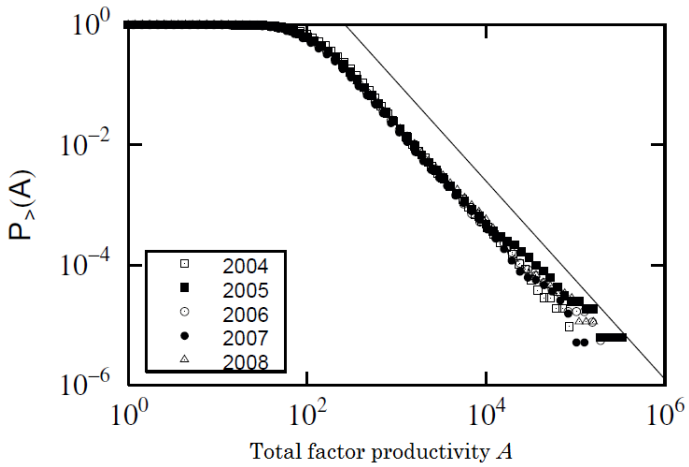

Fig. 5. CDF of total factor productivity for Japanese firms in 2004 to 2008. The straight reference line represents a power law with an exponent of 1.65.

Table I. Inequality in terms of productivity, sales, the number of workers, and tangible fixed assets.

	Productivity	Sales	Number of workers	Tangible fixed assets
20 percent of the total	1.3%	0.019%	0.016%	0.0067%
50 percent of the total	23%	0.21%	0.25%	0.066%
80 percent of the total	43%	2.9%	5.3%	0.90%

holds for almost all countries. This indicates that inequality across firms in terms of A tends to be smaller than inequality in terms of S, K, or L. To show this result in a different way, we count the number of firms that account for 20, 50, 80 percent of the total of A. Similarly, we count the number of firms that account for 20, 50, 80 percent of the total of S, L, and K. The result for Japanese firms in 2008 is presented in Table I, showing that the we need top 23 percent firms to account for 50 percent of the total of A, while we need only 0.21 percent firms to account for 50 percent of the total of S. Even more surprisingly, we need only 0.066 percent firms to account for 50 percent of the total of K.

The result that μ_A is greater than μ_S implies the following. Note that, by construction, A, Z_1 and Z_2 in Eq. (2.14) are independent of each other. Therefore, the four power law exponents, μ_S, μ_A, μ_{Z_1}, and μ_{Z_2}, must satisfy the following:[24),25)]

$$\mu_S = \min\left\{\mu_A, \frac{\mu_{Z_1}}{\theta_1}, \frac{\mu_{Z_2}}{\theta_2}\right\}. \tag{4·3}$$

The empirical result that μ_A tends to be greater than μ_S implies that the tail part of the distribution of sales does *not* stem from the tail part of the distribution of total factor productivity.

Figure 7 shows the power law exponents of total factor productivity for different industries. The three lines represent 2006, 2007, and 2008. We see that μ_A is small

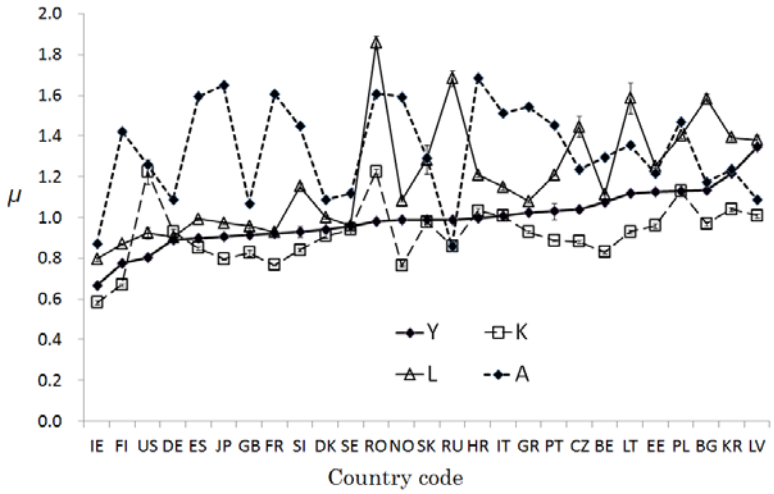

Fig. 6. Power law exponents in 2008 for different countries. The horizontal axis represents the country code, which is given in Table II in the appendix.

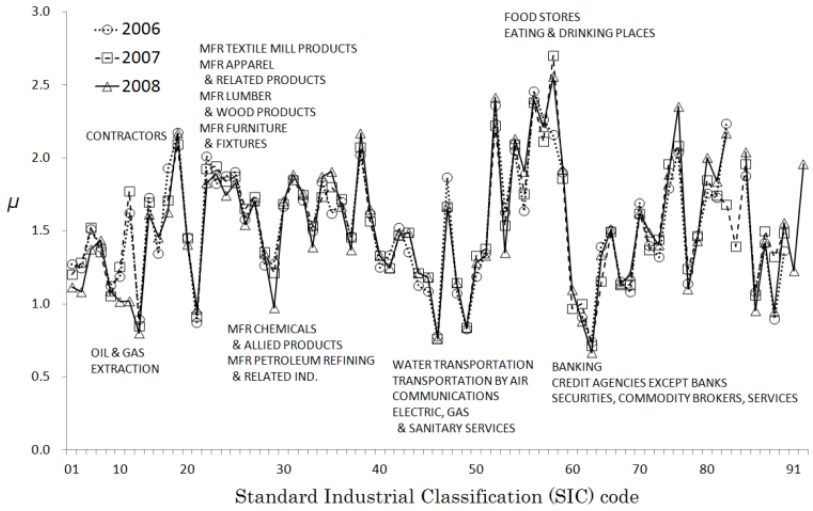

Fig. 7. Power law exponents for different industries. The horizontal axis represents the SIC code, which is given in Table III in the appendix.

for banks, credit agencies, securities, commodity brokers, and services, suggesting that total factor productivity in these industries is significantly unequal across firms. It may be the case that, in these industries, the tail of sales distribution is generated by the tail of productivity distribution. On the other hand, μ_A is relatively large for food stores, restaurants, and automobile repair shops, suggesting that total factor productivity in these industries is not so unequal across firms. It is highly likely that the tail of sales distribution in these industries comes not from the tail of productivity distribution, but from the tail of the number of workers distribution or from the tail of tangible fixed assets distribution.

§5. Production function for value added

So far we have used firm sales as a measure of output. Firm sales data is easy to obtain since it is disclosed by almost all firms in their financial statements, irrespective of which country and which industry a firm belongs to. This is particularly important when one seeks to conduct cross country analysis as we did in the previous sections. However, some economists prefer to use value added as a measure of output, since it may not be appropriate to ignore the role of raw materials and intermediate products in the process of production, at least for some firms belonging to a certain industries. In this section, we check how the results in the previous sections will change when we use value added instead of sales as a measure of output. Unfortunately, value added for each firm is available only for Japanese manufacturing firms. We will use the dataset provided by Teikoku Databank Ltd. for about 21,000 Japanese manufacturing firms in 2008.

Value added and sales are related as:

$$V = S - F, \tag{5·1}$$

where V represents value added and F is intermediate inputs, consisting of the expenditures on raw materials, purchased parts, transportations, outsourcing, and so on. We start with looking at the correlations between the value-added to sales ratio, which is defined by $1 - F/S$, and L or K. The left panel of Figure 8 shows how the distribution of the value-added to sales ratio depends on the value of L. Specifically, we divide the range of L, which is given by $10 \leq L < 10^{3.5}$, evenly into five. For each of the five sub-regions of L, we present a probability density function of the value-added to sales ratio conditional on that L belongs to that sub-range. We see that the five PDFs are almost identical, indicating that the value-added to sales ratio is independent of L. Similarly, the right panel of Figure 8 shows the conditional distributions of the value-added to sales ratio for different values of K. The conditional PDFs are almost identical irrespective of the value of K, suggesting again that the value-added to sales ratio is independent of K.

Given that the value-added to sales ratio is correlated neither with L nor K, we can rewrite Eq. (2.10) as

$$V = A^{'} K^{\alpha} L^{\beta}, \tag{5·2}$$

where $A^{'}$ is defined as $A^{'} = (1 - F/S)A$. Using Eq. (5.2), we estimate total factor productivity as we did in the previous sections. Figure 9 shows the cumulative distribution function of $A^{'}$, together with that of A. We confirm that the tail part of the distribution of $A^{'}$ is characterized by a power law, and that the power law exponent associated with $A^{'}$ is almost identical with the exponent associated with A.

§6. Conclusion

In this paper we have estimated total factor productivity for mega firms from 30 countries. We found that a productivity distribution is characterized by a power

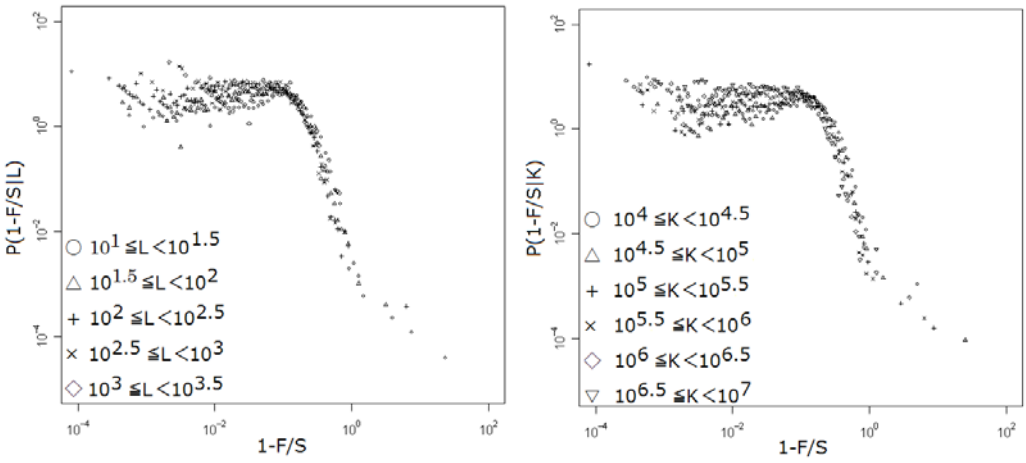

Fig. 8. PDF of the value-added to sales ratio for different values of L (left panel) and for different values of K (right panel). The range of L, $10 \leq L < 10^{3.5}$, is divided evenly into five, while the range of K, $10^4 \leq K < 10^7$, divided evenly into six. The data for Japanese manufacturing firms in 2008 is used.

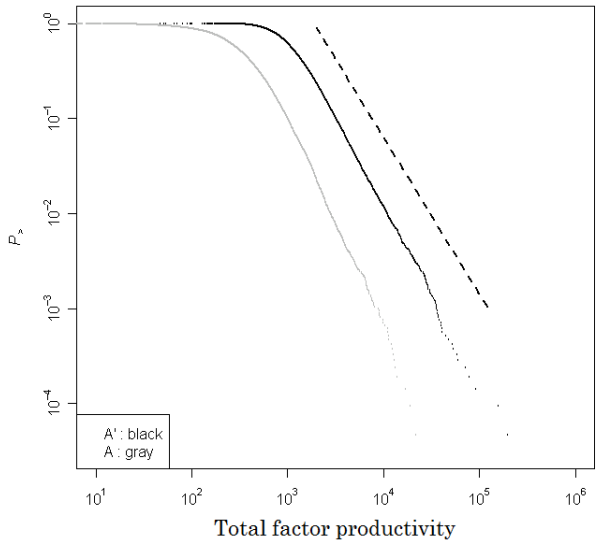

Fig. 9. CDFs of A and A' for Japanese manufacturing firms in 2008. The reference line is a power law with an exponent of 1.65.

law tail, as in the case of other firm size variables, including firm sales, the number of workers, and tangible fixed assets. However, the power law exponent associated with a productivity distribution tends to be greater than those for the other firm size variables. We have confirmed this tendency for different countries, as well as for different industries. This result suggests that the tail part of a sales distribution does not stem from the tail of a productivity distribution, but from the tail of the distributions of the number of workers or tangible fixed assets.

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References

1) V. Pareto, Cours d'Economique Politique (Macmillan, London, 1897).
2) H. A. Simon and C. P. Bonini, The American Economic Review **48** (1958), 67-617.
3) R. E. Quandt, The American Economic Review **56** (1966), 416-432.
4) Y. Ijiri and H. A. Simon, Journal of Political Economy **82** (1974), 315-331.
5) K. Okuyama, M. Takayasu, and H. Takayasu, Physica A **269** (1999), 125.
6) M. H. R. Stanley, S. V. Buldyrev, S. Havlin, R. Mantegna, M.A. Salinger, and H. E. Stanley, Economics Letters **49** (1995), 453-457.
7) T. Mizuno, M. Katori, H. Takayasu, and M. Takayasu, Empirical Science of Financial Fluctuations - The Advent of Econophysics (Springer Verlag, Tokyo, 2002) 321.
8) R. L. Axtell, Science **293** (2001), 1818.
9) S. Fujimoto, A. Ishikawa, T. Mizuno, and T. Watanabe, submitted to Economics.
10) R. N. Mantegna and H. E. Stanley, Nature **376** (1995), 46-49.
11) T. Mizuno, M. Takayasu, and H. Takayasu, Physica A **332** (2004), 403-411.
12) A. Ishikawa, S. Fujimoto, and T. Mizuno, Physica A **390** (2011), 4273-4285.
13) Y. Fujiwara, C. D. Guilmi, H. Aoyama, M. Gallegati, and W. Souma, Physica A **335** (2004), 197-216.
14) A. Ishikawa, S. Fujimoto, T. Watanabe, and T. Mizuno, submitted to EPJ B.
15) Y. Ikeda and W. Souma, Progress of Theoretical Physics Supplement **179** (2009), 93.
16) H. Aoyama, H. Yoshikawa, H. Iyetomi, and Y. Fujiwara, Progress of Theoretical Physics Supplement **179** (2009), 80.
17) T. Watanabe, T. Mizuno, A. Ishikawa, and S. Fujimoto, The Economic Review **62** (2011), 193-208.
18) H. S. Houthakker, The Review of Economic Studied **23** (1955), 27-31.
19) S. Rosen, Economica **45** (1978), 235-250.
20) C. I. Jones, Quarterly Journal of Economics **120** (2005), 517-549.
21) C. W. Cobb and P. H. Douglass, The American Economic Review **18** (1928), 139-165.
22) U.S. Department of Labor, http://www.dol.gov/
23) Y. Malevergne, V. Pisarenko, and D. Sornette, Physical Review E **83** (2011), 036111.
24) A. H. Jessen and T. Mikosch, Publ. de l'Inst. Math. **94** (2006), 171-192.
25) X. Gabaix, Annual Review of Economics **1** (2009), 255-294.

Appendix

—— Country code and Standard Industrial Classification (SIC) code ——

Table II. Country code

BE	Belgium	BG	Bulgaria	CN	China
CZ	Czech	DE	Germany	DK	Denmark
EE	Estonia	ES	Spain	FI	Finland
FR	France	GB	The U.K.	GR	Greece
HR	Croatia	IE	Ireland	IT	Italy
JP	Japan	KR	Korea	LT	Lithuania
LV	Latvia	NL	The Netherlands	NO	Norway
PL	Poland	PT	Portugal	RO	Romania
RS	Serbia	RU	Russia	SE	Sweden
SI	Slovenia	SK	Slovak	US	United States

Table III. Standard Industrial Classification (SIC) code

01-09	Forestry and Fishery
10-19	Mining, Building, Construction
20-29	Food, Textile, Wood, Chemicals
30-39	Rubber, Leather, Metal, Machinery
40-49	Railroad, Transportation, Postal, Telecom, Electricity, Gas, Water Supply
50-59	Wholesale, Retailing, Eatery
60-69	Bank, Finance, Security, Insurance, Real estate, Investor
70-79	Services
80-89	Medical, Legal, Educational, Social Services
90-99	International Affairs and Non-Operating Establishments