ASSET PRICES, TRADING VOLUMES, AND INVESTOR WELFARE IN MARKETS WITH TRANSACTION COSTS

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Abstract. We investigate how an increase in transaction costs affect the equilibrium asset prices and allocations. We find a sufficient condition for an increase in transaction costs to increase buying prices, decrease selling prices, decrease the trading volume, and make all active traders worse off. The sufficient condition is met by a general class of utility functions, which contains all CARA utility functions and even some non-HARA utility functions. As for CRRA utility functions, the class contains all utility functions with CRRA coefficients less than or equal to one. We show that whenever there is an agent with a CRRA coefficient greater than one, an increase in transaction costs may well decrease buying prices and make buyers better off.

Keywords: General equilibrium, asset markets, transaction costs, Tobin tax, constant absolute risk aversion, constant relative risk aversion.

JEL Classification Codes: D51, D53, D61, D63, G11, G12.

1. Introduction

In asset markets, the transaction cost is defined as the difference between the price that the buyer must pay to obtain an asset and the price that the seller can receive by giving up the asset. It can, thus, represent physical or technological costs, brokerage fees, and tax.

While the transaction cost is an impediment to straightforward applications of benchmark results in finance, such as the characterization of optimal portfolios and the pricing of derivative assets, these results have been extended to markets with transaction costs. To name just two notable examples, Constantinides (1986) and Davis and Norman (1990) considered the optimal consumption-investment problem of Merton (1973) when the decision maker incurs transaction costs; and Bensaid et al (1992) considered derivative asset pricing when replicating them requires transaction costs. These results are often stated in comparison with the benchmark results. For example, while Merton's rule of the optimal plan stipulates that, in the case of one risky and one riskless asset, the ratio between the amounts invested in the two assets should be kept constant throughout, Davis and Norman (1990) showed that with transaction costs, the ratio should stay within a wedge, rather than remain constant. In the binomial model of Cox, Ross, and Rubinstein (1979) but with transaction costs, Bensaid et al (1992) showed that it may be

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cheaper to form a portfolio whose payoff dominates that of a derivative asset than to form a portfolio whose payoff perfectly replicates that of the derivative asset, and the optimal trading strategy admits "no-rebalancing intervals" of the number of stocks held.

Although these undertakings are ambitious and important, they have a common drawback. It is that in their models, the asset prices are exogenously specified. While this drawback does not appear to be serious when one is confined to the portfolio choice problem or the derivative asset pricing, it does in fact limit the applicability of the results when one would like to see the consequences of transaction costs from an equilibrium perspective. For example, the introduction of a transaction cost induces investors to shift from the Merton rule to the Davis-Norman rule. In the language of equilibrium theory, this means that the investors change the supply of and the demand for assets. The asset prices, exclusive of transaction costs, would then need to be changed to sustain equilibrium, but these price changes cannot be analyzed in the above-mentioned literature because the asset prices are fixed throughout the analysis. In particular, it is not possible to predict trading volumes or welfare consequences that fully incorporate the investors' reactions to transaction costs in the framework of the above literature.

In this paper, we take up a general equilibrium model of asset markets in which multiple, heterogeneous agents (investors) trade assets incurring, for each unit of assets they trade, transaction costs that are proportional to asset prices. We do not take asset prices as exogenously given. Rather, we take the agents' expected utility functions and initial risks as the primitive data. Then, we determine, for each level of proportional transaction costs, the equilibrium asset prices, and investigate how they depend on the levels of proportional transaction costs. We are interested in whether, as in the case of fixed asset prices, an increase in transaction costs increases the buying price (the asset price plus the transaction cost), decreases the selling price (the asset price minus the proportional transaction cost), and decrease the trading volume; and whether the economic cost of transaction costs (the welfare loss arising from the discrepancy between buying and selling prices) is borne by all actively trading agents.

All our results are concerned with the case of a single consumption period (just as in the classical portfolio choice problem), one risk-free bond, and one risky asset. We assume, without loss of generality, that the transaction costs are incurred on the transactions of the risky asset. First, in Theorem 1, we classify the two "half aggregate" demand functions, one for the buy side and the other for the sell side, according to the signs of their slopes at equilibrium, and show that a small (infinitesimal) increase in proportional transaction costs increases the buying price and decreases the selling price if and only if the two signs are equal. In particular, this theorem is applicable if the two half aggregate demand functions are both downward-sloping.[1] Next, in Theorem 2, we impose a more stringent condition that each individual agent's demand function is downward-sloping globally and show that an increase in proportional transaction costs of arbitrary size increases the buying price, and decreases the selling price, the trading volume, and all actively trading agents' welfare. Theorem 2 builds on Theorem 1, but its proof involves additional intricate arguments to deal with non-differentiability of demand functions, which inevitably arises in the presence of proportional transaction costs.

[1]In the partial equilibrium analysis, the half aggregate demand function of the buyers is called the demand function, and the the half aggregate demand function of the sellers, multiplied by -1, is called the supply function. This case is, thus, nothing but the standard textbook case, where the demand function is downward-sloping and the supply function is upward-sloping at equilibrium.

The assumptions of Theorems 1 and 2 are given in terms of the agents' demand functions. Our third result, Proposition 1 and Theorem 3, gives a sufficient condition, in terms of their Arrow-Pratt measures of absolute risk aversion, for the assumption of Theorem 2 to be met. This condition is satisfied not only by all utility functions exhibiting constant absolute risk aversion (CARA), but also by the utility functions having the coefficient of constant relative risk aversion (CRRA) not exceeding one, and some utility functions that do not even exhibit hyperbolic absolute risk aversion (HARA). Theorem 3, therefore, implies that an increase in transaction costs increases the buying price, decreases the selling price, and decreases the trading volume, even in markets where agents having CARA, CRRA, HARA, and non-HARA utility functions coexist. This is in sharp contrast with Vayanos (1998), who assumed all agents are assumed to have CARA utility functions. We prove this result by drawing much on the analysis in expected utility theory, without deriving closed-form solutions for the equilibrium prices.

Our last result, Proposition 2, shows that the upper bound of the CRRA coefficient equal to one for the applicability of Theorem 3 is in fact tight. Specifically, for any two agents who exhibit CRRA and one of whom has a CRRA coefficient greater than one, there is a distribution of initial risk for the two consumers such that the buying price is lower and the trading volume is higher under a positive but sufficiently small transaction costs than under the zero transaction cost. The crux of this result lies in the way the initial risks are distributed: the agent having a CRRA coefficient greater than one turns out to be the seller of the risky asset. Then his demand function may well be upward-sloping and Theorem 1 implies that both of the buying and selling prices go down.

It is a conventional wisdom that an increase in transaction costs increases the buying price, decreases the selling price, and decreases the trading volume, and hence the economic cost of transaction costs (the welfare loss arising from the discrepancy between the buying and selling prices) is borne by both buyers and sellers. The lesson from Proposition 2 is that it may well be contradicted by a careful equilibrium consideration. It also poses a cautious note on Tobin tax, which Tobin (1978) proposed to levy on currency transactions in order to curb speculative ones, reduce exchange rate volatility, establish the autonomy of monetary policy, and raise the tax revenue. The proponents of Tobin tax, such as Tobin (1978), Stiglitz (1989), Summers and Summers (1989), Eichengreen, Tobin, and Wyplosz (1995), and Krugman (2009), seem to presume that it necessarily increases the buying price, decrease the trading volume, and load the tax burden on all active traders.[2] Yet, Proposition 2 shows that the buying price can go down and the trading volume can go up, and the tax burden can be loaded only on the seller, who is more risk averse than the buyer and, as such, should not be the target of Tobin tax. Moreover, this seemingly pathological phenomenon can arise in the simplest possible model: there are a single consumption period; two states of the world; an Arrow security and a riskless bond; and two fully rational agents having possibly identical CRRA coefficients, identical information, and identical beliefs. Our results should, thus, be contrasted with those of Dow and Rahi (2000), who also showed that the introduction of Tobin tax may make some (and, even, all) traders better off, because the driving force behind their result is that traders, with CARA utility

[2]The opponents of Tobin tax tend to oppose to it for such reasons as the reduction of liquidity and the possibility of tax evasion. McCulloch and Pacillo (2011), Matheson (2011), and Anthony, Bijlsma, Elbourne, Lever, and Zwart (2012) surveyed the literature on the feasibility and revenue forecasts of Tobin tax, and its impact on trading volumes, price volatility, and economic welfare.

functions are asymmetrically informed and Tobin tax may reduce the informativeness of the equilibrium asset prices. The common drawback of our and their models is that they cannot deal with asset price volatilities, excessive or not, because transactions take place just once. Buss, Uppal and Vikov (2011) investigated a model in which transactions take place more than once and the agents' utility functions are recursive ones of Epstein and Zin (1989), including standard time-additive CRRA utility functions. Their analysis is mostly numerical and the pathological phenomenon does not emerge[3] in their example of one bond, one stock, and two consumers with time-additive CRRA utility functions, though both coefficients are greater than one.

The rest of this paper is organized as follows. In Section 2, we give the model of this paper and preliminary results. In Section 3, we present the first two main results, Theorems 1 and 2, on the impact of an increase in proportional transaction costs on asset prices. The sufficient condition of Theorem 2 is implied by a condition presented in stated in Section 4 in terms of the Arrow-Pratt measure of absolute risk aversion. In Section 5, we give a class of examples of CRRA utility functions in which an increase in proportional transaction costs decreases the buying price and increases the trading volume. In Section 6, we summarize our results and suggest directions of future research. All proofs are gathered in the Appendix.

2. Model and preliminary Results

The risk is represented by a standard probability space $(\Omega, \mathscr{F}, P)$.

There are I agents $i = 1, 2, \ldots, I$, each characterized by a utility function u_i defined on some open interval $(\underline{d}_i, \overline{d}_i)$, where $\underline{d}_i \in \{-\infty\} \cup \boldsymbol{R}$, $\overline{d}_i \in \{+\infty\} \cup \boldsymbol{R}$, and $\underline{d}_i < \overline{d}_i$, and an initial risk A_i, a random variable defined on Ω. Assume that u_i is twice continuously differentiable and satisfies $u_i'(x_i) > 0 > u_i''(x_i)$ for every $x_i \in (\underline{d}_i, \overline{d}_i)$, and the so-called Inada condition, that is, $u_i'(x_i) \uparrow \infty$ as $x_i \downarrow \underline{d}_i$, and $u_i'(x_i) \downarrow 0$ as $x_i \uparrow \overline{d}_i$.

Two types of assets, a risky asset and a riskless bond, are traded. The future dividend (value) of the risky asset is denoted by a random variable S defined on Ω. We assume S is not constant, that is, $\operatorname{essinf} S < \operatorname{esssup} S$. We allow S to be correlated with the A_i. The future dividend of the riskless bond is equal to 1.

The relative price of the risky asset with respect to the riskless bond, to be determined at equilibrium, is denoted by π. The non-standard aspect of this model is that agents must incur transaction costs to trade the risky asset, in exogenously specified proportions c_1 and c_0, lying between 0 and 1, to the prices π. To be precise, denote by y^i the number of the risky asset traded by agent i. Of course, if $y_i > 0$, then agent i buys the risky asset, whereas if $y_i = 0$ or if $y_i < 0$, then he does not trade trade or sell it, respectively. If $\pi > 0$ and if $y_i \neq 0$, then agent i must pay the transaction cost $c|y_i|\pi > 0$. Throughout, we shall denote

$$\operatorname{sgn}(y_i) \equiv \begin{cases} +1 & \text{if } y_i > 0, \\ 0 & \text{if } y_i = 0, \\ -1 & \text{if } y_i < 0. \end{cases}$$

Then, the total cost (including transaction costs) or the net revenue (transaction costs subtracted), in terms of the riskless bond, for agent i is equal to $(1 + c\operatorname{sgn}(y_i))\,\pi y_i$. This is the

[3]That is, the buying price of the stock with respect to the bond, when the latter incurs no transaction cost, goes up, the selling price and the trading volume got down.

amount that must be financed by the sales of the riskless bond or can be spent on the purchase on the riskless bond. The maximization problem of agent i can, therefore, be formulated as

$$\max_{y_i} \quad E\left[u_i\left(A_i + y_i\left(S - \left(1 + c\,\mathrm{sgn}(y_i)\right)\pi\right)\right)\right]. \tag{2.1}$$

Whenever there is a unique solution to this problem, we denote it by $f_i(c, \pi)$.

Remark 1. Although we have just formulated the transaction costs as if they were imposed on the risky asset, our formulation can, in fact, accommodates the case where the transaction costs are imposed also on the riskless bond. To see this point, suppose now that the prices of the riskless bond and the risky asset are π^0 and π^1, and the proportional transaction costs of the riskless bond and the risky asset are c^0 and c^1. Then the total cost necessary to form a portfolio (y_i^0, y_i) of the riskless bond and the risky asset is equal to

$$\left(1 + c^0\,\mathrm{sgn}(y_i^0)\right)\pi^0 y_i^0 + \left(1 + c^1\,\mathrm{sgn}(y_i)\right)\pi^1 y_i.$$

The budget constraint stipulates that this value must not exceed zero. Now define

$$c = \frac{c^0 + c^1}{1 + c^0 c^1} \text{ and } \pi = \frac{1 + c^0 c^1}{1 - (c^0)^2}\frac{\pi^1}{\pi^0}.$$

A straightforward calculation shows that the budget constraint is equivalent to requiring that

$$y_i^0 + (1 + c\,\mathrm{sgn}\,(y_i))\pi y_i$$

must not exceed zero. Thus, even if the transaction costs are imposed on the riskless bond, we can assume without loss of generality that the riskfree bond requires no transaction costs and its price is equal to one.

We say that a price π for the risky asset is an *equilibrium price* under the proportional transaction cost c if $\sum_i f_i(c, \pi) = 0$. The consumption that agent i receives at equilibrium is

$$A_i + f_i(c, \pi)\left(S - \left(1 + c\,\mathrm{sgn}(f_i(c, \pi))\right)\pi\right).$$

The sum of these over all agents i is equal to

$$\sum_{i=1}^{I} A_i - c\pi \sum_{i=1}^{I} |f_i(c, \pi)|.$$

That is, by trading assets, the agents give up $c\pi \sum_{i=1}^{I} |f_i(c, \pi)|$ units of the riskfree discount bond in the aggregate. In this paper, we assume that the transaction costs are taken away from the model (to be paid as tax to the government or as commissions to the intermediary that are not modeled here).[4]

In the rest of this section, we explore some useful properties of the demand functions f_i. Define a functions g_i and h_i by

$$g_i(y_i; c, \pi) = E\left[(S - (1 + c)\pi)u_i'(A_i + y_i(S - (1 + c)\pi))\right], \tag{2.2}$$

$$h_i(y_i; c, \pi) = E\left[(S - (1 - c)\pi)u_i'(A_i + y_i(S - (1 - c)\pi))\right]. \tag{2.3}$$

[4]If an agent were a recipient of government subsidies or a shareholder of the intermediary, and thus got back part of $c\pi \sum_{i=1}^{I} |f_i(c, \pi)|$, then the objective function of the maximization problem (2.1) should include the transfer that agent i receives from the government or the intermediary. Although the equilibrium price would in general be different from those analyzed here, it would be the same if all agents have CARA utility functions.

Then, y_i is a solution to the utility maximization problem (2.1) if and only if

$$\begin{cases} g_i(y_i; c, \pi) = 0, & \text{if } y_i > 0, \\ h_i(y_i; c, \pi) = 0, & \text{if } y_i < 0, \\ g_i(0; c, \pi) \leq 0 \leq h_i(0; c, \pi), & \text{if } y_i = 0. \end{cases} \tag{2.4}$$

Remark 2. Suppose that $f_i(c, \pi)$ is well defined (exists). If it is strictly positive, then

$$\frac{1}{1+c} \operatorname{essinf} S < \pi < \frac{1}{1+c} \operatorname{esssup} S.$$

If it is strictly negative, then

$$\frac{1}{1-c} \operatorname{essinf} S < \pi < \frac{1}{1-c} \operatorname{esssup} S.$$

If it is zero, then

$$\frac{1}{1+c} \operatorname{essinf} S < \pi < \frac{1}{1-c} \operatorname{esssup} S.$$

If $c = 0$, then each of these three restrictions can be rewritten as $\operatorname{essinf} S < \pi < \operatorname{esssup} S$. In the subsequent analysis, we assume that $f_i(0, \pi)$ is well defined (exists) whenever $\operatorname{essinf} S < \pi < \operatorname{esssup} S$. Since, then, $E\left[(S - \pi) u_i'(A_i + f_i(0, \pi)(S - \pi))\right] = 0$ by (2.2), (2.3), and (2.4), $P(\{\omega \in \Omega : S(\omega) > \pi\}) > 0$ and $P(\{\omega \in \Omega : S(\omega) < \pi\}) > 0$.

Note that g_i and h_i are continuously differentiable and

$$g_i'(y_i; c, \pi) = E\left[(S - (1+c)\pi)^2 u_i''(A_i + y_i(S - (1+c)\pi))\right] < 0, \tag{2.5}$$

$$h_i'(y_i; c, \pi) = E\left[(S - (1-c)\pi)^2 u_i''(A_i + y_i(S - (1-c)\pi))\right] < 0. \tag{2.6}$$

Thus $g_i(y_i; c, \pi)$ and $h_i(y_i; c, \pi)$ are strictly decreasing function of y. Consulting (2.2) and (2.3), we know that if $f_i(c, \pi) > 0$, then $g_i(0; c, \pi) > 0$; if $f_i(c, \pi) < 0$, then $h_i(0; c, \pi) < 0$; and if $f_i(c, \pi) = 0$, then $g_i(0; c, \pi) \leq 0 \leq h_i(0; c, \pi)$. Since

$$g_i(0; c, \pi) - h_i(0; c, \pi) = -2c\pi\, E\left[u_i'(A_i)\right] \leq 0, \tag{2.7}$$

the converse also holds:

Lemma 1. *If $g_i(0; c, \pi) > 0$, then $f_i(c, \pi) > 0$ and $g_i(f_i(c, \pi); c, \pi) = 0$. If $h_i(0; c, \pi) < 0$, then $f_i(c, \pi) < 0$ and $h_i(f_i(c, \pi); c, \pi) = 0$. Otherwise, $f_i(c, \pi) = 0$. The solution $f_i(c, \pi)$ to the maximization problem (2.1) is unique in all three cases.*

The following lemma establishes the continuity and continuous differentiability of the demand functions f_i.

Lemma 2. (1) *The function f_i is continuous.*

(2) *The (partial) function $f_i(0, \cdot)$ is continuously differentiable and*

$$\frac{\partial f_i}{\partial \pi}(0, \pi) = \frac{E[u_i'(P_i) + f_i(0, \pi)(S - \pi) u_i''(P_i)]}{E[(S - \pi)^2 u_i''(P_i)]}, \tag{2.8}$$

where $P_i = A_i + f_i(0, \pi)(S - \pi)$.

(3) *For every (c, π), if $f_i(c, \pi) > 0$ or $f_i(0, (1+c)\pi) > 0$, then $f_i(c, \pi) = f_i(0, (1+c)\pi)$. Moreover, f_i is continuously differentiable at (c, π) and*

$$\frac{\partial f_i}{\partial c}(c, \pi) = \pi \frac{\partial f_i}{\partial \pi}(0, (1+c)\pi) \text{ and } \frac{\partial f_i}{\partial \pi}(c, \pi) = (1+c) \frac{\partial f_i}{\partial \pi}(0, (1+c)\pi).$$

(4) *For every* (c, π), *if* $f_i(c, \pi) < 0$ *or* $f_i(0, (1-c)\pi) < 0$, *then* $f_i(c, \pi) = f_i(0, (1-c)\pi)$. *Moreover,* f_i *is continuously differentiable at* (c, π) *and*

$$\frac{\partial f_i}{\partial c}(c, \pi) = -\pi \frac{\partial f_i}{\partial \pi}(0, (1+c)\pi) \text{ and } \frac{\partial f_i}{\partial \pi}(c, \pi) = (1-c)\frac{\partial f_i}{\partial \pi}(0, (1-c)\pi).$$

(5) *For every* (c, π), *if* $g_i(0; c, \pi) < 0 < h_i(0; c, \pi)$, *then* f_i *is continuously differentiable at* (c, π) *and*

$$\frac{\partial f_i}{\partial c}(c, \pi) = \frac{\partial f_i}{\partial \pi}(c, \pi) = 0.$$

Part (1) of this lemma guarantees that f_i is continuous on the entire domain, but part (5) guarantees, when $f_i(c, \pi) = 0$, that it is continuously differentiable only on the set of those (c, π) on which $g_i(0; c, \pi) < 0 < h_i(0; c, \pi)$. If $g_i(0; c, \pi) = 0$ or $h_i(0; c, \pi) = 0$, then a small change in (c, π) will induce agent i to remain inactive, while another small change will induce him to trade the risky asset. At such (c, π), f_i is, in general, not differentiable. It is this non-differentiability that calls for an intricate proof for Theorem 2.

3. Impact of transaction costs

In this section, we assess the impact of increasing the proportional transaction cost c on the equilibrium price π for the risky asset. Specifically, we first provide a taxonomy on the impact of a small (infinitesimal) increase in the proportional transaction cost c on the buying price, the selling price, the trading volume, and the agents' welfare. We then find a condition on the individual agents' demand functions for an increase in the proportional transaction cost to always increase the buying price and decrease the selling price, the trading volume, and all agents' welfare.

To start, we say that an equilibrium price π^* of the risky asset under the proportional transaction cost $c^* \in [0, 1)$ is *normal* if $f_i(c^*, \pi^*) \neq 0$ or $g_i(0; c^*, \pi^*) < 0 < h_i(0; c^*, \pi^*)$ for every i. Since g_i and h_i are continuous, this definition implies that every agent who does not trade at a normal equilibrium price π^* under the proportional transaction cost c^* remains not to trade even when c^* and π^* are slightly changed. By Lemma 2, if π^* is a normal equilibrium price under c^*, then f_i is continuously differentiable at (c^*, π^*) for every i.

By an *equilibrium price function around* (c^*, π^*), we mean a function e defined on some open interval V of c^* in $[0, 1)$ and taking values in some open interval W of π^* in $\boldsymbol{R}_{++}$ such that for every $(c, \pi) \in V \times W$, π is an equilibrium price under c if and only if $\pi = e(c)$. The existence of an equilibrium price function around (c^*, π^*) means that there is a locally unique equilibrium price π near π^* under a proportional transaction cost c whenever c is close to c^*.

Theorem 1. *Let* $\pi^* > 0$ *be a normal equilibrium price of the risky asset under the proportional transaction cost* $c^* \in [0, 1)$. *Write* $\mathscr{B} = \{i : f_i(c^*, \pi^*) > 0\}$, $\mathscr{S} = \{i : f_i(c^*, \pi^*) < 0\}$, *and*

$$D_{\mathscr{B}} = \sum_{i \in \mathscr{B}} \frac{\partial f_i}{\partial \pi}(c^*, \pi^*),$$

$$D_{\mathscr{S}} = \sum_{i \in \mathscr{S}} \frac{\partial f_i}{\partial \pi}(c^*, \pi^*).$$

Suppose that $D_{\mathscr{B}} \neq 0$, $D_{\mathscr{S}} \neq 0$, and $D_{\mathscr{B}} + D_{\mathscr{S}} \neq 0$. Then there is a continuously differentiable equilibrium price function e around (c^*, π^*). Write

$$
\begin{aligned}
Q_{\mathscr{B}} &= \frac{\mathrm{d}}{\mathrm{d}c}((1+c)e(c))\Big|_{c=c^*}, \\
Q_{\mathscr{S}} &= \frac{\mathrm{d}}{\mathrm{d}c}((1-c)e(c))\Big|_{c=c^*}, \\
U_i &= \frac{\mathrm{d}}{\mathrm{d}c}E\left[u_i\left(A_i + f_i(c,e(c))S - (1+\mathrm{sgn}(f_i(c,e(c)))c)\pi f_i(c,e(c))\right)\right]\Big|_{c=c^*}, \\
T &= \frac{\mathrm{d}}{\mathrm{d}c}\left(\sum_{i\in\mathscr{B}} f_i(c,e(c))\right)\Bigg|_{c=c^*}.
\end{aligned}
$$

Then, depending on the signs of $D_{\mathscr{B}}$, $D_{\mathscr{S}}$, and $D_{\mathscr{B}} + D_{\mathscr{S}}$, we obtain the signs for $Q_{\mathscr{B}}$, $Q_{\mathscr{S}}$, U_i, and T as in Table 1.

$\mathrm{sgn}\, D_{\mathscr{B}}$	$\mathrm{sgn}\, D_{\mathscr{S}}$	$\mathrm{sgn}\,(D_{\mathscr{B}} + D_{\mathscr{S}})$	$\mathrm{sgn}\, Q_{\mathscr{B}}$	$\mathrm{sgn}\, Q_{\mathscr{S}}$	$\mathrm{sgn}\, U_i\ (i \in \mathscr{B})$	$\mathrm{sgn}\, U_i\ (i \in \mathscr{S})$	$\mathrm{sgn}\, T$
-1	-1	-1	$+1$	-1	-1	-1	-1
$+1$	$+1$	$+1$	$+1$	-1	-1	-1	$+1$
-1	$+1$	-1	-1	-1	$+1$	-1	$+1$
-1	$+1$	$+1$	$+1$	$+1$	-1	$+1$	-1
$+1$	-1	-1	$+1$	$+1$	-1	$+1$	$+1$
$+1$	-1	$+1$	-1	-1	$+1$	-1	-1

Table 1: Sign patterns of changes in buying prices, selling prices, utility levels, and trading volumes

The common characteristic of these sign patterns is that $\mathrm{sgn}\,(D_{\mathscr{B}} D_{\mathscr{S}}) + \mathrm{sgn}\,(Q_{\mathscr{B}} Q_{\mathscr{S}}) = 1$. That is, the buying price goes up and the selling price goes down if and only if the two "half aggregate" demand functions, $\sum_{i\in\mathscr{B}} f_i(c^*, \cdot)$ and $\sum_{i\in\mathscr{S}} f_i(c^*, \cdot)$, are both downward-sloping or both upward-sloping at equilibrium. Of particular interest among these six combinations are the cases where $\mathrm{sgn}\, D_{\mathscr{B}} = \mathrm{sgn}\,(D_{\mathscr{B}} + D_{\mathscr{S}}) = -1$, that is, the aggregate demand function of the buyers and thee aggregate demand function are downward-sloping. The reason is that while it is common to assume that the agents' utility functions exhibit constant or decreasing absolute risk aversion, the aggregate demand for the buyers is downward-sloping whenever it is the case;[5] and if the aggregate demand function $\sum_{i\in\mathscr{B}\cup\mathscr{S}} f_i(c^*, \cdot)$ is upward-sloping, there must be other asset market equilibria for which the sign patterns for $Q_{\mathscr{B}}$, $Q_{\mathscr{S}}$, U_i, and T are different. The first and third rows of Table 1 correspond to these cases. The first row is the textbook case, for which the conventional wisdom is valid. The third row is the case we deal with in Section 5 and for which it is invalid.

The most common way to apply Theorem 1 is to show that $\partial f_i(c,\pi)/\partial\pi < 0$ for every (c,π) and for every i. In such a case, we can derive the following, stronger result. We say that an equilibrium price π under the proportional transaction cost c is *trivial* if $f_i(c,\pi) = 0$ for every i.

[5]This is proved in Kijima and Tamura (2012).

Theorem 2. *Suppose that $S \geq 0$ almost surely and that*

$$\frac{\partial f_i}{\partial \pi}(0, \pi) < 0 \tag{3.9}$$

for every i and every $\pi \in (\operatorname{essinf} S, \operatorname{esssup} S)$.

(1) *For every $c \in [0, 1)$, either there exists a nonempty and compact interval of trivial equilibrium prices (which may be a singleton), or there is a unique equilibrium, which is not trivial, under the proportional transaction cost c.*
(2) *Let c_1 and c_2 be proportional transaction costs with $c_1 \leq c_2$. If there is no nontrivial equilibrium under the proportional transaction cost c_1, then there is no nontrivial equilibrium under the proportional transaction cost c_2.*
(3) *Let V be the set of all proportional transaction costs under which there is a nontrivial equilibrium. For every $c \in V$, let $e(c)$ be a (unique) nontrivial equilibrium price. Then $e(c)$ is a continuous function of c on V, $(1+c)e(c)$ is a continuous and strictly increasing function of c on V, and $(1-c)e(c)$ is a continuous and strictly decreasing function of c on V.*
(4) *Let c_1 and c_2 be proportional transaction costs with $c_1 \leq c_2$. Let $i \in \{1, \ldots, I\}$. If $f_i(c_1, e(c_1)) = 0$, then $f_i(c_2, e(c_2)) = 0$. Moreover, $|f_i(c, e(c))|$ and*

$$E\left[u_i\left(A_i + f_i(c, e(c))(S - (1 + \operatorname{sgn}(f_i(c, e(c))\, c)e(c)))\right)\right]$$

are strictly decreasing function of c on $\{c \in V : f_i(c, e(c)) \neq 0\}$.

Since

$$\frac{\partial f_i}{\partial \pi}(c, \pi) = (1 + \operatorname{sgn}(f_i(c, \pi))c)\frac{\partial f_i}{\partial \pi}(0, (1 + \operatorname{sgn}(f_i(c, \pi))c)\pi),$$

whenever $f_i(c, \pi) \neq 0$, inequality (3.9) implies that $D_{\mathscr{B}} < 0$ and $D_{\mathscr{S}} < 0$ in Theorem 1, and the first row of Table 1 is applicable whenever π^* is a normal equilibrium asset price. Parts (1) and (2) of Theorem 2 that either there are trivial equilibria or there is a unique nontrivial equilibrium, and the latter case applies if and only if the proportional transaction cost is below some threshold. Thus, the set V defined in part (3) is an (possibly empty) interval, and is nonempty if and only if $0 \in V$. Part (3) shows that the (unique) nontrivial equilibrium price depends continuously on proportional transaction costs, and the buying price increases and the selling price decreases as the proportional transaction cost increases. Part (4) shows that each agent's trading volume decreases strictly to zero, after which it never exceeds zero. Note that the domain V of the equilibrium price function e, obtained in part (3), may contain proportional transaction costs under which equilibrium prices are not normal. For such proportional transaction costs, while Theorem 1 is not applicable, Theorem 2 guarantees that the equilibrium asset price depend continuously on proportional transaction costs on the interval of proportional transaction costs under which there is a (unique) nontrivial equilibrium.

Although $e(c)$ is not well defined for those c which have trivial equilibria, $|f_i(c, \pi)|$ is of course well defined and equal to zero for any such c as long as π is a (trivial) equilibrium price of c. By part (4), the trading volume function $t : [0, 1) \to \boldsymbol{R}_+$ defined by

$$t(c) = \begin{cases} \displaystyle\sum_{i=1}^{I} |f_i(c, e(c))| & \text{if } c \in V, \\ 0 & \text{if } c \notin V, \end{cases}$$

is continuous on V. In fact, t is continuous on the entire $[0,1)$. To show this, it suffices to show that for every i, $|f_i(c, e(c))| \to 0$ as $c \uparrow \sup V$, but if there were an i for whom $|f_i(c, e(c))| \not\to 0$, then, based on a method similar to the proof of the right-continuity of e in part (3) above, we can show that there would exist a $c > \sup V$ such that $c \in V$, a contradiction.

4. Monotonicity of asset demand

In this section, we provide a sufficient condition for an agent's demand for the risky asset to decrease as the asset price increases. When this condition is satisfied by every agent, the inequality (3.9) holds. Hence it is also sufficient for the conclusion of Theorem 2 to hold.

Recall that the utility function u_i is defined on an open interval $(\underline{d}_i, \overline{d}_i)$ and $\underline{d}_i \in \{-\infty\} \cup \boldsymbol{R}$. Denote the absolute risk aversion by $R_i(x) = -u_i''(x)/u_i'(x)$. The sufficient condition contains the following condition on u_i.

Assumption 1. There exists an $\alpha_i \geq 0$ such that

$$\alpha_i \leq R_i(x_i) \leq \alpha_i + \frac{1}{x_i - \underline{d}_i} \tag{4.10}$$

for every $x_i \in (\underline{d}_i, \overline{d}_i)$, where, by convention, $1/(x_i - \underline{d}_i) = 0$ if $\underline{d}_i = -\infty$.

This assumption is met by a number of utility functions that are commonly used in finance and economics. First, any utility function that u_i exhibits constant absolute risk aversion (CARA) satisfies this assumption, because α_i can be to be equal to its CARA coefficient. Second, when $\underline{d}_i = 0$ and $\alpha_i = 0$, this assumption is met if and only if the Arrow-Pratt measure of relative risk aversion never exceeds one. As explained in Example 17.F.2 of Mas-Colell, Whinston, and Green (1995), this condition is sufficient for the gross substitute sign pattern of the excess demand function for commodities. But the conclusion of Proposition 1 is not the same, because it is on (excess) demand for a risky asset with arbitrary payoffs, not for commodities. Third, the assumption is satisfied when u_i exhibits hyperbolic absolute risk aversion (HARA) with its coefficient for the hyperbolic term not exceeding one, and it may be met by a utility function u_i that does not exhibit HARA. One such example is given when $\underline{d}_i = 0$, $\overline{d}_i = \infty$, and $R_i(x_i) = 1 + 1/x_i$.

Proposition 1. *If u_i satisfies Assumption 1 and $A_i \geq \underline{d}_i$ almost surely, then $\partial f_i(0, \pi)/\partial \pi < 0$ for every $\pi \in (\operatorname{essinf} S, \operatorname{esssup} S)$.*

This proposition shows that all the conclusions of Theorems 1 and 2 hold if all agents satisfies Assumption 1 and $A_i > \underline{d}_i$ almost surely. The special case for CARA utility functions was proved by Kijima and Tanaka (2012).

Theorem 3. *Suppose that for every i, u_i satisfies Assumption 1 and $A_i \geq \underline{d}_i$ almost surely.*

(1) *Either there exists a nonempty and compact interval of trivial equilibrium prices (which may be a singleton), or there is a unique equilibrium, which is not trivial.*
(2) *Let c_1 and c_2 be proportional transaction costs with $c_1 \leq c_2$. If there is no nontrivial equilibrium under the proportional transaction cost c_1, then there is no nontrivial equilibrium under the proportional transaction cost c_2.*
(3) *Let V be the set of all proportional transaction costs under which there is a nontrivial equilibrium. For every $c \in V$, let $e(c)$ be a (unique) nontrivial equilibrium price. Then*

$e(c)$ is a continuous function of c on V, $(1+c)e(c)$ is a continuous and strictly increasing function of c on V, and $(1-c)e(c)$ is a continuous and strictly decreasing function of c on V.

(4) *Let c_1 and c_2 be proportional transaction costs with $c_1 \leq c_2$. Let $i \in \{1,\dots,I\}$. If $f_i(c_1, e(c_1)) = 0$, then $f_i(c_2, e(c_2)) = 0$. Moreover, $|f_i(c, e(c))|$ and*

$$E\left[u_i\left(A_i + f_i(c, e(c))(S - (1 + \operatorname{sgn}(f_i(c, e(c)))\, c)e(c))\right)\right]$$

are strictly decreasing function of c on $\{c \in V : f_i(c, e(c)) \neq 0\}$.

5. CRRA utilities with decreased buying prices

Proposition 1 and Theorem 3 do not allow any agent to have a CRRA coefficient greater than one. In this section, we give a class of examples in which an agent has a CRRA coefficient greater than, but possibly arbitrarily close to, one and the buying price decreases as the proportional transaction cost goes up from zero. In these examples, therefore, the buyer is better off at equilibrium with positive (but small) proportional transaction costs than without transaction costs, and the burden of transaction costs is borne solely by the sellers. They also show that the upper bound of CRRA, which is equal to one, in Proposition 1 and Theorem 3 is tight.

We let $\Omega = \{1, 2\}$ and $\mathscr{F} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. Let P be the probability measure defined on $\mathscr{F}$ that satisfies $P(\{1\}) = P(\{2\}) = 1/2$. Asset 1 is a so-called Arrow security for state 1, of which the time-1 value S satisfies $S(1) = 1$ and $S(2) = 0$. There are two agents $i = 1, 2$. Agent i has a CRRA coefficient equal to γ_i. This means that $u_i'(x) = x^{-\gamma_i}$ for every $x > 0$.

Proposition 2. *Suppose that $\gamma_1 > 1$. For each $i = 1, 2$ and each $a_i > 0$, there exists an $a_{3-i} > 0$ such that if $A_1(1) = a_1$, $A_1(2) = 0$, $A_2(1) = 0$, and $A_2(2) = a_2$, then there are an equilibrium asset price π^* under zero transaction costs and a continuously differentiable equilibrium price function e around $(0, \pi^*)$ such that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}c}((1+c)e(c))\right|_{c=0} < 0, \tag{5.11}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}c}|f_i(c, e(c))|\right|_{c=0} > 0 \tag{5.12}$$

for each i, and

$$\left.\frac{\mathrm{d}}{\mathrm{d}c}E\left(u_2\left(A_2 + f_2(c, e(c))(S - (1+c)e(c))\right)\right)\right|_{c=0} > 0. \tag{5.13}$$

The crucial aspect of the setting for this proposition is that agent 1 has a CRRA coefficient greater than one and is the seller of the Arrow security. This means that if the selling price were decreased, then, depending on the values a_1 and a_2 of initial risks, agent 1 may decrease his demand for the Arrow security.[6] With the notation of Theorem 1, $D_{\mathscr{S}} > 0$ and the third row of Table 1 is applicable. The consequence is, therefore, that the buying price is decreased, and the trading volume and the buyer's welfare are increased.

The proposition shows that the conventional wisdom, that an increase in transaction costs makes all agent worse off, can be easily invalidated in a two-state, two-agent model as long as there is an agent who is the sole supplier of a risky asset and has a CRRA coefficient

[6] In other words, agent 1 may increase his supply of the Arrow security.

greater than one. The proof, to be given in the Appendix, shows that the equilibrium under the zero transaction cost is normal. By the implicit function theorem, therefore, for every distribution (A_1', A_2') of initial risks sufficiently close to (A_1, A_2) in the proposition, an increase in the proportional transaction cost decreases the buying price. In other words, the conventional wisdom is robustly invalidated with respect to perturbations in initial risks.

More can be said of the equilibria in Proposition 2 if $\gamma_1 = \gamma_2 > 1$, that is, the two agents have the same CRRA coefficient. First, the equilibria are the unique ones when the proportional transaction cost is zero or very low. This implies that there is no equilibrium at which an increase in transaction costs increases the buying price and decreasing the selling price. Indeed, if $\gamma_1 = \gamma_2$, then, under zero transaction cost, the aggregate demand function of the two agents coincides with the demand function of the (representative) agent having the same CRRA coefficient and an initial risk $A_1 + A_2$. Hence there is a unique equilibrium when the transaction cost is zero. Moreover, by the implicit function theorem, there is a unique equilibrium under a proportional transaction cost $c > 0$ whenever c is sufficiently close to 0.

Second, if $\gamma_1 = \gamma_2$, we can also show that an increase in transaction costs decreases the buying price whenever a_2/a_1 is sufficiently close to zero. This condition means that the contingent commodity is much more abundant in the first state than in the second, and clarifies when an increase in transaction costs decreases the buying price. This claim is proved in the Appendix.

Table 2 gathers some numerical examples of the rate of change in the price for the risky asset for various configurations of the CRRA coefficients, which are common for the two agents, and the ratio of the endowments of the contingent commodity in the two states. In the table, the first column lists the common coefficients γ of CRRA for the two agents, the second column lists the ratios a_2/a_1 of the endowments for the contingent commodities in the two states, the third column lists the equilibrium prices for the risky asset, with the riskless bond being the numeraire, when there is no transaction cost, and the fourth column lists the first-order approximations of rates of changes in the buying prices, which are mathematically defined as

$$\frac{1}{\pi^*} \frac{\mathrm{d}}{\mathrm{d}c}((1+c)e(c))\bigg|_{c=0}.$$

This value is equal to one if $a_2/a_1 = 1$, because, then, the two agents are endowed with equal amounts of the contingent commodities. This means that an small increase in the proportional transaction cost does not change the asset price, and hence increases the buying price by the rate equal to the transaction cost itself. By (A.21) in the Appendix,

$$\frac{1}{\pi^*} \frac{\mathrm{d}}{\mathrm{d}c}((1+c)e(c))\bigg|_{c=0} \to 0$$

as $a_2/a_1 \to 0$. For each fixed value $\gamma > 1$ of CRRA coefficients, therefore, we are interested in the value of a_2/a_1 at which the rate of change in the buying price is equal to zero, and the value that minimizes (that is, since the rates we are interested in are negative, maximize the absolute value of) the rate of change in the buying price.

The results are listed in the first three groups of the table. In the first group, the common CRRA coefficient is equal to 1.10, which is close to one, the threshold below which Theorem 3 is applicable. The range of the values of a_2/a_1 for which the proportional transaction cost decreases the buying price is accordingly narrow: the endowment in the second state must be less than 5.5% of the endowment in the first state. The impact on the buying price is also small:

a 1% transaction cost decreases the buying price by 0.068% at most. In the second group, we set the common CRRA coefficient at 2.50. This is the value that Lucas (1994) suggested, in the context of the equity premium of Mehra and Prescott (1985), as an upper bound of CRRA coefficients that are judged as reasonable. The result is that the endowment ratio a_2/a_1 may exceed a half only by a small amount in order for the proportional trasaction cost to decrease the buying price. The common CRRA cofficient in the third group is equal to 10.00, which is the upper bound of the CRRA coefficients used by Mehra and Prescott (1985).[7] Then the maximum endowment ratio is approximately equal to 0.8740, which means that in order for the proportional transaction cost to decrease the buying price, the endowment in the bad state must be less than 87.4% of the endowment in the good state. The impact is maximal when the endowment ratio is equal to 0.770, where a 1% transaction cost decreases the buying price by almost 0.2%.

In the last group, the value of the endowment ratio a_2/a_1 is fixed at 0.9317. The value was choen to match the mean and standard deviation of the annual consumption growth rates in the data set used by Mehra and Prescott (1985). Indeed, in the data set, the mean is 0.018 and the standard deviation is 0.036. Thus, with the equal probability 1/2 for the two states, the endowment ratio a_2/a_1 is given by

$$\frac{a_2}{a_1} = \frac{1 + 0.018 - 0.036}{1 + 0.018 + 0.036} = \frac{0.982}{1.054} \approx 0.9317.$$

By varying the common CRRA coefficients, we see that the threshold, above which an increase in the proportional transaction cost decreases the buying price and makes the buyer better off, is approximately equal to 18.57. This is somewhat disappointing, because tit is much higher than ten, the upper bound of the CRRA coefficients used by Mehra and Prescott (1985). Yet, it might be possible that if a kind of asymmetric information in Dow and Rahi (2000) is introduced into the model, where the buyer is better informed, then an increase in the proportional transaction cost makes the buyer better off even with lower CRRA coefficients.

CRRA	Endowment Ratio	Equilibrium Price	Rate of change in the buying price
1.10	1.0000	0.50000	1.00000
1.10	0.0541	0.03884	−0.00006
1.10	0.0034	0.00192	−0.06809
2.50	1.0000	0.50000	1.00000
2.50	0.5215	0.16416	0.00007
2.50	0.2936	0.04462	−0.18864
10.00	1.0000	0.50000	1.00000
10.00	0.8740	0.20640	0.00003
10.00	0.7770	0.07425	−0.19919
10.00	0.9317	0.33014	0.37531
18.57	0.9317	0.21183	0.00008
30.00	0.9317	0.10691	−0.18906

Table 2: Rates of changes in the buying price

[7] Mehra and Prescott (1985) and Kocherlakota (1996) gave references that estimate agents' CRRA coefficients.

Remark 3. Although the economy of Proposition 2 consists of just two agents, it can be easily modified to consists of many consumers. In fact, if the economy comprises two groups $\mathscr{B}$ and $\mathscr{S}$, each agent in $\mathscr{B}$ has a CRRA coefficient $\gamma_1 \geq 1$ and an initial risk A_i with $A_i(1) > 0 = A_i(2)$, each agent in $\mathscr{S}$ has a CRRA coefficient γ_2 and an initial risk A_i with $A_i(1) = 0 < A_i(2)$, and $\sum_{i \in \mathscr{B}} A_i(1)$ and $\sum_{i \in \mathscr{S}} A_i(1)$ satisfy the condition that a_1 and a_2 satisfy in the proposition, then the conclusion of the proposition holds for this economy. This is because the aggregate demand function of $\mathscr{B}$ coincides with the demand function of the (half representative) agent having a CRRA coefficient γ_1 and an initial risk $\sum_{i \in \mathscr{B}} A_i$, and the aggregate demand function of $\mathscr{S}$ coincides with the demand function of the (half representative) agent having a CRRA coefficient γ_2 and an initial risk $\sum_{i \in \mathscr{B}} A_i$; and the proposition is applicable to this reduced two-agent economy. However, the conclusion of the proposition would not hold if the distributions of CRRA coefficients and initial risks were not perfectly correlated, that is, if there were agents i with a CRRA coefficient γ_1 and an initial risk A_i with $A_i(1) = 0 < A_i(2)$, or agents i with a CRRA coefficient γ_2 and an initial risk A_i with $A_i(1) > 0 = A_i(2)$. In such an economy, the first row of Table 1 in Theorem 1 may be applicable and the selling price may decrease as the proportional transaction cost increases.

6. Conclusion

In this paper, we investigated how an increase in transaction costs affects the equilibrium asset prices and allocations. We found sufficient conditions for an increase in transaction costs to increase the buying price and decrease the selling price, the trading volume, and all active investors' welfare. The sufficient condition is met by a general class of utility functions, which contains some HARA and non-HARA utility functions. As for CRRA utility functions, the class contains all utility functions with CRRA coefficients less than or equal to one, but does not contain any utility function with CRRA coefficient greater than one. By constructing examples, we also showed that whenever there is an agent with a CRRA coefficient greater than one, an increase in transaction costs may well decrease buying prices and make buyers better off.

In the setup of this paper, there are only one consumption period and only one risky asset. It is admittedly very restrictive. We should extend our analysis to the case of multiple (discrete or continuous) trading periods and multiple risky assets, such as an underlying asset and a derivative asset written on it. Developing an algorithm to obtain equilibrium asset prices is also needed, especially in the case of multiple risky assets, to give a practical value to our analysis. Finally, although the level of transaction costs are exogenously specified in our model, it would be interesting to accommodate intermediaries who seek to maximize their own objective functions by choosing a level of transaction costs.

Appendix A. Proofs

Proof of Lemma 2 By inequalities (2.5) and (2.6) and $g_i(y; 0, \pi) = h_i(y; 0, \pi)$, we can apply the implicit function theorem to prove part (2). Parts (3) and (4) follow easily from part (2). As for part (5), if $g_i(0; c, \pi) < 0 < h_i(0; c, \pi)$, then $f_i(c, \pi) = 0$ by Proposition 1 and, since g_i and h_i are continuously differentiable, $g_i(0; c', \pi') < 0 < h_i(0; c', \pi')$ if (c', π') is sufficiently close to (c, π). Again by Proposition 1, $f_i(c', \pi') = 0$. Hence f_i is constant around (c, π) and part (5) follows.

Now that parts (2) through (5) have been proven, it remains to prove that f_i is continuous at (c,π) where $c>0$ and $g_i(0;c,\pi)=0$ or $h_i(0;c,\pi)=0$. Suppose that $c\pi>0$, $g_i(0;c,\pi)=0$, $((c_n,\pi_n))_n$ is a sequence that converges to (c,π), but $\left(f_i(c_n,\pi_n)\right)_n$ does not converges to zero. Since $h_i(0;c,\pi)>0$ by (2.7), $h_i(0;c_n,\pi_n)>0$, and, by Proposition 1, $f_i(0;c_n,\pi_n)\geq 0$, for every sufficiently large n. Thus, by taking a subsequence if necessary, we can assume that there is a $\delta>0$ such that $f_i(c_n,\pi_n)>\delta$ for every n. Since $g_i\left(f_i(c_n,\pi_n);c_n,\pi_n\right)=0$ and g_i is strictly decreasing, $g_i(\delta;c_n,\pi_n)>0$. Since $g_i(\delta;c_n,\pi_n)\to g_i(\delta;c,\pi)$ as $n\to\infty$, $g_i(\delta;c,\pi)\geq 0$. On the other hand, by assumption, $g_i(0;c,\pi)=0$ and, since g_i is strictly decreasing, $g_i(\delta;c,\pi)<0$. This is a contradiction. Hence f_i is continuous at (c,π) where $g_i(0;c,\pi)=0$. The continuity at (c,π) where $h_i(0;c,\pi)=0$ can be analogously proved. ///

Proof of Theorem 1 (1) By (5) of Lemma 2, $\sum_{i=1}^{I}\partial f_i(c^*,\pi^*)/\partial\pi\neq 0$.

By Lemma 2, we can apply the implicit function theorem to the equilibrium condition $\sum_i f_i(c,\pi)=0$ at (c^*,π^*) to show that there is a continuously differentiable equilibrium price function $e:V\to W$ around (c^*,π^*). By Proposition 1, $g_i(0;c^*,\pi^*)>0$ for every $i\in\mathscr{B}$ and $h_i(0;c^*,\pi^*)<0$ for every $i\in\mathscr{S}$. By normality, $g_i(0;c^*,\pi^*)<0<h_i(0;c^*,\pi^*)$ for every $i\notin\mathscr{B}\cup\mathscr{S}$. By continuity, for every c sufficiently close to c^*, $g_i(0;c,e(c))>0$ for every $i\in\mathscr{B}$, $h_i(0;c,e(c))<0$ for every $i\in\mathscr{S}$, and $g_i(0;c,e(c))<0<h_i(0;c,e(c))$ for every $i\notin\mathscr{B}\cup\mathscr{S}$. Again by Proposition 1, $f_i(c,e(c))>0$ for every $i\in\mathscr{B}$, $f_i(c,e(c))<0$ for every $i\in\mathscr{S}$, and $f_i(c,e(c))=0$ for every $i\notin\mathscr{B}\cup\mathscr{S}$. Thus, if V is sufficiently small, then $\mathscr{B}=\{i:f_i(c,e(c))>0\}$ and $\mathscr{S}=\{i:f_i(c,e(c))>0\}$ for every $c\in V$.

Since $\sum_{i=1}^{I}f_i(c,e(c))=0$ for every $c\in V$, by the first claim and Lemma 2,

$$\begin{aligned}
0&=\frac{\mathrm{d}}{\mathrm{d}c}\left(\sum_{i=1}^{I}f_i(c,e(c))\right)\Bigg|_{c=c^*}=\sum_{i=1}^{I}\left(\frac{\partial f_i}{\partial c}(c^*,\pi^*)+\frac{\partial f_i}{\partial\pi}(c^*,\pi^*)e'(c^*)\right)\\
&=\left(\sum_{i\in\mathscr{B}}\frac{\partial f_i}{\partial\pi}(c^*,\pi^*)\right)\left(\frac{\pi^*}{1+c^*}+e'(c^*)\right)+\left(\sum_{i\in\mathscr{S}}\frac{\partial f_i}{\partial\pi}(c^*,\pi^*)\right)\left(-\frac{\pi^*}{1-c^*}+e'(c^*)\right)\\
&=D_{\mathscr{B}}\frac{Q_{\mathscr{B}}}{1+c^*}+D_{\mathscr{S}}\frac{Q_{\mathscr{S}}}{1-c^*}.
\end{aligned}$$

Thus, $\operatorname{sgn}\left(D_{\mathscr{B}}D_{\mathscr{S}}\right)+\operatorname{sgn}\left(Q_{\mathscr{B}}Q_{\mathscr{S}}\right)=0$. Note also that

$$\frac{Q_{\mathscr{B}}}{1+c^*}>\frac{Q_{\mathscr{S}}}{1-c^*}.$$

Thus, if $D_{\mathscr{B}}D_{\mathscr{S}}>0$, then $Q_{\mathscr{B}}>0>Q_{\mathscr{S}}$. If $D_{\mathscr{B}}D_{\mathscr{S}}<0$, consider, for example, the case where $D_{\mathscr{B}}<0<D_{\mathscr{S}}$ and $D_{\mathscr{B}}+D_{\mathscr{S}}<0$. Then $|D_{\mathscr{B}}|>D_{\mathscr{S}}$ and

$$D_{\mathscr{B}}\frac{Q_{\mathscr{B}}}{1+c^*}+D_{\mathscr{S}}\frac{Q_{\mathscr{S}}}{1-c^*}=\frac{Q_{\mathscr{S}}}{1-c^*}D_{\mathscr{S}}-\frac{Q_{\mathscr{B}}}{1+c^*}|D_{\mathscr{B}}|.$$

If $Q_{\mathscr{B}}>0$ and $Q_{\mathscr{S}}>0$, then this would be negative, a contradiction. Hence $Q_{\mathscr{B}}<0$ and $Q_{\mathscr{S}}<0$. The other three cases can be analogously proved. This complete the proof of the signs for $Q_{\mathscr{B}}$ and $Q_{\mathscr{S}}$.

The sign of T follows from $T = Q_{\mathscr{B}} D_{\mathscr{B}}/(1+c^*)$. Finally, by the envelope theorem,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}c} E\left[u_i\left(A_i + f_i(c, e(c))(S - (1+\operatorname{sgn}(f_i(c,e(c)))\,c)e(c))\right)\right]\Big|_{c=c^*} \\
&= \frac{\mathrm{d}}{\mathrm{d}c} E\left[u_i\left(A_i + f_i(c^*, \pi^*)(S - (1+\operatorname{sgn}(f_i(c^*,\pi^*))\,c)e(c))\right)\right]\Big|_{c=c^*} \\
&= - f_i(c^*, \pi^*)\ \frac{\mathrm{d}}{\mathrm{d}c}\left((1+\operatorname{sgn}(f_i(c^*,\pi^*))c)e(c)\right)\Big|_{c=c^*} \\
&\qquad \times E\left[u_i'\left(A_i + f_i(c^*,\pi^*)(S-(1+\operatorname{sgn}(f_i(c^*,\pi^*))\,c^*)\pi^*)\right)\right] \\
&= \begin{cases} -f_i(c^*,\pi^*)\dfrac{Q_{\mathscr{B}}}{1+c^*} E\left[u_i'\left(A_i + f_i(c^*,\pi^*)(S-(1+\operatorname{sgn}(f_i(c^*,\pi^*))\,c^*)\pi^*)\right)\right] & \text{if } i \in \mathscr{B}, \\ -f_i(c^*,\pi^*)\dfrac{Q_{\mathscr{S}}}{1-c^*} E\left[u_i'\left(A_i + f_i(c^*,\pi^*)(S-(1+\operatorname{sgn}(f_i(c^*,\pi^*))\,c^*)\pi^*)\right)\right] & \text{if } i \in \mathscr{B}. \end{cases}
\end{aligned}$$

Since $u_i' > 0$, this implies the signs for the U_i. ///

Proof of Theorem 2 (1) For every i and $c \in [0,1)$, $h_i(0; c, \pi) \to E\left[(S - \operatorname{esssup} S)u_i'(A_i)\right] < 0$ as $\pi \uparrow (1-c)^{-1} \operatorname{esssup} S$. Thus $h_i(0;c,\pi) < 0$ if $\pi < (1-c)^{-1}\operatorname{esssup} S$ is sufficiently close to $(1-c)^{-1}\operatorname{esssup} S$. By Proposition 1, $f_i(c,\pi) < 0$ for every i and every $\pi < (1-c)^{-1}\operatorname{esssup} S$ sufficiently close to $(1-c)^{-1}\operatorname{esssup} S$. Thus $\sum_i f_i(c,\pi) < 0$ for every $\pi < (1-c)^{-1}\operatorname{esssup} S$ sufficiently close to $(1-c)^{-1}\operatorname{esssup} S$. We can analogously show that $\sum_i f_i(c,\pi) > 0$ for every π sufficiently close to $(1+c)^{-1}\operatorname{essinf} S$. By the intermediate value theorem, there is a $\pi \in \left((1+c)^{-1}\operatorname{essinf} S, (1-c)^{-1}\operatorname{esssup} S\right)$ such that $\sum_i f_i(c,\pi) = 0$, that is, there is an equilibrium. There is no equilibrium price outside $\left((1+c)^{-1}\operatorname{essinf} S, (1-c)^{-1}\operatorname{esssup} S\right)$, because if $\pi \geq (1-c)^{-1}\operatorname{esssup} S$, then $0 > h_i(0;c,\pi) \geq g_i(0;c,\pi)$, implying that $f_i(c,\pi) < 0$ for every i (if well defined); and if $\pi \leq (1+c)^{-1}\operatorname{essinf} S$, then $0 < g_i(0;c,\pi) \leq h_i(0;c,\pi)$, implying that $f_i(c,\pi) > 0$ for every i (if well defined).

Let $K_i(c) = \{\pi : f_i(c,\pi) = 0\}$, then, by (2.4),

$$K_i(c) = \left[\frac{E\left[Su_i'(A_i)\right]}{(1+c)E\left[u_i'(A_i)\right]}, \frac{E\left[Su_i'(A_i)\right]}{(1-c)E\left[u_i'(A_i)\right]}\right].$$

Thus the intersection $\bigcap_{i=1}^I K_i(c)$ is a compact interval. If it is nonempty, then every $\pi \in \bigcap_{i=1}^I K_i(c)$ is a trivial equilibrium, and there is no other equilibrium, because if π is to the right of this intersection, then $f_i(c,\pi) \leq 0$ for every i and $f_i(c,\pi) < 0$ for some i, resulting in $\sum_i f_i(c,\pi) < 0$; and if π is to the left of this intersection, then $f_i(c,\pi) \geq 0$ for every i and $f_i(c,\pi) > 0$ for some i, resulting in $\sum_i f_i(c,\pi) > 0$. On the other hand, if $\bigcap_{i=1}^I K_i(c) = \varnothing$, then any equilibrium is nontrivial. Moreover, since $f_i(c,\pi)$ is a strictly decreasing function of π on the entire domain $\left((1+c)^{-1}\operatorname{essinf} S, (1-c)^{-1}\operatorname{esssup} S\right)$, the equilibrium is unique.

(2) If $c_1 \leq c_2$, then $K_i(c_1) \subseteq K_i(c_2)$ for every i. Hence $\bigcap_{i=1}^I K_i(c_1) \subseteq \bigcap_{i=1}^I K_i(c_2)$. Thus, if there is no nontrivial equilibrium under c_1, then $\bigcap_{i=1}^I K_i(c_1) \neq \varnothing$, and hence $\bigcap_{i=1}^I K_i(c_2) \neq \varnothing$, which implies that there is no nontrivial equilibrium under c_2.

(3) First, we prove that e is right-continuous. Let $c^* \in V$ and π^* be an equilibrium price under c^*. Although π^* need not be a normal equilibrium price for the entire economy of I agents, π^* is a normal equilibrium price for the economy consisting only of agents i with $f_i(c^*,\pi^*) \neq 0$. By Lemma 1, if $f_i(c^*,\pi^*) \neq 0$, then $\partial f_i(c^*,\pi^*)/\partial\pi$ and $\partial f_i(0,(1+\operatorname{sgn}(f_i(c^*,\pi^*))c)\pi^*)/\partial\pi$ share

the same sign. Thus, by Theorem 1, there is an equilibrium price function e around (c^*, π^*) for this economy that satisfies $D_{\mathscr{B}} < 0$ and $D_{\mathscr{S}} < 0$. Thus $(1+c)e(c) > (1+c^*)\pi^*$ and $(1-c)e(c) < (1-c^*)\pi^*$ for every $c > c^*$ sufficiently close to c^*. Since $\partial g_i(0; 0, \pi)/\partial \pi < 0$ for every π,

$$g_i(0; c^*, \pi^*) = g_i(0; 0, (1+c^*)\pi^*) > g_i(0; 0, (1+c)e(c)) = g_i(0; c, e(c)).$$

Thus, if $g_i(0; c^*, \pi^*) \leq 0$, then $g_i(0; c, e(c)) < 0$. Similarly, if $h_i(0; c^*, \pi^*) \geq 0$, then $h_i(0; c, e(c)) > 0$. Hence, by Lemma 1, if $f_i(c^*, \pi^*) = 0$, then $f_i(c, e(c)) = 0$. Therefore, $e(c)$ is, in fact, a (unique) nontrivial equilibrium price under c for the entire economy of I agents, and $c \in V$ for every $c > c^*$ sufficiently close to c^*. Thus, e is right-continuous.

Next, we prove that e is left-continuous. To do so, we show that for every i, there is at most one $c \in [0, 1)$ such that $c \in V$ and $g_i(0; c, e(c)) = 0$. Suppose, on the contrary, that there were two proportional transaction costs, c_1 and c_2 with $c_1 < c_2$, such that $c_n \in V$ and $g_i(0; c_n, e(c_n)) = 0$ for both $n = 1, 2$. Since $g_i(0; c_n, e(c_n)) = g_i(0; 0, (1+c_n)e(c_n))$ and $\partial g_i/\partial \pi < 0$, this implies that $(1+c_1)e(c_1) = (1+c_2)e(c_2)$.

Let $\mathscr{B} = \{j : f_j(c_2, e(c_2)) > 0\}$, $\mathscr{S} = \{j : f_j(c_2, e(c_2)) < 0\}$, and $\mathscr{N} = \{j : f_j(c_2, e(c_2)) = 0\}$. Then $\mathscr{B} \neq \varnothing$ and $\mathscr{S} \neq \varnothing$, because $c_2 \in V$.

For every $j \in \mathscr{N}$, $0 \geq g_j(0; c_1, e(c_1)) = g_j(0; c_2, e(c_2))$ by Lemma 1. Thus $f_j(c_1, e(c_1)) \leq 0$ and hence

$$\sum_{j \in \mathscr{N}} f_j(c_1, e(c_1)) \leq 0. \tag{A.14}$$

For every $j \in \mathscr{B}$, $0 < f_j(c_2, e(c_2)) = f_j(0, (1+c_2)e(c_2)) = f_j(0, (1+c_1)e(c_1)) = f_j(c_1, e(c_1))$. Thus

$$\sum_{j \in \mathscr{B}} f_j(c_1, e(c_1)) = \sum_{j \in \mathscr{B}} f_j(c_2, e(c_2)). \tag{A.15}$$

Since $c_1 < c_2$ and $(1+c_1)e(c_1) = (1+c_2)e(c_2)$, $e(c_1) > e(c_2)$ and hence $(1-c_1)e(c_1) > (1-c_2)e(c_2)$. For every $j \in \mathscr{S}$, by (3.9), $0 > f_j(c_2, e(c_2)) = f_j(0, (1-c_2)e(c_2)) > f_j(0, (1-c_1)e(c_1)) = f_j(c_2, e(c_2))$. Thus

$$\sum_{j \in \mathscr{S}} f_j(c_1, e(c_1)) < \sum_{j \in \mathscr{S}} f_j(c_2, e(c_2)). \tag{A.16}$$

By (A.14), (A.15), and (A.16),

$$\begin{aligned}
0 = \sum_{j=1}^{I} f_j(c_1, e(c_1)) &= \sum_{j \in \mathscr{B}} f_j(c_1, e(c_1)) + \sum_{j \in \mathscr{S}} f_j(c_1, e(c_1)) + \sum_{j \in \mathscr{N}} f_j(c_1, e(c_1)) \\
&< \sum_{j \in \mathscr{B}} f_j(c_2, e(c_2)) + \sum_{j \in \mathscr{S}} f_j(c_2, e(c_2)) + \sum_{j \in \mathscr{N}} f_j(c_c, e(c_c)) \\
&= \sum_{j=1}^{I} f_j(c_2, e(c_2)) = 0.
\end{aligned}$$

This is a contradiction. Thus, for every i, there is at most one $c \in [0, 1)$ such that $c \in V$ and $g_i(0; c, e(c)) = 0$.

We can analogously show that for every i, there is at most one $c \in [0,1)$ such that $c \in V$ and $h_i(0; c, e(c)) = 0$. Therefore, there are at most $2I$ c's, for which $e(c)$ is a nontrivial and abnormal equilibrium price under c.

Let's now prove that e is left-continuous. Let $c^* \in V$ and π^* be an equilibrium price under c^*. Since there are only finitely many c's for which $e(c)$ is not a normal equilibrium price, the equilibrium price $e(c)$ of the proportional transaction cost c is normal for every $c \in (c^* - \varepsilon, c^*)$ with $\varepsilon > 0$ sufficiently small. By Theorem 1, $(1-c)e(c)$ is a strictly decreasing function of $c \in (c^* - \varepsilon, c^*)$. Since $(1-c)e(c) \geq 0$, $\lim_{c \uparrow c^*}(1-c)e(c)$ exists, and we write $\pi^* = (1-c^*)^{-1} \lim_{c \uparrow c^*}(1-c)e(c)$. Then $e(c) \to \pi^*$ as $c \uparrow c^*$. Thus, by part (1) of Lemma 2, $f_i(c, e(c)) \to f_i(c^*, \pi^*)$. Hence $\sum_{i=1}^{I} f_i(c, e(c)) \to \sum_{i=1}^{I} f_i(c^*, \pi^*)$. Since $\sum_{i=1}^{I} f_i(c, e(c)) = 0$ for every $c \in (c^* - \varepsilon, c^*)$, $\sum_{i=1}^{I} f_i(c^*, \pi^*) = 0$. Since $c^* \in V$, $\pi^* = e(c^*)$. The left continuity has thus been proved.

Therefore, $(1+c)e(c)$ is a continuous function of c on V. By Theorem 1, has strictly positive derivatives at all but finitely many points. Thus $(1+c)e(c)$ is a strictly increasing function of c on V. Analogously, $(1-c)e(c)$ is a continuous and strictly decreasing function of c on V.

(4) Just as in the proof of Theorem 1, we can show that if $c^* \in V$ and $e(c^*)$ is a normal equilibrium price, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}c}|f_i(c, e(c))|\bigg|_{c=c^*} \\
=&\frac{\partial f_i}{\partial \pi}(c^*, e(c^*))\frac{\operatorname{sgn}(f_i(c^*, e(c^*)))}{1+\operatorname{sgn}(f_i(c^*, e(c^*)))\,c^*}\ \frac{\mathrm{d}}{\mathrm{d}c}\left((1+\operatorname{sgn}(f_i(c^*, e(c^*)))\,c)\,e(c)\right)\bigg|_{c=c^*} \\
=&\begin{cases} \dfrac{\partial f_i}{\partial \pi}(c^*, e(c^*))\dfrac{Q_{\mathscr{B}}}{1+c^*} & \text{if } i \in \mathscr{B}, \\ \dfrac{\partial f_i}{\partial \pi}(c^*, e(c^*))\dfrac{-Q_{\mathscr{S}}}{1-c^*} & \text{if } i \in \mathscr{S} \end{cases}
\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}c}|f_i(c, e(c))|\bigg|_{c=c^*} < 0 \tag{A.17}$$

whenever $f_i(c^*, e(c^*)) \neq 0$.

Based on this result, we can prove the first claim of part (4) as follows. Suppose, on the contrary, that there are c_1 and c_2 such that $c_1 \leq c_2$, $f_i(c_1, e(c_1)) = 0$, and $f_i(c_2, e(c_2)) \neq 0$. Then $c_1 < c_2$, $|f_i(c_1, e(c_1))| = 0$, and $|f_i(c_2, e(c_2))| > 0$. Let c_3 be the largest c such that $c < c_2$ and $f_i(c, e(c)) = 0$. Since $|f_i(c, e(c))| > 0$ for every $c > c_3$ sufficiently close to c_3, there are c_4 and c_5 such that $c_3 < c_4 < c_5$, $e(c)$ is a normal equilibrium price under c for every $c \in [c_4, c_5]$, and $|f_i(c_4, e(c_4))| < |f_i(c_5, e(c_5))|$. Since $|f_i(c, e(c))|$ is a continuously differentiable function of c on $[c_4, c_5]$, we can apply the average value theorem to show that there is a $c_6 \in [c_4, c_5]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}c}|f_i(c, e(c))|\bigg|_{c=c_6} > 0.$$

This contradicts (A.17). Hence if $c_1 \leq c_2$ and $f_i(c_1, e(c_1)) = 0$, then $f_i(c_2, e(c_2)) = 0$.

Thus, the set $\{c \in V : f_i(c, e(c)) \neq 0\}$ is an interval, $|f_i(c, e(c))|$ is a continuous function of c and continuously differentiable at all but finitely many points. Thus, by (A.17), $|f_i(c, e(c))|$ is a strictly decreasing function on this set. Similarly,

$$E\left[u_i\left(A_i + f_i(c, e(c))(S - (1+\operatorname{sgn}(f_i(c, e(c)))\,c)e(c))\right)\right]$$

is a continuous function of c and continuously differentiable at all but finitely many points. Thus, by Theorem 1, it is a strictly decreasing function on this set. ///

PROOF OF PROPOSITION 1 By (2.8) and $u_i'' < 0$, it suffices to prove that

$$B \equiv E\left[u_i'(P_i) + f_i(0,\pi)(S-\pi)u_i''(P_i)\right] > 0,$$

where $P_i = A_i + f_i(0,\pi)(S-\pi)$. Indeed, this follows immediately from the assumption that $u_i' > 0$ if $f_i(0,\pi) = 0$. So suppose that $f_i(0,\pi) \neq 0$. By Remark 2,

$$\begin{aligned}
B &= E\left[u_i'(P_i)\left(1 - f_i(0,\pi)(S-\pi)R_i(P_i)\right)\right] \\
&= E\left[u_i'(P_i)\left(1 - f_i(0,\pi)(S-\pi)\left(R_i(P_i) - \alpha_i\right)\right)\right] - \alpha_i f_i(0,\pi)E\left[u_i'(P_i)(S-\pi)\right] \\
&= E\left[u_i'(P_i)\left(1 - (P_i - A_i)\left(R_i(P_i) - \alpha_i\right)\right)\right].
\end{aligned}$$

Define $\Omega_1 = \{\omega \in \Omega : P_i(\omega) > A_i(\omega)\}$ and $\Omega_2 = \{\omega \in \Omega : P_i(\omega) \leq A_i(\omega)\}$. Again by Remark 2, $P(\Omega_2) > 0$. Since $(P_i(\omega) - A_i(\omega))\left(R_i(P_i(\omega)) - \alpha_i\right) \leq 0$ for every $\omega \in \Omega_2$,

$$\begin{aligned}
&\int_{\Omega_2} u_i'(P_i(\omega))\left(1 - (P_i(\omega) - A_i(\omega))\left(R_i(P_i(\omega)) - \alpha_i\right)\right) \mathrm{d}P(\omega) \\
&\geq \int_{\Omega_2} u_i'(P_i(\omega))\,\mathrm{d}P(\omega) > 0.
\end{aligned}$$

Since $u'(P_i)\left(R_i(P_i) - \alpha_i\right) \geq 0$ and $R_i(P_i) - \alpha_i \leq (P_i - \underline{d}_i)^{-1}$ almost surely,

$$\begin{aligned}
B &= \int_{\Omega_1} u_i'(P_i(\omega))\left(1 - (P_i(\omega) - A_i(\omega))\left(R_i(P_i(\omega)) - \alpha_i\right)\right) \mathrm{d}P(\omega) \\
&\quad + \int_{\Omega_2} u_i'(P_i(\omega))\left(1 - (P_i(\omega) - A_i(\omega))\left(R_i(P_i(\omega)) - \alpha_i\right)\right) \mathrm{d}P(\omega) \\
&> \int_{\Omega_1} u_i'(P_i(\omega))\left(1 - (P_i(\omega) - A_i(\omega))\left(R_i(P_i(\omega)) - \alpha_i\right)\right) \mathrm{d}P(\omega) \\
&\geq \int_{\Omega_1} u_i'(P_i(\omega))\left(1 - \frac{P_i(\omega) - A_i(\omega)}{P_i(\omega) - \underline{d}_i}\right) \mathrm{d}P(\omega) \\
&= \int_{\Omega_1} u_i'(P_i(\omega))\frac{A_i(\omega) - \underline{d}_i}{P_i(\omega) - \underline{d}_i}\,\mathrm{d}P(\omega).
\end{aligned}$$

Since $A_i \geq \underline{d}_i$ and $P_i > \underline{d}_i$ almost surely, the last integral is nonnegative. Thus the proof is completed. ///

PROOF OF PROPOSITION 2 AND THE SUBSEQUENT CLAIM To prove Proposition 2, by Theorem 1, it suffices to show that $f_1(0,\pi^*) < 0$, $f_2(0,\pi^*) >$, $\partial f_1(0,\pi^*)/\partial\pi > 0$, and $\partial f_1(0,\pi^*)/\partial\pi + \partial f_2(0,\pi^*)/\partial\pi < 0$.

For each $i = 1, 2$, define $m_i : (0,1) \to \boldsymbol{R}$ by

$$\begin{aligned}
m_1(\pi) &= \left(\frac{1}{\gamma_1}\frac{1}{1-\pi} - 1\right) + \left(\frac{1}{1-\pi} - 1\right)^{1/\gamma_1}, \\
m_2(\pi) &= \left(\frac{1}{\gamma_2}\frac{1}{\pi} - 1\right) + \left(\frac{1}{\pi} - 1\right)^{1/\gamma_2}.
\end{aligned}$$

Then, m_1 is continuous and strictly increasing, and m_2 is continuous and strictly decreasing. As $\pi \to 0$, $m_1(\pi) \to 1/\gamma_1 - 1$, and as $\pi \to 1$, $m_2(\pi) \to 1/\gamma_2 - 1$. Since $\gamma_1 > 1$, $1/\gamma_1 - 1 < 0$.

Thus, there is a $\pi^\circ \in (0,1)$ such that $m_1(\pi) \lesseqgtr 0$ if and only if $\pi \lesseqgtr \pi^\circ$. Since $m_i(1/2) = 2/\gamma_i$ for each i, $\pi^\circ < 1/2$ and $m_2(\pi) > 0$ for every $\pi \in (0, \pi^\circ]$. Hence $m_2(\pi)/m_1(\pi) \to -\infty$ as $\pi \uparrow \pi^\circ$.

For each $i = 1, 2$, define $k_i : (0,1) \to \boldsymbol{R}$ by

$$k_1(\pi) = \frac{(1-\pi)^{-1/\gamma_1}}{\pi^{1-1/\gamma_1} + (1-\pi)^{1-1/\gamma_1}},$$
$$k_2(\pi) = \frac{\pi^{-1/\gamma_2}}{\pi^{1-1/\gamma_2} + (1-\pi)^{1-1/\gamma_2}}.$$

Then, for each i, k_i is continuously differentiable and strictly-positive-valued. Also, for each i, define $t_i : (0,1) \to \boldsymbol{R}$ by

$$t_i(\pi) = \frac{(\pi(1-\pi))^{-1/\gamma_i}}{\left(\pi^{1-1/\gamma_i} + (1-\pi)^{1-1/\gamma_i}\right)^2} m_i(\pi).$$

Then $k_1'(\pi) = t_1(\pi)$ and $k_2'(\pi) = -t_2(\pi)$ for every π. Moreover, as $\pi \uparrow \pi^\circ$,

$$\frac{k_1(\pi)}{k_2(\pi)} \frac{t_2(\pi)}{t_1(\pi)} \to -\infty.$$

Hence there is a $\pi^* \in (0, \pi^\circ)$ such that

$$\frac{k_1(\pi^*)}{k_2(\pi^*)} \frac{t_2(\pi^*)}{t_1(\pi^*)} < -1. \tag{A.18}$$

When either a_1 or a_2 is given, define the other via

$$a_1 k_1(\pi^*) = a_2 k_2(\pi^*). \tag{A.19}$$

We shall prove that π^* is an equilibrium price for the Arrow security if there is no transaction cost. Note that

$$E[u_1(A_1 + y_1(S - \pi))] = \frac{1}{2} u_1(a_1 + (1-\pi)y_1) + \frac{1}{2} u_1(-\pi y_1)$$

and

$$E[u_2(A_2 + y_2(S - \pi))] = \frac{1}{2} u_1((1-\pi)y_2) + \frac{1}{2} u_2(a_2 - \pi y_2).$$

By the first-order conditions, the the solutions to these problems are given by $f_1(0,\pi) = -a_1 k_1(\pi)$ and $f_2(0,\pi) = a_2 k_2(\pi)$. Hence $f_1(0,\pi^*) < 0$ and $f_2(0,\pi^*) > 0$. Moreover,

$$f_1(0,\pi^*) + f_2(0,\pi^*) = -a_1 k_1(\pi^*) + a_2 k_2(\pi^*) = 0.$$

Thus π^* is an equilibrium price for the Arrow security in the absence of transaction costs. Moreover,

$$\frac{\partial f_1}{\partial \pi}(0,\pi^*) = -a_1 k_1'(\pi^*) = -a_1 t_1(\pi^*) > 0,$$
$$\frac{\partial f_2}{\partial \pi}(0,\pi^*) = \quad a_2 k_1'(\pi^*) = -a_2 t_2(\pi^*) < 0,$$
$$\frac{\partial f_1}{\partial \pi}(0,\pi^*) + \frac{\partial f_2}{\partial \pi}(0,\pi^*) = -a_1 t_1(\pi^*) \left(1 + \frac{k_1(\pi^*)}{k_2(\pi^*)} \frac{t_2(\pi^*)}{t_1(\pi^*)}\right) < 0.$$

This completes the proof of Proposition 2.

To prove the subsequent claim on the case of common CRRA coefficients, it suffices to show that

$$\frac{k_1(\pi)}{k_2(\pi)} \to 0 \tag{A.20}$$

and

$$\frac{k_1(\pi)}{k_2(\pi)}\frac{t_2(\pi)}{t_1(\pi)} \to -\infty \tag{A.21}$$

as $\pi \downarrow 0$. Indeed, then, for all a_1 and a_2 with a_2/a_1 sufficiently close to zero, there is a $\pi^* \in (0,1)$ for which (A.18) and (A.19) hold. Then the argument in Steps 2 and 3 of the proof of Proposition 2 is valid.

To prove (A.20) and (A.21), write $\gamma = \gamma_1 = \gamma_2$. Then

$$\frac{k_1(\pi)}{k_2(\pi)} = \frac{(1-\pi)^{-1/\gamma}}{\pi^{-1/\gamma}} = \left(\frac{\pi}{1-\pi}\right)^{1/\gamma} \to 0$$

as $\pi \downarrow 0$. As for (A.21),

$$\frac{k_1(\pi)}{k_2(\pi)}\frac{t_2(\pi)}{t_1(\pi)} = \left(\frac{\pi}{1-\pi}\right)^{1/\gamma}\frac{m_2(\pi)}{m_1(\pi)} = \frac{\left(\frac{1}{\gamma}\frac{1}{\pi}-1\right)+\left(\frac{1}{\pi}-1\right)^{1/\gamma}}{\left(\frac{1}{\pi}-1\right)^{1/\gamma}}\frac{1}{m_1(\pi)} = \left(\frac{\frac{1}{\gamma}\frac{1}{\pi}-1}{\left(\frac{1}{\pi}-1\right)^{1/\gamma}}+1\right)\frac{1}{m_1(\pi)}.$$

Write $\rho = 1/\pi$, then

$$\frac{\frac{1}{\gamma}\frac{1}{\pi}-1}{\left(\frac{1}{\pi}-1\right)^{1/\gamma}} = \frac{\frac{\rho}{\gamma}-1}{(\rho-1)^{1/\gamma}}$$

The derivative of the numerator with respect to ρ is equal to $1/\gamma$ and that of the denominator is equal to $(1/\gamma)\,(\rho-1)^{1/\gamma-1}$. Since

$$\frac{1/\gamma}{(1/\gamma)\,(\rho-1)^{1/\gamma-1}} = (\rho-1)^{1-1/\gamma} \to \infty$$

as $\rho \uparrow \infty$, L'Hôpital's rule implies that

$$\frac{\frac{1}{\gamma}\frac{1}{\pi}-1}{\left(\frac{1}{\pi}-1\right)^{1/\gamma}} = \frac{\frac{\rho}{\gamma}-1}{(\rho-1)^{1/\gamma}} \to \infty$$

as $\rho \uparrow \infty$. Since $1/m_1(\pi) \to -\gamma/(\gamma-1) < 0$ as $\pi \downarrow 0$, (A.21) follows from this. ///

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