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# On the existence of Walras equilibrium in irreducible economies with satiable and non-ordered preferences

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**Abstract.** Irreducible exchange economies in which consumers' preferences are satiable and non-ordered are considered. A general existence theorem of dividend quasi-equilibrium is proved and by the theorem the existence of Walras equilibrium is proved under weak assumptions of non-satiation.

**Keywords:** dividend equilibrium, Walras equilibrium, irreducibility, satiation.

## 1. Introduction

The purpose of this paper is to prove the existence of Walras equilibrium in an economy with satiable consumers under a set of weaker assumptions. We consider a bounded model of exchange economy where consumers' preferences are non-ordered.[1/] In the classical theory of general equilibrium, in proving the existence of Walras equilibrium, the first step is to prove the existence of quasi-equilibrium and the second step is to show that any quasi-equilibrium is an equilibrium under a certain assumption. That is a well-known procedure which was established by McKenzie (1959) and Debreu (1962). The so-called irreducibility is a famous condition under which any quasi-equilibrium is an equilibrium. In this paper we define the irreducibility in a general formulation for economies with satiable consumers. Following the classical procedure, first we shall prove a basic theorem on the existence of dividend quasi-equilibrium in a very general setting, and next based on the theorem we shall show that a dividend equilibrium exists. The dividend equilibrium, which originates in the concept of 'coupons equilibrium' in Drèze &Müller (1980), was defined by Aumann=Drèze (1986). Furthermore, in order to prove the existence of Walras equilibrium, we shall present two types of non-satiation assumption and show that under each of them any dividend equilibrium is (or, can be viewed as) a Walras equilibrium. Our existence theorems of Walras equilibrium are more general in two points than those in the existing literatures such as Allouch & Le Van (2009), Sato (2010a), and Won & Yannelis (2011). First, we do not assume that the initial endowment of every consumer belongs to the interior of his consumption set. We shall show that only the irreducibility is required even in the case that there are some satiable consumers. Second, our assumptions of non-satiation are weaker and admit that satiation may occur unexceptionally in feasible allocations.

Finally, we will discuss a relation between irreducibility and non-satiation assumptions. We propose a new concept of irreducibility that we will refer to as "generalized irreducibility", and show that it implies both irreducibility and our non-satiation assumptions. One of our conclusions is that the relaxation of non-satiation assumptions can be viewed as the generalization of irreducibility conditions.

This paper is formalized as follows. In section 2, we present a model of exchange economy in which there are satiable consumers and state the basic assumptions. In section 3, we prove an existence theorem of dividend quasi-equilibrium under a very general setting. In section 4, we define the irreducibility condition for an economy with possibly satiable consumers and prove

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1 Unbounded economies have been studied in many papers. For example, see Sato (2010b) which includes a recent good survey on those studies.

the existence of dividend equilibrium in the irreducible economy. In section 5, the existence of Walras equilibrium is proved under two types of non-satiation assumptions. In section 6, some remarks will be made to compare our result with former papers and two examples of exchange economies will be given to show the generality of our results. All the proofs of lemmas are given in Appendix.

## 2. Model

We consider an exchange economy with $L$ commodities and $N$ consumers. The set of consumers is denoted by $I = \{1, \cdots, N\}$. The commodity space is an $L$-dimensional Euclidean space $\mathbf{R}^L$. The consumption set of each consumer $i \in I$ is denoted by $X_i \subset \mathbf{R}^L$ and the initial endowment is by $e_i \in X_i$.

An *allocation* is an $N$-tuple of vectors, $\mathbf{x} = (x_1, \cdots, x_N)$, where $x_i \in X_i$ is an amount of commodities allotted to consumer $i \in I$. An allocation $\mathbf{x} = (x_1, \cdots, x_N)$ $\in X_1 \times \cdots \times X_N$ is said to be *feasible* if $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} e_i$. The set of all feasible allocations is denoted by **A**, i.e.,

$$\mathbf{A} := \{\, \mathbf{x} = (x_1, \cdots, x_N) \in X_1 \times \cdots \times X_N \mid \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} e_i \,\}.$$

Since $e_i \in X_i$ for each $i \in I$, set **A** is non-empty. The preference of each consumer $i \in I$ is denoted by a mapping by $P_i : X_i \to 2^{X_i}$. [2]

Throughout this paper, for each consumer $i \in I$, we assume the following:

(A.1) $X_i$ is a non-empty, closed, and convex subset of $\mathbf{R}^L$.

2 It is known that by defining mapping $P_i$ on set $X_1 \times \cdots \times X_N$ instead of $X_i$, with no mathematical difficulty, we can include cases where consumers' preferences depend on each other.

(A.2) $P_i : X_i \to 2^{X_i}$ is lower hemi-continuous, i.e., if $y^0 \in P_i(x^0)$ and a sequence $\{x^n\}$ converges to $x^0$, then there is a sequence $\{y^n\}$ converging to $y^0$ such that $y^n \in P_i(x^n)$ for all $n$ sufficiently large.

(A.3) For every $x_i \in X_i$, $P_i(x_i)$ is convex and $x_i \notin P_i(x_i)$. [3/]

In addition, we assume the boundedness of the economy:

(A.4) The set $\mathbf{A}$ is bounded, i.e., there is a number $\overline{b} > 0$ such that for any $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$, $\| x_i \| \leq \overline{b}$ for all $i \in I$.

A *dividend vector* is a non-negative vector, $d = (d_1, \cdots, d_N) \in \mathbf{R}^N$, where $d_i$ is an extra income given to consumer $i \in I$.

**Definition 1.** A *dividend quasi-equilibrium* is a triplet $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ of a feasible allocation $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_N) \in \mathbf{A}$, a price vector $\hat{p} \in \mathbf{R}^L$ with $\hat{p} \neq \mathbf{0}$, and a dividend vector $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_N) \in \mathbf{R}^N_+$ such that for each $i \in I$

(1) $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + \hat{d}_i$,

(2) $\hat{p} \cdot y_i \geq \hat{p} \cdot e_i + \hat{d}_i$ for all $y_i \in P_i(\hat{x}_i)$.

A *dividend equilibrium* is a dividend quasi-equilibrium $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ such that, instead of condition (2), $\hat{p} \cdot y_i > \hat{p} \cdot e_i + \hat{d}_i$ for all $y_i \in P_i(\hat{x}_i)$.

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3 It is well known that this assumption can be slightly weaken as follows: For every $x_i \in X_i$, $x_i \notin \mathrm{co}\, P_i(x_i)$. Here, for any set $X \subset \mathbf{R}^L$, co $X$ denotes the convex hull of set $X$.

In the original definition of quasi-equilibrium due to Debreu (1962), instead of (2) of Definition 1, the following condition is required:

$$\hat{p} \cdot y_i > \hat{p} \cdot e_i + \hat{d}_i \text{ for any } y_i \in P_i(\hat{x}_i) \text{ and/or } \hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i = \inf \hat{p} \cdot X_i .$$

For example, when $P_i(x_i)$ is open in $X_i$ for any $x_i \in X_i$ for each $i \in I$, we can easily show that Debreu's condition is equivalent to (2) in Definition 1, and therefore a dividend quasi-equilibrium $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend equilibrium if $\hat{p} \cdot e_i + \hat{d}_i > \inf \hat{p} \cdot X_i$ for all $i \in I$.

**Definition 2.** A *Walras equilibrium* (resp. *quasi-equilibrium*) is a pair $\{\hat{\mathbf{x}}, \hat{p}\}$ of a dividend equilibrium (resp. quasi-equilibrium) $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ such that $\hat{d} = \mathbf{0}$.

For each $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$, we define two sets of consumers as follows:

$$I^S(\mathbf{x}) := \{i \in I \mid P_i(x_i) = \phi\} \text{ and } I^{NS}(\mathbf{x}) := I \setminus I^S(\mathbf{x}).$$

To exclude trivial cases in which every consumer is simultaneously satiated and a dividend equilibrium always exists, we assume the following condition which is the weakest non-satiation assumption.

(A.5) For any $\mathbf{x} \in \mathbf{A}$, $I^{NS}(\mathbf{x}) \neq \phi$.

In fact, assume that $I^{NS}(\hat{\mathbf{x}}) = \phi$ for some $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_N) \in \mathbf{A}$. For any price vector $\hat{p} \in \mathbf{R}^L$ with $\hat{p} \neq \mathbf{0}$, choose a large dividend vector $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_N) \in \mathbf{R}^N$ so that $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + \hat{d}_i$ for all $i \in I$. Then, since $P_i(\hat{x}_i) = \phi$ for all $i \in I$, $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend equilibrium.

## 3. Existence of dividend quasi-equilibrium

First, let us modify consumers' preference relations. For each $i \in I$ and each $x_i \in X_i$, we define a convex cone $K_i(x_i)$ by

$$K_i(x_i) := \{\lambda(y_i - x_i) \mid y_i \in P_i(x_i), \lambda > 0\}.$$

Set $K_i(x_i)$ indicates the desirable directions from $x_i \in X_i$ for consumer $i \in I$.

Moreover, for each $i \in I$, a set $\overline{X}_i$ and a mapping $\overline{P}_i : \overline{X}_i \to 2^{\overline{X}_i}$ are defined in the following way:

$$\overline{X}_i := \{x_i \in X_i \mid \| x_i \| \leq \overline{b} + 1\}.$$

$$\overline{P}_i(x_i) := \{z_i + x_i \mid z_i \in K_i(x_i)\} \cap \overline{X}_i \text{ for each } x_i \in \overline{X}_i.$$

Consider the modified economy in which the consumption set and the preference relation of each consumer $i \in I$ are replaced by $\overline{X}_i$ and $\overline{P}_i : \overline{X}_i \to 2^{\overline{X}_i}$. We should note that for each $i \in I$, set $\overline{X}_i$ is bounded and $\overline{P}_i(x_i)$ is convex for each $x_i \in \overline{X}_i$.

Now, we can prove the following lemma.

**Lemma 3.1.** *Any dividend quasi-equilibrium for the modified economy is a dividend quasi-equilibrium for the original economy.*

By virtue of Lemma 3.1, to prove the existence of a dividend quasi-equilibrium, it suffices only to prove the existence of a dividend quasi-equilibrium for the modified economy. In addition, it is easy to show that mapping $\overline{P}_i$ has the same properties as mapping $P_i$ has, that is, it is lower hemi-continuous, convex-valued, and $x_i \notin \overline{P}_i(x_i)$ for every $x \in \overline{X}_i$. Thus, in what follows, we shall identify $X_i$ with $\overline{X}_i$ and $P_i : X_i \to 2^{X_i}$ with $\overline{P}_i : \overline{X}_i \to 2^{\overline{X}_i}$.

The following theorem on the existence of dividend quasi-equilibrium is a basic and key theorem for our argument.

**Theorem 1.** *Under assumptions (A.1)–(A.5), there exists a dividend quasi-equilibrium. More precisely, there exists a dividend quasi-equilibrium* $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ *such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ *for all* $i \in I^{NS}(\hat{\mathbf{x}})$ *and* $\hat{d}_1 = \cdots = \hat{d}_N$.

Now, let us confine prices to the closed unit ball, $\mathbf{B} = \{p \in \mathbf{R}^n \mid \| p \| \leq 1\}$. For each $i \in I$, define a mapping $\beta_i : \mathbf{B} \to 2^{X_i}$ by:

$$\beta_i(p) := \{y_i \in X_i \mid p \cdot y_i < p \cdot e_i + 1 - \| p \|\} \quad \text{for each } p \in \mathbf{B}.$$ [4/]

Moreover, for each $i \in I$, define a mapping $F_i : \mathbf{B} \times X_i \to 2^{X_i}$ by:

$$F_i(p, x_i) := \begin{cases} \{y_i \in X_i \mid p \cdot y_i < p \cdot x_i\} & \text{when } p \cdot x_i > p \cdot e_i + 1 - \| p \| \\ \beta_i(p) \cap P_i(x_i) & \text{otherwise} \end{cases}$$

for each $(p, x_i) \in \mathbf{B} \times X_i$. This mapping is a modification of the mapping originally constructed by Gale=Mas-Colell (1975). The modification is slight, but crucial since we do not assume that $e_i \in \operatorname{int} X_i$ for each $i \in I$.[5/] The mapping can be applied to cases where consumers' budget sets do not always have interior.

**Lemma 3.2.** *For each* $i \in I$*, mapping* $F_i : \mathbf{B} \times X_i \to 2^{X_i}$ *is convex-valued and lower hemi-continuous*.

We shall follow the usual process to apply the fixed-point theorem that was

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4 The technique of adding extra income $1 - \|p\|$ to consumers is due to Bergstrom (1976) and is used by Shafer (1976) , Mas-Colell (1992), and Kajii (1996).

5 For any set $X \subset \mathbf{R}^L$, int $X$ denotes the interior of set $X$ in $\mathbf{R}^L$.

innovated by Gale=Mas-Colell (1975). Let us define a mapping $F_0 : \mathbf{B} \times \mathbf{R}^L \to 2^{\mathbf{B}}$ by:

$$F_0(p,z) := \{q \in \mathbf{B} \mid q \cdot z > p \cdot z\} \quad \text{for each } (p,z) \in \mathbf{B} \times \mathbf{R}^L. \text{[6/]}$$

Obviously, mapping $F_0$ is convex-valued and lower hemi-continuous.

Now, by using mappings $F_0$, $F_1$, $\cdots$, $F_N$, we can define a convex-valued and lower hemi-continuous mapping $F : \mathbf{B} \times X_1 \times \cdots \times X_N \to \mathbf{B} \times X_1 \times \cdots \times X_N$ by:

$$F(p, x_1, \cdots, x_N) := F_0(p, \sum_{i \in I} x_i - \sum_{i \in I} e_i) \times F_1(p, x_1) \times \cdots \times F_N(p, x_N)$$

for each $(p, x_1, \cdots, x_N) \in \mathbf{B} \times X_1 \times \cdots \times X_N$. Thus, by Gale=Mas-Colell's fixed point theorem (see Appendix), there is a point $(\hat{p}, \hat{x}_1, \cdots, \hat{x}_N) \in \mathbf{B} \times X_1 \times \cdots \times X_N$ such that

$$\text{either } F_0(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i) = \phi \text{ or } \hat{p} \in F_0(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i)$$

and

$$\text{either } F_i(\hat{p}, \hat{x}_i) = \phi \text{ or } \hat{x}_i \in F_i(\hat{p}, \hat{x}_i) \text{ for each } i \in I.$$

From definition of $F_0$, $F_1$, $\cdots$, $F_N$, it follows that $\hat{p} \notin F_0(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i)$ and $\hat{x}_i \notin F_i(\hat{p}, \hat{x}_i)$ for each $i \in I$. Thus, it follows that

$$F_0(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i) = \phi \text{ and } F_i(\hat{p}, \hat{x}_i) = \phi \quad \text{for each } i \in I.$$

Since $F_i(\hat{p}, \hat{x}_i) = \phi$ for each $i \in I$, it follows from the definition of $F_i$ that

$$\hat{p} \cdot \hat{x}_i \le \hat{p} \cdot e_i + 1 - \| \hat{p} \| \text{ and } \beta_i(\hat{p}) \cap P_i(\hat{x}_i) = \phi \text{ for each } i \in I. \tag{3.1}$$

Furthermore, suppose that $\sum_{i \in I} \hat{x}_i \ne \sum_{i \in I} e_i$. Since $F_0(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i) = \phi$, by the

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6 This mapping is first used by Shafer (1976) in a case where preference relations are not necessarily transitive. Mas-Colell (1992) also applied the mapping to a case with possibly satiated consumers.

definition of $F_0$, $q \cdot (\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i) \leq \hat{p} \cdot (\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i)$ for any $q \in \mathbf{B}$. Therefore, $\| \hat{p} \| = 1$ and $\hat{p} \cdot (\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i) > 0$. Thus, (3.1) implies that $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i$ for each $i \in I$, and that $\hat{p} \cdot (\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i) \leq 0$, which is a contradiction. Hence, we can conclude that $\sum_{i \in I} \hat{x}_i = \sum_{i \in I} e_i$.

Suppose that $\|\hat{p}\| = 0$. Then, $\beta_i(\hat{p}) = X_i$ for all $i \in I$, and, by (3.1), $P_i(\hat{x}_i) = \phi$ for all $i \in I$, which contradicts assumption (A.5). Thus, $\|\hat{p}\| \neq 0$.

Now, let $\hat{d}_1 = \cdots = \hat{d}_N = 1 - \| \hat{p} \|$. Then, by (3.1), for each $i \in I$,

$$\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + \hat{d}_i \text{ and } \hat{p} \cdot y_i \geq \hat{p} \cdot e_i + \hat{d}_i \text{ for all } y_i \in P_i(\hat{x}_i). \tag{3.2}$$

Thus, if we let $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_N)$ and $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_N)$, we have shown that $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend quasi-equilibrium.

Finally, since we identify $P_i$ with $\overline{P}_i$, for $i \in I^{NS}(\hat{\mathbf{x}})$ there is a point $y \in P_i(\hat{x}_i)$ which is arbitrarily close to $\hat{x}_i$. Therefore, (3.2) implies that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$. This completes the proof of Theorem 1.

## 4. Equilibrium and quasi-equilibrium

A well-known sufficient condition under which any quasi-equilibrium is an equilibrium is the irreducibility assumption that originates with McKenzie (1956). In case of economies with possibly satiated consumers, the irreducibility can be defined as follows:

**Irreducibility.** Let $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$ and $\{I_1, I_2\}$ be a partition of $I^{NS}(\mathbf{x})$ such that both $I_1$ and $I_2$ are non-empty. Then there is an allocation $\mathbf{y} = (y_1, \cdots, y_N)$

$\in X_1 \times \cdots \times X_N$ such that the following conditions hold:

(i) $\sum_{i \in I_1} (y_i - e_i) + \sum_{i \in I_2} (y_i - x_i) = 0$.

(ii) $y_i \in P_i(x_i)$ for each $i \in I_2$.

The meaning of Irreducibility is that the initial endowment of any group of non-satiated consumers is desired by other non-satiated consumers. Obviously, when $I^S(\mathbf{x}) = \phi$, the above condition is equivalent to the original irreducibility condition for economies where consumers are never satiated.

A weaker condition of irreducibility was considered by Bergstrom (1976). The condition can be defined for economies with possibly satiated consumers in the following fashion:[7/]

(A.6) Let $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$ and $j \in I^{NS}(\mathbf{x})$. If $I^{NS}(\mathbf{x}) \setminus \{j\} \neq \phi$, then there exist an allocation $\mathbf{y} = (y_1, \cdots, y_N) \in X_1 \times \cdots \times X_N$ and a scalar $\theta$ >0 such that

(i) $\theta(y_j - e_j) + \sum_{i \in I^S(\mathbf{x})} (x_i - e_i) + \sum_{i \in I^{NS}(\mathbf{x}) \setminus \{j\}} (y_i - e_i) = 0$ and

(ii) $y_i \in P_i(x_i)$ for each $i \in I^{NS}(\mathbf{x}) \setminus \{j\}$.

Evidently, when $I^S(\mathbf{x}) = \phi$, the above condition is equivalent to the condition of Bergstrom (1976). Moreover, we can prove the following lemma.

**Lemma 4.1.** *Irreducibility implies assumption (A.6).*

In order to prove that any quasi-equilibrium is an equilibrium, we need the following assumptions:

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7 In Bergstrom (1976), the irreducibility condition is assumed only on individually rational allocations in which no consumers are worse off than in their initial endowments. Since in a quasi-equilibrium some consumers might be worse off than in their endowments, we need to include all feasible allocations in defining irreducibility.

(A.7) For each $i \in I$, $P_i(x_i)$ is open in $X_i$ for every $x_i \in X_i$.[8/]

(A.8) For any $\mathbf{x} \in \mathbf{A}$, $\sum_{i \in I^{NS}(\mathbf{x})} e_i \in \operatorname{int} \sum_{i \in I^{NS}(\mathbf{x})} X_i$.

Condition (A.8) is weaker than the assumption that $e_i \in \operatorname{int} X_i$ for all $i \in I$. For example, if there is at least one consumer who is never satiated and if his initial endowment belongs to the interior of his consumption set, then the condition holds.

The following is a fundamental lemma due to Debreu (1962), which is derived from the irreducibility condition.

**Lemma 4.2.** *Under assumptions (A.6) and (A.7), for any dividend quasi-equilibrium* $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$*, if* $\hat{p} \cdot \hat{e}_i > \inf \hat{p} \cdot X_i$ *occurs for some* $i \in I^{NS}(\hat{\mathbf{x}})$*, then it occurs for every* $i \in I^{NS}(\hat{\mathbf{x}})$.

Now, let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend quasi-equilibrium. From (A.8), it follows that $\hat{p} \cdot \hat{e}_i > \inf \hat{p} \cdot X_i$ occurs for some $i \in I^{NS}(\hat{\mathbf{x}})$, and by Lemma 4.2, it occurs for all $i \in I^{NS}(\hat{\mathbf{x}})$. Therefore, $\hat{p} \cdot \hat{e}_i + \hat{d}_i > \inf \hat{p} \cdot X_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$, and by (A.7) we can show that $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend equilibrium. Thus, we have the following theorem.

**Theorem 2.** *Under assumptions (A.6), (A.7), and (A.8), any dividend quasi-equilibrium is a dividend equilibrium.*

By Theorems 1 and 2 we have the following corollary.

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8 As Bergstrom (1976) and Won & Yannelis (2011) showed, this assumption can be weaken in the following way: If $y_i \in P_i(x_i)$ and $z_i \in X_i$, then there exists a number $\theta$>0 such that $(1-\theta)y_i + \theta z_i \in P_i(x_i)$.

**Corollary 1.** *Under assumptions (A.1)-(A.8), there exists a dividend equilibrium. More precisely, there exists a dividend equilibrium* $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ *such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ *for all* $i \in I^{NS}(\hat{\mathbf{x}})$ *and* $\hat{d}_1 = \cdots = \hat{d}_N$ .

## 5. Existence of Walras equilibrium

In order prove the existence of Walras equilibrium, we need an additional assumption which relates satiated and non-satiated consumers. Won and Yannelis (2011) assumed the following condition.

(W-Y) For any $\mathbf{x} \in \mathbf{A}$, $x_i - e_i \in \mathrm{cl}\,[\sum_{j \in I^{NS}(\mathbf{x})} (P_j(x_j) - \{x_j\})]$ for all $i \in I^S(\mathbf{x})$.

For $\mathbf{x} \in \mathbf{A}$, we define a convex cone by

$$K(\mathbf{x}) := \sum_{i \in I^{NS}(\mathbf{x})} K_i(x_i) .$$

The above condition is slightly generalized in the following fashion.

(R.0) For any $\mathbf{x} \in \mathbf{A}$, $x_i - e_i \in \mathrm{cl}\ K(\mathbf{x})$ for all $i \in I^S(\mathbf{x})$.

Obviously, (W-Y) implies (R.0), since $(P_i(x_i) - \{x_i\}) \subset K_i(x_i)$ . Set $K(\mathbf{x})$ indicates the desirable directions for the non-satiated consumers in allocation $\mathbf{x} \in A$. Therefore, condition (R.0) means that in any allocation the direction of satiation point from initial endowment for any satiated consumer is one of the directions which are desirable for the non-satiated consumers.

In what follows, we shall consider two types of non-satiation condition which are weaker than condition (R.0). First we consider the following condition, which is immediately implied by (R.0).

(R.1) For any $\mathbf{x} \in \mathbf{A}$, $\sum_{i \in I^S(\mathbf{x})} (x_i - e_i) \in \mathrm{cl}\ K(\mathbf{x})$.

Now, let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend equilibrium such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$ and $\hat{d}_1 = \cdots = \hat{d}_N$. From the definition of dividend equilibrium, it follows that, for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot y_i > \hat{p} \cdot e_i + \hat{d}_i = \hat{p} \cdot \hat{x}_i$ for any $y_i \in P_i(\hat{x}_i)$. Therefore, $\hat{p} \cdot z > 0$ for any $z \in K_i(\hat{x}_i)$. Hence, by the definition of $K(\hat{\mathbf{x}})$, $\hat{p} \cdot z \geq 0$ for any $z \in \mathrm{cl}\ K(\hat{\mathbf{x}})$. Thus, by (R.1), $\hat{p} \cdot \sum_{i \in I^S(\mathbf{x})} (\hat{x}_i - e_i) \geq 0$.

Since $\sum_{i \in I} (\hat{x}_i - e_i) = 0$, it follows that $\hat{p} \cdot \sum_{i \in I^{NS}(\mathbf{x})} (\hat{x}_i - e_i) \leq 0$. Since $\hat{p} \cdot (\hat{x}_i - e_i) = \hat{d}_i \geq 0$ for all $i \in I^{NS}(\hat{\mathbf{x}})$, we conclude that $\hat{p} \cdot (\hat{x}_i - e_i) = \hat{d}_i = 0$ for all $i \in I^{NS}(\hat{\mathbf{x}})$, and that $\hat{d}_i = 0$ for all $i \in I$. Therefore, $\{\hat{\mathbf{x}}, \hat{p}\}$ is a Walras equilibrium. Thus, by virtue of Corollary 1 we have proved the following theorem.

**Theorem 3.** *Under assumptions (A.1)-(A.8) and (R.1), there exists a Walras equilibrium* $\{\hat{\mathbf{x}}, \hat{p}\}$ *such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ for all $i \in I$.

It should be noted that the existence of a dividend equilibrium with equal dividends is essential in proving Theorem 3.

Next, we assume another weaker form of non-satiation. For $\mathbf{x} \in \mathbf{A}$, we define the following sets:

$$L_i(x_i) := \{\lambda(z + x_i - e_i) \mid z \in K_i(x_i), \lambda > 0\} \text{ for each } i \in I^{NS}(\mathbf{x}).$$

$$L_i(x_i) := \{\lambda(x_i - e_i) \mid \lambda > 0)\} \text{ for each } i \in I^S(\mathbf{x}).$$

(R.2) For any $\mathbf{x} \in \mathbf{A}$, if $\mathbf{0} \notin \mathrm{int} \sum_{i \in I^{NS}(\mathbf{x})} L_i(x_i)$, then $\mathbf{0} \notin \mathrm{int} \sum_{i \in I} L_i(x_i)$.

For consumer $i \in I^{NS}(\mathbf{x})$, set $L(x_i)$ indicates the desirable directions from initial endowment $e_i$. Therefore, condition (R.2) means that in any allocation the desirable directions for the satiated consumers are roughly desirable for the non-satiated consumers in that the desirable directions for both satiated and non-satiated consumers are contained in a common half space of $\mathbf{R}^L$.

**Lemma 5.1.** *Condition (R.0) implies condition (R.2).*

Now, let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend equilibrium such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$. From the definition of dividend equilibrium, it follows that for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot y_i > p \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for any $y_i \in P_i(\hat{x}_i)$. Therefore,

$$\hat{p} \cdot (y_i - \hat{x}_i) > 0 \text{ and } \hat{p} \cdot (\hat{x}_i - e_i) > 0 \text{ for any } y_i \in P_i(\hat{x}_i),$$

$$\text{i.e., } \hat{p} \cdot \lambda(y_i - \hat{x}_i) + \hat{p} \cdot (\hat{x}_i - e_i) > 0 \text{ for any } y_i \in P_i(\hat{x}_i) \text{ and } \lambda > 0,$$

$$\text{i.e., } \hat{p} \cdot (z_i + \hat{x}_i - e_i) > 0 \text{ for any } z_i \in K_i(\hat{x}_i).$$

This implies that for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot w_i \geq 0$ for any $w_i \in L_i(\hat{x}_i)$, and that $\mathbf{0} \notin \operatorname{int} \sum_{i \in I^{NS}(\hat{\mathbf{x}})} L_i(\hat{x}_i)$.

Under assumption (R.2), by Minkowski's separation theorem we have a vector $\bar{p} \in \mathbf{R}^L$ with $\bar{p} \neq \mathbf{0}$ such that $\bar{p} \cdot z \geq 0$ for all $z \in \sum_{i \in I} L_i(\hat{x}_i)$. Namely,

$$\bar{p} \cdot \sum_{i \in I^{NS}(\hat{\mathbf{x}})} \lambda_i (z_i + \hat{x}_i - e_i) + \bar{p} \cdot \sum_{i \in I^{S}(\hat{\mathbf{x}})} \lambda_i (\hat{x}_i - e_i) \geq 0$$

for any $z_i \in K_i(\hat{x}_i)$ and $\lambda_i > 0$. Hence, for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\bar{p} \cdot (\hat{x}_i - e_i) \geq 0$ and $\bar{p} \cdot (y_i - e_i) \geq 0$ for any $y_i \in P_i(\hat{x}_i)$. Moreover, for each $i \in I^{S}(\hat{\mathbf{x}})$, $\bar{p} \cdot (\hat{x}_i - e_i) \geq 0$. Since $\sum_{i \in I} (\hat{x}_i - e_i) = 0$, $\hat{p} \cdot (\hat{x}_i - e_i) = 0$ for all $i \in I$. Thus, we conclude that $\{\hat{\mathbf{x}}, \bar{p}\}$ is a Walras equilibrium such that $\bar{p} \cdot \hat{x}_i = \bar{p} \cdot e_i$ for all $i \in I$. Thus, we have proved the following theorem.

**Theorem 4.** *Let* $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ *be a dividend equilibrium such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ *for all* $i \in I^{NS}(\hat{\mathbf{x}})$. *Then, under assumption (R.2), for some non-zero vector* $\overline{p} \in \mathbf{R}^L$, $\{\hat{\mathbf{x}}, \overline{p}\}$ *is a Walras equilibrium such that* $\overline{p} \cdot \hat{x}_i = \overline{p} \cdot e_i$ for all $i \in I$.

By Corollary 1 and Theorem 4 we have the following corollary.

**Corollary 2.** *Under assumptions (A.1)-(A.8) and (R.2), there exists a Walras equilibrium* $\{\hat{\mathbf{x}}, \hat{p}\}$ *such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ for all $i \in I$.

While the irreducibility condition is a relation among non-satiated consumers, each of conditions (R.0)-(R.2) describes a relation between satiated consumers and non-satiated consumers. However, we can combine conditions (R.0) – (R.2) to the irreducibility condition in the following way, which we call Generalized Irreducibility.

**Generalized Irreducibility.** Let $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$ and $j \in I$. If $I^{NS}(\mathbf{x}) \setminus \{j\} \neq \phi$, then there exist an allocation $\mathbf{y} = (y_1, \cdots, y_N) \in X_1 \times \cdots \times X_N$ and a scalar $\theta$ with $0 < \theta < 1$ such that

(i) $\theta(y_j - e_j) + \sum_{i \in I^S(\mathbf{x}) \setminus \{j\}} (x_i - e_i) + \sum_{i \in I^{NS}(\mathbf{x}) \setminus \{j\}} (y_i - e_i) = 0$,

(ii) $y_i \in P_i(x_i)$ for each $i \in I^{NS}(\mathbf{x}) \setminus \{j\}$,

(iii) $y_j = x_j$ when $j \in I^S(\mathbf{x})$.

We have the following lemma.

**Lemma 5.2.** *Generalized Irreducibility implies conditions (A.6) and (R.0).*

Thus, by this lemma and Theorem 3 (or Corollary 2), we have the following corollary.

**Corollary 3.** *Under assumptions (A.1)-(A.5), (A.7), (A.8), and Generalized Irreducibility, there exists a Walras equilibrium* $\{\hat{\mathbf{x}}, \hat{p}\}$ *such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ *for all* $i \in I$.

## 6. Concluding Remarks and Examples

As for the existence of dividend quasi-equilibrium, Theorem 1 is more general than the result by Allouch & Le Van (2008), (2009, Prop.1, p. 321) since we consider economies with consumers whose preferences are non-ordered. As for the existence of dividend equilibrium, Corollary 1 is more general than the results by Mas-Colell (1992, Thm.1, p.205) and Kajii (1996, Prop.1, p.79) & Le Van (2009, Prop.1, p. 321) since we consider irreducible economies in which consumers have not always positive incomes.

As for the existence of Walras quasi-equilibrium, Theorem 3 and Corollary 2 are neither a special case nor a general case of the results by Allouch & Le Van (2009, Thm.2, p. 323) or by Sato (2010, Thm.2, p541, Thm.3, p.543). While our assumption on consumers' preferences is weaker than their assumptions in the sense that consumers' preferences are non-ordered, the assumptions of non-satiation are quite different and cannot be compared directly with each other. However, our assumption admits that satiation generally occurs in the set of feasible allocations, and our theorem applies to a broader set of economies. In fact, the example of an economy by Sato (2010, Eg.1, p.537) satisfies our assumptions.

In comparison with the result by Won & Yannelis (2011, Thm.4.1, p.249) in which economies with non-ordered preferences are considered, our results are an extension of their result, since we use the assumptions of irreducibility and a weaker assumption of non-satiation. In what follows, we shall show two examples of exchange economy that satisfies weaker assumption of non-satiation than theirs.

In order to show that conditions (R.1) and (R.2) are different, first we shall show that the following example satisfies condition (R.2), but does not satisfy (R.1).

**Example 1.** The economy consists of two kinds of commodities and two consumers, i.e., $L$=2 and $I$={1, 2}. The utility function $U_i$ and the initial endowment $e_i$ of consumer $i$ are as follows:

Consumer 1: $U_1(c_1, c_2)=\min\{c_1, c_2\}$, $e_1=(3, 1)$

Consumer 2: $U_2(c_1, c_2) = -(c_1 - 2)^2 - (c_2 - 2)^2$, $e_2 = (1, 3)$.

Let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a triplet which is defined by

$$\hat{x}_1 = (2, 2), \ \hat{x}_2 = (2, 2), \ \hat{p} = (1-\delta, 1), \text{ and } \hat{d} = (\delta, \delta).$$

Then, $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend equilibrium when $0 \le \delta \le 1$. In particular, when $\delta = 0$, it is a Walras equilibrium that is unique.

In allocation $\hat{\mathbf{x}}$ of Example 1, consumer 2 is satiated, i.e., $I^{NS}(\hat{\mathbf{x}}) = \{1\}$ and $I^S(\hat{\mathbf{x}}) = \{2\}$. We can easily show the following:

$$K_1(\hat{x}_1) = \{(c_1, c_2) \mid c_1 > 0, c_2 > 0\}, \text{ and } \hat{x}_2 - e_2 = (1, -1) \notin \operatorname{cl} K_1(\hat{x}_1).$$

Therefore, condition (R.1) is not met. On the other hand, we can show the following:

$$L_1(\hat{x}_1) = \{(c_1, c_2) \mid c_1 + c_2 > 0, c_2 > 0\}, \ L_2(\hat{x}_2) = \{(c_1, c_2) \mid c_1 > 0, c_1 + c_2 = 0\},$$

$$L_1(\hat{x}_1) + L_2(\hat{x}_2) = \{(c_1, c_2) \mid c_1 + c_2 > 0\}, \text{ and } \mathbf{0} \notin \operatorname{int}[L_1(\hat{x}_1) + L_2(\hat{x}_2)].$$

Therefore, condition (R.2) is met. In any other allocation, no consumers are satiated, and condition (R.2) is automatically satisfied.

In the following second example, condition (R.1) is satisfied, but neither condition (R.0) nor (R.2) are met.

**Example 2.** Add one more consumer, ‘consumer 3’, to the economy of Example 1. The utility function $U_3$ and the initial endowment $e_3$ of consumer 3 are as follows:

Consumer 3: $U_3(c_1, c_2) = -(c_1 - 2)^2 - (c_2 - 2)^2$, $e_3 = (1, 1)$.

Let $\{\hat{\mathbf{x}}, \hat{p}\}$ be a pair which is defined by

$$\hat{x}_1 = (2, 2), \ \hat{x}_2 = (2, 2), \ \hat{x}_3 = (1, 1), \text{ and } \hat{p} = (1, 1).$$

Then, $\{\hat{\mathbf{x}}, \hat{p}\}$ is a Walras equilibrium that is unique.

To show that the economy of Example 2 satisfies condition (R.1), we need to verify that condition (R.1) is met for all feasible allocation of the economy.

(Case 1) Let $\mathbf{x} = (x_1, x_2, x_3)$ be any feasible allocation such that $I^S(\mathbf{x}) = \{2\}$, i.e., $x_1 = (x_{11}, x_{12})$, $x_2 = (2, 2)$, and $x_3 = (x_{31}, x_{32}) \ne (2, 2)$, where $x_{11} + x_{31} = 3$ and $x_{12} + x_{32} = 3$. Allocation $\hat{\mathbf{x}}$ of the Walras equilibrium $\{\hat{\mathbf{x}}, \hat{p}\}$ in Example 2 is a special case of such allocation $\mathbf{x}$. When $x_{11} < x_{12}$, it follows that

$$K_1(x_1) = \{(c_1, c_2) \mid c_1 > 0\}, \text{ and that } x_2 - e_2 = (1, -1) \in K_1(x_1).$$

In case of $x_{11} \ge x_{12}$, since $(2, 2) - x_3 = (x_{11} - 1, x_{12} - 1)$, it follows that

$K_3(x_3) = \{(c_1, c_2) \mid c_1(x_{11} - 1) + c_2(x_{12} - 1) > 0\}$, and that $x_2 - e_2 \in \mathrm{cl}\, K_3(x_3)$. Thus, $x_2 - e_2 \in \mathrm{cl}\,[K_1(x_1) + K_3(x_3)]$, and therefore condition (R.1) is met.

(Case 2) Let $\mathbf{x} = (x_1, x_2, x_3)$ be any feasible allocation such that $I^S(\mathbf{x}) = \{3\}$, i.e., $x_1 = (x_{11}, x_{12})$, $x_2 = (x_{21}, x_{22}) \neq (2, 2)$, and $x_3 = (2, 2)$, where $x_{11} + x_{21} = 3$ and $x_{12} + x_{22} = 3$. Since $x_3 - e_3 = (1, 1)$, it follows that $x_3 - e_3 \in K_1(x_1)$. Thus, $x_3 - e_3 \in \mathrm{cl}\,[K_1(x_1) + K_2(x_2)]$, and condition (R.1) is met in this case, too.

(Case 3) Let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be an allocation such that $I^S(\bar{\mathbf{x}}) = \{2, 3\}$, i.e., $\bar{x}_1 = (1, 1)$, $\bar{x}_2 = (2, 2)$, and $\bar{x}_3 = (2, 2)$. Since $K_1(\bar{x}_1) = \{(c_1, c_2) \mid c_1 > 0, c_2 > 0\}$, it follows that $(x_2 - e_2) + (x_3 - e_3) = (2, 0) \in \mathrm{cl}\, K_1(\bar{x}_1)$. Thus, in allocation $\bar{\mathbf{x}}$, condition (R.1) is met.

Hence, it has been shown that the economy of Example 2 satisfies condition (R.1). However, in allocation $\bar{\mathbf{x}}$, $\bar{x}_2 - e_2 = (1, -1) \notin \mathrm{cl}\, K_1(\bar{x}_1)$, which implies that Example 2 does not satisfy condition (R.0). Furthermore, we have the following.

$L_1(\bar{x}_1) = \{(c_1, c_2) \mid c_2 > 0\}$,

$\bar{x}_2 - e_2 = (1, -1)$, and $L_2(\bar{x}_2) = \{(c_1, c_2) \mid c_1 + c_2 = 0, c_1 > 0\}$,

$\bar{x}_3 - e_3 = (1, 1)$, and $L_3(\bar{x}_3) = \{(c_1, c_2) \mid c_1 = c_2 > 0\}$.

Therefore, $\mathbf{0} \notin \mathrm{int}\, L(\bar{x}_1)$ and $\mathbf{0} \in \mathrm{int}\,[L(\bar{x}_1) + L_2(\bar{x}_2) + L(\bar{x}_3)]$. Thus, in allocation $\bar{\mathbf{x}}$, condition (R.2) is not met. This implies that Example 2 does not satisfy condition (R.2).

## Appendix

**The fixed point theorem** [Gale=Mas-Colell (1975, 1979)]**.** If for each $i = 1, \cdots, M$ $X_i$ is a non-empty convex compact subset of $\mathbf{R}^L$ and $F_i : X_1 \times \cdots \times X_M \to 2^{X_i}$ is a convex-valued and lower hemi-continuous mapping, then there exists a point $(x_1^*, \cdots, x_M^*) \in X_1 \times \cdots \times X_M$ such that either $x_i^* \in F_i(x_1^*, \cdots, x_M^*)$ or $F_i(x_1^*, \cdots, x_M^*) = \phi$ for each $i = 1, \cdots, M$.

**Proof of Lemma 3.1.** Let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend quasi-equilibrium for the modified economy. Namely, the triplet $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ satisfies all the conditions of Definition 1 when each $P_i$ is replaced by $\overline{P}_i$. Since $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_N) \in \mathbf{A}$, by (A.4) we have $\| \hat{x}_i \| \leq \overline{b}$. By construction of $\overline{P}_i$, $\overline{P}_i(\hat{x}_i) = \phi$ if and only if $P_i(\hat{x}_i) = \phi$.

Now, suppose that $\hat{p} \cdot y^0 < \hat{p} \cdot e_i + \hat{d}_i$ for some $y^0 \in P_i(\hat{x}_i)$. Then, for all sufficiently small $\lambda > 0$, $\lambda(y^0 - \hat{x}_i) + \hat{x}_i \in \overline{P}_i(\hat{x}_i)$, and therefore, by the definition of quasi-equilibirum, we have an inequality,

$$\hat{p} \cdot [\lambda(y^0 - \hat{x}_i) + \hat{x}_i] \geq \hat{p} \cdot e_i + \hat{d}_i .$$

Since $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + \hat{d}_i$, by letting $\lambda \to 0$ in the above inequality we have $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$. Thus, the above inequality implies that $\hat{p} \cdot (y^0 - \hat{x}_i) \geq 0$, i.e., $\hat{p} \cdot y^0 \geq \hat{p} \cdot e_i + \hat{d}_i$, a contradiction. Hence, for each $i \in I$, $\hat{p} \cdot y \geq \hat{p} \cdot e_i + \hat{d}_i$ for all $y \in P_i(\hat{x}_i)$. This shows that $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend quasi-equilibrium for the original economy. □

**Proof of Lemma 3.2.** Let $(p^0, x^0) \in \mathbf{B} \times X_i$. The convexity of $F_i(p^0, x^0)$ immediately follows from the definition of $F_i$.

Assume that $y^0 \in F_i(p^0, x^0)$ and a sequence $\{(p^n, x^n)\}$ converges to $(p^0, x^0)$. In case that $p^0 \cdot x^0 > p^0 \cdot e_i + 1 - \| p^0 \|$, from the definition of $F_i$, it follows that $p^0 \cdot y^0 < p^0 \cdot x^0$. Therefore, for all $n$ sufficiently large, $p^n \cdot x^n > p^n \cdot e_i + 1 - \| p^n \|$ and $p^n \cdot y^0 < p^n \cdot x^n$. Define a sequence $\{y^n\}$ by letting $y^n = y^0$ for each $n$.

Then, for all $n$ sufficiently large, $p^n \cdot y^n < p^n \cdot x^n$, i.e., $y^n \in F_i(p^n, x^n)$.

In case that $p^0 \cdot x^0 \le p^0 \cdot e_i + 1 - \| p^0 \|$, from the definition of $F_i$, it follows that $y^0 \in \beta_i(p^0) \cap P_i(x^0)$. Since relations $P_i$ is lower hemi-continuous, there is a sequence $\{y^n\}$ converging to $y^0$ such that $y^n \in P_i(x^n)$ for all $n$ sufficiently large. In addition, since $y^0 \in \beta_i(p^0)$, $p^0 \cdot y^0 < p^0 \cdot e_i + 1 - \| p^0 \|$. Therefore, for all $n$ sufficiently large, $p^n \cdot y^n < p^n \cdot e_i + 1 - \| p^n \|$, i.e., $y^n \in \beta_i(p^n)$. Hence, $y^n \in \beta_i(p^n) \cap P_i(x^n)$ for all $n$ sufficiently large. This implies that $y^n \in F_i(p^n, x^n)$ when $p^n \cdot x^n \le p^n \cdot e_i + 1 - \| p^n \|$. On the other hand, when $p^n \cdot x^n > p^n \cdot e_i + 1 - \| p^n \|$, it follows that $\beta_i(p^n) \subset \{y_i \in X_i \mid p^n \cdot y_i < p^n \cdot x^n\}$, and therefore $y^n \in F_i(p^n, x^n)$. Thus, $y^n \in F_i(p^n, x^n)$ for all $n$ sufficiently large.

This proves the lower hemi-continuity of relation $F_i$. □

**Proof of Lemma 4.1.** Let $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$ and $j \in I^{NS}(\mathbf{x})$. Let $I_1 = \{j\}$ and $I_2 = I^{NS}(\mathbf{x}) \setminus \{j\}$. Then, Irreducibility implies that there is an allocation $\mathbf{z} = (z_1, \cdots, z_N) \in X_1 \times \cdots \times X_N$ such that

$$z_j - e_j + \sum_{i \in I^{NS}(\mathbf{x}) \setminus \{j\}} (z_i - x_i) = 0 \qquad \text{(L4.1.1)}$$

and

$$z_i \in P_i(x_i) \quad \text{for each} \quad i \in I^{NS}(\mathbf{x}) \setminus \{j\}. \qquad \text{(L4.1.2)}$$

Since $\sum_{i \in I} (x_i - e_i) = 0$, by (L4.1.1) we have

$$2\left(\frac{1}{2}(x_j + z_j) - e_j\right) + \sum_{i \in I^{NS}(\mathbf{x}) \setminus \{j\}} (z_i - e_i) + \sum_{i \in I^{S}(\mathbf{x})} (x_i - e_i) = 0.$$

Define an allocation $\mathbf{y} = (y_1, \cdots, y_N) \in X_1 \times \cdots \times X_N$ by:

$$y_j = \frac{1}{2}(x_j + z_j),$$

$$y_i = z_i \quad \text{for each} \quad i \in I^{NS}(\mathbf{x}) \setminus \{j\},$$

and

$$y_i = x_i \quad \text{for each} \quad i \in I^S(\mathbf{x}).$$

Note that $y_j \in X_j$, since $X_j$ is convex. Thus, if we put $\theta$=2, then allocation $\mathbf{y} = (y_1, \cdots, y_N)$ satisfies all the conditions in (A.6). □

**Proof of Lemma 4.2.** Let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend quasi-equilibrium. Define two sets in the following way:

$$I_1 := \{i \in I^{NS}(\hat{\mathbf{x}}) \mid \hat{p} \cdot e_i + \hat{d}_i = \inf \hat{p} \cdot X_i\}.$$

$$I_2 := \{i \in I^{NS}(\hat{\mathbf{x}}) \mid \hat{p} \cdot e_i + \hat{d}_i > \inf \hat{p} \cdot X_i\}.$$

Suppose that $I_2$ were non-empty. Choose $j \in I_1$. Then, (A.6) implies that there are an allocation $(y_1, \cdots, y_N) \in X_1 \times \cdots \times X_N$ and a scalar $\theta$ >0 such that

$$\theta(y_j - e_j) + \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} (y_i - e_i) + \sum_{i \in I^S(\hat{\mathbf{x}})} (\hat{x}_i - e_i) = 0 \tag{L4.2.1}$$

and $y_i \in P_i(\hat{x}_i)$ for each $i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}$. Since $\sum_{i \in I} (\hat{x}_i - e_i) = 0$, by (L4.2.1) we have

$$\theta(y_j - e_j) + (e_j - \hat{x}_j) + \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} (y_i - \hat{x}_i) = 0. \tag{L4.2.2}$$

On the other hand, from the definition of quasi-equilibrium, it follows that $\hat{p} \cdot y_i \geq \hat{p} \cdot e_i + \hat{d}_i \geq \hat{p} \cdot \hat{x}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$ and in particular, by (A.7), $\hat{p} \cdot y_i > \hat{p} \cdot e_i + \hat{d}_i$ for each $i \in I_2$. In addition, as for $j$, since $e_j \in X_j$, we have $\hat{d}_j = 0$ and $\hat{p} \cdot e_j = \hat{p} \cdot \hat{x}_j$. Hence, $\hat{p} \cdot (y_j - e_j) \geq 0$. Thus, we have

$$\theta \hat{p} \cdot (y_j - e_j) + \hat{p} \cdot (e_j - \hat{x}_j) + \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} \hat{p} \cdot (y_i - \hat{x}_i) \geq \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} \hat{p} \cdot (y_i - \hat{x}_i) > 0.$$

This is a contradiction to (L4.2.2). This shows that $I_2 \neq \phi$ implies that $I_1 = \phi$. □

**Proof of Lemma 5.1.** Let $\hat{\mathbf{x}} \in \mathbf{A}$ and assume that $\mathbf{0} \notin \operatorname{int} \sum_{i \in I^{NS}(\hat{\mathbf{x}})} L_i(\hat{x}_i)$. Then, by Minkowski's separation theorem, there is a vector $\hat{p} \in \mathbf{R}^L$ with $\hat{p} \neq \mathbf{0}$ such that $\hat{p} \cdot z \geq 0$ for all $z \in \sum_{i \in I^{NS}(\hat{\mathbf{x}})} L_i(\hat{x}_i)$. Therefore, for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot z_i \geq 0$ for any $z_i \in L_i(\hat{x}_i)$.

Furthermore, by the definition of $L_i(\hat{x}_i)$, we have that

$$\sum_{i \in I^{NS}(\hat{\mathbf{x}})} \hat{p} \cdot (z_i + \hat{x}_i - e_i) \geq 0 \quad \text{for any } z_i \in K_i(\hat{x}_i) \text{ with } i \in I^{NS}(\hat{\mathbf{x}}).$$

Since $\sum_{i \in I} (\hat{x}_i - e_i)$ =0, the above inequality implies that $\hat{p} \cdot z \geq \sum_{i \in I^S(\hat{\mathbf{x}})} \hat{p} \cdot (\hat{x}_i - e_i)$ for any $z \in K(\hat{\mathbf{x}})$. Hence, by (R.0), since $K(\hat{\mathbf{x}})$ is a cone, we have that for each $j \in I^S(\hat{\mathbf{x}})$

$$\hat{p} \cdot \lambda(\hat{x}_j - e_j) \geq \sum_{i \in I^S(\hat{\mathbf{x}})} \hat{p} \cdot (\hat{x}_i - e_i) \quad \text{for any } \lambda>0.$$

By letting $\lambda$ be a number greater than $\# I^S(\hat{\mathbf{x}})$ and adding the above inequalities with respect to $j \in I^S(\hat{\mathbf{x}})$, we can conclude that $\sum_{i \in I^S(\mathbf{x})} \hat{p} \cdot (\hat{x}_i - e_i) \geq 0$. Therefore, the above inequality implies that, for each $i \in I^S(\hat{\mathbf{x}})$, $\hat{p} \cdot \lambda(\hat{x}_i - e_i) \geq 0$ for any $\lambda$>0. Thus, for each $i \in I^S(\hat{\mathbf{x}})$, $\hat{p} \cdot z_i \geq 0$ for any $z_i \in L_i(\hat{x}_i)$.

Thus, we have proved that $\mathbf{0} \notin \operatorname{int} \sum_{i \in I} L_i(\hat{x}_i)$. □

**Proof of Lemma 5.2.** Obviously, Generalized Irreducibility implies condition (A.6). To prove (R.0), let $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$ and $j \in I^S(\mathbf{x})$. Then, by Generalized Irreducibility, there are $y_i \in P_i(x_i)$ for each $i \in I^{NS}(\mathbf{x})$ and a scalar $\theta$ with $0 < \theta < 1$ such that such that

$$\theta(x_j - e_j) + \sum_{i \in I^S(\mathbf{x})\setminus\{j\}} (x_i - e_i) + \sum_{i \in I^{NS}(\mathbf{x})} (y_i - e_i) = 0.$$

Since $\sum_{i \in I} (x_i - e_i) = 0$, we have $-(1-\theta)(x_j - e_j) + \sum_{i \in I^{NS}(\mathbf{x})} (y_i - x_i) = 0$. This implies that $(x_j - e_j) \in \sum_{i \in I^{NS}(\mathbf{x})} K_i(x_i)$. This proves that condition (R.0) holds. □

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