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by

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Abstract

We present a classification of all stationary subgame perfect equilibria of the random proposer model for a three-person cooperative game according to the level of efficiency. The efficiency level is characterized by the number of "central" players who join all equilibrium coalitions. The existence of a central player guarantees asymptotic efficiency. The marginal contributions of players to the grand coalition play a critical role in their expected equilibrium payoffs.

JEL classification: C71, C72, C78
Key words: cooperative game, noncooperative bargaining, three-person game, random proposer, core, marginal contribution

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1 Introduction

The three-person cooperative game with side payments in characteristic-function form is a classical problem of game theory in which three players negotiate about coalition formation and payoff allocations. The game serves as a prototype for the economic analysis of efficiency and equity in resource allocation. Since von Neumann and Morgenstern (1944), various kinds of solutions have been proposed in the literature on cooperative game theory. There has been no consensus among game theorists about what is an appropriate solution for a three-person game and for an n-person cooperative game, in general. This disagreement remains to the present day. It may be argued that the diversity of solutions is a virtue, reflecting the complexity of the real world. However, to apply cooperative game theory to economic analysis, we need a general understanding of when one solution is more suitable than others.

In the last two decades, the noncooperative-game approach to cooperative games has been rapidly growing. Cooperation has been analyzed as a noncooperative equilibrium under a specified procedure of coalition formation. Ray (2007) has provided an excellent review of this field. Among the bargaining games studied well is the random proposer model (Baron and Ferejohn 1989; Okada 1996). The stationary subgame perfect equilibrium (SSPE) of the model provides a noncooperative foundation for various cooperative solutions, including the Nash bargaining solution (Okada 2010), the coalitional Nash bargaining solution (Compte and Jehiel 2010), the core (Yan 2002) for a general game, and the kernel (Montero 2002) and nucleolus (Montero 2006) for a weighted majority game.

The aim of this paper is to characterize all SSPEs of a three-person superadditive game with general parameters played by patient players. To our knowledge, the full structure of the SSPEs of a three-person random proposer game has not yet been reported in the literature.[1] In particular, when a game has an empty core, it is well known that an SSPE outcome must be inefficient,[2] whereas almost all cooperative solutions presume efficiency even in such a case. A complete analysis of a three-person cooperative game helps us to understand why and how efficiency and/or inequality may occur in negotiations among rational players under the condition of complete information.

We consider all SSPEs of a three-person game in terms of the support of every player's mixed strategy, i.e., the set of all coalitions that the player may choose with positive probability. There are 343 possible configurations of supports for players' strategies. These configurations can be classified into different levels of ef-

[1]Recently, Nash (2008) considered a noncooperative bargaining model called the agencies method for a three-person cooperative game and presented some computational results.

[2]It is also well known that an efficient allocation is guaranteed if renegotiation is allowed. See Seidmann and Winter (1998) and Okada (2000), among others.

ficiency, measured by the equilibrium probability of the grand coalition. In the one efficient SSPE, the grand coalition forms with probability one. In an asymptotically efficient SSPE (Compte and Jehiel 2010), the grand coalition will form almost surely. In an inefficient SSPE, the probability of the grand coalition is less than one, and may possibly be zero. We show that the existence of a "central" player who joins all equilibrium coalitions guarantees efficiency (in one configuration) and asymptotic efficiency (in 36 configurations). An inefficient SSPE arises when the core of a game is empty. When the grand coalition may form with positive probability (162 configurations), the expected payoffs of the players are equal to their marginal contributions to the grand coalition except in three configurations.

This paper is organized as follows. Section 2 gives some definitions. Section 3 provides several lemmas useful for the analysis. Section 4 presents a classification of SSPEs. Section 5 concludes the paper.

2 Preliminaries

An n-person game in coalitional form with transferable utility is represented as a pair (N, v), where $N = \{1, 2, \cdots, n\}$ is the set of players. A nonempty subset S of N is called a *coalition* of players. The number of members of S is denoted by s. The *characteristic function* v is a real-valued function that assigns to each coalition S its value $v(S)$. It is assumed that v satisfies (i) $v(\{i\}) = 0$ for all $i \in N$ (zero-normalized), (ii) $v(S \cup T) \geq v(S) + v(T)$ for any two disjoint coalitions S and T (superadditive), and (iii) $v(N) > v(S)$ for every $S \subset N, S \neq N$. The last condition is a regularity one that guarantees that only the grand coalition N maximizes the total value. For $S \subset N$, let R^S denote an s-dimensional Euclidean space with coordinates indexed by the elements of S. Each point in R^S is denoted by $x^S = (x_i^S)_{i\in S}$.

The payoff allocation for a coalition S is a vector $x^S = (x_i^S)_{i\in S}$ of R^S, where x_i^S represents the payoff for player $i \in S$. A payoff allocation x^S for S is *feasible* if $\sum_{i\in S} x_i^S \leq v(S)$. Let X^S denote the set of all feasible payoff allocations for S, and let X_+^S denote the set of all elements in X^S with nonnegative components. For $S \subset N$ and $x \in R^N$, the *excess* of S with respect to x is defined by $e(S, x) = v(S) - \sum_{i\in S} x_i$. For $i \in N$, $m_i = v(N) - v(N - \{i\})$ is player i's *marginal contribution* to the grand coalition N.

As a noncooperative bargaining procedure for a game (N, v), we consider the random proposer model with recognition probability $p = (p_1, \cdots, p_n)$, where $p_i > 0$ for every i. The bargaining rule is simple. Negotiations take place over a possibly infinite number of rounds t $(= 1, 2, \cdots)$ until an agreement is reached. At the start of each round t, one player $i \in N$ is randomly selected as a proposer with probability p_i. Player i proposes a coalition S, with $i \in S$ and a payoff allocation

$x^S \in X^S_+$. All other members of S either accept or reject the proposal (S, x^S) sequentially. The order of responders does not affect the result in any critical way. If all responders accept the proposal, then the game ends with the agreement (S, x^S).[3] All members i of S receive payoffs x^S_i, and the others receive zero payoffs. Otherwise, negotiations continue in the next round $t+1$ with the same rule as in round t. If the game does not stop, all players receive zero payoffs. Let $\delta (0 \leq \delta < 1)$ be the common discount factor for future payoffs. All players have perfect information about the history of play whenever they choose their actions. The bargaining game above is denoted by Γ^δ. The notation Γ denotes the limit as the discount factor δ goes to one in Γ^δ.

A *(behavior) strategy* σ_i for player i in Γ^δ (and also in Γ) is a function that assigns a randomized (mixed) action to every possible move of the player, depending on the history of the game. Under the standard assumptions about v above, it is well known that all responders accept a proposal with probability one in every SSPE of Γ^δ (see Lemma 3.1). A randomized action may occur only in proposing coalitions. For a strategy combination $\sigma = (\sigma_1, \cdots, \sigma_n)$, the expected (discounted) payoff for player i in Γ^δ is defined in the usual way. A strategy σ_i for player i is *stationary* if the (possibly mixed) action of player i at round t is independent of the history before round t.[4] In what follows, the analysis is restricted to a *stationary subgame perfect equilibrium*, in which the equilibrium strategy of every player is stationary.

For an SSPE σ and for every $i \in N$, let v_i be player i's expected payoff; let q_i be the probability distribution over the set of all coalitions S with $i \in S$; let C_i be the *support* of q_i, which is the set of all coalitions including i that q_i assigns a strictly positive probability to; and let θ_i be the conditional probability that player i receives an offer from some other player, given that player i becomes a responder. Note that $\theta_i = (1/(1-p_i)) \sum_{j \in N, j \neq i} p_j \sum_{S: i \in S \in C_j} q_j(S)$. We call the profile $\phi = (v_i, q_i, C_i, \theta_i)_{i \in N}$ the *configuration* of the SSPE σ. Whenever we want to emphasize the dependence of elements of ϕ on δ, we shall add δ to them, as in "v^δ_i" and "q^δ_i." In the following, the collection $(C_i)_{i \in N}$ of supports plays an important role; we call it the *support configuration* of σ.

Definition 2.1. Let $\sigma = (\sigma_1, \cdots, \sigma_n)$ be an SSPE of Γ^δ with a configuration $\phi = (v_i, q_i, C_i, \theta_i)_{i \in N}$.

(1) An SSPE σ of Γ^δ is *efficient* if $\sum_{i \in N} v_i = v(N)$.

(2) A strategy combination $\sigma^* = (\sigma^*_1, \cdots, \sigma^*_n)$ of Γ is an *asymptotically efficient*

[3] Since our aim is to analyze a three-person bargaining game, we assume that only one coalition may form. A general rule used in Okada (1996) allows sequential formation of coalitions.

[4] The players' responses depend surely on the proposal in the present round.

equilibrium with limit payoff $v^* = (v_1^*, \cdots, v_n^*)$ if there exists a sequence $\{\sigma^\delta\}$ of SSPEs of Γ^δ such that the expected payoffs v^δ of σ^δ converge to v^* as δ goes to one, and if $\sum_{i \in N} v_i^* = v(N)$.

(3) An SSPE σ of Γ^δ is *subcoalition-inefficient* if the probability of the grand coalition is zero.

(4) Player i is a *central player* in σ if $\theta_i = 1$, that is, $i \in S$ for every $S \in C_j$ and every $j \in N, j \neq i$.

In the case of the random proposer model Γ^δ, it is known that there is no delay in the agreements in any of the SSPEs whenever $\delta < 1$ (see Lemma 3.1). Because of this fact, the efficiency of an SSPE is determined solely by coalitions formed in equilibrium. Under the regularity assumption $v(N) > v(S)$ for every $S \subset N, S \neq N$, an SSPE is efficient if and only if the grand coalition N forms with probability one. Thus, an efficient SSPE must be the *grand-coalition* SSPE. In an inefficient SSPE, the probability of the grand coalition is strictly smaller than one. The notion of asymptotic efficiency introduced by Compte and Jehiel (2010) describes a situation where the probability of the grand coalition becomes almost equal to one as players become sufficiently patient. Compte and Jehiel proved that the limit payoff in an asymptotically efficient equilibrium is equal to the coalitional Nash bargaining solution (the core allocation maximizing the Nash product). Whereas an efficient SSPE is given by nonrandomized (pure) strategies, an asymptotically efficient equilibrium is genuinely supported by (a sequence of) mixed strategies of players when $\delta < 1$. The probability of any subcoalition S with $v(S) < v(N)$ converges to zero as δ goes to one. In the limit in which δ becomes close to one, the asymptotically efficient equilibrium provides an efficient allocation of payoffs. The limit of the efficient SSPE as $\delta \to 1$ is obviously asymptotically efficient.

In the next section, we shall show that the existence of a central player guarantees asymptotic efficiency of an SSPE. A central player is a player who joins a coalition with probability one. In the efficient SSPE, all players are central. The inefficient SSPEs are divided into two types, according to whether or not the probability of the grand coalition is zero. In a subcoalition-inefficient SSPE, the grand coalition never forms. The Baron and Ferejohn (1989) equilibrium in the majority game is an extreme case of this situation.

3 Lemmas

Here, we present several basic properties of an SSPE that are useful for our analysis. First, we review some known results in the literature (Okada 1996, 2011).

Lemma 3.1.

(1) An SSPE of Γ^δ in behavior strategies exists for every δ ($0 \le \delta < 1$).

(2) For every SSPE σ of Γ^δ, every proposal is accepted in the initial round. In the proposal, all responders j are offered their discounted expected payoffs δv_j.

(3) A strategy combination $\sigma = (\sigma_1, \cdots, \sigma_n)$ is an SSPE of Γ^δ if and only if its configuration $\phi = (v_i, q_i, C_i, \theta_i)_{i \in N}$ satisfies the following conditions, for every $i \in N$:

(i) Every $S \in C_i$ (i.e., $q_i(S) > 0$) is a solution of

$$\max_{i \in T \subset N} \left(v(T) - \sum_{j \in T, j \neq i} \delta v_j \right). \tag{3.1}$$

(ii) $v_i \in R_+$ satisfies

$$v_i = p_i \max_{i \in T \subset N} \left(v(T) - \sum_{j \in T, j \neq i} \delta_j v_j \right) + (1 - p_i)\theta_i \delta_i v_i. \tag{3.2}$$

(4) The grand-coalition SSPE exists if and only if $v(N) \ge v(S)/(1 - \delta \sum_{j \in N-S} p_j)$ for every $S \subset N$. In equilibrium, every player $i \in N$ receives the expected payoff $v_i = p_i v(N)$.

In what follows, we call (3.1) the *optimality condition* of and (3.2) the *payoff equation* of an SSPE.

The grand-coalition (efficient) SSPE is fully characterized by Lemma 3.1(3) for an n-person cooperative game. The next lemma characterizes an asymptotically efficient equilibrium.

Lemma 3.2. Let σ^* be a strategy combination for Γ. If there exists some sequence $\{\sigma^\delta\}$ of SSPEs in Γ^δ such that every σ^δ has at least one central player and $\{\sigma^\delta\}$ converges to σ as $\delta \to 1$, then the following properties hold.

(1) σ^* is an asymptotically efficient equilibrium of Γ.

(2) The limit payoff $v^* = (v_1^*, \cdots, v_n^*)$ of σ^* belongs to the core of (N, v), and $\sum_{i \in S} v_i^* = v(S)$ holds for every S that may form with positive probability in σ^δ for any sufficiently large δ.

(3) For any central player k in σ^δ where δ is sufficiently large, $v_k^* \ge p_k v(N)$.

Proof. (1) Let $\phi^\delta = (v_i^\delta, q_i^\delta, C_i^\delta, \theta_i^\delta)_{i\in N}$ be the configuration of an SSPE σ^δ. We shall omit δ in the elements of ϕ^δ whenever no confusion will arise. For every $i \in N$ and every $S_i \in C_i$, $q_i(S_i)$ denotes the positive probability that player i chooses S_i in σ^δ. Let $x^i(S_i) = (x_j^i(S_i))_{j\in N} \in X_+^N$ be the payoff allocation when player i proposes to S_i. Note that $\sum_{j\in N} x_j^i(S_i) = v(S_i)$. It then holds that

$$\begin{aligned}\sum_{i\in N} v_i &= \sum_{i\in N}\sum_{j\in N} p_j \sum_{S_j\in C_j} q_j(S_j)x_i^j(S_j) = \sum_{j\in N} p_j \sum_{S_j\in C_j} q_j(S_j)\sum_{i\in N} x_i^j(S_j)\\ &= \sum_{j\in N} p_j \sum_{S_j\in C_j} q_j(S_j)v(S_j). \end{aligned}\tag{3.3}$$

Let $k \in N$ be any central player in σ^δ.[5] By definition, $\theta_k = 1$. Let $S_k \in C_k$. It follows from the payoff equation (3.2) that

$$v_k = p_k\left(v(S_k) - \sum_{j\in S_k, j\neq k} \delta v_j\right) + (1-p_k)\delta v_k.$$

This can be rewritten as

$$(1-\delta)v_k = p_k\left(v(S_k) - \sum_{j\in S_k} \delta v_j\right). \tag{3.4}$$

It follows from the optimality condition (3.1) that

$$v(S_k) - \sum_{j\in S_k}\delta v_j \geq v(N) - \sum_{j\in N}\delta v_j. \tag{3.5}$$

Noting (3.3), it follows from (3.4) and (3.5) that

$$(1-\delta)v_k \geq p_k\left(v(N) - \delta\sum_{j\in N} p_j \sum_{S_j\in C_j} q_j(S_j)v(S_j)\right).$$

This can be rewritten as

$$v_k \geq \frac{p_k}{1-\delta}\sum_{j\in N} p_j \sum_{S_j\in C_j} q_j(S_j)(v(N) - \delta v(S_j)). \tag{3.6}$$

By way of contradiction, suppose that σ^* is not asymptotically efficient. Then, there exists some $j \in N$ and some $S_j \in C_j, S_j \neq N$, such that

$$\lim_{\delta\to 1} q_j^\delta(S_j) > 0.$$

[5] By choosing a subsequence (if necessary), we can assume without loss of generality that every σ^δ has at least one central player in common.

Since S_j is a proper subset of N, $v(N) > v(S_j)$ by assumption. The right-hand side of (3.6) then becomes infinite as $\delta \to 1$. This contradicts the assertion that v_k^* is bounded from above. This proves (1).

(2) Since σ^* is asymptotically efficient by (1), its limit payoff v^* satisfies $\sum_{i \in N} v_i^* = v(N)$. Since $q_i^\delta(N) > 0$ for every $i \in N$ and every sufficiently large δ, the optimality condition (3.1) for an SSPE σ^δ implies

$$v(N) - \sum_{j \in N} \delta v_j^\delta \geq v(S) - \sum_{j \in S} \delta v_j^\delta \tag{3.7}$$

for every $S \subset N$. As $\delta \to 1$, (3.7) implies that

$$0 \geq v(S) - \sum_{j \in S} v_j^*.$$

Thus, $v^* = (v_1^*, \cdots, v_n^*)$ belongs to the core of (N, v). If the coalition S is proposed with positive probability in σ^δ for sufficiently large δ, equality holds in (3.7) by the optimality condition (3.1). Thus, as $\delta \to 1$ in (3.7), we obtain $\sum_{i \in S} v_i^* = v(S)$.

(3) Finally, it follows from (3.6) that $v_k^\delta \geq p_k v(N)$ for every δ. This proves that $v_k^* \geq p_k v(N)$. Q.E.D.

Lemma 3.2 is closely related to the results of Compte and Jehiel (2010, Proposition 1 and Claim C). These authors showed that if there exists some "key" player who belongs to all binding coalitions in the coalitional Nash bargaining solution for some reduction parameter of v, then a sufficient condition (called P1) for asymptotic efficiency holds whenever the core is nonempty. In contrast to their approach, we define a central player in terms of the support configuration of an SSPE, and give a direct proof that the existence of a central player guarantees asymptotic efficiency. This approach enables us to classify all SSPEs for $n = 3$ according to the number of central players.

The final lemma presented here demonstrates some properties of the excess of a coalition with respect to the supports of an SSPE.

Lemma 3.3. Let σ be an SSPE of Γ^δ with expected payoffs v_i and supports C_i for all $i \in N$. For $S \subset N$, let $e(S, \delta v)$ be the excess of S with respect to $\delta v = (\delta v_1, \cdots, \delta v_n)$.

(1) For all S and T in C_i, $e(S, \delta v) = e(T, \delta v)$.

(2) For $j \in S \in C_i$ and $i \in T \in C_j$, $e(S, \delta v) = e(T, \delta v)$.

(3) If there exists an order $(i_1, \cdots, i_n)$ over N such that every player i's support C_i includes some S_i satisfying

$$i_j \in S_{i_{j-1}} \cap S_{i_j} \quad \text{for all} \quad j = 1, \cdots, n,$$

(with $i_0 = i_n$), then $e(S_{i_1}, \delta v) = \cdots = e(S_{i_n}, \delta v)$.

Proof. All of these results follow from the optimality condition (3.1) for an SSPE σ given in Lemma 3.1.

(1) Since $S, T \in C_i$, we have

$$v(S) - \sum_{j \in S, j \neq i} \delta v_j = v(T) - \sum_{j \in T, j \neq i} \delta v_j.$$

This yields $e(S, \delta v) = e(T, \delta v)$.

(2) Since $S \in C_i$ and $i \in T$, we have

$$v(S) - \sum_{j \in S, j \neq i} \delta v_j \geq v(T) - \sum_{j \in T, j \neq i} \delta v_j.$$

This yields $e(S, \delta v) \geq e(T, \delta v)$. Similarly, since $T \in C_j$ and $j \in S$, we have $e(T, \delta v) \geq e(S, \delta v)$. Thus, $e(S, \delta v) = e(T, \delta v)$.

(3) Since $i_j \in S_{i_{j-1}}$ and $S_{i_j} \in C_{i_j}$, it holds that $e(S_{i_{j-1}}, \delta v) \leq e(S_{i_j}, \delta v)$. By varying j from 1 to n, we have

$$e(S_{i_n}, \delta v) \leq e(S_{i_1}, \delta v) \leq \cdots \leq e(S_{i_n}, \delta v).$$

This proves (3). Q.E.D.

4 A Classification of SSPEs: $n = 3$

We can classify all SSPEs in a three-person cooperative game according to their support configurations $C = (C_1, C_2, C_3)$. Table 4.1 gives a list of all seven possible supports for each player's equilibrium strategy.[6] There are 353 ($= 7 \times 7 \times 7$) possible support configurations. We characterize the limit payoff of an SSPE for each configuration of supports as the discount factor δ converges to one. For the sake of analysis, we assume the uniform distribution $(1/3, 1/3, 1/3)$ for the recognition probabilities p_i for each player $i = 1, 2, 3$. A similar analysis can be applied to a general distribution.

[6] We have simplified the set notation $\{1, 2, 3\}$ to 123 in Table 4.1. Similar notation is used in this section.

Table 4.1 List of all possible supports of three players' strategies

C_1			C_2			C_3		
123,	12,	13	123,	12,	23	123,	13,	23
123,	12		123,	12		123,	13	
123,		13	123,		23	123,		23
	12,	13		12,	23		13,	23
123			123			123		
	12			12			13	
		13			23			23

We classify all possible support configurations into four cases, according to the number of central players.

Case 1. All three players are central, i.e., $C_1 = C_2 = C_3 = \{123\}$ (one type).

In this case, the SSPE is the grand-coalition SSPE characterized by Lemma 3.1(4). The grand-coalition SSPE exists if and only if $v(123) \geq (3/(3-\delta))v(S)$ for every two-person coalition S. The expected payoff v_i of every player $i = 1, 2, 3$ is $v(123)/3$. As the discount factor δ goes to one, the equilibrium allocation converges to the equity allocation $(v(123)/3, v(123)/3, v(123)/3)$, regardless of the proposer. The equity allocation must belong to the core, i.e., $v(123)/3 \geq v(S)/2$ for every two-person coalition S.

Case 2. Only two players are central (nine types).

Table 4.2 shows a list of all possible configurations of supports when only players 1 and 2 are central. Notice that each player's support C_i must include the grand coalition 123, since the SSPE is asymptotically efficient by Lemma 3.2. In Table 4.2, the configuration $C_1 = C_2 = C_3 = \{123\}$ considered in case 1 is excluded. Thus, there are three types: (i) $C_1 = C_2 = \{123, 12\}, C_3 = \{123\}$, (ii) $C_1 = \{123, 12\}, C_2 = C_3 = \{123\}$, and (iii) $C_2 = \{123, 12\}, C_1 = C_3 = \{123\}$. In total, there are nine types of configurations, considering permutations of players.

Table 4.2 List of players' supports when only players 1 and 2 are central ($C_1 = C_2 = C_3 = \{1, 2, 3\}$ is excluded)

C_1		C_2		C_3
123,	12	123,	12	123
123		123		

In each type of SSPE, the payoff equation (3.2) for $i = 1, 2, 3$ gives

$$3v_1 = v(123) - \delta v_2 - \delta v_3 + 2\delta v_1, \tag{4.1}$$
$$3v_2 = v(123) - \delta v_1 - \delta v_3 + 2\delta v_2, \tag{4.2}$$
$$3v_3 = v(123) - \delta v_1 - \delta v_2 + 2\delta\theta_3 v_3, \tag{4.3}$$

where θ_3 is the conditional probability that player 3 joins a coalition, given that that player becomes a responder. Equations (4.1) and (4.2) solve $v_1 = v_2$ for any $\delta < 1$. Since $123, 12 \in C_1$ or C_2, Lemma 3.3(1) implies $e(123, \delta v) = e(12, \delta v)$. This yields

$$v_3 = \frac{v(123) - v(12)}{\delta}. \tag{4.4}$$

Equations (4.1) and (4.4) with $v_1 = v_2$ solve

$$v_1 = v_2 = \frac{v(12)}{3 - \delta}. \tag{4.5}$$

Thus, the limit payoff $v^* = (v_1^*, v_2^*, v_3^*)$ of an SSPE as $\delta \to 1$ is given by

$$v_1^* = v_2^* = \frac{v(12)}{2}, \quad v_3^* = v(123) - v(12). \tag{4.6}$$

The optimality conditions (3.1) for $i = 1$ and 2 imply $e(123, \delta v) \geq e(13, \delta v)$ and $e(123, \delta v) \geq e(23, \delta v)$, respectively. As $\delta \to 1$, these conditions yield

$$\frac{v(12)}{2} \leq v(123) - v(13), \ \frac{v(12)}{2} \leq v(123) - v(23). \tag{4.7}$$

Substituting (4.4) and (4.5) into (4.3) solves

$$\theta_3 = \frac{1}{2(v(123) - v(12))} \left[\frac{3 - \delta}{\delta} v(123) - \frac{9 - 3\delta - 2\delta^2}{\delta(3 - \delta)} v(12) \right].$$

¿From $\theta_3 < 1$, this yields $((3 - \delta)/3)v(123) < v(12)$. As $\delta \to 1$, we obtain

$$v(123) - v(12) \leq \frac{v(123)}{3}. \tag{4.8}$$

In summary, (4.6), (4.7), and (4.8) show the following bargaining outcome in the limit in which the discount factor δ is almost equal to one. When the equity allocation does not belong to the core, player 3, whose marginal contribution $v(123) - v(12)$ is smaller than the equity allocation $v(123)/3$, receives not $v(123)/3$ but that player's marginal contribution. The two other players, 1 and 2, split the surplus $v(12)$ equally. They are central players; that is, all equilibrium coalitions

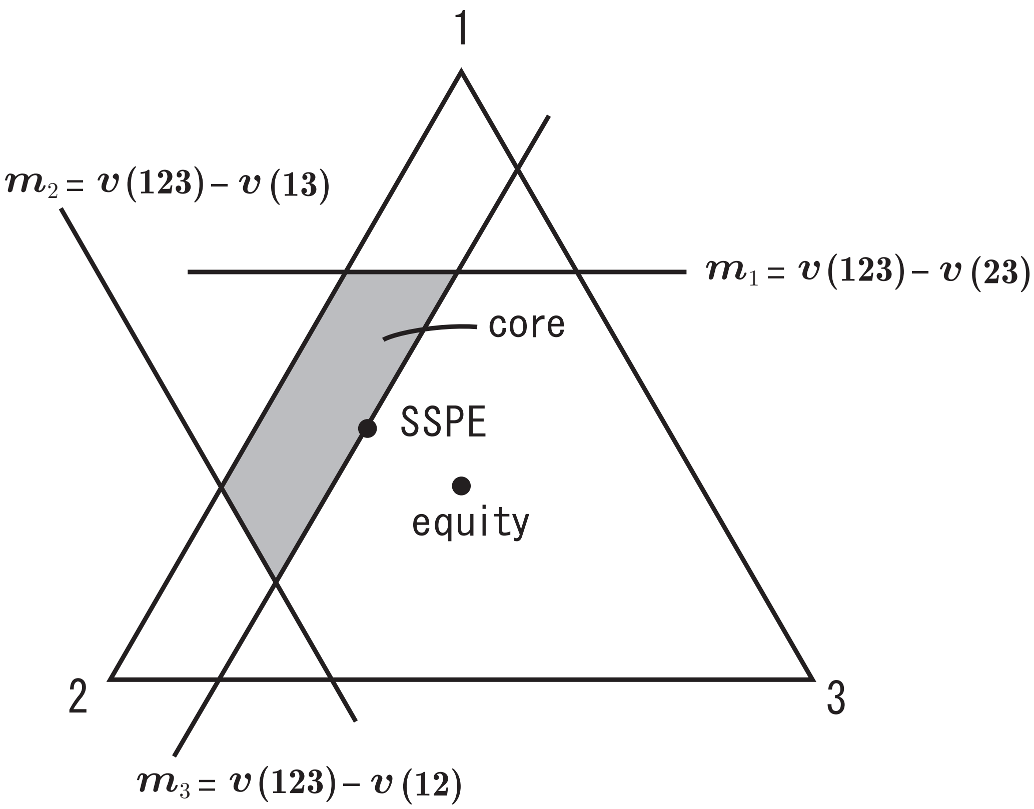

Figure 4.1 The SSPE allocation with two central players.

include them. The limit payoff allocation (4.6) is equal to the coalitional Nash bargaining solution of Compte and Jehiel (2010). Figure 4.1 illustrates the payoff allocation.

Case 3. Only one player is central (27 types).

Table 4.3 shows a list of all possible configurations of supports when only player 1 is central. Notice that the support of each player must include the grand coalition 123, since the SSPE is asymptotically efficient. There are nine possible configurations in this subcase.[7]

Table 4.3 All configurations of players' supports where only player 1 is central

C_1	C_2, C_3	
$C_1 \ni 123$	$C_2 = \{123, 12\}$,	$C_3 = \{123, 13\}$
$C_1 \ni 123, 13$	$C_2 = \{123, 12\}$,	$C_3 = \{123\}$
$C_1 \ni 123, 12$	$C_2 = \{123\}$,	$C_3 = \{123, 13\}$
$C_1 = \{123, 12, 13\}$	$C_2 = \{123\}$,	$C_3 = \{123\}$

In all nine possible configurations in Table 4.3, we have $123, 12 \in C_i$ and $123, 13 \in C_j$ for some $i, j \in N$ (including the case $i = j$). It then follows from Lemma 3.3 that $e(123, \delta v) = e(12, \delta v) = e(13, \delta v)$. These equations yield

$$\delta v_2 = v(123) - v(13), \tag{4.9}$$

$$\delta v_3 = v(123) - v(12). \tag{4.10}$$

Since player 1 is a central player, the payoff equations for an SSPE yield (4.1), (4.3), and

$$3v_2 = v(123) - \delta v_1 - \delta v_3 + 2\delta\theta_2 v_2 \tag{4.11}$$

(instead of (4.2)).

¿From (4.1), (4.9), and (4.10), it follows that

$$v_1 = \frac{v(12) + v(13) - v(123)}{3 - 2\delta}. \tag{4.12}$$

It can be seen without much difficulty that (4.11) and $\theta_2 < 1$ imply

$$(3 - \delta)v(123) < \delta v(12) + (3 - \delta)v(13). \tag{4.13}$$

[7] In Table 4.3, the first row shows that the support C_1 of player 1 must include the grand coalition 123. There are four possible configurations in this case. The second and third rows contain two possible configurations each. In total, there are 9 (= 4 + 2 + 2 + 1) possible configurations.

Similarly, it follows from (4.3) and $\theta_3 < 1$ that

$$(3-\delta)v(123) < (3-\delta)v(12) + \delta v(13). \tag{4.14}$$

For $\delta < 1$, the optimality condition (3.1) implies

$$v(123) - \delta v_1 - \delta v_3 \geq v(23) - \delta v_3.$$

Substituting (4.12) into this inequality, we have

$$(3-\delta)v(123) \geq \delta v(12) + \delta v(13) + (3-2\delta)v(23). \tag{4.15}$$

Finally, as $\delta \to 1$, we obtain the limit payoff of an SSPE as

$$v_1^* = v(12) + v(13) - v(123), \quad v_2^* = v(123) - v(13), \quad v_3^* = v(123) - v(12) \tag{4.16}$$

from (4.9), (4.10), and (4.12), and obtain

$$v(123) - v(13) \leq \frac{v(12)}{2}, \tag{4.17}$$
$$v(123) - v(12) \leq \frac{v(13)}{2}, \tag{4.18}$$
$$2v(123) \geq v(12) + v(13) + v(23) \tag{4.19}$$

from (4.13), (4.14), and (4.15), respectively. Equation (4.19) is a well-known condition for the core to be nonempty in a three-person game. Under (4.17) and (4.18), the limit payoff (4.16) becomes the coalitional Nash bargaining solution.

The last two cases, 4 and 5, without central players, correspond to a game with an empty core. Case 4 deals with the case where the grand coalition may form with positive probability. Case 5 examines a subcoalition-inefficient SSPE where the grand coalition never forms.

Case 4. An SSPE without central players where the grand coalition may form with positive probability (162 types).

We shall show that all support configurations satisfy

$$e(123, \delta v) = e(12, \delta v) = e(23, \delta v) = e(13, \delta v), \tag{4.20}$$

except for the three configurations

$$C_1 = \{123, 12, 13\},\ C_2 = \{23\},\ C_3 = \{23\}, \tag{4.21}$$
$$C_1 = \{13\},\ C_2 = \{123, 12, 23\},\ C_3 = \{13\}, \tag{4.22}$$
$$C_1 = \{12\},\ C_2 = \{12\},\ C_3 = \{123, 13, 23\}. \tag{4.23}$$

Without any loss of generality, we assume $123 \in C_1$, and shall show that (4.20) holds for all possible configurations except (4.21). Equations (4.22) and (4.23) are obtained by similar arguments in other cases, where $123 \in C_2$ and $123 \in C_3$, respectively. Consider the following four cases.

(i) $C_1 = \{123, 12, 13\}$. In this case, $e(123, \delta v) = e(12, \delta v) = e(13, \delta v)$ by Lemma 3.3.(1). Suppose that $23 \in C_2$. If $13 \in C_3$, then $e(12, \delta v) = e(23, \delta v) = e(13, \delta v)$ by Lemma 3.3(3) with the order $(1, 2, 3)$, and thus (4.20) holds. If $123 \in C_3$, then $e(123, \delta v) = e(23, \delta v)$, since $23 \in C_2$, and thus (4.20) holds. Equation (4.21) remains. Suppose that $23 \notin C_2$. We must then have $23 \in C_3$ so that player 1 is not a central player. If $123 \in C_2$, then $e(123, \delta v) = e(23, \delta v)$ from Lemma 3.1.(2), and thus (4.20) holds. If $12 \in C_2$, then $e(13, \delta v) = e(23, \delta v) = e(12, \delta v)$ from Lemma 3.3(3), and thus (4.20) holds.

(ii) $C_1 = \{123, 12\}$. In this case, $e(123, \delta v) = e(12, \delta v)$. Suppose that $23 \in C_2$. We must have $13 \in C_3$, so that player 2 is not a central player. Then $e(12, \delta v) = e(23, \delta v) = e(13, \delta v)$ by Lemma 3.3(3). Thus, (4.20) holds. Suppose that $23 \notin C_2$. In this case, we have $C_2 = \{123, 12\}$, $C_2 = \{123\}$ or $C_2 = \{12\}$. In either case, we must have $13, 23 \in C_3$ so that neither 1 nor 2 is a central player. Thus, $e(13, \delta v) = e(23, \delta v)$. If $123 \in C_2$, then $e(123, \delta v) = e(23, \delta v)$ by Lemma 3.3(2), and thus (4.20) holds. Finally, consider the case $C_2 = \{12\}$. If $C_3 = \{123, 13, 23\}$, then (4.20) holds, since $C_1 = \{123, 12\}$. If $C_3 = \{13, 23\}$, then we have $e(12, \delta v) \geq e(23, \delta v) \geq e(123, \delta v) = e(12, \delta v)$, and thus (4.20) holds.

(iii) $C_1 = \{123, 13\}$. We must have $12 \in C_2$, so that player 3 is not a central player. This must induce $23 \in C_3$ so that player 1 is not a central player. Then (4.20) holds, from Lemma 3.3(3).

(iv) $C_1 = \{123\}$. Suppose that $C_2 = \{123, 12, 23\}$. We have $e(123, \delta v) = e(12, \delta v) = e(23, \delta v)$. We must have $13 \in C_3$, so that 2 is not a central player. Thus, $e(123, \delta v) = e(13, \delta v)$ from Lemma 3.3(2), since $C_1 = \{123\}$. This yields (4.20). Neither $C_2 = \{123\}$, $C_2 = \{23\}$, nor $C_2 = \{123, 23\}$ is possible, otherwise 3 would become a central player. Suppose that $C_2 = \{12, 23\}$. We must have $13 \in C_3$, so that 2 is not a central player. Then $e(123, \delta v) = e(12, \delta v) = e(13, \delta v)$ from Lemma 3.3(2). We have $e(12, \delta v) = e(23, \delta v)$, since $C_2 = \{12, 23\}$, and thus (4.20) holds. Finally, suppose that $C_2 = \{123, 12\}$ or $\{12\}$. We must then have $13, 23 \in C_3$, so that neither 1 nor 2 is a central player. We have $e(123, \delta v) = e(12, \delta v)$ and $e(123, \delta v) = e(13, \delta v)$, since $12 \in C_2$ and $13 \in C_3$, respectively. Since $13, 23 \in C_3$, we have $e(13, \delta v) = e(23, \delta v)$. Thus, (4.20) holds.

By solving (4.20), we obtain

$$\delta v_1 = v(123) - v(23), \delta v_2 = v(123) - v(13), \delta v_3 = v(123) - v(12). \tag{4.24}$$

All players' discounted expected payoffs are equal to their marginal contributions. In every SSPE where (4.20) applies, the optimality condition is trivially satisfied.

It must hold that $0 \le \theta_i < 1$ for each $i = 1, 2, 3$, and

$$\frac{3}{2} < \theta_1 + \theta_2 + \theta_3 < 3.^8 \tag{4.25}$$

The payoff equation for an SSPE for $i = 1$ is given by

$$3v_1 = v(123) - \delta v_2 - \delta v_3 + 2\delta\theta_1 v_1. \tag{4.26}$$

Equations (4.24) and (4.26) solve

$$\theta_1 = \frac{(3+\delta)v(123) - \delta v(12) - 3v(23) - \delta v(13)}{2\delta(v(123) - v(23))}. \tag{4.27}$$

Letting $\delta \to 1$, the constraints $\theta_1 < 1$ and $\theta_1 \ge 0$ yield

$$2v(123) \le v(12) + v(23) + v(13), \tag{4.28}$$
$$4v(123) \ge v(12) + 3v(23) + v(13), \tag{4.29}$$

respectively, and we have the limit payoff of an SSPE,

$$v_1^* = v(123) - v(23), v_2^* = v(123) - v(13), v_3^* = v(123) - v(12). \tag{4.30}$$

Similarly to (4.29), we have

$$4v(123) \ge v(12) + v(23) + 3v(13), \tag{4.31}$$
$$4v(123) \ge 3v(12) + v(23) + v(13). \tag{4.32}$$

The three cases (4.21)–(4.23) remain. Since the analysis is similar, we shall solve the case of (4.21) only. When (4.21) holds, we have $\theta_1 = 0$ and $e(123, \delta v) = e(12, \delta v) = e(13, \delta v)$. Thus, $\delta v_2 = v(123) - v(13)$ and $\delta v_3 = v(123) - v(12)$. Since $\theta_1 = 0$, the payoff equation for $i = 1$ is $3v_1 = v(12) - \delta v_2$. This solves

$$v_1 = \frac{v(12) + v(13) - v(123)}{3}. \tag{4.33}$$

The payoff equation for $i = 2$ is $3v_2 = v(23) - \delta v_3 + 2\delta\theta_2 v_2$. Substituting v_1, v_2, and v_3, this yields

$$\theta_2 = \frac{(3+\delta)v(123) - \delta v(12) - \delta v(23) - 3v(13)}{2\delta(v(123) - v(13))}. \tag{4.34}$$

[8] The first inequality can be derived as follows. Let r_1 and r_1' be the probabilities that player 1 chooses coalitions 12 and 13, respectively, let r_2 and r_2' be the probabilities that player 2 chooses coalitions 12 and 23, respectively, and let r_3 and r_3' be the probabilities that player 3 chooses coalitions 23 and 13, respectively. Since $\theta_1 = 1 - (r_2' + r_3)/2$, $\theta_2 = 1 - (r_1' + r_3')/2$, and $\theta_3 = 1 - (r_1 + r_2)/2$, we have $\theta_1 + \theta_2 + \theta_2 = 3 - ((r_1 + r_1') + (r_2 + r_2') + (r_3 + r_3'))/2 > 3/2$.

Letting $\delta \to 1$, the constraints $\theta_2 < 1$ and $\theta_2 \geq 1/2$ (by (4.25)) yield (4.28) and

$$3v(123) \geq v(12) + v(23) + 2v(13), \tag{4.35}$$

respectively. Interchanging 2 with 3, we obtain

$$3v(123) \geq 2v(12) + v(23) + v(13). \tag{4.36}$$

Finally, as $\delta \to 1$, the optimality condition $e(23, \delta v) \geq e(123, \delta v)$ yields

$$4v(123) \leq v(12) + 3v(23) + v(13). \tag{4.37}$$

We can summarize our analysis of case 4 as follows. In the limit as δ goes to one, every player i's expected payoff is equal to that player's marginal contribution $m_i = v(123) - v(jk)$, $i \neq j, k$, in most configurations of supports (159 cases) if m_i is greater than or equal to $(v(123) - m_j - m_k)/3$. In the remaining three cases, two players receive their marginal contributions and the other player receives $(v(123) - m_j - m_k)/3$ more than his or her marginal contribution.

Case 5. A subcoalition-inefficient SSPE without central players (18 types).

Table 4.4 shows a list of all possible support configurations. The configurations in Table 4.4 can be divided into four subcases according to the values of θ_i ($i = 1, 2, 3$). In this case, note that $\theta_1 + \theta_2 + \theta_3 = 3/2$ (see footnote 5). In every configuration, all two-person coalitions (i.e., 12, 23, and 13) belong to the support of some players. Because of this fact, the condition for optimality of an SSPE implies $e(12, \delta v), e(23, \delta v), e(13, \delta v) \geq e(123, \delta v)$. Thus, the limit expected payoff v_i of every player $i = 1, 2, 3$ as $\delta \to 1$ is greater than or equal to that player's marginal contribution $m_i = v(123) - v(jk)$ ($i \neq j, k$).

Subcase (i). $\theta_1 = \theta_2 = \theta_3 = 1/2$ (Nos. 13 and 18).

From Lemma 3.3(3), $e(12, \delta v) = e(23, \delta v) = e(13, \delta v)$. Together with this, it can be seen without much difficulty that the payoff equations

$$3v_1 = v(12) - \delta v_2 + \delta v_1, \tag{4.38}$$

$$3v_2 = v(23) - \delta v_3 + \delta v_2, \tag{4.39}$$

$$3v_3 = v(13) - \delta v_1 + \delta v_3 \tag{4.40}$$

imply that $v(12) = v(23) = v(13)$ and

$$v_1 = v_2 = v_3 = \frac{1}{3} v(12). \tag{4.41}$$

The optimality condition $e(12, \delta v) \geq e(123, \delta v)$ implies $v(12) \geq (3/(3 + \delta))v(123)$. As $\delta \to 1$, we obtain

$$v(12) = v(23) = v(13) \geq \frac{3}{4} v(123). \tag{4.42}$$

Table 4.4 List of all possible configurations of an SSPE without central players where the grand coalition never forms

	C_1	C_2	C_3
1	12, 13	12, 23	13, 23
2	12, 13	12, 23	13
3	12, 13	12, 23	23
4	12, 13	12	13, 23
5	12, 13	12	23
6	12, 13	23	13, 23
7	12, 13	23	13
8	12, 13	23	23
9	12	12, 23	13, 23
10	12	12, 23	13
11	12	12	13, 23
12	12	23	13, 23
13	12	23	13
14	13	12, 23	13, 23
15	13	12, 23	13
16	13	12, 23	23
17	13	12	13, 23
18	13	12	23

An SSPE is possible in this case only in a symmetric game where each two-person coalition is productive relative to the grand coalition. All players receive equal expected payoffs $v(12)/3$, and thus the payoff allocation in each two-person coalition is unequal in that a proposer receives a payoff twice as large as a responder.

Subcase (ii). $\theta_i = 1/2$ for only one $i = 1, 2, 3$ (Nos. 5, 7, 10, 12, 16, 17).

We consider only the configuration $C_1 = \{12, 13\}, C_2 = \{12\}, C_3 = \{23\}$ (No. 5) in Table 4.4. Other configurations can be solved in the same way. It follows from Lemma 3.3 that $e(12, \delta v) = e(23, \delta v) = e(13, \delta v)$. Together with this, it can be seen without much difficulty that the payoff equations

$$3v_1 = v(12) - \delta v_2 + \delta v_1, \tag{4.43}$$

$$3v_2 = v(12) - \delta v_1 + 2\theta_2 \delta v_2, \tag{4.44}$$

$$3v_3 = v(23) - \delta v_2 + 2\theta_3 \delta v_3 \tag{4.45}$$

imply

$$\begin{aligned}
v_1 &= \frac{v(12) + v(13) - v(23)}{3}, \\
\delta v_2 &= \frac{\delta v(12) + (3-\delta)v(23) - (3-\delta)v(13)}{3}, \\
\delta v_3 &= \frac{-(3-\delta)v(12) + (3-\delta)v(23) + \delta v(13)}{3}, \\
\theta_2 &= \frac{\delta v_1 + 3v_2 - v(12)}{2\delta v_2}, \\
\theta_3 &= \frac{\delta v_2 + 3v_3 - v(23)}{2\delta v_3}.
\end{aligned}$$

In the limit as $\delta \to 1$, we examine the constraints $0 \leq \theta_i < 1$ $(i = 2, 3)$[9] and the optimality condition $e(12, \delta v) \geq e(123, \delta v)$. It can be shown that

$$\begin{aligned}
v(12) + v(13) &\geq v(23), \\
v(12) + 5v(23) - 5v(13) &\geq 0, \\
-5v(12) + 5v(23) + v(13) &\geq 0, \\
v(12) + 2v(23) + v(13) &\geq 3v(123).
\end{aligned}$$

The limiting expected payoffs for players are given by

$$\begin{aligned}
v_1^* &= \frac{v(12) + v(13) - v(23)}{3}, \quad v_2^* = \frac{v(12) + 2v(23) - 2v(13)}{3}, \\
v_3^* &= \frac{-2v(12) + 2v(23) + v(13)}{3}.
\end{aligned}$$

[9] There is another constraint, $\theta_2 + \theta_3 = 1$, which we omit for simplicity of exposition.

Subcase (iii). $\theta_i = 0$ for some $i = 1, 2, 3$ (Nos. 8, 11, 15).

Consider the configuration $C_1 = \{12, 13\}, C_2 = \{23\}, C_3 = \{23\}$ (No. 5) in Table 4.4, where $\theta_1 = 0$. From the payoff equation $3v_1 = v(12) - \delta v_2$ and the optimality condition $e(12, \delta v) = e(13, \delta v)$, it follows that $\delta v_2 = v(12) - 3v_1$ and $\delta v_3 = v(13) - 3v_1$. We obtain v_1 by solving a quadratic equation constructed from $\theta_2 + \theta_3 = 3/2$.[10]

Subcase (iv). Others (Nos. 1, 2, 3, 4, 6, 9, 14).

In all configurations, the optimality conditions $e(12, \delta v) = e(23, \delta v) = e(13, \delta v)$ hold. Thus, $\delta v_2 = v(23) - v(13) + \delta v_1$ and $\delta v_3 = v(23) - v(12) + \delta v_1$. We obtain v_1 by solving a cubic equation constructed from $\theta_1 + \theta_2 + \theta_3 = 3/2$.

We summarize the result of a three-person cooperative game in the following proposition and Table 4.5.

Proposition. Let (N, v) be a three-person superadditive game, where $N = \{1, 2, 3\}$ and $m_i = v(N) - v(N - \{i\})$ is player i's marginal contribution to the grand coalition, and let Γ be the random proposer game for (N, v) with a uniform recognition probability. The limit of the expected payoffs for an SSPE in Γ when the discount factor goes to one can be classified as follows.

1. The equal allocation, where the probability of the grand coalition is one.

2. The coalitional Nash bargaining solution, where the probability of the grand coalition converges to one. In equilibrium, there exists at least one player who joins all possible coalitions.

3. The marginal contributions (m_1, m_2, m_3), where $m_i \geq n_i \equiv (v(\{1, 2, 3\}) - m_j - m_k)/3$ for all $i = 1, 2, 3$ and $j, k \neq i$.

4. The vector (n_1, m_2, m_3) (and two permutations), where $n_1 \geq m_1$, $m_2 \geq n_2$, and $m_3 \geq n_3$.

5. Allocations within two-person coalitions.

In the first two cases, the limit of the SSPE payoff belongs to the core of the game. In the remaining cases, the core is empty.

When the players are sufficiently patient, the SSPEs of the random proposer game can be classified according to the level of efficiency, i.e., the equilibrium probability of the grand coalition. The efficiency level is characterized by the number

[10] We can compute $\theta_2 = (3v(12) + \delta v(13) - \delta v(23) - (9 + 3\delta)v_1)/(2\delta v(12) - 6\delta v_1)$ and $\theta_3 = (\delta v(12) + 3v(13) - \delta v(23) - (9 + 3\delta)v_1)/(2\delta v(13) - 6\delta v_1)$. Since it is cumbersome to derive a general formula for the expected equilibrium payoffs in subcases (iii) and (iv), we have omitted this derivation.

of central players who join all equilibrium coalitions (Table 4.5). The efficient SSPE (Okada 1996) has the full number of central players, and an asymptotically efficient equilibrium (Compte and Jehiel 2010) has at least one central player. These (asymptotically) efficient equilibria exist only when the core of a game is nonempty. When the core is empty, an SSPE must be inefficient. There are two types of inefficient SSPE, depending on whether or not the probability of the grand coalition is positive.

In a three-person game, the equal allocation $v(123)/3$ and the marginal contributions m_i to the grand coalition for every player i play a critical role in the expected payoffs for players in equilibrium. If the equal allocation $v(123)/3$ is smaller than all players' marginal contributions m_i (or, equivalently, the equal allocation belongs to the core), then the SSPE expected payoffs are given by the equal allocation. In this case, all players are central. If the equal allocation exceeds the marginal contribution for some player, then that player must be noncentral, and that player receives his or her marginal contribution. The remaining players split the surplus equally. In an inefficient SSPE where the probability of the grand coalition is positive, every player's expected payoff is equal to their marginal contribution m_i if it exceeds the threshold $(v(\{1,2,3\}) - m_j - m_k)/3$ for $j, k \neq i$. In a subcoalition-inefficient SSPE where the probability of the grand coalition is zero, all players' expected payoffs are not less than their marginal contributions.

Table 4.5 Classification of SSPEs ($n = 3$)[a]

Case	No. of central players	Efficiency level	Core	Marginal contribution
1	3	Efficient	Nonempty	$v_i = v(123)/3 \leq m_i$: $i = 1,2,3$
2	2	Asymptotically efficient	Nonempty	$v_i \leq m_i$: central
3	1	Asymptotically efficient	Nonempty	$v_j = m_j$: noncentral
4	0	Inefficient	Empty	$v_i = m_i$: $i = 1,2,3$
5	0	Subcoalition-inefficient	Empty	$v_i \geq m_i$: $i = 1,2,3$

[a] In case 4, the three allocations (v_1, m_2, m_3), (m_1, v_2, m_3), and (m_1, m_2, v_3) are possible, where $v_i \geq m_i$ for $i = 1, 2, 3$. They correspond to (4.21), (4.22), and (4.23), respectively.

5 Concluding Remarks

The classification of the SSPEs of the random proposer model for a three-person game reveals a variety of bargaining outcomes regarding the level of efficiency.

When the core is nonempty, the grand coalition forms almost surely, and the payoff allocation is characterized by the coalitional Nash bargaining solution (Compte and Jehiel 2010). When the core is empty, the equilibrium is inefficient. Our analysis of a three-person game shows that although no single cooperative solution appropriately describes bargaining behavior, the concepts of the Nash bargaining solution, the core, and the marginal contribution are closely related to an SSPE allocation of the random proposer model. The noncooperative analysis of a three-person cooperative game is a cornerstone of a promising research program to unite two different approaches in game theory.

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