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Abstract

We introduce a multivariate GARCH model that incorporates realized measures of volatility and covolatility. The realized measures extract information about the current level of volatility and covolatility from high-frequency data, which is particularly useful for the modeling of return volatility during periods with rapid changes in volatility and covolatility. When applied to market returns in conjunction with returns on an individual asset, the model yields a dynamic model of the conditional regression coefficient that is known as the *beta*. We apply the model to a large set of assets and find the conditional betas to be far more variable than is usually found with rolling-window regressions based exclusively on daily returns. In the empirical part of the paper we examine the cross-sectional as well as the time variation of the conditional beta series during the financial crises.

Keywords: Financial Volatility; Beta; Realized GARCH; High Frequency Data.

JEL Classification: G11, G17, C58

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1 Introduction

Relatively accurate measurements of volatility and covolatility can be computed from high frequency data, and such statistics are commonly referred to as *realized measures*. Incorporating realized measures when modeling the dynamic properties of volatility, such as done with GARCH models, is very beneficial. The reason is that returns yield very weak signals about latent volatility, whereas realized measures provide accurate measurements. The latter is particularly useful during times with rapid changes in volatility and covolatility.

In this paper we propose a multivariate GARCH-type model that utilizes and models realized measures of volatility and covolatility. The model has hierarchical structure where the "market" return is modeled with a univariate Realized GARCH model, see Hansen and Huang (2012) and Hansen et al. (2012). A multivariate structure is constructed by modeling "individual" returns conditional on the past and contemporary market variables (return and volatility). The resulting model has the structure of a conditional CAPM model that enables us to extract the "betas" and study their dynamic properties. Moreover, the model is complete in the sense that all observables (returns, realized volatilities and realized correlations) are modeled. The latter enables us to infer the distribution of multi-period returns including the joint distribution of "market" returns and "individual" returns over longer horizons.

The main contributions of our paper are the following. We propose a flexible and tractable framework that enables the modeling of a potentially large set of assets. Unlike conventional multivariate GARCH models, which can suffer from the curse of dimensionality and estimation issues, we avoid such issues by incorporating realized measures and the use of measurement equations. Measurement equations tie realized measures to the latent volatility quantities, which induces a useful regularization of the model. This particular structure was chosen for a number of reasons. First, the model provides good empirical fit for the wide range of assets used in our empirical study; Second, the structure of the model is amenable to a deeper analysis of secondary quantities such as betas; Third, the model is simple to estimate, which is particularly important when a large set of assets are to be analyzed as is the case in our empirical analysis.

The proposed model structure has a hierarchical structure where the market return and a corresponding realized measure forms the core of the model. The model can be extended to an arbitrary large set of individual returns, by adding a conditional model for an individual return and two realized measures, one being a realized measure of return volatility, the other being a realized measure of the correlation between the individual return and the market. This yield a flexible model with a dynamic covariance structure that is constantly revised by using the information contained in the realized measures.

The concept of realized betas is not new. Bollerslev and Zhang (2003) carry out a large scale estimation of the Fama-French three-factor model using high-frequency (5-minute) data on 6,400 stocks over a period of 7 years. Their analysis showed that high-frequency data can improve the pricing accuracy of asset pricing models. Their approach differs from ours in important ways. For instance, they model raw realized factor loadings and use simple time series processes to forecast these. So there is no explicit link between realized and conditional moments of returns in their framework. Nor do they explicitly account for the measurement error (the sampling error) in the realized quantities. Another related paper is Andersen et al. (2006) who study the time variation in realized variances, covariances, and betas using daily returns to construct quarterly realized measures. They find evidence of long memory in the time series for variance and covariances, while the realized beta time series is less persistent and seemingly a short-memory process, which is indicative of fractional cointegration between realized volatility and realized covariance. Other related studies include: Barndorff-Nielsen and Shephard (2004*a*) who established asymptotic results for realized beta and Dovonon et al. (2013) who established the theory for bootstrap inference. MSE-optimal estimation of realized betas was analyzed in Bandi and Russell (2005) and Patton and Verardo (2012) studied the impact of news on betas. The importance of separating jump and continuous component of returns in relation to betas as highlighted in Todorov and Bollerslev (2010) and Tsay and Yeh (2011) allow the dynamic beta to vary within the day.

The use of realized volatility measures in this context yields valuable insight about the degree of time-variation in the betas, which has been up for debate in the literature. The studies by Ferson and Harvey (1991, 1993), and Shanken (1990) specify parametric relationships between betas and proxies for the state of the economy and find support for time-varying betas. Gomes et al. (2003) provide a theoretical justification for a time-varying conditional beta specification in the context of a dynamic general equilibrium production economy. Conditional betas have been modeled by means of conventional GARCH models by Braun et al. (1995) and Bekaert and Wu (2000), among others. Lewellen and Nagel (2006) argue that variation in betas would have to be "implausibly large" to explain important asset-pricing anomalies. In our empirical analysis we do find a substantial amount of time-variation in the conditional betas, this is particularly the case during the global financial crises period. We find the variation in betas to be substantial, even over short periods of time, such as a quarter. Figure 5 in this paper is a good illustration of this point.

The research devoted to high-frequency volatility measures was spurred by Andersen and Bollerslev (1998), who documented that the sum of squared intraday returns, known as the realized variance, provides an accurate measurement of daily volatility. The theoretical foundation of realized variance was developed in Andersen, Bollerslev, Diebold and Labys (2001) and Barndorff-Nielsen and Shephard (2002). Currently a large number of related estimators, such as realized bipower variation, realized kernels, multiscale estimators, preaveraging estimators and Markov chain estimators have been proposed to deal with issues such as jumps and market microstructure frictions, see Barndorff-Nielsen and Shephard (2004*b*), Barndorff-Nielsen, Hansen, Lunde and Shephard (2008), Zhang (2006), Jacod et al. (2009), Hansen and Horel (2009) and references therein. The multivariate extensions of the concept of realized volatility is theoretically developed in Barndorff-Nielsen and Shephard (2004*a*). Estimators that are robust to noise and/or asynchronous observations have been proposed by Hayashi and Yoshida (2005), Voev and Lunde (2007), Griffin and Oomen (2011), Christensen et al. (2010), and Barndorff-Nielsen et al. (2011). In this paper we will rely on the multivariate kernel estimator by Barndorff-Nielsen, Hansen, Lunde and Shephard (2011) that guarantees positive semi-definite estimates of the realized variance-covariance matrices we need.

While volatility is unobservable, the use of realized measures allows us to construct precise ex-post volatility proxies. Currently, a growing body of research investigates the extend to which realized measures can be used to specify better volatility models and improve the accuracy of volatility forecasts. Hansen and Lunde (2010) categorize the existing approaches into two broad classes: reduced-form and model-based. Reduced-form volatility forecasts are based on a time series model for the series of realized measures, while a model-based forecast rests on a parametric model for the return distribution. Model-based approaches effectively build on GARCH models in which a realized measure is included as an exogenous variable in the GARCH equation, see e.g. Engle (2002). A complete framework that jointly specifies models for returns and realized measures of volatility was first proposed by Engle and Gallo (2006), who refer to their model as the Multiplicative Error Model (MEM). A simplified MEM structure was proposed in Shephard and Sheppard (2010), who estimated their referred to this model as the HEAVY model. The realized GARCH model by Hansen et al. (2012) involves a different approach to the joint modeling of returns and realized volatility measures. A key component of the Realized GARCH model is a measurement equation that links the realized measure with the underlying conditional variance. This idea is generalized to the multivariate framework in this paper, where we introduce measurement equations for the realized measures of correlations.

The rest of the paper is structured as follows. The model and the underlying theory is presented in Section 2, and we discuss estimation of the model in Section 3. In section 4 we show how multistep predictions of volatilities and correlations as well as forecasts of return densities can be obtained with our model. Section 5 contains the empirical application of the model, and Section 6 concludes.

2 A Hierarchical Realized GARCH Framework

Broadly speaking, our objective is the same as that of existing multivariate GARCH models, which is to model the conditional distribution of a vector of returns. But unlike conventional GARCH models we also model the realized measures of volatility and covolatility and make extensive use of these in the modeling of returns. The realized measures are highly informative about local (in time) levels of volatility and covolatility which is the main reason these variables are beneficial in this context. By tying all individual return series to the market return, we are implicitly imposing a factor structure on the volatility, where the variation in the correlation structure is driven by time-variation in the correlations between the market return and the individual assets. This keeps the model relatively simple and parsimonious, facilitates estimation, and makes it easy to relate key variables in the model to (dynamic) betas.

Our model has a hierarchical structure. The core of our framework is a marginal model for the market return and its realized measure of volatility. Individual returns, their realized measures of volatility and covolatility (with the market) are then modeled conditionally on market variables. The marginal model we use for the market-specific time series is the Realized EGARCH model by Hansen and Huang (2012), see also Hansen et al. (2012, section 6.3), which shares certain features with the EGARCH model by Nelson (1991). The conditional model we use in this paper is new.

Initially, we present the Realized Beta GARCH model in the simplest situation with a bivariate vector of returns (the market return and an individual asset return) and the corresponding 2×2 matrix realized volatility measures. Subsequently we discuss the straight forward extension to an arbitrary number of individual assets.

2.1 Notation and Modeling Strategy

Let $r_{0,t}$ and $x_{0,t}$ denote the market return and a corresponding realized measure of volatility, respectively. Similarly, we use the notation $r_{1,t}$ and $x_{1,t}$ for the same variables associated with an individual asset return, and use $y_{i,t}$ to denote a realized measure of correlation, where $y_{it} \in (-1, 1)$.

In this context with two returns, two realized measures of volatility, and a realized measure of correlation we have five observable variables to model. The natural filtration is given by

$$\mathcal{F}_t = \sigma(\mathcal{X}_t, \mathcal{X}_{t-1}, \ldots) \quad \text{with} \quad \mathcal{X}_t = (r_{0,t}, r_{1,t}, x_{0,t}, x_{1,t}, y_{1,t})'.$$

The structure of our model will take advantage of the simple decomposition of the conditional density,

$$f(r_{0,t}, x_{0,t}, r_{1,t}, x_{1,t}, y_{1,t}|\mathcal{F}_{t-1}) = f(r_{0,t}, x_{0,t}|\mathcal{F}_{t-1}) f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}), \tag{1}$$

which serves to illustrate the hierarchical structure of our model. We will adopt the Realized EGARCH model as our specification of the first term, $f(r_{0,t}, x_{0,t}|\mathcal{F}_{t-1})$. The individual asset variables (r_{1t}, x_{1t}, y_{1t}) will be modeled with a novel structure that conditions on contemporary market variables. The specification for the second conditional density, $f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1})$, defines how the time series associated with the individual asset evolves conditional on contemporary market variables. Our specification of this conditional density has a structure that is similar to that of the univariate Realized GARCH model, but has some important adaptations for the modeling of the correlation structure. This structure is very convenient because it avoids the need for introducing realized measures of the correlation measures between the individual assets, because it is implicitly assumed that these correlations are characterized through the correlations between the individual returns and the market return. In our empirical analysis, we investigate the validity of this assumption.

In practice, the estimation proceeds by first estimating the model for the market data $(r_{0,t}, x_{0,t})$ and then estimating each conditional model for $(r_{i,t}, x_{i,t}, y_{i,t})$ separately for $i = 1, 2, \ldots, n$, where n is the number of assets. This can be done for a very large number of assets. For instance, in the empirical analysis we estimated the Realized Beta GARCH model for about 600 assets.

2.2 Realized EGARCH Model for Market Returns

The Realized EGARCH model for market returns and realized measures of volatility is given by the following three equations

$$r_{0,t} = \mu_0 + \sqrt{h_{0,t}} z_{0,t}, \tag{2}$$

$$\log h_{0,t} = a_0 + b_0 \log h_{0,t-1} + c_0 \log x_{0,t-1} + \tau_0(z_{0,t-1}) \tag{3}$$

$$\log x_{0,t} = \xi_0 + \varphi_0 \log h_{0,t} + \delta_0(z_{0,t}) + u_{0,t}, \tag{4}$$

where model $z_{0,t} \sim \text{iid}N(0,1)$, and $u_{0,t} \sim \text{iid}N(0, \sigma^2_{u0})$. As is the case in conventional GARCH models, $h_{0,t}$, denotes a conditional variance, $h_{0,t} = \text{var}(r_{0,t}|\mathcal{F}_{t-1})$, the key difference being that the information set, $\mathcal{F}_t$, is richer than in the conventional framework. The normality of $u_{0,t}$ is not critical for estimation, but can be motivated by findings in Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2001) and Andersen et al. (2003), who document that realized volatility is approximately log-normal. Furthermore, Andersen, Bollerslev, Diebold and Ebens (2001) find that returns standardized by realized volatility are approximately normally distributed.

The functions $\tau(z)$ and $\delta(z)$ are called leverage functions because they model aspects related to the *leverage effect*, which refers to the dependence between returns and volatility. Hansen et al. (2012) found that a simple second-order polynomial form provides a good empirical fit. We will adopt this structure in our framework, and set $\tau(z) = \tau_1 z + \tau_2(z^2 - 1)$ and $\delta(z) = \delta_1 z + \delta_2(z^2 - 1)$. This leads to a GARCH equation that is somewhat similar to that of an EGARCH model. An important difference is that we also utilize the realized measure x_{t-1} in this equation to model the dynamic variation in volatility.

We refer to the first two equations, (2) and (3), as the *return equation* and the *GARCH equation*, respectively. These two equations define a GARCH-X model, similar to those that were estimated by Engle (2002), Barndorff-Nielsen and Shephard (2007), and Visser (2011). See also **?** for additional variants of the GARCH-X model and some related models.

The third equation, (4) called the *measurement equation*, completes the model. Tying the realized measure, x_t, to the conditional variance, h_t, is motivated by the fact that the GARCH equation trivially implies that

$$\log(r_t - \mu)^2 = \log h_t + \log z_t^2.$$

Since the realized measure, x_t, is similar to r_t^2 in the sense of being a measurement of volatility (just far more accurate), it is natural to expect that $\log x_t \approx \log h_t + f(z_t) + \text{error}_t$. Because we may compute realized measures of volatility over a shorter period of time than the one spanned by the return (e.g., if we use only data from the trading session, which often excludes the overnight period), some flexibility in the specification may be required motivating the "intercept" ξ_0 and the "slope" φ_0. So long as $x_{0,t}$ is roughly proportional to $h_{0,t}$, we should expect $\varphi_0 \simeq 1$, and $\xi_0 < 0$, which is always the case empirically.

Note that we do not follow the conventional GARCH notation, because we want to reserve the notation "β" for

$$\beta_{1,t} = \text{cov}(r_{1,t}, r_{0,t}|\mathcal{F}_{t-1})/\text{var}(r_{0,t}|\mathcal{F}_{t-1}), \tag{5}$$

We are particularly interested in the dynamic properties and the cross-sectional variation of $\beta_{i,t}$, where $i = 1, \ldots, N$ with N being the number of individual assets in our analysis.

2.3 Conditional Model for Individual Asset Returns, Volatility, and Co-volatility

To extend the framework to a joint model for the market returns/volatility and another asset's return/volatility and their correlation, we shall formulate a model for the time series associated with the individual asset, conditional on contemporaneous "market" variables, i.e., a specification for $f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1})$. We utilize a further decomposition of this conditional density, specifically

$$f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) = f(r_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1})f(x_{1,t}, y_{1,t}|r_{1,t}, r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}).$$

The first part, $f(r_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1})$, is modeled with three equations. The first two have the Realized EGARCH structure as above,

$$r_{1,t} = \mu_1 + \sqrt{h_{1,t}} z_{1,t}, \tag{6}$$

$$\log h_{1,t} = a_1 + b_1 \log h_{1,t-1} + c_1 \log x_{1,t-1} + d_1 \log h_{0,t} + \tau_1(z_{1,t-1}). \tag{7}$$

The difference between the two GARCH equations, (3) for the market return and (7) for the asset return, is the presence of the term, $d_1 \log h_{0,t}$. This term relates the market conditional variance to the conditional variance of the individual asset under consideration. Note that $h_{0,t}$ is $\mathcal{F}_{t-1}$-measurable, so that $h_{1,t}$ can be interpreted as the conditional variance of $r_{1,t}$ w.r.t. $\mathcal{F}_{t-1}$. The parameter d_1 can be interpreted as a spillover effect that measures the extend to which the market's volatility affects the volatility of the individual asset while accounting for the asset-specific volatility dynamics.

To capture the dependence between market returns and individual returns, we introduce the conditional covariance

$$\rho_{1,t} = \mathrm{cov}(z_{0,t}, z_{1,t}|\mathcal{F}_{t-1}).$$

It follows directly that $\rho_{1,t}$ is the conditional correlation between $r_{0,t}$ and $r_{1,t}$, so that the beta for asset 1 is given by

$$\beta_{1,t} = \frac{\rho_{1,t}\sqrt{h_{0,t}h_{1,t}}}{h_{0,t}} = \rho_{1,t}\sqrt{h_{0,t}/h_{1,t}}.$$

For the dynamic modeling of ρ_t we shall use of the Fisher transformation (also known as the inverse hyperbolic tangent, arctanh), $\rho \mapsto F(\rho) \equiv \frac{1}{2}\log\frac{1+\rho}{1-\rho}$, which is a one-to-one mapping from $(-1,1)$ into $\mathbb{R}$. The GARCH equation for the transformed correlations is given by

$$F(\rho_{1,t}) = a_{10} + b_{10}F(\rho_{1,t-1}) + c_{10}F(y_{1,t-1}).$$

The measurement equations for the two realized measures in the conditional model are:

$$\log x_{1,t} = \xi_1 + \varphi_1 \log h_{1,t} + \delta_1(z_{1,t}) + u_{1,t}, \tag{8}$$

and

$$F(y_{1,t}) = \xi_{10} + \varphi_{10}F(\rho_{1,t}) + v_{1,t}, \tag{9}$$

and the covariance structure for the error terms in the three measurement equations is denoted by

$$\Sigma = \mathrm{var}\begin{pmatrix} u_{0,t} \\ u_{1,t} \\ v_{1,t} \end{pmatrix} = \begin{bmatrix} \sigma^2_{u_0} & \sigma_{u_0,u_1} & \sigma_{u_0,v_1} \\ \bullet & \sigma^2_{u_1} & \sigma_{u_1,v_1} \\ \bullet & \bullet & \sigma^2_{v_1} \end{bmatrix}.$$

Hence, we allow the error terms to be correlated across measurement equations, which we find to be empirically important.

2.4 The Extensions to Multiple Individual Assets

We have specified the model structure for a market return and a single individual assets (along with their corresponding realized volatility variables). Next we discuss the extension to multiple individual assets. Fortunately, the existing structure is amendable to this extension, albeit some additional assumptions are needed before certain interpretations carry over to the general context. First we need to redefine the natural filtration, $\mathcal{F}_t = \sigma(\mathcal{X}_t, \mathcal{X}_{t-1}, \ldots)$, to be defined by the full set of variables,

$$\mathcal{X}_t = (r_{0,t}, r_{1,t}, \ldots, r_{N,t}, x_{0,t}, x_{1,t}, \ldots, x_{N,t} y_{1,t} \ldots, y_{N,t})'.$$

The conditional model for the individual asset is assumed to be invariant to this enhancement of the information set. This implicitly assumes that the dynamic variation in correlations between individual assets is fully explained by the individual assets correlation with the market return. Put differently: The variation in the $N+1 \times N+1$ conditional covariance matrix is fully described by the $N+1$ conditional variances and the N conditional correlations. This structure has testable implications that we return to in our empirical section.

2.4.1 Variables for Model Diagnostics

For model diagnostics, in particular the validity of the single-factor structure, we define conditional studentized residuals

$$\hat{w}_{i,t} = \frac{\hat{z}_{i,t} - \hat{\rho}_{i,t}\hat{z}_{0,t}}{\sqrt{1-\hat{\rho}_{i,t}^2}}, \; i = 1, \ldots, N.$$

So far the model structure has been silent about the dependence structure across the population equivalents of these residuals,

$$w_{i,t} = \frac{z_{i,t} - \rho_{i,t} z_{0,t}}{\sqrt{1-\rho_{i,t}^2}}, \; i = 1, \ldots, N,$$

and the same is true for the conditional error terms, $(u_{i,t}, v_{i,t}|u_{0,t})$, across individual assets. We cast light on this dependence structure in our empirical section.

3 Estimation

In this section, we define the quasi log-likelihood function and exploit its structure to simplify the estimation problem. We have five observed variables, $(r_{0,t}, x_{0,t}, r_{1,t}, x_{1,t}, y_{1,t})$, and we consider their joint density conditional on past information, $\mathcal{F}_{t-1}$. Without loss of generality we can decompose this "joint" density as stated in (1), and, for the purpose of estimation, we adopt Gaussian specifications for the "marginal" and "conditional" densities, $f(r_{0,t}, x_{0,t}|\mathcal{F}_{t-1})$ and $f(r_{i,t}, x_{i,t}, y_{i,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1})$, $i = 1, \ldots, N$. Moreover, we assume that the studentized returns, $(z_{0,t}, z_{1,t})$, are independent of the error terms in the three measurement equations, $(u_{0,t}, u_{1,t}, v_{1,t})$. This enables us to decompose the quasi log likelihood function into four terms as we discuss below.

3.1 The Marginal Model for Market Variables

The marginal model is essentially that of Hansen et al. (2012), which implicitly entails a further decomposition of the conditional density,

$$f(r_{0,t}, x_{0,t}|\mathcal{F}_{t-1}) = f_{r_0}(r_{0,t}|\mathcal{F}_{t-1}) f_{x_0|r_0}(x_{0,t}|r_{0,t}, \mathcal{F}_{t-1}).$$

The two densities are given from $r_{0,t} \sim N(\mu_0, h_{0,t})$ and $\log x_{0,t} \sim N(\xi_0 + \varphi_0 \log h_{0,t} + \tau_0(z_{0,t}), \sigma^2_{u_0})$, which leads to the following two contribution to (minus two times) the log-likelihood function,

$$\begin{aligned}
\ell_{z_0} &= \sum_{t=1}^{T} \log h_{0,t} + \frac{(r_{0,t} - \mu_0)^2}{h_{0,t}} &&= \sum_{t=1}^{T} \log h_{0,t} + z^2_{0,t}. \\
\ell_{u_0} &= \sum_{t=1}^{T} \log \sigma^2_{u_0} + \frac{(\log x_{0,t} - \xi_0 - \varphi_0 \log h_{0,t} - \tau_0(z_{0,t}))^2}{\sigma^2_{u_0}} &&= \sum_{t=1}^{T} \log \sigma^2_{u_0} + \frac{u^2_{0,t}}{\sigma^2_{u_0}}.
\end{aligned}$$

3.2 The Conditional Model for Individual Assets

Next we consider the likelihood contributions from the conditional model. The conditional model also permits a further decomposition of the conditional density,

$$\begin{aligned}
f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) &= f_{r_1|r_0,x_0}(r_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) \\
&\quad \times f_{x_1,y_1|r_1,r_0,x_0}(x_{1,t}, y_{1,t}|r_{1,t}, r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}).
\end{aligned}$$

The first term is the density of the individual asset return conditional on the contemporaneous market variables (and the past). Due to the Gaussian specification we only need to derive the conditional mean and variance of $r_{1,t}$ in order to compute the appropriate likelihood term. The assumed independence between $(z_{0,t}, z_{1,t})$ and $u_{0,t}$ and the iid assumptions imply that

$$\mathrm{E}[g(r_{1,t})|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) = \mathrm{E}[g(r_{1,t})|z_{0,t}, u_{0,t}, \mathcal{F}_{t-1}) = \mathrm{E}[g(r_{1,t})|r_{0,t}, \mathcal{F}_{t-1}),$$

for any function g for which the conditional mean is well defined. Hence,

$$\mathrm{var}(r_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) = \mathrm{var}(r_{1,t}|r_{0,t}, \mathcal{F}_{t-1}) = h_{1,t} - (\rho_{1,t}\sqrt{h_{0,t}h_{1,t}})^2/h_{0,t} = (1-\rho_{1,t}^2)h_{1,t},$$

since $\mathrm{cov}(r_{1,t}, r_{0,t}|\mathcal{F}_{t-1}) = \rho_{1,t}\sqrt{h_{0,t}h_{1,t}}$. Next the conditional mean of $r_{1,t}$ is

$$\begin{aligned}\mathrm{E}(r_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) &= \mu_1 + \beta_{1,t}(r_{0,t} - \mu_0) = \mu_1 + \frac{\rho_{1,t}\sqrt{h_{0,t}h_{1,t}}}{h_{0,t}}(r_{0,t} - \mu_0) \\ &= \mu_1 + \rho_{1,t}\sqrt{h_{1,t}}z_{0,t},\end{aligned}$$

So that the contribution to (minus two times) the log-likelihood function from this conditional density is,

$$\ell_{z_1|z_0} = \sum_{t=1}^{T} \log[(1-\rho_{1,t}^2)h_{1,t}] + \frac{\left(r_{1,t}-\mu_1-\rho_{1,t}\sqrt{h_{1,t}}z_{0,t}\right)^2}{(1-\rho_{1,t}^2)h_{1,t}}.$$

The last likelihood term, $\ell_{u_1,v_1|u_0}$, which relates to the two measurement equations is associated with the conditional density, $f_{x_1,y_1|r_1,r_0,x_0}(x_{1,t}, y_{1,t}|r_{1,t}, r_{0,t}, x_{0,t}, \mathcal{F}_{t-1})$. First we note that the conditional distribution of $(u_{1,t}, v_{1,t})$ given $(u_{0,t}, z_{0,t}, z_{1,t})$ is Gaussian with mean

$$\begin{pmatrix} \sigma_{u_1,u_0}/\sigma_{u_0}^2 \\ \sigma_{v_1,u_0}/\sigma_{u_0}^2 \end{pmatrix} u_{0,t},$$

and variance

$$\Omega = \begin{bmatrix} \sigma_{u_1}^2 & \sigma_{u_1,v_1} \\ \bullet & \sigma_{v_1}^2 \end{bmatrix} - \begin{bmatrix} \sigma_{u_1,u_0} \\ \sigma_{v_1,u_0} \end{bmatrix} \frac{1}{\sigma_{u_0}^2} \begin{bmatrix} \sigma_{u_0,u_1} & \sigma_{u_0,v_1} \end{bmatrix}.$$

So it does not depend on $(z_{0,t}, z_{1,t})$ due to the assumed independence. The implication is that

$$f_{,x_1,y_1|r_1,r_0,x_0}(x_{1,t}, y_{1,t}|r_{1,t}, r_{0,t}, x_{0,t}, \mathcal{F}_{t-1}) = f_{,x_1,y_1|r_1,r_0,x_0}(x_{1,t}, y_{1,t}|u_{0,t}, \mathcal{F}_{t-1}),$$

and that the last term in (minus two times) the log-likelihood is given by

$$\ell_{u_1,v_1|u_0} = \sum_{t=1}^{T} \log\det\Omega + U_{1,t}'\Omega^{-1}U_{1,t},$$

where we have defined

$$U_{1,t} = \begin{pmatrix} u_{1,t} \\ v_{1,t} \end{pmatrix} - \begin{pmatrix} \sigma_{u_1,u_0}/\sigma_{u_0}^2 \\ \sigma_{v_1,u_0}/\sigma_{u_0}^2 \end{pmatrix} u_{0,t}.$$

3.3 Simplification in Estimation

To simplify the estimation we can concentrate the likelihood function with respect to the covariance matrix of $(u_{0,t}, u_{1,t}, v_{1,t})$. Let $\hat{u}_{0,t}$, $\hat{u}_{1,t}$ and $\hat{v}_{1,t}$ be the residuals of the three measurement equations. The Gaussian likelihood implies that the maximum likelihood estimators of the variance-covariance parameters are given by

$$\hat{\sigma}_{u_0}^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_{0,t}^2, \quad \hat{\sigma}_{u_1,u_0} = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_{1,t}\hat{u}_{0,t}, \quad \hat{\sigma}_{v_1,u_0} = \frac{1}{T}\sum_{t=1}^{T}\hat{v}_{1,t}\hat{u}_{0,t},$$

and

$$\hat{\Omega} = \frac{1}{T}\sum_{t=1}^{T}\hat{U}_{1,t}\hat{U}_{1,t}', \qquad \text{where} \quad \hat{U}_{1,t} = \begin{pmatrix} \hat{u}_{1,t} \\ \hat{v}_{1,t} \end{pmatrix} - \begin{pmatrix} \hat{\sigma}_{u_1,u_0}/\hat{\sigma}_{u_0}^2 \\ \hat{\sigma}_{v_1,u_0}/\hat{\sigma}_{u_0}^2 \end{pmatrix} \hat{u}_{0,t}.$$

The reduces the number of free parameters that the likelihood has to be maximized over, to $\theta = (\theta_0', \theta_1')'$, where

$$\theta_0 = (\mu_0, \omega_0, a_0, b_0, c_0, \tau_{01}, \tau_{02}, \xi_0, \varphi_0, \delta_{01}, \delta_{02}, h_{0,1})',$$

is the vector of (remaining) parameters in the market model, and

$$\theta_1 = (\mu_1, \omega_1, a_1, b_1, c_1, d_1, \tau_{11}, \tau_{12}, \xi_1, \varphi_1, \delta_{11}, \delta_{12}, a_{10}, b_{10}, c_{10}, \xi_{10}, \varphi_{10}, h_{1,1}, \rho_{1,1})',$$

is the vector of (remaining) parameters in the conditional model. Here we follow the convention and threat the initial values for the latent variables, $h_{0,1}$, $h_{1,1}$, and $\rho_{1,1}$, as were they unknown parameters.

These parameters are now estimated by maximizing

$$\ell(\theta) = -\frac{1}{2}\left(\ell_{z_0}(\theta_0) + \ell_{u_0}(\theta_0) + \ell_{z_1|z_0}(\theta_1) + \ell_{u_1,v_1|u_0}(\theta_1)\right),$$

where $\ell_{z_0}(\theta_0) = \sum_{t=1}^{T}[\log h_{0,t}(\theta_0)+z_{0,t}^2(\theta_0)]$, $\ell_{u_0}(\theta_0) = T[\log \hat{\sigma}_{u_0}^2(\theta_0)+1]$, $\ell_{u_1,v_1|u_0}(\theta) = T[\log\det\hat{\Omega}(\theta)+2]$, and

$$\ell_{z_1|z_0}(\theta) = \left(\sum_{t=1}^{T}\log\{[1-\rho_{1,t}^2]h_{1,t}(\theta)\} + \frac{(z_{1,t}(\theta)-\rho_{1,t}(\theta)z_{0,t}(\theta))^2}{1-\rho_{1,t}^2(\theta)}\right).$$

In practice this amounts to the following procedure:

1. Given initial values for θ_0, the time series for $z_{0,t}$ and $h_{0,t}$ are computed iteratively. First, $z_{0,1} = (r_{0,1}-\mu_0)/\sqrt{h_{0,1}}$, then for $t = 2,\ldots,T$ we compute

$$h_{0,t}(\theta_0) = \exp\left\{a_0 + b_0\log h_{0,t-1} + c_0\log x_{0,t-1} + \tau_0(z_{0,t-1})\right\},$$

and $z_{0,t}(\theta_0) = \frac{r_{0,t}-\mu_0}{\sqrt{h_{0,t}(\theta_0)}}$. This produces the first term of the log-likelihood function, $\ell_{z_0}(\theta_0) = \sum_{t=1}^{T}\log h_{0,t}(\theta_0) + z_{0,t}^2(\theta_0)$.

2. Next, we compute $u_{0,t}(\theta_0) = \log x_{0,t} - \xi_0 - \varphi_0\log h_{0,t} - \tau_0(z_{0,t})$ for $t = 1,\ldots,T$, which yields the second term of the log-likelihood function, $\ell_{u_0}(\theta_0) = T\left[\log\sigma_{u_0}^2(\theta_0)+1\right]$, where $\sigma_{u_0}^2(\theta_0) = \frac{1}{T}\sum_{t=1}^{T}u_{0,t}^2(\theta_0)$.

3. Now we turn to the conditional model. We compute $z_{1,1}(\theta_1) = (r_{1,1}-\mu_1)/\sqrt{h_{1,1}}$ and then for $t = 2,\ldots,T$, we proceed with

$$h_{1,t}(\theta_1) = \exp\left\{a_1 + b_1\log h_{1,t-1} + c_1\log x_{1,t-1} + d_1\log h_{0,t} + \tau_1(z_{1,t-1})\right\},$$

and $z_{1,t}(\theta_1) = \frac{r_{1,t}-\mu_1}{\sqrt{h_{1,t}}}$. The notation above suppress that $h_{1,t}$, and hence $z_{1,t}$, depend on the market parameters, θ_0 (unless $d_1 = 0$). This is implicit since $h_{0,t} = h_{0,t}(\theta_0)$ depends on θ_0, and a similar dependence on θ_0 arises below through $z_{0,t}$ and $u_{0,t}$. To make this dependence explicit we shall add the argument, m_{θ_0}, to the likelihood terms below, which is short for the market variables, $\{z_{0,t}(\theta_0), h_{0,t}(\theta_0), u_{0,t}(\theta_0)\}$.

Independently of $h_{1,t}$ and $z_{1,t}$, we can compute:

$$\rho_{1,t}(\theta_1) = F^{-1}\left\{a_{10} + b_{10}F(\rho_{1,t-1}) + c_{10}F(y_{1,t-1})\right\}$$

recursively, for $t = 2,\ldots,T$. So the third likelihood term is given by

$$\ell_{z_1|z_0}(\theta_1;\mathrm{m}_{\theta_0}\}) = \sum_{t=1}^{T}\log\{(1-\rho_{1,t}^2(\theta_1))h_{1,t}(\theta_1)\} + \frac{(z_{1,t}(\theta_1)-\rho_{1,t}(\theta_1)z_{0,t})^2}{1-\rho_{1,t}^2(\theta_1)}.$$

4. The last step involves the two measurement equations in the conditional model, whose residuals are computed by

$$\begin{aligned} u_{1,t}(\theta_1) &= \log x_{1,t} - \xi_1 - \varphi_1 \log h_{1,t} - \delta_1(z_{1,t}), \\ v_{1,t}(\theta_1) &= F(y_{1,t}) - \xi_{1,0} - \varphi_{1,0} F(\rho_{1,t}). \end{aligned}$$

Next we get $\sigma_{u_1,u_0}(\theta_1) = T^{-1}\sum_{t=1}^{T} u_{1,t}(\theta_1)u_{0,t}$ and $\sigma_{v_1,u_0}(\theta_1) = T^{-1}\sum_{t=1}^{T} v_{1,t}(\theta_1)u_{0,t}$, that is the sample covariances of the measurement errors (that also depend on θ_0 through $u_{0,t} = u_{0,t}(\theta_0)$). This leads to the last likelihood term,

$$\ell_{u_1,v_1|u_0}(\theta_1; \mathrm{m}_{\theta_0}) = T(\log\det\Omega(\theta_1; \mathrm{m}_{\theta_0}) + 2),$$

where

$$\Omega(\theta_1; \mathrm{m}_{\theta_0}) = \frac{1}{T}\sum_{t=1}^{T} U_{1,t}U_{1,t}',$$

with

$$U_{1,t} = U_{1,t}(\theta_1; \mathrm{m}_{\theta_0}) = \begin{pmatrix} u_{1,t}(\theta_1) \\ v_{1,t}(\theta_1) \end{pmatrix} - \begin{pmatrix} \sigma_{u_1,u_0}(\theta_1)/\sigma_{u_0}^2 \\ \sigma_{v_1,u_0}(\theta_1)/\sigma_{u_0}^2 \end{pmatrix} u_{0,t}.$$

3.4 Hierarchical Approach to Estimation of Large Systems

When estimating a large system, it is advantageous to use a two-step procedure that the hierarchical structure is well suited for. First we estimate the market model by maximizing

$$-\frac{1}{2}\left\{\sum_{t=1}^{T}\left[\log h_{0,t}(\theta_0) + z_{0,t}^2(\theta_0)\right] + T\left[\log\left(T^{-1}\sum_{t=1}^{T} u_{0,t}^2(\theta_0)\right) + 1\right]\right\}.$$

Then in a second step, where we take $\{(h_{0,t}, z_{0,t}, u_{0,t})\}$ as given, which amount to dropping the argument m_{θ_0} in the expressions of the previous section (steps 3 and 4). So with the two-step procedure, we estimate θ_i by maximizing

$$-\frac{1}{2}\sum_{t=1}^{T}\log\{(1-\rho_{i,t}^2(\theta_i))h_{i,t}(\theta_i)\} + \frac{(z_{i,t}(\theta_i) - \rho_{i,t}(\theta_i)z_{0,t})^2}{1-\rho_{i,t}^2(\theta_i)} + T(\log\det\Omega(\theta_i) + 2),$$

for each of the individual assets, $i = 1, \ldots, N$.

4 Forecasting

In this section we discuss how multistep predictions of volatilities and correlations as well as return density forecasts can be obtained with our model. Denote $\tilde{h}_{0,t} \equiv \log h_{0,t}$, $\tilde{h}_{i,t} \equiv \log h_{i,t}$ and $\tilde{\rho}_{i,t} \equiv F(\rho_{i,t})$. Point forecasts turn out to be very easy to obtain owing to the fact that the vector $(\tilde{h}_{0,t}, \tilde{h}_{i,t}, \tilde{\rho}_{i,t})$ can be represented as a VARMA(1,1) system. Substituting each of the measurement equations (4), (8) and (9) into the equations for the corresponding conditional moments one obtains

$$\begin{aligned}
\tilde{h}_{0,t+1} &= a_0 + c_0\xi_0 + (b_0 + c_0\varphi_0)\tilde{h}_{0,t} + c_0\delta_0(z_{0,t}) + \tau_0(z_{0,t}) + c_0 u_{0,t} \\
\tilde{h}_{i,t+1} &= a_i + c_i\xi_i + (b_i + c_i\varphi_i)\tilde{h}_{i,t} + d_i\tilde{h}_{0,t+1} + c_i\delta_i(z_{i,t}) + \tau_i(z_{i,t}) + c_i u_{i,t}. \\
\tilde{\rho}_{i,t+1} &= a_{i0} + c_{i0}\xi_{i0} + (b_{i0} + c_{i0}\varphi_{i0})\tilde{\rho}_{i,t} + c_{i0}v_{i,t}
\end{aligned} \tag{10}$$

Let $V_t = (\tilde{h}_{0,t}, \tilde{h}_{i,t}, \tilde{\rho}_{i,t})'$, then by substituting the equation for $\tilde{h}_{0,t+1}$ into that for $\tilde{h}_{i,t+1}$, one can show that

$$V_{t+1} = C + AV_t + B\varepsilon_t,$$

where $\varepsilon_t = (\delta_0(z_{0,t}), \tau_0(z_{0,t}), \delta_i(z_{i,t}), \tau_i(z_{i,t}), u_{0,t}, u_{i,t}, v_{i,t})'$ and

$$C = \begin{bmatrix} a_0 + c_0\xi_0 \\ a_1 + c_i\xi_i + d_i(a_0 + c_0\xi_0) \\ a_{i0} + c_{i0}\xi_{i0} \end{bmatrix}, \qquad A = \begin{bmatrix} b_0 + c_0\varphi_0 & 0 & 0 \\ d_i(b_0 + c_0) & b_i + c_i\varphi_i & 0 \\ 0 & 0 & b_{i0} + c_{i0}\varphi_{i0} \end{bmatrix},$$

$$B = \begin{bmatrix} c_0 & 1 & 0 & 0 & c_0 & 0 & 0 \\ d_i c_0 & d_i & c_i & 1 & d_i c_0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{i0} \end{bmatrix}.$$

The innovation process, ε_t, is a martingale difference sequence but is slightly heterogeneous. Time-variation in the distribution of ε_t arises from (and is fully described by) $\rho_{i,t} = \mathrm{corr}(z_{0,t}, z_{i,t}|\mathcal{F}_{t-1})$. It follows that $\mathrm{E}(V_{t+k}|V_t) = A^k V_t + \sum_{j=0}^{k-1} A^j C$ which can be used to produce a k-step ahead forecast of V_{t+k}. Forecast of the conditional distribution of $V_{t+k}|\mathcal{F}_t$, which can be used to deduce unbiased forecasts of the non-transformed variables, e.g., $h_{0,t} = \exp(\tilde{h}_{0,t})$, can be obtained by simulation or bootstrap methods. In the simulation approach, we first generate

$$\eta_t = \begin{pmatrix} z_{0,t} \\ \tilde{z}_{i,t} \\ u_{0,t} \\ u_{i,t} \\ v_{i,t} \end{pmatrix} \sim N_5 \left(0, \begin{bmatrix} I_2 & 0 \\ 0 & \Sigma \end{bmatrix} \right), \qquad t = 1, \ldots, n.$$

Given an initialization for $\rho_{i,0}$, one can produce the entire time series $\{\tilde{\rho}_{i,t}\}$ from $\{v_{i,t}\}$ using (10). Next one can define $z_{i,t} = \rho_{i,t} z_{0,t} + \sqrt{1 - \rho_{i,t}^2} w_{i,t}$, which has the proper correlation with $z_{0,t}$, and thus finally ε_t can be computed.

Alternatively, a bootstrap approach can be used if the Gaussian assumption concerning the distribution of η_t is questionable. From the estimated model we can obtain residuals, $(\hat{\eta}_1, \ldots, \hat{\eta}_n)$, from which we can draw resamples instead of sampling from the Gaussian distribution. Time series for V_t can now be generated from the bootstrapped residuals $\{\hat{\eta}_t^*\}$ in the same manner as with simulated $\{\eta_t\}$.

To simulate the time series for larger systems is straight forward using the bootstrap of the residuals from the estimated structure. Simulations would require one to take an explicit stand about the correlation structure of $w_{i,t}$-variables and the correlation structure of the errors in the various measurement equations.

5 Empirical Analysis

5.1 Data Description

The model is estimated for a large cross section of assets. We included any asset that was a constituent of the S&P 500 index at some point between January 19, 2006 and June 25, 2010, albeit excluding assets for which we had less than 1000 daily observations during our sample period from January 3, 2002 to the end of 2009. This results in a total of 594 time series with distinct PERMNO (see below). Thus our sample period spans a total of 2,008 trading days and the sample size for each of the individual stocks ranges from 1,000 to 2,008 observations.

Our data were constructed by merging information from the TAQ dataset and the CRSP daily stock files that were accessed through the WRDS research service. The former provides the high-frequency data used for our construction of realized measures of volatility, and the latter has the opening and closing prices that are properly adjusted for stock splits and dividends. The TAQ database uses ticker symbols as stock identifiers which can be problematic for a comprehensive analysis such as this one. The reason is that about 10% of the companies in our sample have traded under different ticker symbols during the sample period and, more importantly, some ticker symbols represent very different companies at different point in time. Relying on tickers as identifiers can result in data for two or more companies being mixed up. The CRSP data identifies companies using the CRSP Permanent Company Numbers (PERMNOs) and we can use these to track changes in companies ticker symbol, thus ensuring that the proper high-frequency data are extracted from TAQ. This is achieved as follows: First, we match the ticker symbols of the S&P 500 constituents to the CRSP dataset and obtain their PERMNOs. Second, we extract the ticker symbols that were associated with each PERMNO over the sample period. This information is then used to extract high-frequency data from the TAQ, from which our realized measures of volatility are constructed, and the the daily data from the CRSP are appended to the time series of realized measures. The high-frequency transaction data are cleaned according to the filtering algorithm described in Barndorff-Nielsen et al. (2009), and the multivariate realized kernel by Barndorff-Nielsen et al. (2011) is used as our realized measures of volatility and co-volatility. We use the exchange traded fund, SPY, as a proxy for the market index in our empirical analysis, making the total number of assets in our analysis 595.

5.2 Empirical Results

A summary of the estimation results is presented in in Table 1 and Figure 1.[1] The first row in Table 1 contains the estimates for the marginal model for the market return, as defined by equations (2-4), and the rest of the table presents a summary of the estimation results for the 594 conditional models, each defined by equations: (6-9). To conserve space the cross-sectional statistics for the estimates of Σ are omitted, but some selected estimates of Σ will be presented in Table 2.

[1]The results reported are the estimates when imposing the restrictions $\varphi_i = 1$, $i = 0, 1, \ldots, N$, which did not result a significant reduction of the log-likelihood function, see Hansen and Huang (2012) for a discussion on this. The initial values for the latent variables, $h_{0,t}$, $h_{i,t}$ and $\rho_{i,t}$, are treated as parameters, and their estimated values are reported as $h_{0,1}$, $h_{i,1}$ and $\rho_{i,1}$, respectively.

Table 1: Parameter estimates of the Realized GARCH Beta model.

	Volatility parameters											*Correlation Parameters*					
	$h_{0,1}$	μ_0	a_0	b_0	c_0		τ_{01}	τ_{02}	ξ_0	δ_{01}	δ_{02}						
SPY	1.333	0.001	0.175	0.601	0.377		-0.092	0.020	-0.469	-0.104	0.018						
	$h_{i,1}$	μ_i	a_i	b_i	c_i	d_i	τ_{i1}	τ_{i2}	ξ_i	δ_{i1}	δ_{i2}	$\rho_{i,1}$	a_{i0}	b_{i0}	c_{i0}	ξ_{i0}	ϕ_{i0}
Mean	7.747	0.023	0.208	0.585	0.335	0.045	-0.041	0.010	-0.346	-0.030	0.064	0.353	0.036	0.708	0.285	-0.099	0.982
Median	3.374	0.018	0.193	0.592	0.340	0.034	-0.041	0.009	-0.352	-0.032	0.063	0.350	0.036	0.701	0.278	-0.066	0.915
Min	0.010	-0.157	-0.024	0.350	0.093	-0.023	-0.084	-0.019	-0.963	-0.100	0.011	-0.180	-0.103	0.536	0.027	-1.661	0.420
Min(1)	0.024	-0.130	-0.011	0.354	0.146	-0.019	-0.080	-0.018	-0.895	-0.091	0.011	-0.114	-0.097	0.544	0.032	-1.522	0.445
Min(2)	0.119	-0.121	-0.000	0.381	0.166	-0.016	-0.079	-0.014	-0.880	-0.088	0.012	-0.113	-0.096	0.549	0.043	-1.516	0.488
Min(3)	0.120	-0.082	0.006	0.382	0.176	-0.014	-0.078	-0.012	-0.860	-0.085	0.018	-0.054	-0.095	0.556	0.048	-1.379	0.505
Min(4)	0.166	-0.079	0.017	0.391	0.187	-0.013	-0.078	-0.010	-0.779	-0.080	0.018	-0.042	-0.089	0.558	0.051	-0.955	0.514
Min(5)	0.221	-0.077	0.023	0.397	0.204	-0.012	-0.076	-0.010	-0.760	-0.080	0.020	-0.018	-0.086	0.559	0.051	-0.952	0.518
1%	0.231	-0.073	0.023	0.402	0.204	-0.012	-0.076	-0.009	-0.759	-0.079	0.021	0.003	-0.082	0.562	0.060	-0.942	0.529
5%	0.691	-0.042	0.063	0.455	0.256	-0.002	-0.065	-0.005	-0.577	-0.060	0.033	0.076	-0.035	0.601	0.121	-0.407	0.603
95%	25.57	0.102	0.402	0.701	0.407	0.125	-0.016	0.028	-0.094	0.002	0.095	0.659	0.107	0.834	0.484	0.148	1.463
99%	54.62	0.140	0.522	0.756	0.442	0.163	-0.006	0.035	-0.016	0.016	0.112	0.750	0.139	0.887	0.600	0.227	2.539
Max(-5)	56.98	0.149	0.523	0.762	0.447	0.164	-0.005	0.036	-0.012	0.019	0.113	0.754	0.140	0.890	0.600	0.229	2.560
Max(-4)	74.39	0.153	0.525	0.763	0.447	0.166	-0.005	0.037	-0.009	0.019	0.113	0.764	0.143	0.905	0.602	0.238	2.599
Max(-3)	76.10	0.164	0.537	0.769	0.449	0.173	-0.005	0.037	-0.005	0.023	0.116	0.772	0.157	0.925	0.615	0.239	2.897
Max(-2)	100.4	0.168	0.565	0.789	0.459	0.175	0.000	0.038	-0.002	0.026	0.120	0.779	0.185	0.939	0.631	0.242	2.988
Max(-1)	139.4	0.220	0.649	0.828	0.473	0.190	0.001	0.039	0.056	0.026	0.128	0.804	0.193	0.958	0.640	0.247	3.331
Max	207.7	0.229	0.800	0.887	0.483	0.203	0.020	0.048	0.082	0.037	0.154	0.854	0.205	0.963	0.647	0.252	3.956

A summary of the estimated parameters. First row has the estimates from the market return model, (2-4), and information about the cross section of estimates for the 594 conditional models (6-9) are presented in the remaining rows.

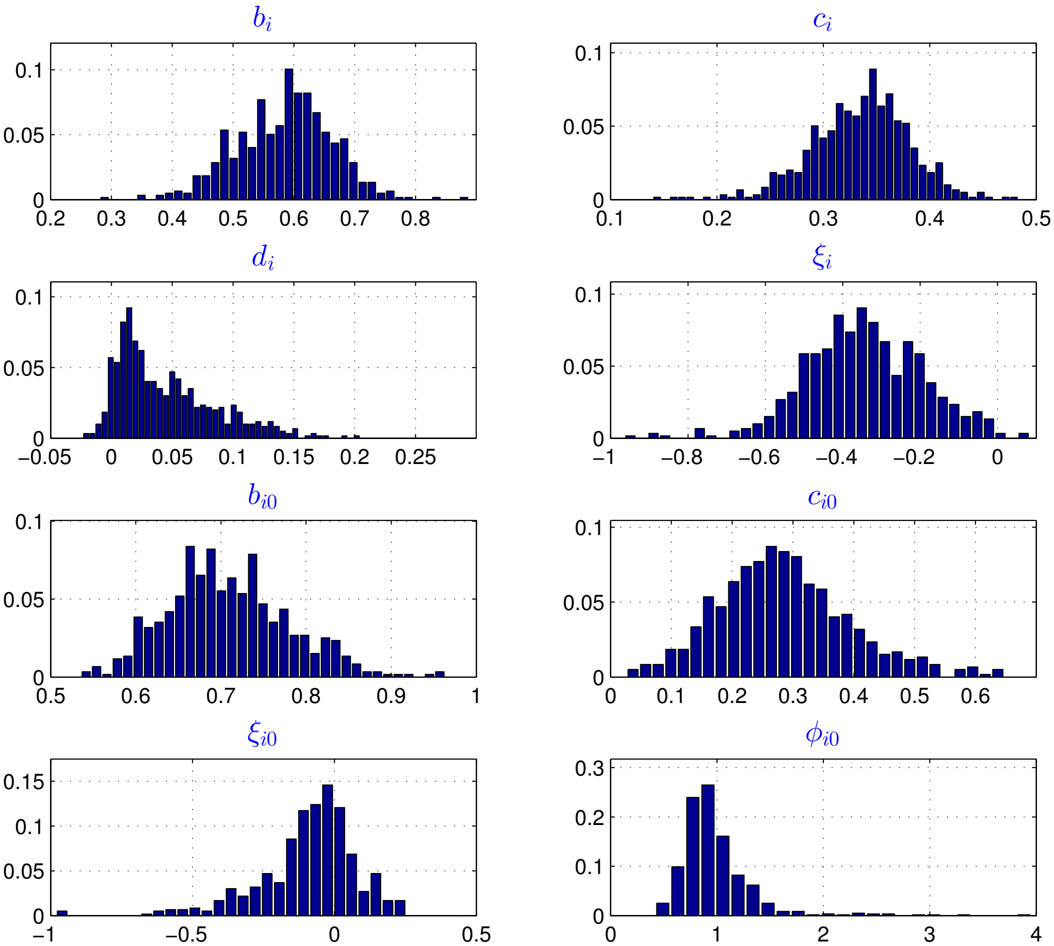

Figure 1: Histograms of the 594 parameter estimates for some selected parameters.

The parameter c_i, which captures the effect of the lagged realized measure on the conditional variance, is large and significant while the GARCH parameter, b_i, is much smaller than is usually the case for conventional GARCH models. This reason for this is that the realized measure is far more informative about volatility than the squared return, which makes the model far more adaptive to abrupt changes in volatility, which in turn, leads to a better empirical fit and more accurate forecast. The negative estimates of τ_{i1} and positive estimates of τ_{i2} indicate the presence of a leverage effect, see Hansen et al. (2012) for the relation of these leverage functions to the news impact curve. Examining the parameters of the measurement equation, we find that ξ_i is negative. This is to be expected because the realized measures is computed over the open-to-close period, which only capture a fraction of daily (close-to-close) volatility. The conditional model for the individual stocks has the additional parameter, d_i, in the GARCH equation. This parameter measures the spillover effect from market volatility to individual stock volatility. The mean and the median of this coefficient is positive, and so is the vast majority of the individual estimates. Altogether this shows that market volatility tend to have a positive contemporaneous effect on individual asset volatility.[2]

The cross sectional variation of parameter estimates are presented in the histogram plots in Figure 1. Both Table 1 and Figure 1 show that the parameter estimates are quite stable in our cross-section of stocks. Only $\hat{\phi}_{i0}$ is estimated to have an extreme value some cases, but even in these cases, we have verified that the estimated conditional variances and correlations are in agreement with their corresponding realized measures.

Table 2 presents the estimates of Σ for six selected assets. The upper left element in these matrices is the estimated variance of $u_{0,t}$ in the measurement equation for the realized measures associated with the market return. This point estimate varies slightly across the six matrices due to variation in the sample period for the six assets. Note that the measurement error variance for the individual assets to be larger than that of the market. This is to be expected because the realized measure for the market return is based on a larger number of high-frequency data. Also note that there is substantial correlation across the measurements error, in particular for the realized measures of volatility.

Table 2: Selected Point Estimates of the Measurement Error Variance.

$$\begin{array}{ccc}
\text{CVX} & \text{UTX} & \text{WMB} \\
\begin{bmatrix} 0.112 & & \\ 0.071 & 0.121 & \\ 0.016 & 0.011 & 0.015 \end{bmatrix} &
\begin{bmatrix} 0.113 & & \\ 0.060 & 0.161 & \\ 0.016 & 0.003 & 0.017 \end{bmatrix} &
\begin{bmatrix} 0.112 & & \\ 0.055 & 0.197 & \\ 0.015 & 0.000 & 0.017 \end{bmatrix} \\
\text{EK} & \text{SNV} & \text{MSFT} \\
\begin{bmatrix} 0.112 & & \\ 0.051 & 0.224 & \\ 0.015 & 0.001 & 0.018 \end{bmatrix} &
\begin{bmatrix} 0.114 & & \\ 0.078 & 0.217 & \\ 0.018 & 0.010 & 0.025 \end{bmatrix} &
\begin{bmatrix} 0.112 & & \\ 0.076 & 0.138 & \\ 0.017 & 0.010 & 0.014 \end{bmatrix}
\end{array}$$

The estimated measurement error variance matrix, $\Sigma = \mathrm{var}(u_{0,t}, u_{i,t}, v_{i,t})$, for six selected assets.

In Figure 2 we present the realized variance of CVX and SPY against the model-implied conditional variance.[3] Clearly, the conditional variance tracks the realized series closely but has less high-frequency variation. Naturally, this relation is largely imposed by the models structure, because the measurement equation implies a (noisy) relationship between the conditional variance and realized measure. The apparent downward bias of the realized measure is due to

[2]Note that a row in Table 1 does not present the estimates for a particular stock. For example, the 1% quantile of b_0 and c_0 may not be estimates for the same asset.

[3]The plots for all stocks are available from the authors upon request.

the fact that it is computed over a fraction of the day (the about 6.5 hours where assets are actively traded). This aspect of the realized measures explains that the coefficients ξ_0 and ξ_1 are negative.

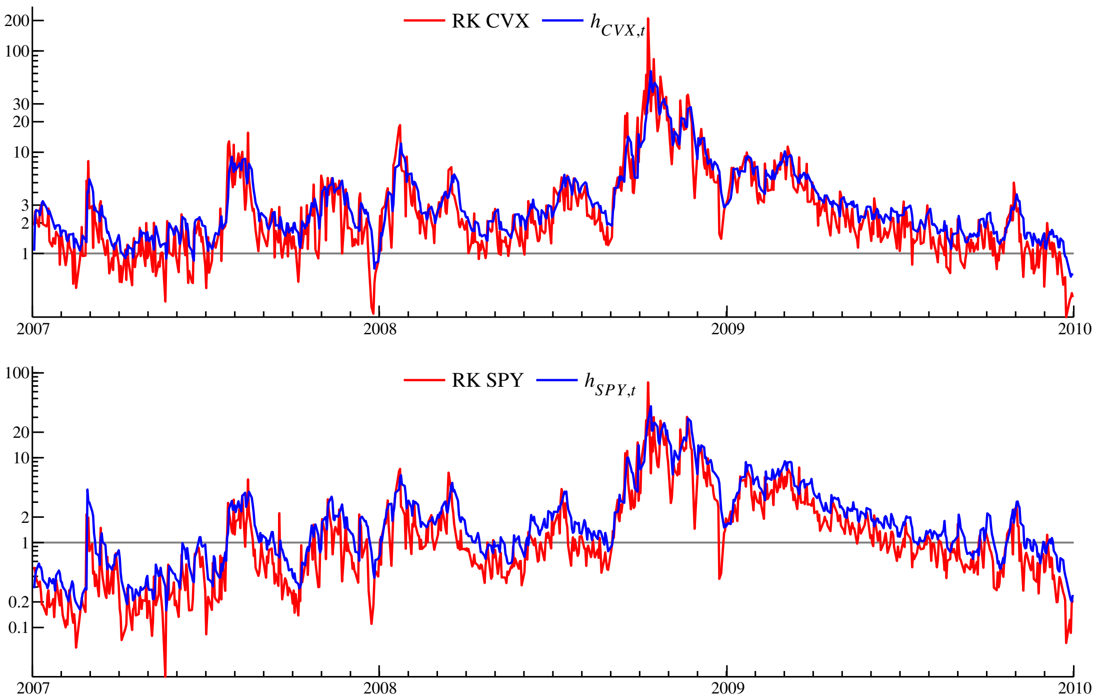

Figure 2: Realized kernel (RK) variance and conditional variance of CVX (upper panel) and SPY (lower panel) over the period 2007 – 2009.

We turn next to the model-implied betas, given by

$$\hat{\beta}_t = \hat{\rho}_t \sqrt{\hat{h}_{1,t}/\hat{h}_{0,t}}, \tag{11}$$

where $\hat{\rho}_t = \rho_t(\hat{\theta})$ and $\hat{h}_{i,t} = h_{i,t}(\hat{\theta})$, $i = 0, 1$, denote the estimated quantities. The time series can be contrasted to the realized betas

$$\tilde{\beta}_t = y_{1,t} \sqrt{x_{1,t}/x_{0,t}}, \tag{12}$$

that are computed exclusively from high-frequency data on day t.

The model-implied betas take into account the presence of measurement error in the realized quantities as well as the dynamic linkages between realized measures and conditional moments. To get an idea of the time variation of $\hat{\beta}_t$ in our model compared to its raw realized counterpart, we continue with our previous example, and present graphic results for the realized and the

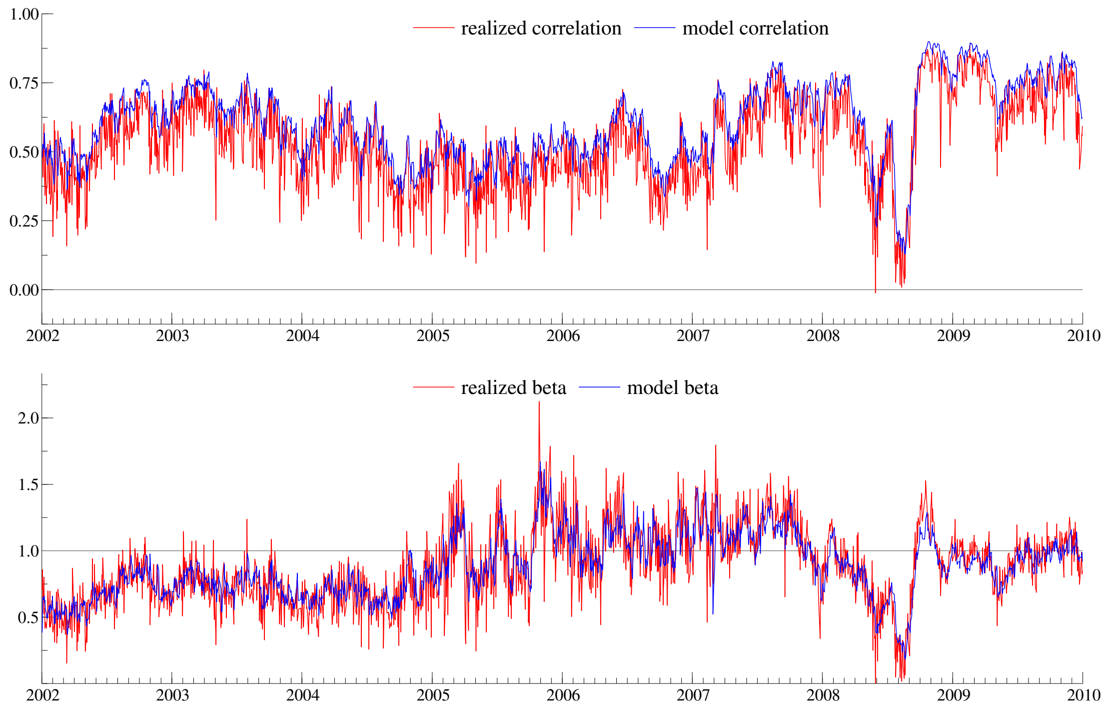

Figure 3: Realized and conditional correlation (upper panel) and beta (lower panel) of CVX (perm number 14,541) for the period 2002 – 2010.

conditional beta and correlation of CVX in Figure 3. The correlation changes rapidly during the sample period, which carries over to the the systematic risk of CVX, as defined by its beta. In fact the beta for CVX ranges from about 0 to more than 1.5 over this period.

In Figure 4 we present quantile time series plots of the cross sectional variation in the conditional correlation and beta during the financial crisis, where the time of the collapse of Lehman Brothers is clearly identified. It is, perhaps, the period leading up to the collapse of Lehman Brothers that stands out the most. On July 15th the SEC temporarily prohibited naked short selling in the securities of Fannie Mae and Freddie Mac. The time of this announcement coincides with some major changes in the cross sectional distribution of correlations and betas, although we do not claim any causal relation in this matter. In the subsequent period correlations decreased (on average) and the cross sectional distribution became increasingly left-skewed correlations. This might suggest that assets became somewhat more susceptible to idiosyncratic shocks and less to market-wide shocks. So it is perhaps surprising that the distribution of conditional betas became more right-skewed and increased. The explanation is that individual asset volatility increased relatively more than market volatility, to an extend that the relative increase more than offset the reduction in average correlation. The mechanics of this is easily understood from the definition of the conditional beta, $\beta_t = \rho_{i,t}\sqrt{h_{i,t}/h_{0,t}}$. After this initial chaotic period correlations started to increase and the variation in betas decreased. Eventually, correlations peaked around mid-November with a median value of over 70% well above the 55% value at the beginning of June.

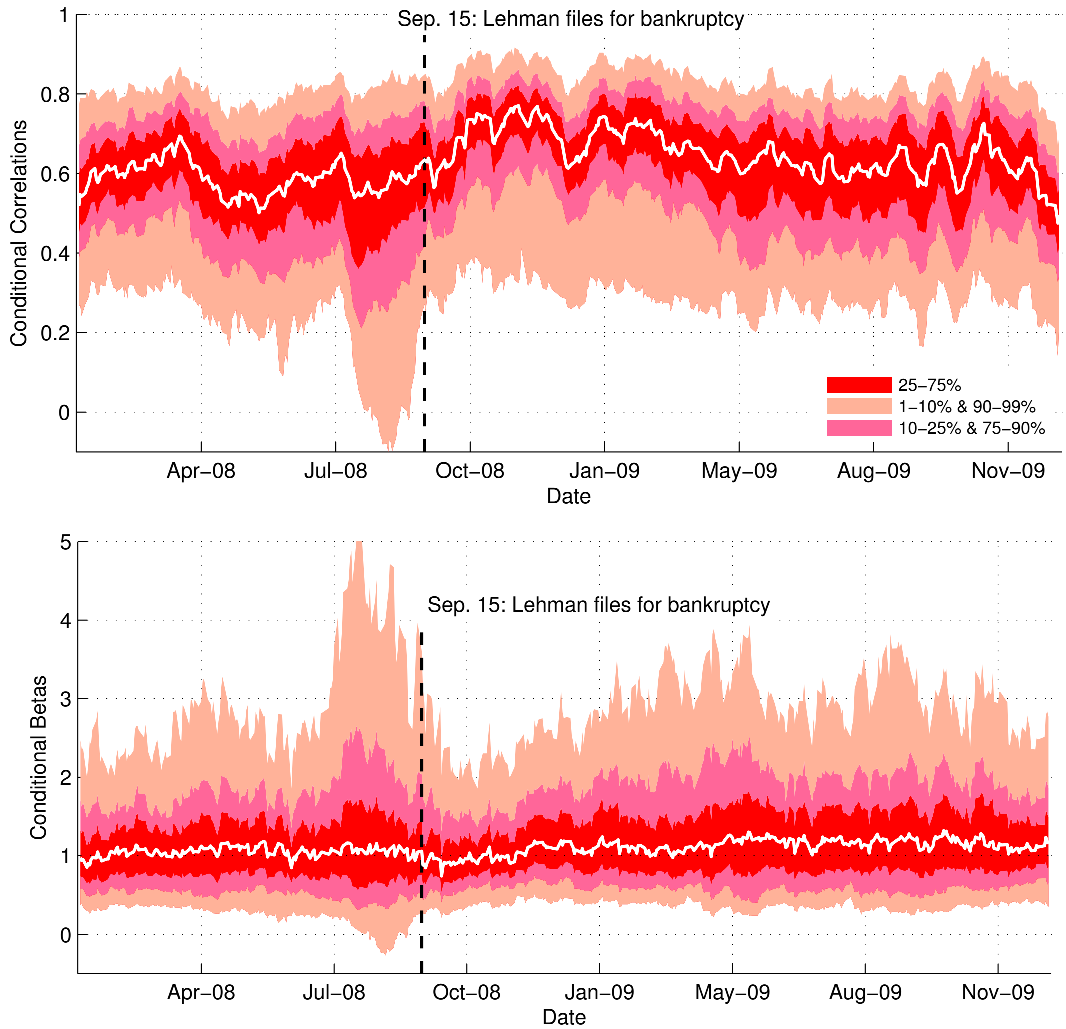

Figure 4: Quantile time series plot of conditional realized GARCH correlations for the period 06.2008 – 12.2008.

It is important to understand that the high degree of variation that we find in the betas cannot simply be attributed to variation in the realized quantities. In fact, the main source of this variation is driven by daily returns. The reason is simply that it is time variation in the dependence structure in daily returns that causes the realized measures to be found to be useful predictors in the GARCH equations. Variation of this magnitude would be close to impossible to obtain with standard approaches using rolling window OLS techniques based on daily returns. Based on rolling window estimated betas of this kind, Lewellen and Nagel (2006) concluded that time variation in beta is insufficient to explain certain asset pricing "anomalies". Given the large variation we observed in the systematic risk of individual companies, it could be interesting to revisit this question.

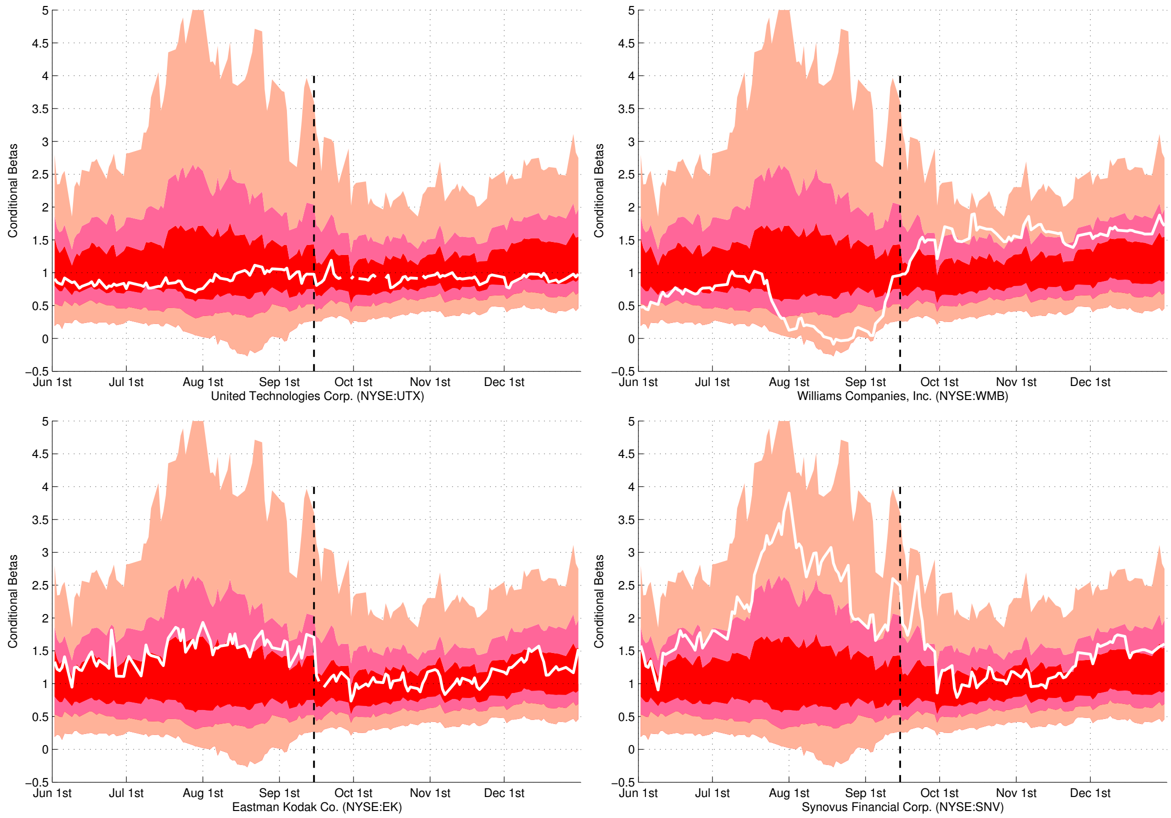

Figure 5: Quantile time series plot of conditional realized GARCH betas for the period 06.2008 – 12.2008.

In Figure 5 we have replaced the median line of Figure 4 with the time series of conditional betas for four selected stocks, and the four panels covers the second half of. Our objective is to demonstrate the substantial variation that some betas display. An interesting example is Williams Companies (WMB) that moved from the lower 10% to the upper 10% in the fall of 2008. The example of SNV shows how some financial companies got relatively more risky as the financial crises approached in the early fall of 2008. Finally, we included UTX and EK to show that the betas of some companies are relatively stable.

5.3 Residual Correlations and Test for Constant Correlations

The Realized Beta GARCH model implies that the correlation between the individual studentized returns, z_{it} and z_{jt}, is time varying. Recall the decomposition

$$z_{i,t} = \rho_{i,t} z_{0,t} + z_{i,t} - \rho_{i,t} z_{0,t} = \rho_{i,t} z_{0,t} + \sqrt{1-\rho_{i,t}^2}\, w_{i,t},$$

where $w_{i,t}$ and $z_{0,t}$ are uncorrelated, both have mean zero and unit variance, and in the likelihood analysis we modeled both as standard Gaussian random variables. It follows that

$$\mathrm{corr}(z_{i,t}, z_{j,t}) \quad = \quad \rho_{i,t}\rho_{j,t} + \sqrt{(1-\rho_{i,t}^2)(1-\rho_{j,t}^2)}\mathrm{E}(w_{i,t} w_{j,t}),$$

which is time varying unless $\mathrm{E}(w_{i,t} w_{j,t})$ behaves in a rather unlikely way that offsets the variation in $\rho_{i,t}$ and $\rho_{j,t}$. We have not stated explicit assumptions about the correlation, $\mathrm{E}(w_{i,t} w_{j,t})$, which induces additional dependence between $z_{i,t}$ and $z_{j,t}$, beyond that inherited from their correlations with the market return. This additional channel for dependence is ignored in our estimation (in order to make the estimation of large systems feasible). A non-zero correlation between $w_{i,t}$ and $w_{j,t}$ is evidence that the Realized Beta GARCH model does not fully characterize the complete system, so that the estimated model will need to be enhanced to capture such effects. It would also suggest that the estimation is inefficient to some extent, albeit this is to be expected with a relatively simple estimation procedure in a highly complex model.

In this section we study the magnitude of $\mathrm{E}(w_{i,t} w_{j,t})$ and the potential evidence of time-variation in this correlations. Since our model implies time variation in the correlation between $z_{i,t}$ and $z_{j,t}$ we shall evaluate the empirical evidence of this.

First we consider a test for constant correlation that is based on the general theory by Nyblom (1989). This is the underlying framework of several test for parameter constancy including that of Hansen (1992) (linear regression models) and that of Hansen and Johansen (1999) (cointegration VAR).

Consider a bivariate process (x_t, y_t) of studentized variables, $\mathrm{E}(x_t) = \mathrm{E}(y_t) = 0$ and $\mathrm{E}(x_t^2) = \mathrm{E}(y_t^2) = 1$; So that the correlation is given by

$$\rho_t = \mathrm{E}(x_t y_t).$$

We are to construct tests for constant correlation and zero correlation. The maintained hypothesis is that the partial sum

$$W_T(u) \equiv T^{-\frac{1}{2}} \sum_{s=1}^{\lfloor uT \rfloor} (x_s y_s - \rho_s), \qquad u \in [0,1],$$

satisfies a functional central limit theorem, so that $W_T(u) \Rightarrow \sigma_W B(u)$ where $B(u)$ is a standard Brownian motion, and σ_W^2 is the long-run variance of $x_t y_t - \rho_t$.

Under the null hypothesis, $H_0 : \rho_t = \rho$ (constant correlation) it follows that

$$NB_c = \frac{T^{-1} \sum_{t=1}^{T} \left(T^{-1/2} \sum_{s=1}^{t} (x_s y_s - \bar{\rho}) \right)^2}{\hat{\sigma}_W^2} \xrightarrow{d} \int_0^1 B_b(u)^2 \mathrm{d}u,$$

where $B_b(u) = B(u) - uB(1)$ is a standard Brownian bridge, $\bar{\rho} = T^{-1} \sum_{t=1}^{T} x_t y_t$ and $\hat{\sigma}_W^2$ is some consistent estimator of σ_W^2. Under the null hypothesis $H_0 : \rho_t = 0$ (zero correlation) we have

$$NB_0 = \frac{T^{-1} \sum_{t=1}^{T} \left(T^{-1/2} \sum_{s=1}^{t} x_s y_s \right)^2}{\hat{\sigma}_W^2} \xrightarrow{d} \int_0^1 B(u)^2 \mathrm{d}u,$$

where $\hat{\sigma}_W^2 \xrightarrow{p} \sigma_W^2$. In the absence of serial dependence we can use the estimator $\hat{\sigma}_W^2 = T^{-1} \sum_{t=1}^{T} (x_t y_t - \bar{\rho})^2$, which is consistent for σ_W^2 under both null hypotheses. The 5% critical values of these limit distributions are 0.462 and 1.656, respectively, see Nyblom (1989).

In our application we shall apply the test for constant correlation to $z_{i,t} z_{j,t}$ and $w_{i,t} w_{j,t}$, and we apply the test for zero correlation to $w_{i,t} w_{j,t}$.

5.3.1 Empirical Results Concerning Residual Correlation Structure

With 594 stock in our cross section there are 176,121 distinct correlation series to look at. To handle this we aggregate the correlation estimation and test results by industrial segmentation. We employ the sector definition given by the Global Industry Classification Standard (GICS) that is the industry taxonomy developed by Morgan Stanley Capital International (MSCI) and Standard & Poor's. The GICS structure consists of 10 sectors, 24 industry groups, 68 industries and 154 sub-industries and it is used as a basis for S&P and MSCI financial market indexes. To make our analysis as clear as possible we aggregate to sector level.

To match our stocks to the ten GICS sector we pair TAQ with Standard & Poor's CapitalIQ database that contains continuously updated GICS classifications for a large set of publicly listed companies assigned by S&P's analysts. These GICS classifications reflect those used by many wealth and investment managers and financial institutions. To match CUSIP and

Ticker identifiers from TAQ to the GICS identifiers, TAQ stock identifiers are first matched by CUSIP, and then double checked for a matched with company names from CapitalIQ. In cases without a match from this procedure, Ticker's are used. If this procedure does not provide a match, CapitalIQ Equity Listings report is used to check for inactive listings and these are again matched according to exchange tickers. If none of the above procedures achieve a positive match, CapitalIQ's business description is used to identify company name changes and a final match is attempted. The above series of matching procedures match all considered TAQ identifiers with available GICS classifications. The 10 sectors are listed in Table 3 along with the number of companies and summery statistics for their betas within each of the sectors.

Table 3: Sector Statistics.

Sector	Company Counts	Min beta	Median	Max Beta
Energy	46	0.299 (0.174)	1.009 (0.145)	2.396 (0.510)
Materials	37	0.382 (0.104)	0.873 (0.117)	2.347 (0.836)
Industrials	63	0.264 (0.133)	0.931 (0.144)	2.293 (0.427)
Consumer Discretionary	103	0.163 (0.156)	0.912 (0.136)	2.395 (0.664)
Consumer Staples	46	0.336 (0.127)	0.970 (0.135)	2.188 (0.422)
Healthcare	64	0.358 (0.147)	1.072 (0.158)	2.586 (0.632)
Financials	101	0.309 (0.127)	1.037 (0.141)	2.633 (0.665)
Information Technology	86	0.335 (0.156)	0.944 (0.145)	2.496 (0.734)
Telecommunication Services	9	0.727 (0.166)	1.082 (0.186)	1.620 (0.307)
Utilities	40	0.284 (0.140)	0.988 (0.175)	2.332 (0.579)

The table gives summary statistics of the sectoral aggregation. The third to fifth columns give the time series average of the minimum, median and maximum beta for each sector. Standard deviations are given in parenthesis.

Next we turn to the constancy of correlations within and and across sectors. These results are presented in Tables 4-6. Table 4 gives the sample average of the unconditional correlations for residuals sorted by industry classification (GICS). The upper panel is for the studentized returns, $\hat{z}_{i,t}$ and $\hat{z}_{j,t}$, and serve as a benchmark measure. It is interesting to compare these to the numbers in the lower panel that are based on $\hat{w}_{i,t}$ and $\hat{w}_{j,t}$. The difference between these numbers show us how much of the correlations between individual stocks that could be attributed to our market factor. The immediate impression that one gets is that the market in most cases account for most of the co-variation between the stocks we consider. That is certainly the case for all the cross sector combinations (the off diagonals of the lower panel are close to zero). However, for stocks in the same sector we see from the numbers on the diagonal that a substantial amount of correlation is left unexplained by the market (the diagonal entries of the lower panel). The implication is that we need to look for more factors when modeling the co-variation between stocks in the same sector. For now we will simply investigate some statistical properties of the residual co-variation and leave the modeling for future work.

Table 4: Unconditional Correlations (Sorted by GICS)

	Energy	*Materials*	*Industrials*	*Consumer Discretionary*	*Consumer Staples*	*Healthcare*	*Financials*	*Information Technology*	*Telecommun. Services*	*Utilities*
Energy	0.598	0.324	0.276	0.203	0.158	0.180	0.231	0.217	0.196	0.298
Materials		0.417	0.376	0.308	0.240	0.242	0.341	0.297	0.273	0.285
Industrials			0.400	0.330	0.258	0.262	0.356	0.320	0.287	0.286
Consumer Discretionary				0.322	0.234	0.232	0.326	0.281	0.251	0.238
Consumer Staples					0.260	0.207	0.260	0.203	0.213	0.232
Healthcare						0.260	0.258	0.227	0.215	0.221
Financials							0.429	0.301	0.289	0.300
Information Technology								0.356	0.267	0.229
Telecommun. Services									0.369	0.251
Utilities										0.487

	Energy	*Materials*	*Industrials*	*Consumer Discretionary*	*Consumer Staples*	*Healthcare*	*Financials*	*Information Technology*	*Telecommun. Services*	*Utilities*
Energy	0.483	0.091	0.015	-0.035	-0.044	-0.018	-0.056	-0.022	-0.026	0.099
Materials		0.160	0.082	0.042	0.008	0.004	0.023	0.018	0.006	0.021
Industrials			0.104	0.058	0.021	0.020	0.026	0.036	0.012	0.013
Consumer Discretionary				0.098	0.032	0.020	0.049	0.033	0.010	-0.005
Consumer Staples					0.105	0.036	0.021	-0.015	0.016	0.040
Healthcare						0.096	0.012	0.009	0.010	0.022
Financials							0.145	0.002	0.011	0.024
Information Technology								0.130	0.023	-0.023
Telecommun. Services									0.187	0.029
Utilities										0.341

Average sample correlations for residuals sorted *by industry classification (GICS). Upper panel is for $\hat{z}_{i,t}$ and $\hat{z}_{j,t}$ and the numbers in the lower panel are based on $\hat{w}_{i,t}$ and $\hat{w}_{j,t}$.*

Table 5: Testing for Constant Correlations (Sorted by GICS)

	Energy	Materials	Industrials	Consumer Discretionary	Consumer Staples	Healthcare	Financials	Information Technology	Telecommun. Services	Utilities
Energy	0.649	0.781	0.854	0.636	0.589	0.423	0.718	0.794	0.679	0.570
Materials		0.619	0.676	0.619	0.470	0.341	0.612	0.609	0.616	0.588
Industrials			0.767	0.665	0.616	0.415	0.623	0.628	0.626	0.683
Consumer Discretionary				0.645	0.470	0.387	0.647	0.575	0.595	0.536
Consumer Staples					0.507	0.453	0.516	0.552	0.577	0.674
Healthcare						0.367	0.409	0.369	0.410	0.475
Financials							0.738	0.575	0.570	0.439
Information Technology								0.682	0.562	0.513
Telecommun. Services									0.778	0.597
Utilities										0.474

	Energy	Materials	Industrials	Consumer Discretionary	Consumer Staples	Healthcare	Financials	Information Technology	Telecommun. Services	Utilities
Energy	0.722	0.435	0.200	0.373	0.248	0.286	0.608	0.090	0.244	0.354
Materials		0.386	0.221	0.207	0.150	0.125	0.260	0.135	0.072	0.125
Industrials			0.147	0.159	0.110	0.094	0.168	0.132	0.063	0.142
Consumer Discretionary				0.182	0.107	0.079	0.130	0.128	0.137	0.142
Consumer Staples					0.173	0.153	0.141	0.204	0.143	0.183
Healthcare						0.166	0.107	0.093	0.113	0.130
Financials							0.382	0.131	0.086	0.411
Information Technology								0.260	0.087	0.094
Telecommun. Services									0.444	0.097
Utilities										0.268

Rejection frequencies for the NB_c test for constant correlation using a 5% significance level. The upper table are the results for $\hat{z}_{i,t}\hat{z}_{j,t}$ and the lower table are those for $\hat{w}_{i,t}\hat{w}_{j,t}$.

In Table 5 we apply the NB_c test for constant correlation to our residual series. We report the rejection frequencies for a 5% significance level. In the upper panel we present the frequencies for the product of the studentized returns, $\hat{z}_{i,t}\hat{z}_{j,t}$. For example, in the case of the 46 energy companies there are 1,035 tests and the null hypothesis of constant correlation is rejected for almost 65% of these test. In fact the constant correlation of across studentized returns is frequently rejected across the board. Once we account for the market factor, and test the hypothesis of constant correlation for the idiosyncratic studentized returns, w_{it}, we observe that the rejection frequencies are much smaller, these frequencies are presented in the lower panel of Table 5. One exception of the energy sector, where non-constant correlation is quite prevalent. Both for the correlation with other energy stocks and with assets from other sectors. That is the what we get from the rejection frequencies for the NB_c applied to the $\hat{w}_{i,t}\hat{w}_{j,t}$ series reported in the lower panel. Especially, for the cross-sector combinations we have that the market factor in many cases could fully account for the time-varying correlation between individual stocks. Still, it seems that within some sectors (such as Energy, Materials, Financials, and Telecom) there is a need for a sector factor that is allowed to correlate in a time-varying fashion with the individuals stocks.

One key message to take away from Table 5 is that the evidence of time-varying correlations across sectors is greatly reduced by accounting their associations with market returns. Within sectors there is substantial residual time-variation and, evidently, there is a need for additional factors, such as sector specific factors if a larger fraction of the time-variation is to be accounted for.

Table 6: Testing for Zero Correlations (Sorted by GICS)

	Energy	*Materials*	*Industrials*	*Consumer Discretionary*	*Consumer Staples*	*Healthcare*	*Financials*	*Information Technology*	*Telecommun. Services*	*Utilities*
Energy	1.000	0.605	0.304	0.343	0.389	0.153	0.484	0.277	0.326	0.868
Materials		0.943	0.746	0.454	0.223	0.183	0.344	0.176	0.186	0.226
Industrials			0.744	0.545	0.236	0.210	0.298	0.335	0.145	0.142
Consumer Discretionary				0.692	0.299	0.189	0.427	0.259	0.149	0.170
Consumer Staples					0.736	0.283	0.259	0.298	0.130	0.435
Healthcare						0.706	0.157	0.270	0.123	0.162
Financials							0.816	0.238	0.111	0.443
Information Technology								0.853	0.265	0.281
Telecommun. Services									0.889	0.267
Utilities										1.000

Rejection frequencies for the NB_0 test for zero correlation applied to $\hat{w}_{i,t}\hat{w}_{j,t}$.

Table 6 presents the tests for zero correlation. The table reports the rejection frequencies for the NB_0 test applied to the $\hat{w}_{i,t}\hat{w}_{j,t}$ series. Given the results in Table 5 it is not surprising that the test is frequently rejected for assets belonging in the same sector. Across sectors the zero-correlation is also frequently rejected. By introducing sector specific factors it may be possible to explain the correlation structure of stocks within the same sector. Since additional factors would change the definition of the residual studentized returns, it is also plausible that sector specific factors could mitigate the residual correlation we find for assets in different sectors. We shall pursue this issue in future research.

6 Conclusion

In this paper we propose a multivariate GARCH model that utilizes realized measures of volatility and correlation, and entails a complete modeling of their dynamic properties. The model builds on a self-contained system of equations that link realized measures to the appropriate population quantities of volatility and covolatility. The structure implies a dynamic model of the conditional betas, that are popular measures of risk in finance. The proposed framework permits for leverage effects and spillover effects between the assets and the market volatility. In this respect the model combines the flexibility of the GARCH modeling framework with the statistical precision in volatility measurement resulting from the use of high-frequency data.

Importantly, the Realized Beta GARCH model has a hierarchical structure that makes it easy to apply to vast number of assets. The model has a structure where the entire correlation structure is driven by time-varying volatilities and time-varying correlation between each of the assets and the market return.

Our empirical study revealed some interesting features of the cross-sectional variation of the conditional betas, as well as their time-series variation. In particular, we find that the betas exhibit far more variation at a daily frequency – variation that is largely concealed in the rolling-window estimates of β that one can obtain with regression methods using daily returns.

We have proposed Nyblom-type test for constant and zero correlation. In our empirical analysis we found that the Realized Beta GARCH model explains a great deal of the time variation in the correlation structure, but the Nyblom tests revealed significant residual variation in the correlation structure, in particular between assets within the same sector. For this reason, it will be interesting to consider a generalize structure where additional sector specific correlation factors used. We shall pursue this generalization in future research.

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