

Designing an Optimal Public Pension System¹

Takao Fujii *Associate Professor, Kobe University*

Fumiaki Hayashi *Post Graduate Studies, Kobe University*

Jun Iritani *Professor, Fukuyama University*

Kazumasa Oguro *Associate Professor, Hitotsubashi University*

This version: January 7, 2013

¹ We would like to express our sincere gratitude for the invaluable advice received from Professor Takashi Kamihigashi (Kobe University) as well as Senior Economist Ryo Ishida (Policy Research Institute, Ministry of Finance) during the writing of the first draft of this paper.

Abstract

This paper uses a two-period overlapping generations model in order to provide a theoretical design for an optimal public pension system based on a partial equilibrium analysis. Household preferences only depend on two periods consumption and leisure and is homogeneous of degree m with respect to consumption in the working and retired periods. We present characteristic features of an optimal public pension system in this paper. First, differences in the population growth rate do not affect the relative level of the optimal net lifetime burden rate of each generation. Second, if $m \neq 0$ or $m < 1$, the optimal public pension system can be expressed explicitly. Third, the difference between the market time-preference rate and social time-preference rate provides a crucial insight into the optimal burden rate of each generation.

Keywords: Overlapping generations model, public pension, optimal burden rate

JEL codes: D30, D60, D90, H21, H60

1. Introduction

Although considerable research exists on public pension plans, such as the studies by Feldstein (1995, 1998) of the possibility for social welfare improvement based on a transition process from the pay-as-you-go system to the funded system, most work in this area theoretically and empirically

examines the effectiveness and fairness of existing and reformed plans. By contrast, few studies have been carried out on designing an optimal pension plan (but see Oguro, 2008). The present paper thus uses a simplified overlapping generations (OLG) model in order to analyze the characteristics of an optimal public pension system that maximizes the sum of indirect utilities of each generation under its intertemporal budget constraint. The presented theoretical analysis thus aims to bridge this gap in the current body of knowledge on this topic.

Little research exists specifically on optimal public pension plans owing to analytical difficulties. In particular, the characteristics of the two macroeconomic models applied to such research, namely the representative household model and OLG model, hinder the analysis of optimal taxation. Well-known arguments in favor of the former are offered by Barro (1979, 1999) and Bohn (1990) on tax smoothing and by Judd (1999) regarding dynamic optimal taxation. However, as the representative household model does not usually involve generation alternation, it is difficult to analyze a public pension plan (e.g., a pay-as-you-go system), which is an intergenerational income distribution policy.

The OLG models proposed by Samuelson (1958) and Diamond (1965) assume generation alternation and thus facilitate the consideration of public pension plans. However, unlike the representative household model, the key agents in these models increase by number of generations, further complicating the theoretical analysis. Therefore, most research carried out since the seminal

achievements presented by Auerbach and Kotlikoff (1987) tends to adopt a multi-period OLG model for the empirical analysis in order to adapt the influence of current pension plans and reforms to the utility of each generation.

To avoid the summation of indirect utilities diverging to infinity it is necessary for the sequence of utilities in this paper to be discounted using the time-preference rate. We designate this rate the “social time-preference rate” in order to differentiate it from the interest rate, which we term the “market time-preference rate.” In the first step of designing our optimal pension plan, we suggest three simplifications to the traditional OLG model, which can cope with an infinite number of households and goods. First, we ignore the general equilibrium and instead adopt a partial equilibrium analysis. Second, although the pay-as-you-go pension systems in most countries consist of two parts (e.g., the basic pension and the income proportional pension), we recognize only income proportional payments for the sake of simplicity. Third, we use a typical two-period OLG model (e.g., working period and retired period) and assume that each household’s utility depends on two periods of consumption and labor. Thus, we assume that the utility function is homogeneous of degree m with respect to consumption in the working and retired periods.

Assuming the first simplification above allows us to set wage income and the interest rate as exogenous variables. By contrast, wage income and the interest rate become a function of the pension benefit rate and premium rate in the general equilibrium model, rendering optimal pension

plan characteristics extremely difficult to analyze. By considering the second simplification, the analysis of the optimal pension plan becomes the determination of the net lifetime burden rate of each generation, as pension benefits and premiums are a fixed proportion of lifetime wage income under the lifetime budget constraint of households. Therefore, the proposed optimal pension plan design resembles the discussions of Barro (1979, 1999), Bohn (1990), and Judd (1999). Finally, the third simplification clarifies the social welfare function, as shown in Lemmas 1 and 2 in Section 2.

The following points summarize the characteristics of the optimal pension plan proposed in this paper:

(1) The differences in population growth rates do not affect the relative level of the net lifetime burden rate of each generation.

(2) If $m \neq 0$ or $m < 1$,

(2-1) The optimal pension system can be expressed explicitly.

(2-2) The optimal pension plan decreases the net lifetime burden rate of the generation with the higher growth rate of lifetime wage income.

(2-3) If the social time-preference rate is sufficiently larger than the market time-preference rate, then an increase in the population growth rate of some generations would reduce the net lifetime burden rate of each generation under an optimal pension system.

(2-4) If the social time-preference rate is larger (smaller) than that of the market, the net lifetime

burden rate of future generations would be closer to 100% (increase the current generation's net lifetime burden, while reducing the net lifetime burden of future generations).

(2-5) If the social time-preference rate is equal to that of the market, the net lifetime burden rate of future generations under an optimal pension plan would converge to the same level (i.e., the smoothing of the net lifetime burden of each generation holds at a certain point in time).

(3) If $m \geq 1$, with the possible exception of one generation, the net lifetime burden of each generation would become either 100% or 0%.

Point (1) is crucial for the pension plan design presented herein. Although Japan's declining birthrate implies an increasing pension burden on future generations from a conventional viewpoint, (1) suggests that when comparing the net lifetime burden of one generation to that of another, the relative level of the net burden should remain unaffected by a change in the population growth rate under an optimal pension system. By contrast, as described in (2-3), the population growth rate affects the net lifetime burden rate for all generations. Further, (2-2) indicates that a "regressive" plan is desirable for the net lifetime burden of each generation under an optimal pension system.

This study's analysis is based on the assumption that each generation behaves selfishly without displaying dynastic altruism. Barro (1974) argues that when there exist inheritance transfers among generations and intergenerational altruism, the utility of each generation is influenced not only by its consumption but also by the descendants' utility, meaning that the OLG model is fundamentally

equivalent to the representative household model. However, Takayama, Aso, Miyaji, and Kamiya (1996) and Horioka (2002) show that intergenerational altruism is rarely evident in the analysis of inheritance distribution.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 defines the optimal pension plan as the maximization of social welfare. Section 4 analyzes the relationship between the funded system and the net lifetime burden rate of each generation in order to demonstrate that the argument for the optimal plan under the pay-as-you-go system can be fully applied to that under the funded system. Section 5 concludes.

2. Consumer Choices and Government Budget

2.1 Consumer Decisions

Each generation lives for two periods (i.e., working period and retired period). The utility for the generation born in period t is as follows:

$$u(c_{1,t}, c_{2,t+1}, h_t), \quad u: R_+^3 \rightarrow R \quad (1)$$

Here, $c_{1,t}$ and $c_{2,t+1}$ represent the consumption at period t and at $t+1$, respectively, while h_t represents leisure. In addition, we assume that the utility function satisfies Assumption 1:

Assumption 1: u is continuous in the domain. For interior points, u is a strictly increasing and

strictly quasi-concave function, and homogeneous of degree m with consumption levels $(c_{1,t}, c_{2,t+1})$.

Except for the assumption of homogeneous of degree m with the consumptions above, the utility function is satisfied with the normal desirable characteristics. We also assume $m \neq 0$. If $0 < m < 1$, the concave function $(c_{1,t}^\alpha c_{2,t+1}^\beta h_t^\gamma, \alpha + \beta + \gamma < 1)$ in the Cobb–Douglas form can be included in the same utility function class. The symbol τ_t represents the pension premium rate for wage income (w_t) and ξ_{t+1} represents the pension benefit rate, expressing the pension to be received in multiples of the premium rate (τ_t). The upper limit of leisure provided during the working period is assumed to be 1. Further, by representing the interest rate as r , the budget constraint of generation t will be the following:

$$c_{1,t} + s_t = (1 - \tau_t)w_t(1 - h_t); \quad c_{2,t+1} = (1 + r)s_t + \xi_{t+1}\tau_t w_t(1 - h_t),$$

where s_t is the saving. These can be consolidated into a single budget as:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r} + (1 - \theta_t)\tau_t w_t h_t = (1 - \theta_t)w_t \quad (2)$$

Here, θ_t represents the net lifetime burden rate of generation t as follows:

$$\theta_t = \tau_t - \frac{\xi_{t+1}\tau_t}{1+r}$$

This can be rewritten as $\xi_{t+1}\tau_t = (\tau_t - \theta_t)(1+r)$. Under the condition that r , w_t , and θ_t are given, each generation maximizes its utility through the selections of $c_{1,t}$, $c_{2,t+1}$, and h_t . Then, if written

without the time index t , this becomes the following:

$$\max u(c_1, c_2, h), \text{ subject to } c_1 + \frac{c_2}{1+r} + (1-\theta)zwh = (1-\theta)w \quad (3)$$

We denote the solution of (3) as $c_1((1-\theta)w, r)$, $c_2((1-\theta)w, r)$, and $h((1-\theta)w, r)$. Further, it is

obvious that the following condition holds:

$$\forall t, \forall t' \begin{cases} c_{1,t}((1-\theta)w, r) = c_{1,t'}((1-\theta)w, r) \\ c_{2,t+1}((1-\theta)w, r) = c_{2,t'+1}((1-\theta)w, r) \\ h_t((1-\theta)w, r) = h_{t'}((1-\theta)w, r) \end{cases}$$

(4)

This means that the functional form is the same when time is also expressed as $c_{1,t}((1-\theta)w, r)$,

$c_{2,t+1}((1-\theta)w, r)$, and $h_t((1-\theta)w, r)$. The papers of Oguro (2008) and Oguro, Nakakarumai, and

Takama (2007) show that when $m=1$, the function h_t is not dependent on w_t and that the

functions $c_{1,t}$ and $c_{2,t+1}$ are separable from w_t . In the same manner, we also show that a similar

lemma also holds when $m \neq 0$. As shown below, this simplifies the formula of the social welfare

function.

Lemma 1: If the utility function satisfies Assumption 1, then solution h_t becomes a function of the

interest rate (r) alone and the solutions $c_{1,t}$ and $c_{2,t+1}$ are separable from $(1-\theta_t)w_t$. In other

words, the following holds:

$$\begin{cases} h_t((1-\theta)w_t, r) = h_t(1, r) \\ c_{1,t}((1-\theta)w_t, r) = (1-\theta)w_t c_{1,t}(1, r) \\ c_{2,t+1}((1-\theta)w_t, r) = (1-\theta)w_t c_{2,t+1}(1, r) \end{cases} \quad (5)$$

Proof: In order to solve the problem of (3), we considered the following problem:

$$\max_{\phi_1, \phi_2, h} u(\phi_1, \phi_2, h), \text{ subject to } \phi_1 + \frac{\phi_2}{1+r} + h = 1 \quad (6)$$

Let us denote solutions to the problem (3) and to (6) as (c_1^*, c_2^*, h^*) and $(\hat{\phi}_1, \hat{\phi}_2, \hat{h})$, respectively.

As the pair $(c_1^*/(1-\theta)w, c_2^*/(1-\theta)w, h^*)$ satisfies the budget in (6), the relation

$$u(\hat{\phi}_1, \hat{\phi}_2, \hat{h}) \geq u(c_1^*/(1-\theta)w, c_2^*/(1-\theta)w, h^*) = u(c_1^*, c_2^*, h^*)/\{(1-\theta)w\}^m \text{ holds.}$$

Conversely, as the pair $((1-\theta)w\hat{\phi}_1, (1-\theta)w\hat{\phi}_2, \hat{h})$ satisfies the budget in (3),

$$u(c_1^*, c_2^*, h^*) \geq u((1-\theta)w\hat{\phi}_1, (1-\theta)w\hat{\phi}_2, \hat{h}) = u(\hat{\phi}_1, \hat{\phi}_2, \hat{h})\{(1-\theta)w\}^m \text{ holds and, therefore, we can obtain}$$

$$u(c_1^*, c_2^*, h^*) = u((1-\theta)w\hat{\phi}_1, (1-\theta)w\hat{\phi}_2, \hat{h}). \text{ Since the utility function is strictly convex, the solution to}$$

maximization is unique. Therefore, the following holds:

$$c_1^* = (1-\theta)w\hat{\phi}_1, \quad c_2^* = (1-\theta)w\hat{\phi}_2, \quad h^* = \hat{h}$$

As (6) does not include $(1-\theta)w$ as a parameter, we can confirm the following:

$$c_i((1-\theta)w, r) = c_i^* = (1-\theta)w\hat{\phi}_i = (1-\theta)wc_i(1, r), \quad i=1, 2$$

$$h((1-\theta)w, r) = h^* = \hat{h} = h(1, r)$$

The above establishes the proof for (5).

Expressing utility as indirect utility and using Lemma 1, we obtain the following:

$$V_t(\theta_t, r) = u(c_{1,t}((1-\theta_t)w_t, r), c_{2,t+1}((1-\theta_t)w_t, r), h_t((1-\theta_t)w_t, r)) = \{(1-\theta_t)w_t\}^m \times \mu$$

$$\text{where } \mu \equiv u(c_1(1, r), c_2(1, r), h(1, r)) \quad (7)$$

When $m < 1$, indirect utility is a strictly increasing function of $1 - \theta$, and becomes a strictly concave function. In addition, in an environment that regards r as fixed, it becomes a function of disposable income $((1 - \theta)w_t)$ alone.

2.2 Government Budget

An economy starts at period $t = 0$, when the working and retired generations already exist. In period $t = 0$, the government determines income transfer G to the retired generation, which is funded by tax revenue and the issuance of government bonds (B_0). At the same time, the government establishes a pay-as-you-go pension system with τ_t and ξ_{t+1} ($t=0, 1, 2, \dots$). τ_t represents the pension premium rate for the working generation in period t . ξ_{t+1} is the pension benefit rate that the working generation t receives upon retiring expressed as a percentage of the pension premium paid by this generation. The amount of government bonds issued is an endogenous variable that is determined in order to satisfy the government budget constraint. By contrast, a pension plan with τ_t and ξ_{t+1} ($t=0, 1, 2, \dots$) is assumed to be given. Thus, denoting the population of generation t as L_t , the government budget constraint is the following at period $t = 0$:

$$G = \tau_0 w_0 (1 - h_0) L_0 + B_0 \quad (8)$$

This can be considered to be the definition of B_0 . When $t \geq 1$, the government budget constraint is:

$$\xi_t \tau_{t-1} w_{t-1} (1 - h_{t-1}) L_{t-1} + (1 + r) B_{t-1} = \tau_t w_t (1 - h_t) L_t + B_t \quad (9)$$

Each B_t is endogenously determined so that equations (8) and (9) hold. By defining such endogenous variables, we can use τ_t and ξ_{t+1} as the policy parameters.

Assumption 2: Under the condition that the interest rate (r), wage income growth rate (g_t), and population growth rate (n_t) in period t , as well as the initial values (w_0 and L_0), are given, wage income in period t and the population of generation t are represented as follows:

$$w_t = w_0 \prod_{j=0}^{t-1} (1 + g_j), \quad L_t = L_0 \prod_{j=0}^{t-1} (1 + n_j).$$

In addition, there exists a positive real number (q) such that $0 < q < 1$, and the following condition exists:

$$q(1+r) > (1+g_t)(1+n_t) \quad \text{for } t=1, 2, 3, \dots$$

First, this assumption signifies the following. If $(1+r) < (1+g_t)(1+n_t)$ holds at any given time, the national income growth rate ($g_t + n_t$) is greater than the interest rate (r) on government bonds. Thus, adopting a pay-as-you-go pension system increases the utility of the current generation without decreasing that of the future generation. In addition, the discounted value of national income in the future period will diverge to infinity. In this case, it is unnecessary to consider the burden on the pension plan; by definition, this type of economy does not require the careful consideration of an optimal public pension system.

Next, w_t is assumed to converge to a positive value: $\lim_{t \rightarrow \infty} w_t = w > 0$.

This assumption is not considered to be a strict condition, because the price of goods in our model is normalized to be unity.

Note: The following relationship holds: $\theta_t > 0$ if and only if $1 > \xi_t/(1+r)$, for $t=1, 2, 3, \dots$. We must consider the case $1 \leq \xi_t/(1+r)$ when $\theta_t \leq 0$.

Based on the above considerations, the following lemma is achieved. In this lemma, h_t is the demand function of leisure ($h_t((1-\theta_t)w_t, r)$).

Lemma 2: Suppose that the pension premium rate ($\tau_t, t=1, 2, 3, \dots$) satisfies budgets (8) and (9), and that the No Ponzi Game condition holds as follows:

$$\lim_{t \rightarrow \infty} \frac{B_{t+1}}{(1+r)^{t+1}} = 0 \quad (10)$$

Then, if the government budget constraints (8) and (9) hold, the following also holds:

$$\sum_{t=0}^{\infty} \frac{\theta_t I_t}{(1+r)^t} = G \quad (11)$$

$$\text{where } \theta_t = \tau_t - \frac{\xi_{t+1} \tau_t}{1+r} \text{ and } I_t = w_t(1-h_t)L_t \text{ (} t=1, 2, 3, \dots \text{)}.$$

Conversely, if there are values ($\theta_t \leq 1, t=1, 2, 3, \dots$) that satisfy (11), then there exist τ_t and B_t ($t=1, 2, 3, \dots$) such that (8) and (9) are satisfied, and the No Ponzi Game condition (10) holds.

Proof: (Necessity) By rewriting (9), we obtain the following:

$$(\tau_{t-1} - \theta_{t-1})w_{t-1}(1 - h_{t-1})L_{t-1} + B_{t-1} = \tau_t w_t(1 - h_t)L_t / (1 + r) + B_t / (1 + r)$$

Then, reorganizing by writing the total wage income of generation t as $I_t = w_t(1 - h_t)L_t$, we obtain

the following:

$$\theta_{t-1}I_{t-1} = \tau_{t-1}I_{t-1} + B_{t-1} - \tau_t I_t / (1 + r) - B_t / (1 + r)$$

From the above equations with $t=1, 2, 3, \dots, n$ and equation (8), we obtain the following:

$$\left\{ \begin{array}{l} 0 = G - \tau_0 I_0 - B_0 \\ \theta_0 I_0 = \tau_0 I_0 + B_0 - \frac{\tau_1 I_1}{1+r} - \frac{B_1}{1+r} \\ \frac{\theta_1 I_1}{1+r} = \frac{\tau_1 I_1}{1+r} + \frac{B_1}{1+r} - \frac{\tau_2 I_2}{(1+r)^2} - \frac{B_2}{(1+r)^2} \\ \frac{\theta_2 I_2}{(1+r)^2} = \frac{\tau_2 I_2}{(1+r)^2} + \frac{B_2}{(1+r)^2} - \frac{\tau_3 I_3}{(1+r)^3} - \frac{B_3}{(1+r)^3} \\ \vdots \\ \frac{\theta_n I_n}{(1+r)^n} = \frac{\tau_n I_n}{(1+r)^n} + \frac{B_n}{(1+r)^n} - \frac{\tau_{n+1} I_{n+1}}{(1+r)^{n+1}} - \frac{B_{n+1}}{(1+r)^{n+1}} \end{array} \right. \quad (12)$$

Adding the above results in:

$$\sum_{t=0}^n \frac{\theta_t I_t}{(1+r)^t} = G - \frac{\tau_{n+1} I_{n+1}}{(1+r)^{n+1}} - \frac{B_{n+1}}{(1+r)^{n+1}}$$

In addition, the following also holds:

$$\frac{\tau_{n+1} I_{n+1}}{(1+r)^{n+1}} \leq \frac{\tau_{n+1} w_{n+1} L_{n+1}}{(1+r)^{n+1}} = \tau_{n+1} w_0 L_0 \prod_{t=0}^n \frac{(1+g_t)(1+n_t)}{(1+r)}$$

Therefore, by Assumption 2 we know $(1+g_t)(1+n_t)/(1+r) < q < 1$. This implies that the above term

converges to zero as n tends to infinity. Furthermore, by using the No Ponzi Game condition (10),

the government budget constraint can be summarized as:

$$\sum_{t=0}^{\infty} \frac{\theta_t I_t}{(1+r)^t} = G$$

(Sufficiency) The initial income transfer G is given and we set $\theta_t \leq 1$ ($t=1, 2, 3, \dots$) to satisfy equation (11). If $\theta_t = 1$, we can satisfy $\theta_t = \tau_t - \xi_{t+1} \tau_t / (1+r)$ by using $\tau_t = 1$ and $\xi_t = 0$. If $\theta_t < 1$, we can define τ_t so as to satisfy $\tau_t < 1$ and $\theta_t = \tau_t - \xi_{t+1} \tau_t / (1+r)$, by choosing a sufficiently small positive value for ξ_t .

Thus, we can obtain τ_t and ξ_t for each period that satisfies $\theta_t = \tau_t - \xi_{t+1} \tau_t / (1+r)$, from the given value of θ_t . In this case, we define B_t as follows:

$$B_0 = G - \tau_0 I_0$$

$$B_{t+1} = (\tau_t - \theta_t) I_t (1+r) - \tau_{t+1} I_{t+1} + (1+r) B_t, \quad t=1, 2, 3, \dots$$

Then, we obtain the following:

$$G = \tau_0 I_0 + B_0$$

$$\theta_t I_t = B_t - \frac{B_{t+1}}{1+r} + \tau_t I_t - \frac{\tau_{t+1} I_{t+1}}{1+r}, \quad t=1, 2, 3, \dots$$

Therefore, the following also holds:

$$\sum_{t=0}^n \frac{\theta_t I_t}{(1+r)^t} = G - \frac{\tau_{n+1} I_{n+1}}{(1+r)^{n+1}} - \frac{B_{n+1}}{(1+r)^{n+1}}$$

The above equation satisfies the government budget, including bonds, for each period. Then, because $\tau_{n+1} I_{n+1} / (1+r)^{n+1} \rightarrow 0$ holds if $n \rightarrow \infty$ and θ_t satisfies $\sum_{t=0}^{\infty} \theta_t I_t / (1+r)^t = G$, the No Ponzi

Game condition is satisfied as follows:

$$\lim_{t \rightarrow \infty} \frac{B_{t+1}}{(1+r)^{t+1}} = 0 \quad \blacksquare$$

This lemma does not necessarily guarantee that the set of $(\theta_t)_{t=0}^{\infty}$, which satisfies (11), is non-null.

Therefore, we propose the following hypothesis:

Assumption 3: $\sum_{t=0}^{\infty} I_t / (1+r)^t > G$ holds.

By contrast, as shown below, the series $\sum_{t=0}^{\infty} I_t / (1+r)^t$ is finite.

3. Maximization of Social Welfare

By using the indirect utility of each generation, we define the social welfare function as follows:

$$W(\theta_0, \theta_1, \theta_2, \dots) = \sum_{t=0}^{\infty} \beta_t V_t(\theta_t, r), \text{ where } \beta_0 = L_0 \text{ and } \beta_t = \frac{L_0 \prod_{j=0}^{t-1} (1+n_j)}{(1+R)^t}, \quad t=1, 2, 3, \dots$$

Here, R represents the social time-preference rate. Applying the results from (7) from Section 2, we

obtain the following:

$$W(\theta_0, \theta_1, \theta_2, \dots) = \mu \sum_{t=0}^{\infty} \beta_t \{(1-\theta_t)w_t\}^m, \quad \mu \equiv u(c_1(1,r), c_2(1,r), h(1,r))$$

In addition,

$$I_t = w_t(1-h_t((1-\theta_t)w_t, r))L_t = w_t(1-h(1,r))L_t, \quad t=1, 2, 3, \dots$$

allows us to regard I_t as an exogenous variable. Based on this setting, we can consider the

following problem:

$$\max_{\theta_0, \theta_1, \theta_2, \dots} W(\theta_0, \theta_1, \theta_2, \dots) \quad \text{subject to} \quad \sum_{t=0}^{\infty} \frac{\theta_t I_t}{(1+r)^t} = G$$

which addresses an optimal public pension system. By using $\gamma_t = I_t / (1+r)^t$ ($t=1, 2, 3, \dots$), the

above problem is rewritten as follows:

$$\max_{\theta_0, \theta_1, \theta_2, \dots} \mu \sum_{t=0}^{\infty} \beta_t \{(1-\theta_t)w_t\}^m \quad \text{subject to} \quad \sum_{t=0}^{\infty} \gamma_t \theta_t = G \quad (13)$$

For this problem, we allow $\theta_t < 0$.

Assumption 4: There exists the solution $(\theta_t^*)_{t=0}^{\infty}$ in (13).

This assumption requires R to be an appropriately positive value. We analyze the case with $m < 1$

and $m \neq 0$ in Section 3.1 and the case with $m \geq 1$ in Sections 3.2 and 3.3.

3.1 The case with $m < 1$ and $m \neq 0$

First, $\partial V(\theta_t, r) / \partial \theta_t = -\mu m (1-\theta_t)^{m-1} w_t^m$ holds. As $\lim_{\theta_t \rightarrow 1} \partial V(\theta_t, r) / \partial \theta_t = -\infty$, there are no t 's satisfying

$\theta_t^* = 1$.² Therefore, the solution in (13) is an interior solution and must satisfy the following by

² In the case of $m < 0$, the utility value will be negative because of the monotonic and homogeneous characteristics of the utility function. Therefore, we must keep $\mu < 0$ and $\mu m > 0$ in mind. A utility function of $m < 0$, for example,

$$\sum_{t=s}^n (1+\delta)^{-(i-s)} \frac{C_s^{1-1/\nu}}{1-1/\nu}, \quad \nu < 1$$

can be used often in a pension simulation. When $m > 0$, it holds that $\mu > 0$ and $\mu m > 0$.

denoting the Lagrange multiplier as δ :

$$\mu m w_t \beta_t \{(1-\theta_t)w_t\}^{m-1} = \delta \gamma_t, \quad t=1, 2, 3, \dots$$

By canceling out μ and δ , we obtain the following:³

$$\frac{w_t}{w_{t+1}} \frac{\beta_t \{(1-\theta_t)w_t\}^{m-1}}{\beta_{t+1} \{(1-\theta_{t+1})w_{t+1}\}^{m-1}} = \frac{\gamma_t}{\gamma_{t+1}},$$

$$\frac{1-\theta_t^*}{1-\theta_{t+1}^*} = (1+g_t) \left(\frac{1+R}{1+r} \right)^{1/(1-m)}, \quad t=1, 2, 3, \dots$$

This results in

$$\theta_t^* < \theta_{t+1}^* \quad \text{if and only if} \quad (1+g_t) \left(\frac{1+R}{1+r} \right)^{1/(1-m)} < 1 \quad (14)$$

In general, a comparison of two points in time t and t' ($t < t'$), results in the following:

$$\frac{1-\theta_t^*}{1-\theta_{t'}^*} = \left(\frac{1+R}{1+r} \right)^{(t'-t)/(1-m)} \prod_{j=t}^{t'-1} (1+g_j) \quad (15)$$

These results signify the following concerning a comparison of generational burdens.

Theorem 1 When $m < 1$ and $m \neq 0$ hold, the following characteristics hold in a comparison of optimal generational burdens:

(i) Assume that there is no change in the wage growth rate. If R is larger than r (r is larger than R), the larger the value, a higher (lower) net lifetime burden rate of the relative future generation (generation t'), in comparison to the net lifetime burden rate of the past generation

³ Attention must be paid to the following equalities as well:

$$\frac{w_t}{w_{t+1}} = \frac{1}{1+g_t}, \quad \frac{\gamma_t}{\gamma_{t+1}} = \frac{1+r}{(1+g_t)(1+n_t)}, \quad \frac{\beta_t}{\beta_{t+1}} = \frac{1+R}{1+n_t}$$

(generation t), is desirable.

(ii) Assume that the interest rate is equal to the social time-preference rate. A relatively low net lifetime burden rate of the generation with the higher wage income increase rate is desirable.

(iii) The relative level of the net lifetime burden rate of each generation is not influenced by the population growth rate.

The significance of (iii) is critical, as it asserts that a change in the population growth rate should not affect the relative burden of the public pension system compared with the net lifetime burdens for each generation. In addition, (ii) argues that the burden on the generation that has a low wage income increase rate should be increased and affirms that “a regressive system for pension burden is desirable.”

Two items need to be noted concerning Theorem 1. First, it indicates that population growth does not influence the relative value of optimal burden rates. In other words, it implies that population growth may influence the absolute value of the burden. Second, Theorem 1 indicates that (i) holds if there exists an optimal solution; therefore, it cannot be determined at this point whether the inequality “ $r > R$ ” itself is consistent or not with a configuration of the problem.

The relative level of the net lifetime burden rate does not always clarify the degree of influence from the change in the population growth rate or explain what the results of (i) reveal about optimization.

Let us thus pursue an explicit form of solution in order to clarify these factors. We know that (15)

holds for $t=1, 2, 3, \dots$. Then we obtain the following:

$$1 - \theta_t^* = \alpha_t (1 - \theta_0^*), \quad \alpha_t = \frac{1}{\prod_{j=0}^{t-1} (1 + g_j)} \left(\frac{1+r}{1+R} \right)^{t/(1-m)} = \frac{w_0}{w_t} \left(\frac{1+r}{1+R} \right)^{t/(1-m)}$$

Here, $\alpha_0 = 1$. Using the constraints of an optimized pension plan, we define α as follows:

$$\alpha = \sum_{t=0}^{\infty} \gamma_t \alpha_t$$

In this case, we have

$$\sum_{t=0}^{\infty} \gamma_t - G = \sum_{t=0}^{\infty} \gamma_t (1 - \theta_t^*) = \sum_{t=0}^{\infty} \gamma_t \alpha_t (1 - \theta_0^*)$$

and, therefore, we have the explicit solution of the optimal burden:

$$\theta_0^* = 1 - \frac{\sum_{j=0}^{\infty} \gamma_j - G}{\alpha} \tag{16}$$

$$\theta_t^* = 1 - \alpha_t \frac{\sum_{j=0}^{\infty} \gamma_j - G}{\alpha}, \quad t=0, 1, 2, 3, \dots \tag{17}$$

At this point, we assume that values for α and $\gamma = \sum \gamma_j$ exist. Subsections 3.1.1 and 3.1.2

consider the existence of α and γ . By setting $x = ((1+r)/(1+R))^{1/(1-m)}$, we can obtain:

$$\begin{aligned} \frac{\partial \alpha}{\partial n_i} &= \frac{I_0}{1+n_i} \left(\prod_{j=0}^i \frac{1+n_j}{1+r} x^{i+1} + \prod_{j=0}^i \frac{1+n_j}{1+r} x^{i+2} + \dots \right) \\ \frac{\partial \gamma}{\partial n_i} &= \frac{I_0}{1+n_i} \left(\prod_{j=0}^i \frac{(1+g_j)(1+n_j)}{1+r} + \prod_{j=0}^i \frac{(1+g_j)(1+n_j)}{1+r} + \dots \right) \end{aligned}$$

Compared with the corresponding entries, we find:

$$\prod_{j=0}^{t-1} \frac{1+n_j}{1+r} x^t - \prod_{j=0}^{t-1} \frac{(1+g_j)(1+n_j)}{1+r} = \prod_{j=0}^{t-1} \frac{1+n_j}{1+r} \left(x^t - \prod_{j=0}^{t-1} (1+g_j) \right)$$

We assume for each $t = i + 1, i + 2, \dots$

$$x \leq \left(\prod_{j=0}^{t-1} (1+g_j) \right)^{1/t} = (1+\bar{g}_t)$$

Here, \bar{g}_t represents the average growth rate of t years of real wages. In this case, we obtain the following:

$$0 < \frac{\partial \alpha}{\partial n_i} \leq \frac{\partial \gamma}{\partial n_i}$$

Specifically, assume that x is not large and that the growth rate of wage income for generation $i + 1$ and later is high. Then, an increase in n_i enlarges γ more than it enlarges α , which boosts $(\gamma - G)/\alpha$. Accordingly, the burden rate is comprehensively reduced. The following theorem summarizes these results:

Theorem 2 When $m < 1$ and $m \neq 0$ hold, the following characteristics hold for the net lifetime burden rate of each generation.

(iv) The optimal net lifetime burden rates of the public pension system for each generation are expressed in equations (16) and (17).

(v) Let $x = ((1+r)/(1+R))^{1/(1-m)}$ and \bar{g}_{i+1} be the average real wage growth rate from period 0 to generation i . If $x < 1 + \bar{g}_{i+j}, j = 1, 2, 3, \dots$ hold, the growth rate increase for generation i will reduce

each net lifetime burden rate.

Theorems 1 and 2 allow us to perform a relative generational comparison of an optimal pension system with the characteristics of the net lifetime burden rate. Next, we shall examine movements of the optimal pension system through time.

3.1.1 Value of $\sum_j \gamma_j$

Here, we consider the value of $\sum_j \gamma_j$ which appears in equations (16) and (17). According to

Assumption 2, we have

$$\sum_{j=0}^{\infty} \gamma_j = I_0 + \frac{I_1}{1+r} + \frac{I_2}{(1+r)^2} + \dots < w_0 L_0 (1+q+q^2+\dots) = \frac{w_0 L_0}{1-q} < \infty$$

and thus we know $\sum_j \gamma_j - G$ is a finite number.

3.1.2 Values of α and α_t

As the solution $(\theta_t^*)_{t=0}^{\infty}$ satisfies budget (11),

$$\sum_{j=0}^{\infty} \gamma_j - G = \lim_{t \rightarrow \infty} (1 - \theta_0^*) \sum_{j=0}^t \gamma_j \alpha_j$$

is valid and $\theta_0^* \leq 1$ holds. Suppose $\theta_0^* = 1$. The right-hand side of the equation is zero. As the left-hand side is a positive value, this is a contradiction. Accordingly, $\theta_0^* < 1$ must be valid. In addition, when $\alpha = \sum_t \gamma_t \alpha_t = \infty$, the right-hand side becomes an infinite value and contradicts

Assumption 2. Thus, $\alpha < \infty$ must also be valid.

Next, we confirm the conditions that validate $\alpha < \infty$. To begin with,

$$\alpha = \sum_{t=0}^{\infty} \alpha_t \gamma_t = I_0 \left(1 + \sum_{t=1}^{\infty} \left(\frac{1+r}{1+R} \right)^{t/(1-m)} \prod_{j=0}^{t-1} \frac{1+n_j}{1+r} \right)$$

Now, putting $x = ((1+r)/(1+R))^{1/(1-m)}$, $a_0 = 1$, $a_t = \prod_{j=0}^{t-1} (1+n_j)/(1+r)$, $t = 1, 2, 3, \dots$ we can

rewrite the above equality as follows:

$$\frac{\alpha}{I_0} = a_0 + a_1 x + a_2 x^2 + \dots$$

The right-hand side of this equation is a power series. According to Cauchy–Hadamard’s theorem,

the convergence radius k of this power series can be expressed as:⁴

$$\frac{1}{k} = \limsup_{t \rightarrow \infty} |a_t|^{1/t}$$

In addition, the inequality

$$a_t^{1/t} = \left(\frac{(1+n_0) \times \dots \times (1+n_{t-1})}{(1+r)^t} \right)^{1/t} \leq q$$

holds for a sufficiently large t .⁵ Therefore, $k \geq 1/q (> 1)$. Two cases can be distinguished as follows:

(A) if $x > k$, then $\alpha = \infty$.

(B) if $x < k$, then $0 < \alpha < \infty$.

The possibility of $\alpha = \infty$, which was excluded from Theorem 2. This indicates that (A) above must

also be eliminated. Furthermore, we have

⁴ See Yoshida (1965, p. 15).

⁵ We are grateful to Ryo Ishida for pointing out that the sufficiently large t restriction is essential.

$$\frac{\alpha_t}{\alpha} = \frac{(w_0 / w_t)x^t}{I_0(a_0 + a_1x + a_2x^2 + \dots)}$$

Assuming w_t converges to w , and using the results from (B), we have three differing cases of parameters that determine the temporal value of an optimal pension,

(case 1) If $k > x > 1$, then $\alpha_t / \alpha \rightarrow \infty$

(case 2) If $1 = x$, then $\alpha_t / \alpha \rightarrow \left\{ w(1-h) \sum_{j=0}^{\infty} a_j \right\}^{-1}$

(case 3) If $1 > x > 0$, then $\alpha_t / \alpha \rightarrow 0$

By expressing the average value of period t as \bar{n}_t for the change in the population growth rate from period 0 to $t-1$, we obtain the following:

$$((1+n_0) \times \dots \times (1+n_{t-1}))^{1/t} = 1 + \bar{n}_t, t = 1, 2, 3, \dots$$

If $\bar{n}_t, t = 1, 2, \dots$ converges to \bar{n} , we obtain the following:

$$k = \frac{1+r}{1+\bar{n}} > \frac{1}{q}$$

3.1.3 Value of θ_t^*

The burden rates for each generation in (case 1) to (case 3) above are described below. r is the market interest rate, which thus expresses the market time-preference rate. Accordingly, x is the ratio of the market to social time-preference rates.

Let us consider (case 1) $k > x > 1$. Here, the following apply:

$$\theta_0^* > 0, \exists \bar{t}: \forall t \geq \bar{t}, \theta_t^* < 0, \lim_{t \rightarrow \infty} \theta_t^* = -\infty$$

Since $x^{1-m} = (1+r)/(1+R)$, $x > 1$ indicates $r > R$. At this point, the social time-preference rate is lower than the market time-preference rate and the market may be more short-sighted than society.

In this case, the results show that it is preferable to place a larger burden on the current generation and reduce the burden on future generations by offering subsidies.

Next, let us consider (case 2) $x=1$. This case implies the market and social time-preference rates coincide and that society trusts the market. The burden rate θ_t^* for each generation is as follows:

$$\theta_0^* = 1 - \frac{\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{(1+n_j)(1+g_j)}{(1+r)} - G/I_0}{\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{(1+n_j)}{(1+r)}}$$

$$\theta_t^* = 1 - \frac{1}{\prod_{j=0}^{t-1} (1+g_j)} \frac{\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{(1+n_j)(1+g_j)}{(1+r)} - G/I_0}{\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{(1+n_j)}{(1+r)}}$$

Based on this, according to the cases $g_t > 0$, $g_t = 0$, or $g_t < 0$, the following

$$\theta_t^* < \theta_{t+1}^*, \theta_t^* = \theta_{t+1}^*, \theta_t^* > \theta_{t+1}^*$$

must hold respectively. Furthermore, θ_t^* converges to a positive value θ^* .

$$\theta^* = 1 - \frac{1}{\prod_{j=0}^{\infty} (1+g_j)} \frac{\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{(1+n_j)(1+g_j)}{(1+r)} - G/I_0}{\sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \frac{(1+n_j)}{(1+r)}}$$

Finally, let us consider (case 3) $1 > x > 0$. This indicates that the social time-preference rate is higher than that of the market and that society is more short-sighted than the market. The burden rate θ_t^* for each generation is as follows. At this point,

$$\theta_t^* \rightarrow 1 \text{ as } t \rightarrow \infty$$

and it is thus preferable to place the burden on future generations.

Theorem 3 When $m < 1$ and $m \neq 0$ hold, the following characteristics hold for the time series value of the optimal burden rate for each generation.

(vi) When the market time-preference rate exceeds the social time-preference rate, a negative value is preferable for the net lifetime burden rate after a certain generation in order to increase future consumption.

(vii) When the market and social time-preference rates match, the optimal burden rates will converge at a certain value.

(viii) When the market time-preference rate falls below the social time-preference rate, a nearly 100% burden rate is preferable for future generations.

Of the above results, (vii) can be considered to extend the tax smoothing theorems proposed by Barro (1979, 1999) and Bohn (1990). First, their research assumes $R = r$, which is equivalent to the

assumption in (vii) of $x = 1$. Second, G is assigned to the initial period in the current model, which is completely plausible in this discussion even when G is considered to be the total public expenditure of each period (discounted value). Finally, if the limit rate of burden $\frac{1 - (\sum \gamma_j - G) \left\{ w(1-h) \sum_{j=0}^{\infty} a_j \right\}^{-1}}$ is a positive value and time t is sufficiently large, the value of θ_t^* lies in a small positive interval, signifying burden smoothing.

3.2 The case with $m > 1$

Until this subsection, the constraint proposed in equation (13) was $\theta_t \leq 1$, but here we presume a series of t for $0 \leq \theta_t \leq 1$. First, let us consider the case in which $m > 1$. Here, we define the following:

$$b_t = \frac{\beta_t W_t^m}{\gamma_t}, t = 1, 2, \dots$$

Let $\theta_t, t = 0, 1, 2, 3, \dots$ be burden rates satisfying constraint (13). We also assume $0 < \theta_t < 1$ and $0 < \theta_{t'} < 1$ for θ_t and $\theta_{t'}$, respectively. Thus, without any loss of generality, we can assume

$$b_t(1 - \theta_t)^{1-m} \geq b_{t'}(1 - \theta_{t'})^{1-m} (> 0)$$

holds. At this time, we assume a series of sufficiently small positive numbers for $\varepsilon > 0$, based on the following:

$$\begin{cases} \tilde{\theta}_t = \theta_t - \frac{\gamma_{t'}}{\gamma_t} \varepsilon \\ \tilde{\theta}_{t'} = \theta_{t'} + \varepsilon \\ \tilde{\theta}_j = \theta_j \quad \text{if } j \neq t, t' \end{cases} \quad (18)$$

This operation is called ε -conversion. Then, because

$$\gamma_t \tilde{\theta}_t + \gamma_{t'} \tilde{\theta}_{t'} = \gamma_t \left(\theta_t - \frac{\gamma_{t'}}{\gamma_t} \varepsilon \right) + \gamma_{t'} (\theta_{t'} + \varepsilon) = \gamma_t \theta_t + \gamma_{t'} \theta_{t'},$$

$\tilde{\theta}_t, j=0, 1, 2, 3, \dots$ satisfies the constraint in (13). According to Taylor's theorem, some form of

$0 < \eta < 1$ exists, resulting in the following:

$$\frac{1}{\mu} (W(\tilde{\theta}_0, \tilde{\theta}_1, \dots) - W(\theta_0, \theta_1, \dots)) = m \varepsilon \gamma_{t'} \{ b_t (1 - \theta_t + \eta \varepsilon \gamma_{t'} / \gamma_t)^{m-1} - b_{t'} (1 - \theta_{t'} - \eta \varepsilon)^{m-1} \}$$

Because we already have $m > 1$ then the relation

$$b_{t'} (1 - \theta_{t'} - \eta \varepsilon)^{m-1} < b_{t'} (1 - \theta_{t'})^{m-1} \leq b_t (1 - \theta_t)^{m-1} < b_t (1 - \theta_t + \eta \varepsilon \gamma_{t'} / \gamma_t)^{m-1}$$

holds for a sufficiently small $\varepsilon > 0$. Therefore, since " $\mu > 0$ " holds in this case, we also obtain

$$W(\tilde{\theta}_0, \tilde{\theta}_1, \dots) - W(\theta_0, \theta_1, \dots) > 0$$

The solution in (13), $\theta_t^*, t = 0, 1, 2, 3, \dots$, must satisfy⁶:

$$\# \{i | 1 > \theta_i^* > 0\} \leq 1$$

This indicates that, with one possible exception, the optimal net lifetime burden rates are either 1 or

0, implying that the solution is an end-point solution.

Theorem 4 When $m > 1$ holds, generational optimal burden rates $\theta_t^*, t = 0, 1, 2, \dots$, with the possible exception of one, are either 1 or 0.

3.3 The case with $m = 1$

⁶ Let A be a set. The symbol " $\#A$ " represents the number of elements in the set A .

Next, we presume a series of t for $0 \leq \theta_t \leq 1$. Define

$$b_t = \frac{\beta_t w_t}{\gamma_t}, t=0, 1, 2, \dots$$

Below, we study this case from two perspectives: $r \neq R$ and $r = R$.

3.3.1 When $r \neq R$

The relation

$$\frac{\beta_t w_t}{\gamma_t} = \frac{L_0 \frac{\prod_{j=0}^{t-1} (1+n_j)}{(1+R)^t} w_0 \prod_{j=0}^{t-1} (1+g_j)}{I_t / (1+r)^t} = \left(\frac{1+r}{1+R} \right)^t \frac{1}{1-h}$$

holds. In other words, $b_t = \beta_t w_t / \gamma_t$, $t=0, 1, 2, \dots$ is strictly monotonic. Through monotonic increases (decreases), the result diverges to infinity (converges to zero). In addition, the fact

$(1+n_t)(1+g_t)/(1+r) < q < 1$ assumed in Assumption 2 leads us to

$$\gamma_t = \gamma_{t-1} \frac{(1+n_{t-1})(1+g_{t-1})}{1+r} < \gamma_{t-1}$$

Accordingly, γ_t , $t = 1, 2, \dots$ is strictly decreasing. At the same time, $\lim_{t \rightarrow \infty} \gamma_t = 0$.

Lemma 3 Let θ_t ; $t=0, 1, 2, \dots$ be a series of burden rates that satisfy the budget $\sum_{t=1}^{\infty} \gamma_t \theta_t = G$. If

the relation

$$b_t = \frac{\beta_t w_t}{\gamma_t} > b_{t'} = \frac{\beta_{t'} w_{t'}}{\gamma_{t'}}, \quad 0 < \theta_t \leq 1, \quad 0 \leq \theta_{t'} < 1$$

holds for two distinct periods t and t' , then welfare will increase as the ε of ε -conversion is

increased until either $\tilde{\theta}_t = 0$ or $\tilde{\theta}_{t'} = 1$ holds.

Proof Assuming $\theta_t; t=0, 1, 2, \dots$ satisfies the budget $\sum_{t=1}^{\infty} \gamma_t \theta_t = G$, t and t' can be expressed by

$$b_t = \frac{\beta_t w_t}{\gamma_t} > b_{t'} = \frac{\beta_{t'} w_{t'}}{\gamma_{t'}}, \quad 0 < \theta_t \leq 1, \quad 0 \leq \theta_{t'} < 1$$

Because $b_{t'} > 0$, we define $\tilde{\theta}_k; k=0, 1, 2, \dots$ according to ε -conversion. These $\tilde{\theta}_k; k=0, 1, 2, \dots$

satisfy the budget in the same manner as previously explained. We also obtain

$$\begin{aligned} & \beta_t w_t (1 - \tilde{\theta}_t) + \beta_{t'} w_{t'} (1 - \tilde{\theta}_{t'}) - \{\beta_t w_t (1 - \theta_t) + \beta_{t'} w_{t'} (1 - \theta_{t'})\} \\ &= \beta_t w_t (\theta_t - \tilde{\theta}_t) + \beta_{t'} w_{t'} (\theta_{t'} - \tilde{\theta}_{t'}) \\ &= \varepsilon \gamma_{t'} (b_t - b_{t'}) > 0 \end{aligned}$$

Therefore, welfare will increase as ε increases until either $\tilde{\theta}_t = 0$ or $\tilde{\theta}_{t'} = 1$ holds.

According to Lemma 3, in order to maximize welfare, the range of burden rates to be studied should

be limited to generations that have a burden rate of either 1 or 0, with one possible exception.

According to Assumption 3, for any natural number n ,

$$G < \sum_{t=0}^n \gamma_t + \sum_{t=n+1}^{\infty} \gamma_t$$

holds. The sets of natural numbers U and Z together with number \bar{t} , are defined below.

First, we look at $b_t; t=0, 1, 2, \dots$ that is monotonically increasing:

$$I = \{n \in N \mid \sum_{t=0}^n \gamma_t \geq G\}$$

$I \neq \emptyset$ is self-evident. Let the first element of Set I be \bar{t} . U and Z are defined as follows:

$$U = \{0, 1, \dots, \bar{t}\}, \quad Z = \{\bar{t} + 1, \bar{t} + 2, \dots\}.$$

Next, we examine b_t ; $t=0, 1, 2, 3, \dots$ that is monotonically decreasing:

$$J = \{n \in N \mid \sum_{t=n}^{\infty} \gamma_t \geq G\}$$

As $J \neq \emptyset$, we denote the last element of set J as \bar{t} . The last element exists since a sequence,

$s_n = \sum_{t=n}^{\infty} \gamma_t$, $n=0, 1, 2, \dots$, is monotonic and converging to zero and since $s_0 > G$. Z and U are

defined as $Z = \{0, 1, \dots, \bar{t} - 1\}$ and $U = \{\bar{t}, \bar{t} + 1, \dots\}$, respectively.

The following then holds for the U and Z defined above:

$$\text{if } t \in U \text{ and } t' \in Z \text{ then } b_t < b_{t'} \quad (19)$$

$$\text{if } t \in U \setminus \{\bar{t}\}, \text{ then } b_t < b_{\bar{t}} \quad (20)$$

Lemma 4 Using the U , Z , and \bar{t} obtained above, we define

$$\begin{aligned} \hat{\theta}_j &= 0, & \text{if } j \in Z \\ &= 1, & \text{if } j \in U \setminus \{\bar{t}\} \\ &= \frac{G - \sum_{h \in U \setminus \{\bar{t}\}} \gamma_h}{\gamma_j}, & \text{if } j = \bar{t} \end{aligned}$$

This is the solution for equation (13).

Proof for Lemma 4

Step 1 $(\hat{\theta}_j)_{j=0}^{\infty}$ satisfies the budget constraint.

Based on the definitions of U , Z , and \bar{t} ,

$$\sum_{j \in U} \gamma_j \geq G > \sum_{j \in U \setminus \{\bar{t}\}} \gamma_j$$

holds. Therefore, the definition of $\hat{\theta}_{\bar{t}}$ results in:

$$G = \sum_{j \in U \setminus \{\bar{t}\}} \gamma_j + \hat{\theta}_{\bar{t}} \gamma_{\bar{t}}, \quad 0 < \hat{\theta}_{\bar{t}} \leq 1$$

Accordingly, $(\hat{\theta}_{\bar{t}})_{j=0}^{\infty}$ satisfies the budget constraint.

Step 2 Assume the solution to welfare maximization is θ_t^* and $t=0, 1, 2, 3, \dots$. If $P = \{t \mid \theta_t^* > 0\}$,

then either $P \subset U$ or $U \subset P$ holds.

From Lemma 3, we know that $(\theta_j^*)_{j=0}^{\infty}$ has a value of 0 or 1, with one possible exception. We

assume that t and $t' \in N$ exist and that $t \in U \setminus P$ and $t' \in P \setminus U$ are valid. Here, since $t \in U$

and $t' \in Z$, we obtain the following using equation (19):

$$b_{t'} > b_t, \quad 1 \geq \theta_{t'}^* > 0, \quad \theta_t^* = 0$$

Welfare can be increased by focusing on t and t' , and creating θ_k^* ; $k = 0, 1, 2, \dots$ to $\tilde{\theta}_k^*$; $k = 0, 1,$

$2, \dots$ according to ε -conversion. This results in a contradiction. Accordingly, the following must be

valid:

$$P \subset U \text{ or } U \subset P$$

Step 3 $P \subset U$ holds.

If $U = P$ then $P \subset U$ holds. Then, it suffices for us to examine the case $U \subset P$ and $U \neq P$.

It is clear that

$$\sum_{t \in U \setminus \{\bar{t}\}} \gamma_t + \hat{\theta}_t \gamma_t = \sum_{t \in U} \hat{\theta}_t \gamma_t = G = \sum_{t \in P} \theta_t^* \gamma_t \quad (21)$$

A certain $t \in U$ exists and satisfies $1 \geq \hat{\theta}_t > \theta_t^* > 0$. When $t' \in P \setminus U$ holds, then $t \in U$ and $t' \in Z$, and thus we obtain the following using equation (19):

$$b_{t'} > b_t \text{ and } 1 > \theta_{t'}^* > 0 \text{ and } 1 \geq \theta_t^* > 0$$

According to Lemma 3, this contradicts optimization.

Step 4 $P = U$

Suppose $P \neq U$. The case $P \subset U \setminus \{\bar{t}\}$ leads us to $\sum_{t \in P} \theta_t^* \gamma_t < G$. This is also a contradiction.

Then $\bar{t} \in P$. Furthermore, if $P \subset U \setminus \{t\}$ and $t \in U$ hold for some t , we obtain $\theta_t^* = 0$ and $\theta_t^* > \hat{\theta}_t$. When considering t, \bar{t}

$$b_{\bar{t}} > b_t, \quad 1 \geq \theta_{\bar{t}}^* > 0 \text{ and } \theta_t^* = 0$$

is confirmed using equation (20). According to Lemma 3, this contradicts optimization.

These steps indicate $\theta_t^* = \hat{\theta}_t, t = 0, 1, 2, \dots$ ■

3.3.2 When $r = R$

The following holds:

$$\frac{\beta_t w_t}{\gamma_t} = b; t = 1, 2, 3, \dots, \text{ where } b = 1/(1-h).$$

Let $\theta_t, t = 0, 1, 2, \dots$ be arbitrary burden rates satisfying the constraint $\sum_{t=0}^{\infty} \theta_t \gamma_t = G$. We obtain the following result:

$$\begin{aligned} W(\theta_0, \theta_1, \dots) &= \sum_{t=0}^{\infty} \beta_t w_t (1 - \theta_t) = \sum_{t=0}^{\infty} b \gamma_t (1 - \theta_t) \\ &= \sum_{t=0}^{\infty} b \gamma_t - \sum_{t=0}^{\infty} b \gamma_t \theta_t = b \sum_{t=0}^{\infty} \gamma_t - bG \end{aligned}$$

This implies social welfare reaches a constant level. Accordingly, regardless of the burden rate used, social welfare does not change. Finally, we summarize the above results into one theorem.

Theorem 5 If $m = 1$, the optimal net lifetime burden rate shows the following characteristics.

When the social time-preference rate and interest rate match ($R = r$), any resulting burden rate is optimal. When the social time-preference rate and interest rate differ ($R \neq r$), it is desirable to satisfy the budget by assigning a burden rate of zero to the generation that has the highest current income value $\beta_t w_t / \gamma_t$ compared with its contribution to welfare (when the burden rate is zero) and a burden rate of 100% to the generation that has the lowest such income, while sequentially increasing the rate.

4. Application to a Funded System

Until this section, we have focused on the pay-as-you-go approach that is used to manage the

Japanese pension plan system. In this section, we consider the consequences of introducing a funded system into the current structure.

First, the household budget constraint is depicted as follows:

$$c_{1t} + s_t = (1 - \tau_t)w_t(1 - h_t), \quad c_{2t+1} = (1 + r)s_t + (1 + r - d_{t+1})\tau_t w_t(1 - h_t)$$

Although pension assets are represented by $\tau_t w_t(1 - h_t)$, this system only allows a pension to be received from the principal and interest after a certain burden amount $d_{t+1}\tau_t w_t(1 - h_t)$ has been deducted. These various factors can be summarized in one budget using the following expression:

$$c_{1t} + \frac{c_{2t+1}}{1 + r} + \left(1 - \frac{\tau_t d_{t+1}}{1 + r}\right)w_t h_t = \left(1 - \frac{\tau_t d_{t+1}}{1 + r}\right)w_t$$

Define $\theta_t = \tau_t d_{t+1} / (1 + r)$. The parameter θ_t is a substantial net lifetime burden rate of the public pension. For a given r , w_t , and θ_t , a generation maximizes its utility by selecting c_{1t} , c_{2t+1} , and h_t . In other words, by dropping the index, the expression becomes

$$\max u(c_1, c_2, h), \text{ subject to } c_1 + \frac{c_2}{1 + r} + (1 - \theta)w h = (1 - \theta)w .$$

Thus, this maximization problem is similar to problem (3) in the pay-as-you-go pension system. The difference between the pay-as-you-go and funded systems lies in that of government budget. The government budget in funded system can be financed by two agencies, namely government bonds agency and pension funds agency. The process of the former works as follows:

$$G_0 = B_0$$

$$(1+r)B_{t-1} = B_t + d_t \tau_{t-1} w_{t-1} (1-h_{t-1}) L_{t-1}, \quad t=1,2,3,\dots$$

First, in period 0 government bonds B_0 are issued and the amount of G_0 is transferred to the pension agency. From this point onwards, new bonds are issued up to the amount B_t and the pension agency uses the returns on these funds $d_t \tau_{t-1} w_{t-1} (1-h_{t-1}) L_{t-1}$ as income to pay the principal and interest from the previous bond issuance $(1+r)B_{t-1}$.

On the other hand, the pension agency's budget is shown in the following expression,

$$P_0 + A_0 = G_0 + \tau_0 w_0 (1-h_0) L_0$$

$$(1+r) \tau_{t-1} w_{t-1} (1-h_t) L_{t-1} + A_t = (1+r) A_{t-1} + \tau_t w_t (1-h_t) L_t, \quad t=1, 2, \dots$$

In period 0, we assume that the government pays pension benefits P_0 and funds A_0 using G_0 from the government bond financing program and the revenue derived from pension premiums $\tau_0 w_0 (1-h_0)$. In subsequent periods, the pension agency incurs expenditure in the form of pension benefits and investment profits, which it must repay to the government, as well as asset returns $(1+r-d_t) \tau_{t-1} w_{t-1} (1-h_t) + d_t \tau_{t-1} w_{t-1} (1-h_t) + A_t$, which it pays from the asset investments and pension premiums $(1+r) A_{t-1} + \tau_t w_t (1-h_t)$.

Combining these budgets into a single undifferentiated sum, we have the following:

$$P_0 + A_0 = B_0 + \tau_0 w_0 (1-h_0) L_0$$

$$\begin{aligned}
& (1+r-d_t)\tau_{t-1}w_{t-1}(1-h_{t-1})L_{t-1} + A_t + (1+r)B_{t-1} \\
& = (1+r)A_{t-1} + \tau_t w_t (1-h_t)L_t + B_t \quad t=1, 2, \dots
\end{aligned}$$

We can rewrite the expression to show the net debt as $\hat{B}_t = B_t - A_t$, $t = 0, 1, 2, \dots$, which results in

$$P_0 = \hat{B}_0 + \tau_0 w_0 (1-h_0)L_0$$

$$(1+r-d_t)\tau_{t-1}w_{t-1}(1-h_{t-1})L_{t-1} + (1+r)\hat{B}_{t-1} = \tau_t w_t (1-h_t)L_t + (1+r)\hat{B}_t$$

$$(\tau_{t-1} - \theta_t)w_{t-1}(1-h_{t-1})L_{t-1} + \hat{B}_{t-1} = \frac{\tau_t w_t (1-h_t)L_t}{1+r} + \frac{\hat{B}_t}{1+r}$$

Define $I_t = w_t(1-h_t)L_t$, we arrive at

$$\left\{ \begin{array}{l}
0 = P_0 - \hat{B}_0 - \tau_0 I_0 \\
\theta_0 I_0 = \tau_0 I_0 - \frac{\tau_1 I_1}{1+r} + \hat{B}_0 - \frac{\hat{B}_1}{1+r} \\
\frac{\theta_1 I_1}{1+r} = \frac{\tau_1 I_1}{1+r} - \frac{\tau_2 I_2}{(1+r)^2} + \frac{\hat{B}_1}{1+r} - \frac{\hat{B}_2}{(1+r)^2} \\
\dots \dots \dots \\
\frac{\theta_n I_n}{(1+r)^n} = \frac{\tau_n I_n}{(1+r)^n} - \frac{\tau_{n+1} I_{n+1}}{(1+r)^{n+1}} + \frac{\hat{B}_n}{(1+r)^n} - \frac{\hat{B}_{n+1}}{(1+r)^{n+1}}
\end{array} \right.$$

which finally results in

$$\sum_{t=0}^n \frac{\theta_t I_t}{(1+r)^t} = P_0 - \frac{\tau_{n+1} I_{n+1}}{(1+r)^{n+1}} + \frac{\hat{B}_{n+1}}{(1+r)^{n+1}}$$

This is the same equation as obtained in the proof Theorem 2. And thus we know that the same result

as in Theorem 2 holds in the funded system. Accordingly, the argument put forward earlier can

also be applied to a funded system.

5. Conclusion

This paper used a two-period OLG model in order to put forward a theoretical design for an optimal public pension system based on a partial equilibrium analysis. Household preferences only depended on consumption and leisure. The following findings suggest that an optimal public pension system shares similar characteristics if the utility function of households is homogeneous of degree m with respect to consumption in the working and retired periods. First, differences in the population growth rate do not affect the relative level of the optimal net lifetime burden rate of each generation. Second, if $m \neq 0$ or $m < 1$, the optimal public pension system can be expressed explicitly. Thus, the difference between the market time-preference rate and social time-preference rate provides a crucial insight into the optimal burden rate of each generation.

One limitation of this study is that we make assumptions based on certain preferences, such as homogeneity. The presented findings would be more worthwhile if it were possible to show these results more generally. We will carry out this task in subsequent research.

References

Auerbach, A. J. and L. J. Kotlikoff (1987) *Dynamic Fiscal Policy*, Cambridge: Cambridge University Press.

Barro, R. J. (1974) “Are Government Bonds Net Wealth?”, *Journal of Political Economy*, Vol. 82, No. 6, pp. 1095–1117.

— (1979) “On the Determination of the Public Debt”, *Journal of Political Economy*, Vol. 87, pp. 940–971.

— (1999) “Notes on Optimal Debt Management”, *Journal of Applied Economics*, Vol. 2, pp. 281–289.

Bohn, H. (1990) “Tax Smoothing with Financial Instruments”, *American Economic Review*, Vol. 80, pp. 1217–1230.

Diamond, P. A. (1965) “National Debt in A Neo-classical Growth Model”, *American Economic Review*, Vol. 55, pp. 1125–1150.

Feldstein, M. (1995) “Would Privatizing Social Security Raise Economic Welfare?”, *NBER Working Paper*, No. 5281.

— (1998) “The Effect of Privatizing Social Security on Economic Welfare: Appendix to the Introduction”, in M. Feldstein ed., *Privatizing Social Security*, The University of Chicago Press, pp. 1–29.

Horioka, C. Y. (2002) “Are the Japanese Selfish, Altruistic, or Dynastic?”, *The Japanese Economic Review*, Vol. 53, No. 1, pp. 26–54.

Judd, K. L. (1999) “Optimal taxation and spending in general competitive growth models”, *Journal of Public Economics*, Vol. 71, pp. 1–26.

Oguro K. (2008) “The Inter-generational Gap in Social Security: Smoothing the Net Lifetime Burden of Each Generation as a Solution”, *Quarterly of Social Security Research*, Vol. xx, pp. 224–233 (in Japanese).

Oguro, K., H. Nakakarumai, and S. Takama (2007) “The Inter-generational Gap in Social Security: Smoothing the Net Lifetime Burden of Each Generation as a Solution—Supplemental Introduction of ‘Funding Account’ for Long-term Care Insurance,” Ministry of Finance, Policy Research Institute, Discussion Paper Series 07A-5 (in Japanese).

Samuelson, P. A. (1958) “An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money”, *Journal of Political Economy*, Vol. 66, pp. 467–482.

Takayama, N., Y. Aso, T. Miyaji, and Y. Kamiya (1996) “Reality of the Accumulation of Household Assets and the Successive Inheritance”, in N. Takayama, C. Y. Horioka, and K. Ota, eds., *The Savings of Aging Society and the Successive Inheritance*, Nippon Hyoron Sha Co., Ltd. (in Japanese).

Yoshida Y. (1965) *Theory of Functions*, Iwanami Zensho (in Japanese).