Exchange Rates and Fundamentals: Closing a Two-country Model

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Abstract

In an influential paper, Engel and West (2005) claim that the near random-walk behavior of nominal exchange rates is an equilibrium outcome of a present-value model of a partial equilibrium asset approach when economic fundamentals follow exogenous first-order integrated processes and the discount factor approaches one. Subsequent empirical studies further confirm this proposition by estimating a discount factor that is close to one under distinct identification schemes. In this paper, I argue that the unit market discount factor implies the counterfactual joint equilibrium dynamics of random-walk exchange rates and economic fundamentals within a canonical, two-country, incomplete market model. Bayesian posterior simulation exercises based on post-Bretton Woods data from Canada and the United States reveal difficulties in reconciling the equilibrium random-walk proposition within the two-country model; in particular, the market discount factor is identified as being much lower than one.

Key Words: Exchange rates; Present-value model; Economic fundamentals; Random walk; Two-country model; Incomplete markets; Cointegrated TFPs; Debt elastic risk premium; Backus-Smith puzzle.

JEL Classification Number: E31, E37, and F41

1. Introduction

Few equilibrium models for nominal exchange rates systematically beat a naïve random-walk counterpart in terms of out-of-sample forecast performance. Since the study of Meese and Rogoff (1983), this robust empirical property of nominal exchange rate fluctuations has stubbornly resisted theoretical challenges to understand the behavior of nominal exchange rates as an equilibrium outcome. Recently developed open-economy dynamic stochastic general equilibrium (DSGE) models also suffer from this problem. Infamous as the disconnect puzzle, open-economy DSGE models fail to generate random-walk nominal exchange rates along an equilibrium path because their exchange rate forecasts are closely related to other macroeconomic fundamentals.

In a recent paper, Engel and West (2005, hereafter EW) establish the near random-walk behavior of nominal exchange rates within a partial equilibrium asset approach.[1] Their model implies that equilibrium nominal exchange rates are given as the present discounted values of the expected future economic fundamentals. If economic fundamentals are integrated of order one (hereafter I(1)) and the discount factor approaches one, a nominal exchange rate then follows a near random-walk process in equilibrium. This equilibrium random-walk property is attributable to the fact that only the Beveridge-Nelson trend components in the I(1) economic fundamentals are reflected in present-value calculation at the limit of the unit discount factor. Because the Beveridge-Nelson permanent component is a random walk, the current economic fundamentals lack the power to forecast future depreciation rates even along an equilibrium path.[2]

Because the assumed non-stationarity of economic fundamentals seems to hold without question, subsequent studies within the literature have focused on the empirical validity of the assumption that the discount factor is close to one. Examining data on different currencies and spanning distinct sample periods, Sarno and Sojli (2009) and Balke et al. (2013, hereafter BMW) identify a

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[1] Engel (2014) provides the most recent survey on past studies on nominal exchange rates.

[2] Nominal exchange rates, therefore, need to Granger-cause future economic fundamentals, not vice versa. The empirical exercises of EW based on vector autoregressions (VARs) provide solid evidence for this implication of Granger-causality across different currencies. The cross-sectional and panel regressions by Sarno and Schmeling (2014) also confirm the hypothesis that nominal exchange rates have predictive power for nominal economic fundamentals.

discount factor based on partial equilibrium asset models similar to that of EW and infer that the estimated discount factor is indeed distributed near to one. In particular, the Bayesian unobservable component (UC) model of BMW estimates money demand shocks as a dominant underlying driver of a long sample of the British pound/U.S. dollar rate. This empirical fact supports the conjecture of EW that persistent unobservable economic fundamentals play a significant role in near random-walk nominal exchange rates.

Nason and Rogers (2008, hereafter NR) attempt to generalize EW's proposition more rigorously and preserve the random-walk property of nominal exchange rates within a two-country dynamic stochastic general equilibrium (DSGE) model that includes incomplete international financial markets. NR rely only on a subset of the first-order necessary conditions (FONCs) of the proposed two-country model, i.e., the utility-based, uncovered interest rate parity (UIP) condition, money demand functions, and purchasing power parity (PPP) condition, to construct the present value model (DSGE-PVM) of nominal exchange rates. In their DSGE-PVM, an equilibrium nominal exchange rate is given as the present discounted values of the expected future values of fundamentals that consist of cross-country consumption and money supply differentials. As claimed in EW, if these fundamentals are I(1), the nominal exchange rate behaves like a near random-walk at the limit of the unit market discount factor.

Utilizing the cross-equation restrictions (CERs) of the DSGE-PVM and specifying the exogenous I(1) processes of the economic fundamentals, NR estimate a restricted UC model for the bilateral exchange rate between Canada and the United States. Their Bayesian posterior inferences using post-Bretton Woods data confirm EW's proposition, finding that the market discount factor is close to one. Moreover, they observe that permanent shocks to the consumption and money supply differentials dominate the historical movements of the bilateral exchange rate.[3]

In this paper, I go beyond the theoretical and empirical achievements of NR. My challenge of reconciling random-walk exchange rates within a two-country general equilibrium model begins

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[3]This empirical result is consistent with the argument known as the PPP puzzle (Rogoff 1996) because, by incorporating price stickiness, many open-economy DSGE models emphasize the role of mean-reverting monetary policy shocks as the main force driving nominal exchange rates.

by arguing that NR somehow stop short before closing their two-country model. There are three concerns in their empirical exercise based on the DSGE-PVM. First, NR construct their DSGE-PVM by taking the log-linear approximations of the stochastically de-trended FONCs around the stable, deterministic, steady state of the model. The incompleteness of the international financial market in their two-country model, in which only state non-contingent bonds are traded by representative households across the two countries, might lead endogenous variables to exhibit permanent unit-root dynamics. In this case, there is no guarantee that a stable, deterministic, steady state will exist.[4]

Second, NR's specification of an I(1) consumption differential is inconsistent with a balanced growth path of the two-country model endowed with a single consumption good. The source of the non-stationary consumption differential is their presumption that the cross-country differential in the total factor productivity (TFP) is I(1). In the exercise of NR, each country's endogenous variables are stochastically de-trended with its own TFP. The de-trended market-clearing condition of a single consumption good, which is equivalent to the de-trended resource constraint for the global economy, then depends on the two-country TFP differential. In this case, the non-stationary TFP differential makes the de-trended resource constraint violate the balanced growth restriction.

Finally, the third concern is that NR omit the Euler equations for the optimal intertemporal consumption allocations of the two countries and treat the consumption differential as an exogenous random variable. The omitted CERs that the Euler equations impose on the consumption differential, however, might result in the serious misidentification and misevaluation of the two-country general equilibrium model as the true data-generating mechanism of random-walk exchange rates. The third concern is primarily relevant once I recognize that each country's consumption is determined by the permanent income hypothesis (PIH) and depends substantially on the I(1) endowment and the size of the market discount factor as well. In fact, to my best knowledge, no past study in this literature has taken into consideration the endogeneity of economic fundamentals when estimating the market discount factor. The joint determination of nominal exchange rates and economic

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[4]See the detailed discussions of Ghironi (2006) and Boileu and Normandin (2008) regarding the non-stationarity problem inherent to incomplete asset market models.

fundamentals within a single two-country model, hence, might lead to a statistical inference on the discount factor that is sharply different from those in the past studies.

To address these three concerns, I investigate a canonical, single-good, two-country, endowment economy model in which incomplete international financial markets are utilized as a device for intertemporal consumption-smoothing. The model used in this paper is quite stylized but similar to that of NR except with regard to two important aspects. The first is that the model contains a debt-elastic risk premium. As characterized by Schmitt-Grohé and Uribe (2003) in a small open-economy model and Boileu and Normandin (2008) in a two-country international business cycle model, a debt-elastic risk premium has served as a popular instrument to induce the stationarity of the net foreign asset distribution.[5] I show that by introducing a wedge between the world and country-specific interest rates, the debt-elastic risk premium alters the UIP condition and makes the resulting present-value model of the nominal exchange rate different from the DSGE-PVM in NR.

The second aspect that differentiates this paper's model from that of NR is that the stochastic trends in both countries appear to be independent in the short run but comove in the long run. In this model, the exogenous endowment processes of the two countries consist of both permanent and transitory components. I then allow the stochastic trends of the two countries, which are interpreted as their TFPs, to be cointegrated, as emphasized in recent papers by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013) in the context of international business cycles. In this case, because the TFP differential is stationary in population, a balanced growth path is guaranteed to exist in equilibrium. Moreover, if the speed of technological diffusion the cointegration restriction reflects is set sufficiently slow, the TFP differential is empirically identified as an I(1) process with a finite sample. This conjecture is consistent with the empirical finding of NR that a unit root in the cross-country consumption differential cannot be rejected.

Harnessing all the FONCs of the model to endogenously determine the nominal exchange

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[5]A non-exhaustive list of studies that adopt a debt-elastic risk premium as a device to avoid the non-stationarity problem in open-economy DSGE models includes Nason and Rogers (2006), Adolfson et al. (2007), Kano (2009), Justiniano and Preston (2010), García-Cicco et al. (2010), and Bodenstein (2011).

rate along the unique equilibrium path, I theoretically show that the expected equilibrium currency return is characterized by a linear function of the de-trended net foreign asset position and other transitory components. When the market discount factor approaches one, this dependence of the expected currency return on the transitory components of the model vanishes asymptotically. Therefore, the near random-walk property of the equilibrium exchange rate indeed holds even after the two-country model is closed suitably. Importantly, the model generates a tractable approximated analytical solution of equilibrium random-walk exchange rates in cases with two symmetric countries. The resulting closed-form solution reveals that the exchange rate is primarily driven by a permanent shock to the money supply differential, among other stationary shocks. This stringent theoretical prediction echoes the findings of NR. However, in contrast to the claim of NR, a permanent but cointegrated TFP shock cannot be a significant driver of the random-walk nominal exchange rate because the TFP differential should be stationary to close the two-country model.

In addition, the investigation in this paper goes even further. I also characterize the equilibrium paths of two other endogenous relative variables, the consumption and interest rate differentials, through deriving approximated analytical solutions. The resulting closed-from representation of relative consumption reveals that at the limit of the unit market discount factor, the consumption differential is correlated perfectly with the PPP deviation, i.e., the real exchange rate (RER). This implication stems from two theoretically crucial facts. First, consumption in each country does not rely on any monetary shocks due to the classical dichotomy of this flexible price model. Second, at the limit of the unit discount factor, no country-specific endowment shock has a significant impact on the present discounted values of expected future endowment differentials because of the balanced growth restriction. The resulting homogeneity of the permanent income calculation across the two countries makes their consumption identical. Consequently, neither permanent nor transitory idiosyncratic endowment shock matters for the two-country consumption differential. Only the relative price, i.e., the RER, has an immediate effect on the consumption differential. The resulting perfect correlation between relative consumption and the RER has been recognized as a major empirical difficulty related to a broad class of international business cycle models since that

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of Backus and Smith (1993).[6]

The close-form solution of the interest rate differential uncovers that the relative interest rate is dominated by transitory monetary disturbances, i.e., transitory shocks to the money supply and demand differentials. Because in this model the unit discount factor means the zero nominal interest rate at the steady state, at the limiting case with the unit discount factor the money demand function becomes perfectly flat, i.e., the liquidity trap emerges around the steady state. To explain the actual data variations in the interest rate differential together with those in the exchange rate and the money supply differential, the implied liquidity trap, however, requires counterfactually large volatilities of transitory monetary disturbances.

A macroeconometrician who tries to fit the model to both the exchange rate and economic fundamentals then faces a serious trade-off. If she or he fits the model to the near random-walk exchange rate, the market discount factor should be close to one. The model, however, tends to impose three unrealistic theoretical restrictions on the data — a permanent money supply differential shock as the dominant driver of random-walk exchange rates, the infamous Backus and Smith problem of an implausibly strong connection between relative consumption and the RER, and counterfactually large volatilities of transitory nominal shocks required by the liquidity trap. If she or he tries to avoid these counterfactual restrictions by sufficiently lowering the discount factor, the model loses its ability to mimic the near random-walk behavior of nominal exchange rates.

An obvious empirical question then is how seriously the statistical inference on the market discount factor is affected by this theoretical trade-off. To address this question, I estimate a UC model that is fully restricted by the proposed two-country model by a Bayesian posterior simulation method. Given relevant prior distributions of the model's structural parameters, the same post-Bretton Woods data for Canada and the United States investigated in NR then finds that the market discount factor is *a posteriori* distributed around 0.525. Notice that this size of the market discount factor is far below the size close to one that is statistically inferred by many recent empirical

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[6]Because the model used in this paper does not include non-tradable goods, the RER is not determined endogenously in this model, as in the two-country incomplete market model of Benigno and Thoenissen (2008). Rather, the RER is defined as the exogenous shocks to the PPP deviation, as specified in EW and BMW.

studies under different identification strategies. The observation of this paper, hence, implies the theoretical trade-off mentioned above is indeed severe: it is still a quite difficult task to explain data variations in the nominal exchange rate and the corresponding macroeconomic fundamentals jointly and consistently within a canonical, open-economy, general equilibrium framework once such a stylized, two-country, incomplete market model is closed correctly.

The remainder of this paper is organized as follows. In the next section, I introduce the two-country incomplete market model employed in this paper. Section 3 then derives and discusses the equilibrium random-walk property of nominal exchange rates and the Backus and Smith puzzle of a perfect correlation between relative consumption and the RER at the limit of the unit market discount factor. After reporting the main results of the Bayesian exercises in section 4, I conclude in section 5.

2. A two-country incomplete market model

2.1. The model

In this paper, I investigate a canonical incomplete market model with two countries, the home (h) and foreign (f) countries. Each country is endowed with a representative household whose objective is the lifetime money-in-utility

$$\sum_{j=0}^{\infty} \beta^j E_t \left\{ \ln C_{i,t+j} + \phi_{i,t+j} \ln \left(\frac{M_{i,t+j}}{P_{i,t+j}} \right) \right\}, \quad 0 < \beta < 1, \quad \text{for } i = h, f,$$

where $C_{i,t}$, $M_{i,t}$, and $P_{i,t}$ represent the ith country's consumption, money stock, and price index, respectively. The money-in-utility function is subject to a persistent money demand shock $\phi_{i,t}$. The representative households in the home and foreign countries maximize their lifetime utility functions subject to the home budget constraint

$$B_{h,t}^h + S_t B_{h,t}^f + P_{h,t} C_{h,t} + M_{h,t} = (1 + r_{h,t-1}^h) B_{h,t-1}^h + S_t (1 + r_{h,t-1}^f) B_{h,t-1}^f + M_{h,t-1} + P_{h,t} Y_{h,t} + T_{h,t}, \quad (1)$$

and its foreign counterpart

$$\frac{B_{f,t}^h}{S_t} + B_{f,t}^f + P_{f,t}C_{f,t} + M_{f,t} = (1 + r_{f,t-1}^h)\frac{B_{f,t-1}^h}{S_t} + (1 + r_{f,t-1}^f)B_{f,t-1}^f + M_{f,t-1} + P_{f,t}Y_{f,t} + T_{f,t}, \quad (2)$$

respectively, where $B_{i,t}^l$, $r_{i,t}^l$, $Y_{i,t}$, $T_{i,t}$, and S_t denote the ith country's holdings of the lth country's nominal bonds at the end of time t, the ith county's returns on the lth country's bonds, the ith country's output level, the ith country's government transfers, and the level of the bilateral nominal exchange rate, respectively. Each country's output $Y_{i,t}$ is given as an exogenous endowment following a stochastic process $Y_{i,t} = y_{i,t}A_{i,t}$, where $y_{i,t}$ is the transitory component and $A_{i,t}$ is the permanent component. Below, I interpret the permanent component $A_{i,t}$ as the TFP in the underlying production technology.

The first-order necessary conditions (FONCs) of the home country's household are given by the budget constraint (1), the Euler equation

$$\frac{1}{P_{h,t}C_{h,t}} = \beta(1 + r_{h,t}^h)E_t\left(\frac{1}{P_{h,t+1}C_{h,t+1}}\right), \quad (3)$$

the utility-based uncovered parity condition (UIP)

$$(1 + r_{h,t}^h)E_t\left(\frac{1}{P_{h,t+1}C_{h,t+1}}\right) = \frac{(1 + r_{h,t}^f)}{S_t}E_t\left(\frac{S_{t+1}}{P_{h,t+1}C_{h,t+1}}\right), \quad (4)$$

and the money demand function

$$\frac{M_{h,t}}{P_{h,t}} = \phi_{h,t}\left(\frac{1 + r_{h,t}^h}{r_{h,t}^h}\right)C_{h,t}. \quad (5)$$

The foreign country's FONC counterparts are the budget constraint (2), the Euler equation

$$\frac{1}{P_{f,t}C_{f,t}} = \beta(1 + r_{f,t}^f)E_t\left(\frac{1}{P_{f,t+1}C_{f,t+1}}\right), \quad (6)$$

the utility-based uncovered parity condition (UIP)

$$(1 + r_{f,t}^h)E_t\left(\frac{1}{S_{t+1}P_{f,t+1}C_{f,t+1}}\right) = \frac{(1 + r_{f,t}^f)}{S_t}E_t\left(\frac{1}{P_{f,t+1}C_{f,t+1}}\right), \quad (7)$$

and the money demand function

$$\frac{M_{f,t}}{P_{f,t}} = \phi_{f,t}\left(\frac{1 + r_{f,t}^f}{r_{f,t}^f}\right)C_{f,t}. \quad (8)$$

Each country's government transfers the seigniorage to the household as a lump-sum. Hence, the government's budget constraint is

$$M_{i,t} - M_{i,t-1} = T_{i,t}, \quad \text{for } i = h, f.$$

The money supply $M_{i,t}$ is specified to consist of permanent and transitory components, $M_{i,t}^\tau$ and $m_{i,t}$: $M_{i,t} \equiv m_{i,t} M_{i,t}^\tau$ for $i = h, f$.

To close the model within an incomplete international financial market, I allow for a debt-elastic risk premium in the interest rates faced only by the home country:

$$r_{h,t}^l = r_{w,t}^l [1 + \psi\{\exp(-B_{h,t}^l/M_{l,t}^\tau + \bar{d}) - 1\}], \quad \bar{d} \le 0, \quad \psi > 0, \quad \text{for} \quad l = h, f \qquad (9)$$

where $r_{w,t}^l$ is the equilibrium world interest rate of the lth country's bond. The risk premium is given as an externality: The household does not take into account the effect of the debt position on the risk premium when maximizing the lifetime utility function. On the other hand, I do not attach a risk premium to the foreign country's interest rates: $r_{f,t}^l = r_{w,t}^l$ for $l = h, f$.

Following EW and BMW, I assume throughout this paper that purchasing power parity (PPP) holds only up to a persistent PPP deviation shock $\ln q_t$:

$$S_t P_{f,t} = P_{h,t} q_t.$$

The market-clearing conditions of the two countries' bond markets are

$$B_{h,t}^h + B_{f,t}^h = 0 \quad \text{and} \quad B_{h,t}^f + B_{f,t}^f = 0,$$

i.e., along an equilibrium path, the world net supply of nominal bonds is zero on a period-by-period basis.

As in NR, I assume that the logarithms of the total factor productivity (TFP) and the permanent component of the money supply, $\ln A_{i,t}$ and $\ln M_{i,t}^\tau$, are I(1) for $i = h, f$, and the cross-country differential in the permanent component of money supply, $\ln M_{h,t}^\tau - \ln M_{f,t}^\tau$, is also I(1):

Assumption 1: $\ln A_{i,t}$ and $\ln M_{i,t}^\tau$ are I(1) for $i = h, f$.

Assumption 2: $\ln M_{h,t}^\tau - \ln M_{f,t}^\tau$ is I(1).

Following Assumptions 1 and 2, I specify each country's monetary growth rate $\Delta \ln M_{i,t}^{\tau}$ to be an independent AR(1) process:

$$\Delta \ln M_{i,t}^{\tau} = (1 - \rho_M) \ln \gamma_M + \rho_M \Delta \ln M_{i,t-1}^{\tau} + \epsilon_{M,t}^{i}, \quad \text{for } i = h, f.$$

where $\ln \gamma_M$ and ρ_M are the mean and AR root, respectively, of the money supply growth rate common to the two countries.

Importantly, I do not make NR's assumption that the cross-country TFP differential, $\ln a_t \equiv \ln A_{h,t} - \ln A_{f,t}$, is I(1). Rather, I assume that the TFP differential is integrated of order zero (I(0)). This deviation from NR's key assumption stems from the fact that an I(1) TFP differential is inconsistent with the stationarity of the stochastically de-trended model and the deterministic steady state of the resulting equilibrium-balanced growth path, as I will show below in more detail. Notice that Assumption 1 and the stationary TFP differential jointly imply that the TFP of the home country must be cointegrated with that of the foreign country:

Assumption 3: *$\ln A_{h,t}$ and $\ln A_{f,t}$ are cointegrated with the cointegrated vector $[1, -1]$ and have the error correction models (ECMs)*

$$\Delta \ln A_{h,t} = \ln \gamma_A - \frac{\lambda}{2}(\ln A_{h,t-1} - \ln A_{f,t-1}) + \epsilon_{A,t}^{h},$$
$$\Delta \ln A_{f,t} = \ln \gamma_A + \frac{\lambda}{2}(\ln A_{h,t-1} - \ln A_{f,t-1}) + \epsilon_{A,t}^{f}, \tag{10}$$

where $\gamma_A > 1$ is the common drift term and $\lambda \in [0, 1)$ is the adjustment speed of the error correction mechanism.

The cointegration restriction that Assumption 3 imposes on the two countries' TFPs is adopted by recent open-economy DSGE studies by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013). ECMs (10) imply that the cross-country TFP differential is I(0) because

$$\ln a_t = (1 - \lambda) \ln a_{t-1} + \epsilon_{A,t}^{h} - \epsilon_{A,t}^{f}.$$

Importantly, if the adjustment speed λ is sufficiently close to zero, the cross-country TFP differential

can be realized near I(1), as maintained by NR.

The stochastic process of the logarithm of the transitory output component for each country, $\ln y_{i,t}$, is specified as the following AR(1) process:

$$\ln y_{i,t} = (1 - \rho_y) \ln y_i + \rho_y \ln y_{i,t-1} + \epsilon^i_{y,t},$$

for $i = h, f$. Similarly, the stochastic process of the logarithm of the transitory money supply component for each country, $\ln m_{i,t}$, is specified as the following AR(1) process:

$$\ln m_{i,t} = (1 - \rho_m) \ln m_i + \rho_m \ln m_{i,t-1} + \epsilon^i_{m,t},$$

for $i = h, f$. The three other structural shocks, the home and foreign money demand shocks $\phi_{h,t}$ and $\phi_{f,t}$, respectively, and the PPP shock q_t, follow persistent stationary processes. Specifically, they are characterized by AR(1) processes in terms of the following logarithm:

$$\ln \phi_{i,t} = (1 - \rho_\phi) \ln \phi + \rho_\phi \ln \phi_{i,t-1} + \epsilon^i_{\phi,t},$$

for $i = h, f$ and

$$\ln q_t = \rho_q \ln q_{t-1} + \epsilon_{q,t}.$$

Throughout this paper, I assume that all structural shocks are distributed independently.

2.2. The log-linear approximation of the stochastically de-trended system

Define stochastically de-trended variables as $c_{i,t} \equiv C_{i,t}/A_{i,t}$, $p_{i,t} \equiv P_{i,t}A_{i,t}/M^\tau_{i,t}$, $b^l_{i,t} \equiv B^l_{i,t}/M^\tau_{l,t}$, $\gamma^i_{A,t} \equiv A_{i,t}/A_{i,t-1}$, $\gamma^i_{M,t} \equiv M^\tau_{i,t}/M^\tau_{i,t-1}$, and $s_t \equiv S_t M^\tau_{f,t}/M^\tau_{h,t}$. Taking the stochastic de-trending of the FONCs, (1)-(9), I construct the stochstically de-trended system of the FONCs, as reported in accompanying online appendix A. The resulting ten equations determine the ten endogenous variables $c_{h,t}$, $c_{f,t}$, $p_{h,t}$, s_t, $b^h_{h,t}$, $b^f_{h,t}$, $r^h_{h,t}$, $r^f_{h,t}$, $r^h_{w,t}$, and $r^f_{w,t}$, given nine exogenous variables $\gamma^h_{M,t}$, $\gamma^f_{M,t}$, $\gamma^h_{A,t}$, $\gamma^f_{A,t}$, a_t, $m_{h,t}$, $m_{f,t}$, $y_{h,t}$, and $y_{f,t}$.

Let $\hat{x}$ denote a percentage deviation of any variable x_t from its deterministic steady state value x^*, $\hat{x} \equiv \ln x_t - \ln x^*$.[7] Also, let $\tilde{x}$ denote a deviation of x from its deterministic steady state,

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[7]Online appendix A provides the deterministic steady state.

$\tilde{x} = x - x^*$.[8] The log-linear approximation of the stochastically de-trended home budget constraint is

$$p_h^*(c_h^* - y_h)\hat{p}_{h,t} + p_h^* c_h^* \hat{c}_{h,t} - p_h^* y_h \hat{y}_{h,t} + \tilde{b}_{h,t}^h + \bar{d}(1 - \beta^{-1})s^* \hat{s}_t + s^* \tilde{b}_{h,t}^f$$

$$= \beta^{-1}\bar{d}[(1 + \hat{r}_{h,t-1}^h) - \hat{\gamma}_{M,t}^h] + s^*\beta^{-1}\bar{d}[(1 + \hat{r}_{h,t-1}^f) - \hat{\gamma}_{M,t}^f] + \beta^{-1}\tilde{b}_{h,t-1}^h + s^*\beta^{-1}\tilde{b}_{h,t-1}^f; \quad (11)$$

that of the home Euler equation is

$$\hat{p}_{h,t} + \hat{c}_{h,t} + (1 + \hat{r}_{h,t}^h) = E_t(\hat{p}_{h,t+1} + \hat{c}_{h,t+1} + \hat{\gamma}_{M,t+1}^h); \quad (12)$$

that of the home UIP condition is

$$E_t\hat{s}_{t+1} - \hat{s}_t = (1 + \hat{r}_{h,t}^h) - (1 + \hat{r}_{h,t}^f) - E_t(\hat{\gamma}_{M,t+1}^h - \hat{\gamma}_{M,t+1}^f); \quad (13)$$

and that of the home money demand function is

$$\hat{p}_{h,t} + \hat{c}_{h,t} - \hat{m}_{h,t} = \frac{1}{r^*}(1 + \hat{r}_{h,t}^h) - \hat{\phi}_{h,t}. \quad (14)$$

The foreign country's counterparts are the log-linear approximation of the stochastically de-trended foreign budget constraint

$$p_h^*(c_f^* - y_f)(\hat{p}_{h,t} + \hat{q}_t - \hat{a}_t) + p_h^* c_f^* \hat{c}_{f,t} - p_h^* y_f \hat{y}_{f,t} - \tilde{b}_{h,t}^h - \bar{d}(1 - \beta^{-1})s^* \hat{s}_t - s^* \tilde{b}_{h,t}^f$$

$$= -\beta^{-1}\bar{d}[(1 + \hat{r}_{w,t-1}^h) - \hat{\gamma}_{M,t}^h] - s^*\beta^{-1}\bar{d}[(1 + \hat{r}_{w,t-1}^f) - \hat{\gamma}_{M,t}^f] - \beta^{-1}\tilde{b}_{h,t-1}^h - s^*\beta^{-1}\tilde{b}_{h,t-1}^f; \quad (15)$$

that of the foreign Euler equation

$$\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t - (1 + \hat{r}_{w,t}^f) = E_t(\hat{s}_{t+1} - \hat{p}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1} + \hat{a}_t - \hat{\gamma}_{M,t+1}^f); \quad (16)$$

that of the foreign UIP condition

$$E_t\hat{s}_{t+1} - \hat{s}_t = (1 + \hat{r}_{w,t}^h) - (1 + \hat{r}_{w,t}^f) - E_t(\hat{\gamma}_{M,t+1}^h - \hat{\gamma}_{M,t+1}^f); \quad (17)$$

and that of the home money demand function

$$\hat{s}_t + \hat{m}_{f,t} - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = -\frac{1}{r^*}(1 + \hat{r}_{w,t}^f) + \hat{\phi}_{f,t}. \quad (18)$$

The log-linear approximations of the home country's interest rates are

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[8]In particular, for an interest rate r_t, $(1 + \hat{r}_t) = (r_t - r^*)/(1 + r^*)$.

$$(1 + \hat{r}_{h,t}^h) = (1 + \hat{r}_{w,t}^h) - \psi(1 - \kappa)\tilde{b}_{h,t}^h, \quad \text{and} \quad (1 + \hat{r}_{h,t}^f) = (1 + \hat{r}_{w,t}^f) - \psi(1 - \kappa)\tilde{b}_{h,t}^f. \tag{19}$$

Notice that the home interest rates (19) redefine the home UIP condition (13) as

$$E_t\hat{s}_{t+1} - \hat{s}_t = (1 + \hat{r}_{w,t}^h) - (1 + \hat{r}_{w,t}^f) - \psi(1 - \kappa)(\tilde{b}_{h,t}^h - \tilde{b}_{h,t}^f) - E_t(\hat{\gamma}_{M,t+1}^h - \hat{\gamma}_{M,t+1}^f).$$

Comparing the above home UIP condition with the foreign UIP condition (17) implies that the home and foreign bonds are perfectly substitutable along the equilibrium path. Hence, the equilibrium condition $\tilde{b}_t \equiv \tilde{b}_{h,t}^h = \tilde{b}_{h,t}^f$ holds.[9]

3. Random-walk exchange rates, the Backus and Smith anomaly, and the liquidity trap

3.1. Equilibrium random-walk property of nominal exchange rates

I will now show that the equilibrium random-walk property of the exchange rate holds in this two-country model. To prove this proposition, I first derive the DSGE-PVM of the exchange rate as an equilibrium condition. Let c_t, m_t, and ϕ_t denote the de-trended consumption ratio, the transitory money supply ratio, and the money demand shock ratio between the two countries, $c_t \equiv c_{h,t}/c_{f,t}$, $m_t \equiv m_{h,t}/m_{f,t}$, and $\phi_t \equiv \phi_{h,t}/\phi_{f,t}$, respectively. Furthermore, let M_t^τ denote the ratio of the permanent money supplies of the home and foreign countries $M_{h,t}^\tau/M_{f,t}^\tau$; let M_t foreign money supplies of the home to the foreign countries $M_{h,t}/M_{f,t} \equiv m_t M_t^\tau$; and let C_t denote the ratio of the consumptions of the home and foreign countries $C_{h,t}/C_{f,t}$. Below, the steady state value of the nominal market discount factor is denoted by $\kappa \equiv 1/(1+r^*) = \beta/\gamma_M$. The home and foreign money demand functions, (14) and (18), and the home interest rates (19) yield the following interest rate differential:

$$(1 + \hat{r}_{w,t}^h) - (1 + \hat{r}_{w,t}^f) = r^*(\hat{s}_t + \hat{c}_t - \hat{m}_t + \hat{\phi}_t - \hat{q}_t + \hat{a}_t) + \psi(1 - \kappa)\tilde{b}_t. \tag{20}$$

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[9]Online appendix B characterizes the equilibrium transitory dynamics of the model for a simplified case including two symmetric countries.

Substituting the interest rate differential (20) into the foreign UIP condition (17) leads to the expectational difference equation of the de-trended exchange rate $\hat{s}_t$:

$$\hat{s}_t = \kappa E_t \hat{s}_{t+1} - (1-\kappa)\hat{c}_t + (1-\kappa)(\hat{m}_t - \hat{\phi}_t + \hat{q}_t - \hat{a}_t) + \kappa E_t(\hat{\gamma}^h_{M,t+1} - \hat{\gamma}^f_{M,t+1}) - \psi\kappa(1-\kappa)\tilde{b}_t.$$

After unwinding stochastic trends, the above expectational difference equation can be rewritten as

$$\ln S_t = \kappa E_t \ln S_{t+1} + (1-\kappa)\ln M_t - (1-\kappa)\ln C_t - (1-\kappa)(\ln \phi_t - \ln q_t) - \psi\kappa(1-\kappa)\tilde{b}_t.$$

Solving this expectational difference equation by forward iterations under a suitable transversality condition provides the DSGE-PVM of this model:

$$\ln S_t = (1-\kappa)\sum_{j=0}^{\infty}\kappa^j E_t\left(\ln M_{t+j} - \ln C_{t+j} - \psi\kappa\tilde{b}_{t+j} - \ln \phi_{t+j} + \ln q_{t+j}\right). \tag{21}$$

If the fundamental $\ln M_t - \ln C_t$ is I(1), so is the exchange rate. Moreover, the exchange rate should be cointegrated with the fundamentals. To signify this property, the DSGE-PVM (21) can be rewritten as

$$\ln S_t - \ln M_t + \ln C_t = \sum_{j=1}^{\infty}\kappa^j E_t\left(\Delta \ln M_{t+j} - \Delta \ln C_{t+j}\right)$$

$$- (1-\kappa)\sum_{j=0}^{\infty}\kappa^j E_t\left(\psi\kappa\tilde{b}_{t+j} + \ln \phi_{t+j} - \ln q_{t+j}\right). \tag{22}$$

Since the RHS of eq.(22) is I(0), the exchange rate $\ln S_t$ and the I(1) fundamental $\ln M_t - \ln C_t$ are cointegrated. NR hypothesize the cointegration relation among $\ln S_t$, $\ln M_t$, and $\ln C_t$ based on their DSGE-PVM. The model in this paper theoretically restricts the stationarity of the consumption differential $\ln C_t$ because of Assumption 3 due to the requirement of closing the two-country model.[10]

EW and NR, however, reject the cointegration relation between the exchange rate and fundamentals in actual data for major currencies. In particular, EW suggest other unobservable I(1) components that the standard asset approach does not identify as primary reasons for the failure of the cointegration hypothesis (22). Notice that in the DSGE-PVM (21), the equilibrium exchange rate also depends on the present discounted values of expected future de-trended net foreign asset positions $\tilde{b}_t$, the relative money demand shock $\ln \phi_t$, and the PPP shock $\ln q_t$. As shown in online

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[10]If the adjustment speed of the error correction mechanism of both countries' TFPs, λ, is sufficiently slow, the maintained stationarity of the consumption differential is unlikely to be detected with a finite sample.

appendix B in a case including symmetric countries, the stationarity of the de-trended international bond holding $\tilde{b}_t$ relies on the sizes of the debt elasticity of the risk premium ψ as well as the market discount factor κ: if either ψ is sufficiently close to zero or κ approaches one, $\tilde{b}_t$ follows a near-I(1) process. Moreover, as stated by BMW, the relative money demand shock and the PPP shock could be unobservable near I(1) components. The theoretical result of (22) can be interpreted as the empirical failure of cointegration among the exchange rate and economic fundamentals, which is consistent with EW's proposal.

NR claim that the DSGE-PVM (21) implies an error-correction representation of the currency return $\Delta \ln S_t$, in which $\Delta \ln S_t$ depends on the lagged error correction term $\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1}$. Their argument also holds even in this model. Online appendix C shows that after rearranging the DSGE-PVM (21) in several steps and using the cointegration relation (22), the currency return is

$$\Delta \ln S_t = \frac{1-\kappa}{\kappa}\left(\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} + \ln \phi_{t-1} - \ln q_{t-1}\right) + \psi(1-\kappa)\tilde{b}_{t-1} + u_{s,t}, \qquad (23)$$

where $u_{s,t}$ is the i.i.d., rational expectations error

$$u_{s,t} = (1-\kappa)\sum_{j=0}^{\infty}\kappa^j (E_t - E_{t-1})(\ln M_{t+j} - \ln C_{t+j} - \psi\kappa\tilde{b}_{t+j} + \ln q_{t+j} - \ln \phi_{t+j}).$$

Recall that the DSGE-PVM (21) is constructed as an equilibrium condition from some of the model's FONCs. The general equilibrium property of the model, however, imposes another restriction on the present value of the future fundamentals in the DSGE-PVM (21). Note that combining the log-linearized Euler equations of the home and foreign countries, (12) and (16), with those of the home country's interest rates (19), yields the first-order expectational difference equation of $\ln S_t - \ln M_t + \ln C_t - \ln q_t$:

$$\ln S_t - \ln M_t + \ln C_t - \ln q_t = \kappa E_t(\ln S_{t+1} - \ln M_{t+1} + \ln C_{t+1} - \ln q_{t+1})$$
$$+ \kappa\rho_M\hat{\gamma}_{M,t} + \kappa(\rho_m - 1)\ln m_t - (1-\kappa)\ln \phi_t,$$

where $\hat{\gamma}_{M,t} \equiv \hat{\gamma}_{M,t}^h - \hat{\gamma}_{M,t}^f$ is the money supply growth rate differential. Because κ is less than one, the difference equation above has the unique forward solution

$$\ln S_t = \ln M_t - \ln C_t + \ln q_t + \frac{\kappa \rho_M}{1 - \kappa \rho_M} \hat{\gamma}_{M,t} - \frac{\kappa(1 - \rho_m)}{1 - \kappa \rho_m} \ln m_t - \frac{1 - \kappa}{1 - \kappa \rho_\phi} \ln \phi_t \qquad (24)$$

under a suitable transversality condition.

Imposing the CER (24) on the error-correction process (23) provides the equilibrium currency return

$$\Delta \ln S_t = \psi(1 - \kappa)\tilde{b}_{t-1} + \frac{(1 - \kappa)\rho_M}{1 - \kappa \rho_M} \hat{\gamma}_{M,t-1}$$
$$+ \frac{(1 - \kappa)(1 - \rho_\phi)}{1 - \kappa \rho_\phi} \ln \phi_{t-1} - \frac{(1 - \kappa)(1 - \rho_m)}{1 - \kappa \rho_m} \ln m_{t-1} + u_{s,t}. \qquad (25)$$

Equation (25) clearly shows that any dependence of the currency return on past information emerges through the persistence of the net foreign asset position, the money supply growth differential, the transitory money demand shock differential, and the transitory money supply differential.

The important implication of the equilibrium currency return equation (25) is that the logarithm of the exchange rate follows a Martingale difference sequence at the limit of $\kappa \to 1$ because

$$\lim_{\kappa \to 1} E_t \Delta \ln S_{t+1} = 0.$$

Therefore, in this paper, the exchange rate behaves like a random walk when the market discount factor approaches one along the equilibrium path of the two-country model. The equilibrium currency return equation (25) exhibits no dependence of the currency return on past information in this case. Hence, the equilibrium random walk property of the exchange rate, as found in EW and NR, is also preserved in this model.[11]

In the limiting case with the unit market discount factor, the equilibrium currency return is dominated by the i.i.d. rational expectations error $u_{s,t}$. An advantage of working with a structural two-country model is that the rational expectations error $u_{s,t}$ is now fully interpretable as a linear combination of structural shocks. To see this, note that the rational expectations error $u_{s,t}$ in equilibrium is represented by

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[11] A caveat of the above result is that in this model, κ is given as a function of structural parameters β and γ_M: $\kappa = \beta / \gamma_M$. If $\gamma_M > 1$, as found in the postwar data on money growth rates in Canada and the United States, the admissible range of β between zero and one implies that κ is strictly less than one. In this paper, I assume that the limit of $\kappa \to 1$ is well approximated by the limit of $\beta \to 1$ because γ_M takes a value that is very close to one.

$$u_{s,t} = (E_t - E_{t-1})\Delta \ln S_t = \epsilon_{M,t} + (E_t - E_{t-1})\hat{s}_t,$$

where $\epsilon_{M,t} \equiv \epsilon_{M,t}^h - \epsilon_{M,t}^f$ and denote the relative permanent money supply shock. It is not straightforward, however, to calculate the equilibrium surprise of the de-trended exchange rate $(E_t - E_{t-1})\hat{s}_t$. Online appendix B shows that in the special case of two symmetric countries, assuming $\bar{d} = 0$ and $y_h = y_f$, the equilibrium de-trended exchange rate is determined by a linear function of $\tilde{b}_{t-1}$, $\ln a_t$, $\ln m_t$, $\ln \phi_t$, $\ln y_t$, $\ln q_t$, and $\hat{\gamma}_{M,t}$:

$$\hat{s}_t = \frac{\beta\eta - 1}{\beta p_h^* y^*}\tilde{b}_{t-1} - \frac{1 - \beta\eta}{1 - \beta\eta(1-\lambda)}\ln a_t + \frac{1 - \kappa}{1 - \kappa\rho_m}\ln m_t - \frac{1 - \kappa}{1 - \kappa\rho_\phi}\ln \phi_t$$
$$- \frac{1 - \beta\eta}{1 - \beta\eta\rho_y}\ln y_t + \frac{1 - \beta\eta}{1 - \beta\eta\rho_q}\ln q_t + \frac{\kappa\rho_M}{1 - \kappa\rho_M}\hat{\gamma}_{M,t}, \quad (26)$$

where the constant η, which is less than one, approaches one at the limit of $\kappa \to 1$.[12] Hence, the surprise in the de-trended exchange rate between times t and $t - 1$ is

$$(E_t - E_{t-1})\hat{s}_t = -\frac{1 - \beta\eta}{1 - \beta\eta(1-\lambda)}\epsilon_{A,t} + \frac{1 - \kappa}{1 - \kappa\rho_m}\epsilon_{m,t} - \frac{1 - \kappa}{1 - \kappa\rho_\phi}\epsilon_{\phi,t} - \frac{1 - \beta\eta}{1 - \beta\eta\rho_y}\epsilon_{y,t}$$
$$+ \frac{1 - \beta\eta}{1 - \beta\eta\rho_q}\epsilon_{q,t} + \frac{\kappa\rho_M}{1 - \kappa\rho_M}\epsilon_{M,t}.$$

where $\epsilon_{A,t} \equiv \epsilon_{A,t}^h - \epsilon_{A,t}^f$, $\epsilon_{m,t} \equiv \epsilon_{m,t}^h - \epsilon_{m,t}^f$, $\epsilon_{\phi,t} \equiv (\epsilon_{\phi,t}^h - \epsilon_{\phi,t}^f)$, and $\epsilon_{y,t} \equiv \epsilon_{y,t}^h - \epsilon_{y,t}^f$ denote shocks to the relative TFP, the relative transitory money supply, the relative transitory money demand, and the relative transitory income. The rational expectations error is then given as an explicit linear function of the structural shocks:

$$u_{s,t} = \frac{1}{1 - \kappa\rho_M}\epsilon_{M,t} - \frac{1 - \beta\eta}{1 - \beta\eta(1-\lambda)}\epsilon_{A,t} + \frac{1 - \kappa}{1 - \kappa\rho_m}\epsilon_{m,t} - \frac{1 - \kappa}{1 - \kappa\rho_\phi}\epsilon_{\phi,t}$$
$$- \frac{1 - \beta\eta}{1 - \beta\eta\rho_y}\epsilon_{y,t} + \frac{1 - \beta\eta}{1 - \beta\eta\rho_q}\epsilon_{q,t}.$$

Notice that at the limit of $\kappa \to 1$, the model also implies the subjective discount factor $\beta \to 1$ under a positive deterministic money supply growth rate, $\gamma_M > 1$, which is close to one. In this limiting case, observe that the permanent monetary shock $\epsilon_{M,t}$ surely dominates the rational

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[12] As defined in online appendix B, the constant η is one of the two roots of the expectational difference equation of the de-trended net foreign asset position $\tilde{b}_t$. A simple calculation shows that the equilibrium currency return (25) can be derived directly from the CER (26) once the approximated relation $\hat{s}_t \approx \ln S_t + \ln A_t - \ln M_t^\tau$ is recognized.

expectations error $u_{s,t}$ and, as a result, the random walk of the exchange rate.

$$\lim_{\kappa \to 1} \Delta \ln S_t = \lim_{\kappa,\beta,\eta \to 1} u_{s,t} = \frac{1}{1 - \rho_M} \epsilon_{M,t}.$$

Therefore, no transitory shock matters for the total variations in the random-walk exchange rate.

In contrast to the empirical result of NR, which depends on a more flexible reduced-form specification of the consumption differential, no permanent TFP shock can be a primary driver of the random-walk exchange rate. This result is due to the cointegration of the two-country TFPs: No discrepancy between the two countries' TFPs can be permanent in order to guarantee the equilibrium-balanced growth path. The model's theoretical implication of the dominant role of the permanent money supply shock on the random-walk exchange rate, hence, is too restrictive to trace out the actual data variations in the bilateral nominal exchange rate, at least between Canada and the United States.

3.2. Backus and Smith's puzzle at the limit

This model, moreover, has another unrealistic implication on the consumption differential equilibrium dynamics $\ln C_t$ when the discount factor approaches one. At the limit of the unit discount factor, Backus and Smith's (1993) problem of a perfect correlation between relative consumption and the RER emerges even under incomplete international financial markets. To observe this property, taking the first difference of the CER (24) yields

$$\Delta \ln C_t = -\Delta \ln S_t + \frac{(1-\kappa)\rho_M}{1 - \kappa\rho_M} \hat{\gamma}_{M,t-1} + \frac{1}{1 - \kappa\rho_M} \epsilon_{M,t}$$
$$+ \frac{1-\kappa}{1 - \kappa\rho_m} \Delta \ln m_t - \frac{1-\kappa}{1 - \kappa\rho_\phi} \Delta \ln \phi_t + \Delta \ln q_t.$$

Substituting the equilibrium currency return (25) into the above equation and exploiting the rational expectations error $u_{s,t}$ leads to the following consumption differential dynamics:

$$\Delta \ln C_t = \Delta \ln q_t - \psi(1-\kappa)\tilde{b}_{t-1} + \frac{1-\beta\eta}{1 - \beta\eta(1-\lambda)} \epsilon_{A,t} + \frac{1-\beta\eta}{1 - \beta\eta\rho_y} \epsilon_{y,t} - \frac{1-\beta\eta}{1 - \beta\eta\rho_q} \epsilon_{q,t}. \quad (27)$$

Notice, therefore, that except through the net foreign asset position, no monetary shock directly matters for the change in the equilibrium consumption differential: As in the standard international business cycle model, only real shocks to the endowments and the PPP deviation affect the

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equilibrium consumption allocation between the two countries.

Taking the limit of equation (27) above with respect to κ results in

$$\lim_{\kappa,\beta,\eta\to 1} \Delta \ln C_t = \Delta \ln q_t.$$

Thus, relative consumption becomes unrelated to any shocks to the endowments of the two countries but is rather perfectly correlated with the exogenous RER. The intuition behind this result is quite straightforward. In this incomplete market model with the PIH households, consumption in each country is determined by splitting the global aggregate endowment across both countries in each period. The portion of the global aggregate endowment allocated to one country is simply given as the present discounted values of the expected future relative endowments of this country to the other. Because the endowment differential is stationary due to the balanced growth restriction, the unit discount factor at the limit makes the portion converge to a constant; in particular, one-half in the case of two symmetric countries. Consumption in both countries, hence, responds to any endowment shocks in the same fashion. As the result, with the discount factor being close to one, relative consumption depends neither on permanent nor transitory endowment shocks. The only shock that can affect the relative consumption is in the corresponding relative price, i.e., the RER.[13]

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[13]More precisely, from online appendix B, the consumption logarithms of the home and foreign countries in terms of the home currency can be solved as

$$2\ln C_{h,t} = \ln Y_{h,t} + \ln q_t Y_{f,t} + \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)}\ln a_t + \frac{1-\beta\eta}{1-\beta\eta\rho_y}\ln y_t - \frac{1-\beta\eta}{1-\beta\eta\rho_q}\ln q_t + \frac{1-\beta\eta}{\beta p_h^* y^*}\tilde{b}_{t-1},$$

$$2\ln q_t C_{f,t} = \ln Y_{h,t} + \ln q_t Y_{f,t} - \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)}\ln a_t - \frac{1-\beta\eta}{1-\beta\eta\rho_y}\ln y_t + \frac{1-\beta\eta}{1-\beta\eta\rho_q}\ln q_t - \frac{1-\beta\eta}{\beta p_h^* y^*}\tilde{b}_{t-1}.$$

Each country's consumption depends on the log-linearized global aggregate endowment $\ln Y_{h,t} + \ln q_t Y_{f,t}$, the log-linearized country-specific portion of the aggregate endowment $\frac{1-\beta\eta}{[1-\beta\eta(1-\lambda)]}\ln a_t + \frac{1-\beta\eta}{(1-\beta\eta\rho_y)}\ln y_t - \frac{1-\beta\eta}{(1-\beta\eta\rho_q)}\ln q_t$, and the wealth effect of the net foreign asset position $\frac{1-\beta\eta}{\beta p_h^* y^*}\tilde{b}_{t-1}$. If the discount factor approaches one, both the log-linearized country-specific portion and the wealth effect of the net foreign asset position disappear and the log consumption levels become

$$\ln C_{h,t} = \frac{1}{2}(\ln Y_{h,t} + \ln Y_{f,t}) + \frac{1}{2}\ln q_t, \quad \ln C_{f,t} = \frac{1}{2}(\ln Y_{h,t} + \ln Y_{f,t}) - \frac{1}{2}\ln q_t.$$

Relative consumption then turns out to be correlated perfectly with the RER because

$$\ln C_{h,t} - \ln C_{f,t} = \ln q_t.$$

The two country model of this paper also characterizes the analytical closed-form solutions of the nominal interest rates along the equilibrium path. Online appendix B shows that the equilibrium home and foreign interest rates are

$$(1 + \hat{r}^h_{h,t}) = (1 - \kappa) \left(\frac{\rho_M}{1 - \kappa\rho_M} \hat{\gamma}^h_{M,t} - \frac{1 - \rho_m}{1 - \kappa\rho_m} \hat{m}_{h,t} + \frac{1 - \rho_\phi}{1 - \kappa\rho_\phi} \hat{\phi}_{h,t} \right),$$

$$(1 + \hat{r}^f_{f,t}) = (1 - \kappa) \left(\frac{\rho_M}{1 - \kappa\rho_M} \hat{\gamma}^f_{M,t} - \frac{1 - \rho_m}{1 - \kappa\rho_m} \hat{m}_{f,t} + \frac{1 - \rho_\phi}{1 - \kappa\rho_\phi} \hat{\phi}_{f,t} \right).$$

Hence, the home and foreign interest rates are determined by the money supply growth shock $\hat{\gamma}^j_{M,t}$, the transitory money supply shock $\hat{m}_{j,t}$, and the money demand shock $\hat{\phi}_{j,t}$ for $j = h, f$. Because the AR root of the money supply growth ρ_M is expected close to zero, the main determinants of the nominal interest rates are supposed to be transitory monetary shocks, $\hat{m}_{j,t}$ and $\hat{\phi}_{j,t}$.

Recall that in this model the unit market discount factor $\kappa = 1$ means the zeros steady state nominal interest rate $r^* = 0$. The above equilibrium interest rates then show that at the limit of the unit market discount factor, each of the home and foreign nominal interest rates $r^h_{h,t}$ and $r^f_{f,t}$ is insensitive to the domestic transitory monetary shocks. This is exactly the situation of the liquidity trap in which the money demand functions (14) and (18) are perfectly flat. The difficulty due to the liquidity trap should be that the transitory monetary shocks have to have extremely large volatilities to explain the actual data variations in the nominal interest rates. In particular, an extremely volatile transitory money supply shock might result in a counterfactual decomposition of the actual money supply into the permanent and transitory components and worsen the overall fit of the model to data.

4. A Bayesian unobserved component approach

This section empirically explores the question of how significantly the tension emerging at the limit of the unit market discount factor among the three theoretical implications — the random-walk exchange rate, the dominance of permanent money supply differential shocks in the variations in the random-walk exchange rate, and the perfect correlation between the relative consumption and

the RER — affects posterior inferences in relation to the market discount factor. For this specific purpose, I simplify the estimation exercise as much as possible by adopting the symmetric version of the two-country model, in which the same structural parameters are shared by both countries. This paper then takes a Bayesian UC approach to the proposed structural two-country model.

4.1. The restricted UC model and posterior simulation strategy

Under the symmetric case with $\bar{d} = 0$, FONCs (11)-(18) are degenerated to the following three expectational difference equations with respect to the three endogenous variables $\hat{s}_t$, $\hat{c}_t$, and $\tilde{b}_t$, given the six exogenous variables $\hat{\gamma}_{M,t}$, $\hat{m}_t$, $\hat{a}_t$, $\hat{y}_t$, $\hat{\phi}_t$, and $\hat{q}_t$:

$$\hat{s}_t = \kappa E_t \hat{s}_{t+1} - (1-\kappa)\hat{c}_t + (1-\kappa)(\hat{m}_t - \hat{\phi}_t + \hat{q}_t - \hat{a}_t) + \kappa E_t \hat{\gamma}_{M,t+1} - \psi\kappa(1-\kappa)\tilde{b}_t,$$

$$\hat{s}_t + \hat{c}_t - \hat{q}_t + \hat{a}_t = \kappa E_t(\hat{s}_{t+1} + \hat{c}_{t+1} - \hat{q}_{t+1} + \hat{a}_{t+1}) + (1-\kappa)(\hat{m}_t - \hat{\phi}_t) + \kappa E_t \hat{\gamma}_{M,t+1},$$

$$\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + p_h^* y^*(\hat{y}_t - \hat{c}_t), \tag{28}$$

where $y^* = y/4$ and $y = y_h = y_f$. Let $\mathbf{X}_t$ denote an unobserved state vector defined as

$$\mathbf{X}_t = [\hat{s}_t\ \hat{c}_t\ E_t\hat{s}_{t+1}\ E_t\hat{c}_{t+1}\ \tilde{b}_t\ \hat{\gamma}_{M,t}\ \hat{a}_t\ \hat{m}_t\ \hat{y}_t\ \hat{q}_t\ \hat{\phi}_t]'.$$

Furthermore, let ϵ_t and ω_t denote random vectors consisting of structural shocks and rational expectations errors: $\epsilon_t \equiv [\epsilon_{M,t}\ \epsilon_{A,t}\ \epsilon_{m,t}\ \epsilon_{y,t}\ \epsilon_{q,t}\ \epsilon_{\phi,t}]'$ and $\omega_t \equiv [\hat{s}_t - E_{t-1}\hat{s}_t\ \hat{c}_t - E_{t-1}\hat{c}_t]'$, respectively. In particular, for empirical investigation purposes, I presume that the structural shock vector ϵ_t is normally distributed, with a mean of zero and a diagonal variance-covariance matrix Σ: $\epsilon_t \sim$ $i.i.d.N(\mathbf{0}, \Sigma)$ with $diag(\Sigma) = [\sigma_M^2\ \sigma_A^2\ \sigma_m^2\ \sigma_y^2\ \sigma_q^2\ \sigma_\phi^2]'$.

Accompanied by the stochastic processes of the exogenous forcing variables, the linear rational expectations model (28) then implies that

$$\Gamma_0 \mathbf{X}_t = \Gamma_1 \mathbf{X}_{t-1} + \Phi_0 \omega_t + \Phi_1 \epsilon_t,$$

where Γ_0, Γ_1, Φ_0, and Φ_1 are the corresponding coefficient matrices. Applying Sims's (2001) QZ algorithm to the linear rational expectations model above yields a unique solution as the following stationary transition equation of the unobservable state vector:

$$\mathbf{X}_t = \mathbf{F}\mathbf{X}_{t-1} + \Phi\epsilon_t, \tag{29}$$

where $\mathbf{F}$ and Φ are confirmable coefficient matrices.

To construct this paper's UC model, I further expand the unobservable state vector $\mathbf{X}_t$ by the permanent money supply differential $\ln M_t^\tau$ to obtain the augmented state vector $\mathbf{Z}_t$: $\mathbf{Z}_t \equiv [\mathbf{X}_t' \ \ln M_t^\tau]'$. The stochastic process of $\ln M_t^\tau$ and the state transition (29) then imply the following non-stationary transition of the expanded state vector $\mathbf{Z}_t$:

$$\mathbf{Z}_t = \mathbf{G}\mathbf{Z}_{t-1} + \Psi\epsilon_t, \quad \epsilon_t \sim i.i.d.N(\mathbf{0}, \Sigma), \tag{30}$$

where $\mathbf{G}$ and Ψ are confirmable coefficient matrices.

In this paper, I explore time-series data on the log of the consumption differential $\ln C_t$, the log of the output differential $\ln Y_t$, the log of the money supply differential $\ln M_t$, the interest rate differential $r_t \equiv r_{h,t}^h - r_{f,t}^f$, and the log of the bilateral exchange rate $\ln S_t$. Let $\mathbf{Y}_t$ denote the information set that consists of these five time series: $\mathbf{Y}_t \equiv [\ln C_t \ \ln Y_t \ \ln M_t \ r_t \ \ln S_t]'$. It is then straightforward to show that the information set $\mathbf{Y}_t$ is linearly related to the unobservable state vector $\mathbf{Z}_t$ as

$$\mathbf{Y}_t = \mathbf{H}\mathbf{Z}_t, \tag{31}$$

where $\mathbf{H}$ is a confirmable coefficient matrix. The transition equation, the unobserved state (30), and the observation equation (31) jointly consist of a non-stationary state-space representation of the two-country model, which is the restricted UC model estimated in this paper.[14]

Given the data set $\mathbf{Y}^T \equiv \{\mathbf{Y}_t\}_{t=0}^T$, applying the Kalman filter to the UC model provides model likelihood $\mathbf{L}(\mathbf{Y}^T|\theta)$, where θ is the structural parameter vector of the two-country model. Multiplying the likelihood by a prior probability of the structural parameters, $\mathbf{p}(\theta)$, is proportional to the corresponding posterior distribution $\mathbf{p}(\theta|\mathbf{Y}^T) \propto \mathbf{p}(\theta)\mathbf{L}(\mathbf{Y}^T|\theta)$ through the Bayes law. The posterior distribution $\mathbf{p}(\theta|\mathbf{Y}^T)$ is simulated by the random-walk Metropolis-Hastings algorithm, as

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[14]The state-space form of the model, (30) and (31), decomposes the I(1) difference-stationary information set $\mathbf{Y}_t$ into permanent and transitory components exploiting the theoretical restrictions provided by the two-country model. Recursion of the Kalman filter for a non-stationary state-space model is explained in detail by Hamilton (1994).

implemented by Schorfheide (2000), Bouakez and Kano (2006), and Kano (2009).

4.2. Data and prior construction

The two countries that I empirically examine in this paper are Canada and the United States as the model's home and foreign countries, respectively. I examine post-Bretton Woods quarterly data for these two countries because they satisfy the model assumptions. The data span the period from Q1:1973 to Q4:2007. All the data included in the information set $\mathbf{Y}^T$, except nominal exchange rates, are seasonally adjusted annual rates.[15]

Table 1 reports the prior distributions of the structural parameters of the two-country model, $\mathbf{p}(\theta)$. Since the main goal of this paper's empirical investigation is to draw a posterior inference on the market discount factor $\kappa \equiv \beta/\gamma_M$, I elicit a uniform prior distribution of κ and let the data tell the posterior position of κ given the identification of the restricted UC model. In so doing, on the one hand, the prior distribution of the mean gross monetary growth rate, γ_M, is intended to tightly cover its sample counterparts in both countries through the Gamma distribution, with a mean of 1.015 and standard deviation of 0.005.[16] On the other hand, the prior distribution of the subjective discount factor β is uniformly distributed between zero and one. As a result, the prior distribution of the market discount factor κ is well approximated as the uniform distribution spread over the support of the unit interval.

To guarantee the stationarity of the de-trended net foreign asset position $\tilde{b}_t$, the debt elasticity of the home risk premium ψ should be positive. I therefore set the prior distribution of ψ to the Gamma distribution, with a mean of 0.010 and standard deviation of 0.001. Closing the model also requires the technological diffusion speed λ to be positive but less than one. This necessary condition for the equilibrium-balanced growth path elicits the prior distribution of λ as the Beta distribution, with a mean of 0.010 and standard deviation of 0.001. The slow technological diffusion that the prior mean of λ implies is intended to capture the slow-moving time-series properties observed in

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[15]Online appendix D provides a detailed description of the source and construction of the data examined in this paper.

[16]The sample mean of the M1 money supply's gross growth rate is 1.016 for Canada and 1.014 for the United States.

the actual consumption and output differentials between Canada and the United States. The prior distribution of the mean monetary demand shock ϕ follows the Gamma distribution, with a mean of 1.000 and small standard deviation of 0.010. By doing so, I assume *a priori* that the monetary demand shock has no effect on the deterministic steady state.

I admit a small persistence of the permanent money growth rate by setting the prior distribution of the AR(1) coefficient ρ_M to the Beta distribution, with a mean of 0.100 and standard deviation of 0.010. The PPP deviation shock, i.e., the RER shock, is presumed to be very persistent, as observed by many past empirical studies on the RER. The AR(1) coefficient of the RER, ρ_q, is then accompanied by the Beta prior distribution, with a mean of 0.850 and standard deviation of 0.100. This prior distribution mimics fairly well the posterior distribution of the same structural parameter reported in Figure 3 of BMW, who used a long annual sample of data from the United Kingdom and the United States.[17] On the other hand, there is no robust empirical consensus on the extent of the persistence of the money demand shock. Hence I allow the prior distribution of the AR(1) coefficient of the money demand shock, ρ_ϕ, to be distributed around 0.850 following the Beta distribution, with a mean of 0.850 and a large standard distribution of 0.100. The resulting 95 % coverage, indeed, is [0.607 0.983], which also covers the corresponding posterior distribution displayed within Figure 3 of BMW. Furthermore, to better identify the permanent components of the money supplies and TFPs of both countries, I assume that the corresponding transitory components are white noise by setting the prior mass points of the AR(1) coefficients ρ_m and ρ_y to zero. Following NR, I also allow for the deterministic time trend in the exchange rate, γ_S, with the normal prior distribution with the zero mean and the large standard deviation of 1.500. Finally, the prior standard deviations of all the structural shocks are assumed to share the identical inverse-Gamma distribution, with a mean of 0.010 and standard deviation of 0.010. This prior distribution of Σ yields a higher marginal likelihood among small perturbations. Below, I refer to this prior specification as the Benchmark model.

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[17]In fact, the 95 % interval of [0.607 0.983] includes the most inferences on RER persistence established in major past studies (see, e.g., Rogoff 1996 and Lothian and Taylor 2000).

The second, third, and fourth columns of Table 2 describe the posterior distributions of the structural parameters under the Benchmark model. The most striking posterior inference conveyed by these columns is that the market discount factor κ is identified as being far below one. As displayed in the first row, the data pin down the location of κ very tightly around the posterior mean of 0.512, with a standard deviation of 0.028. This posterior distribution of the market discount factor is too low to guarantee the second necessary condition of the equilibrium random-walk exchange rate established by EW and NR, i.e., that the market discount factor is sufficiently close to one. The other significant result in Table 2 relates to the posterior inferences on the money demand differential shock, ρ_ϕ and σ_ϕ: The data show a more persistent and volatile money demand differential shock compared to the prior specification of the Benchmark model. Notice that the posterior mean of ρ_ϕ is 0.997 and almost 10 % larger than its prior mean value; the posterior mean of σ_ϕ is 0.027 and 17 % larger than its prior mean value. The very persistent money demand differential shock provides evidence that such a structural shock could play a significant role in actual exchange rate movements.

Does this lower market discount factor deteriorate the model's fit to actual exchange rate movements? The answer is clearly no, although the equilibrium currency return depends slightly on past economic fundamentals. The estimated Benchmark model is indeed successful in explaining the historical trajectory of the exchange rate. Figure 1(a) plots the actual depreciation rate of the Canadian dollar against the United States dollar as the solid black line. The same figure also displays the 95 % Bayesian highest probability density (HPD) interval of the in-sample prediction of the depreciation rate by the Benchmark model (the dashed blue lines).[18] The HPD interval is very narrow: the Benchmark model yields a sharp in-sample prediction of the depreciation rate. Indeed, the HPD interval includes the actual depreciation rate at almost all the sample periods. Hence, the model tracks the actual depreciation rate fairly well.

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[18]The in-sample prediction of the depreciation rate is calculated by feeding the Kalman smoothers of all the structural shocks into the restricted UC model (30) and (31) evaluated at each posterior draw of the structural parameters.

Which structural shocks are the main drivers of the successful in-sample fit of the Benchmark model to the depreciation rate? To answer this important question, I calculate the same in-sample prediction of the depreciation rate with the Benchmark model as in Figure 1(a), but shutting down one structural shock at a time. Along with the actual depreciation rate (the solid black line), each window in Figure 2 corresponding to a particular structural shock exhibits the HPD interval of the in-sample prediction of the depreciation rate with the Kalman smoother of the corresponding structural shock excluded (the dashed blue lines); the upper-left window corresponds to the prediction with the TFP differential shock $\epsilon_{A,t}$ excluded; the upper-middle window corresponds to the prediction with the transitory money supply differential shock $\epsilon_{m,t}$ excluded; the upper-right window corresponds to the prediction with the transitory output differential shock $\epsilon_{y,t}$ excluded; the lower-left window corresponds to the prediction with the PPP deviation shock $\epsilon_{q,t}$ excluded; the lower-center window corresponds to the prediction with the money demand differential shock $\epsilon_{\phi,t}$ excluded; and, finally, the lower-right window corresponds to the prediction with the permanent money supply differential shock $\epsilon_{M,t}$ excluded. If the corresponding structural shock plays a major role in the successful in-sample prediction of the depreciation rate observed in Figure 1(a), shutting down such a shock will deteriorate the in-sample fit of the Benchmark model to the depreciation rate significantly.

The six windows in Figure 2(a) clearly reveal that the most important structural shock for the near-random-walk exchange rate between Canada and the United States is identified as the very persistent money demand differential shock in conjunction with the permanent money supply differential shock. This inference about the main driver of nominal exchange rates echoes the findings of the past studies by EW, BMW, and Sarno and Schmelling (2014): economic fundamentals of near random-walk exchange rates should be unobservable and nominal such as a money demand shock. Nevertheless, it is important to note that the estimated low discount factor allows the TFP differential shock to contribute to actual exchange rate fluctuations, although to a much smaller degree than the permanent money supply and the money demand differential shocks. In contrast to the observation of NR, the TFP shock plays only a minor role in data variations in the nominal

exchange rate.

The same historical decomposition of the in-sample prediction of the Benchmark model into the structural shocks is also applicable to the two endogenous economic fundamentals, the consumption differential and the TB differential. Each window of Figure 3(a) (Figure 4a) corresponding to a particular structural shock displays the 95% HPD interval of the in-sample prediction of the consumption growth differential (the TB differential) with the Kalman smoother of the corresponding structural shock excluded (the dashed blue lines), respectively. Observe in the upper-left window of Figure 3(a) the dominant role that the TFP differential shock plays in the actual consumption growth differential. The Benchmark model identifies that the other structural shocks are unlikely to have any significant effect on the variations in the consumption growth differential at all. The upper-middle window of Figure 4(a) then shows evidence that the transitory money supply differential shock primarily drives the actual TB differential data. This result is consistent with the theoretical implication of the model for the equilibrium interest rate differential, given a highly persistent money demand differential shock.

4.4. Understanding lower discount factors: The High Discount Factor model

Why does the Benchmark model result in such a lower discount factor? To understand this question, I conduct an alternative Bayesian posterior simulation exercise. In this exercise, I intend to fix the discount factor close to one and observe how the empirical performance of the model changes relative to that of the Benchmark model. In so doing, I replace the uniform prior distribution of β in the Benchmark model with a more informative Beta distribution, with a mean of 0.999 and standard deviation of 0.001, and stay with the same prior distributions of the remainder of the structural parameters as in the Benchmark model. I refer to this new specification as the High Discount Factor (HDF) model.

The fifth, sixth, and seventh columns of Table 2 correspond to the posterior distributions of the structural parameters under the HDF model. Observe that the resulting posterior distributions of both the market and subjective discount factors are much closer to one, with posterior means

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of 0.932 and 0.998, respectively. Crucial changes in the posterior distributions of the structural parameters from the Benchmark model, then, are recognized as significant increases in the posterior means of the standard deviations of the three monetary shocks, σ_M, σ_m, and σ_ϕ. The HDF model, which suffers from the liquidity trap at the deterministic steady state, requires counterfactually greater volatilities in all the monetary shocks to explain the data.

An important difference between the Benchmark and HDF models is related to the overall fit to the data. First, the last row of Table 2 reports the estimated marginal likelihood for each model.[19] The HDF model yields a smaller marginal likelihood of 1873.109 compared to that of the Benchmark model, which was 2143.269. The difference in the marginal likelihoods of the two models is so significant that I conclude that forcing the discount factor to be close to one makes the HDF model's overall fit to the data much worse than that of the Benchmark model. Figure 5 more clearly reveals the source of this significant deterioration of the HDF model compared to the Benchmark model with respect to the marginal likelihood. This figure plots the 95 % HPD intervals of the one-period-ahead forecast errors of the Benchmark and HDF models toward the actual data as the dashed blue and dotted red lines, respectively.[20] The figure clearly shows the greatest difficulty for the HDF model relative to the Benchmark model relates to its fit to the money supply differential.

Why does the HDF model fail to explain the money supply differential? Remember the model's implication at the limit of the unit discount factor: in contrast to the Benchmark model, with a lower discount factor, the exchange rate data should be explained exclusively by either the permanent money supply differential shock or the persistent money demand differential shock or both. The permanent money supply differential shock, however, needs to predict the actual trajectory of the money supply differential as well. These two restrictions on the permanent money supply differential shock then force (i) the persistent money demand differential shock to play a dominant role in explaining the exchange rate and (ii) the white noise transitory money supply differential shock to act as a significant driver of the money supply differential. Recall that the

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[19]This paper estimates the marginal likelihoods by using Geweke's (1999) modified harmonic mean estimator. A marginal likelihood is the probability of data $\mathbf{Y}^T$ conditional on an underlying model. In general, the higher the marginal likelihood is, the better the underlying model's overall fit to the data.

[20]The forecast errors of the two models are calculated through the Kalman filter forward recursion.

unit discount factor at the limit also implies that the transitory money supply differential shock dominates the TB differential data. Under the liquidity trap of the HDF, this implication for the TB differential requires the transitory money supply differential shock to be counterfactually volatile. Because the money supply differential consists of the permanent and transitory components, the permanent component should also be greatly volatile. This volatile permanent component of the money supply differential then leads to the worse prediction of the HDF model toward the money supply differential data.

The dotted red line displayed in Figure 1(b) indicates the 95 % HDF interval of the in-sample prediction on the depreciation rate implied by the HDF model. Furthermore, Figures 2(b), 3(b), and 4(b) exhibit the historical decompositions of the in-sample predictions of the depreciation rate, the consumption growth differential, and the TB differential, as the counterparts of Figures 2(a), 3(b) and 4(b) for the Benchmark model, respectively. These in-sample predictions convey four properties of the HDF model: (i) the HDF model tracks the actual near random-walk exchange rate to the almost same degree as the Benchmark model, (ii) the permanent money supply differential shock and the persistent money demand differential shock jointly and dominantly explain actual exchange rate movements; and (iii) the PPP deviation shock, not the TFP shock as in the Benchmark model, is the dominant driver of the consumption growth differential; and (iii) not only the transitory money supply differential shock but also the persistent money demand differential shock explains the TB differential.[21] The first and second properties echo the main finding of the Benchmark model. The third property, however, represents the drawback of the HDF model. As seen in section 3, with a high discount factor, the consumption differential almost perfectly matches the exogenous PPP deviation shock, of which the exchange rate becomes independent. In the HDF model, the PPP deviation shock, hence, acts as a free latent variable to dominantly explain the consumption differential. This third property, however, is counterfactual.

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[21]The HDF model identifies a smaller AR coefficient of the money demand differential shock ρ_ϕ. The implied smaller persistence of the money demand differential shock results in the larger role this shock plays in the TB differential than that identified by the Benchmark model.

5. Conclusions

In this paper, I try to reconcile the random-walk property of nominal exchange rates with a canonical two-country endowment model including incomplete international financial markets. The main challenge undertaken in this paper is to establish the joint equilibrium dynamics of nominal exchange rates and economic fundamentals, both of which should be endogenously determined by the two-country model. After closing the model correctly by allowing the TFPs of both countries to be cointegrated, I discover the equilibrium random-walk property of exchange rates when the cross-country money supply differential contains a permanent component and the market discount factor approaches one. The assumption for the equilibrium random-walk exchange rate that the discount factor is close to one, however, implies unrealistic restriction restrictions — permanent money supply differential shocks as the dominant driver of random-walk exchange rates, the Backus and Smith puzzle of a perfect correlation between relative consumption and the RER, and the counterfactually large volatilities of monetary disturbances due to the steady state liquidity trap.

Bayesian posterior simulation exercises based on post-Bretton Woods data from Canada and the United States reveal a major difficulty in reconciling the random-walk exchange rate and the economic fundamentals with the proposed two-country model. Indeed, under the benchmark identification of the model, the data updates the value of the market discount factor to far below one. Investigating the model with a specification in which the market discount factor is *a priori* set sufficiently high, I empirically confirm the theoretical conjecture that the posterior inference of a low market discount factor stems from the fact that the model suffers from the Backus and Smith puzzle and that it fails to explain the actual money supply differential.

This paper's findings of such a low discount factor are in sharp contrast to those of high market discount factors in past empirical studies such as NR, Sarno and Sojli (2009), and BMW. Because these past studies did not jointly consider the endogenous determination of economic fundamentals with nominal exchange rates, the general equilibrium consideration sought by this paper is relevant to better understanding of the near random-walk behavior of nominal exchange rates within structural open-economy models. Identifying an open economy DSGE model that can reconcile the

joint equilibrium dynamics of random-walk exchange rates and economic fundamentals under an empirically plausible market discount factor value is a serious open question to be addressed. Since the most crucial difference between the empirical exercise in this paper and that of NR's is in the differing stochastic treatments of the TFP differential and, as a result, the consumption differential, it would be a promising research direction to search for a model-consistent way of allowing the TFP differential to be I(1) without violating the balanced growth restriction.

Furthermore, the model of this paper is absent from a more realistic specification of a monetary policy framework such as the inflation targeting policy introduced by the Bank of Canada in 1991. Because the inflation targeting policy affects the way of the market participants to form long-run expectations of inflation, incorporating such a monetary policy framework into the model changes its CERs significantly. An open question, then, is how to admit an I(1) economic fundamental within the inflation targeting policy framework to preserve random walk exchange rates. A very persistent trend inflation, as investigated by Cogley and Sbordone (2008), might be a plausible candidate of an I(1) nominal economic fundamental. I leave these challenging questions as valuables for future studies on open-economy macroeconomics to undertake.

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Appendices not intended for publication

Appendix A. Stochastically de-trended system

The stochastically de-trended versions of the FONCs of the home country, (1) (3), (4), and (5), are

$$p_{h,t}c_{h,t} + b^h_{h,t} + s_t b^f_{h,t} = \frac{(1+r^h_{h,t-1})b^h_{h,t-1}}{\gamma^h_{M,t}} + \frac{(1+r^f_{h,t-1})s_t b^f_{h,t-1}}{\gamma^f_{M,t}} + p_{h,t}y_{h,t},$$

$$\frac{1}{p_{h,t}c_{h,t}} = \beta(1+r^h_{h,t})E_t\left(\frac{1}{\gamma^h_{M,t+1}p_{h,t+1}c_{h,t+1}}\right),$$

$$s_t(1+r^h_{h,t})E_t\left(\frac{1}{p_{h,t+1}c_{h,t+1}\gamma^h_{M,t+1}}\right) = (1+r^f_{h,t})E_t\left(\frac{s_{t+1}}{p_{h,t+1}c_{h,t+1}\gamma^f_{M,t+1}}\right),$$

$$\frac{m_{h,t}}{p_{h,t}} = \phi_{h,t}c_{h,t}\left(\frac{1+r^h_{h,t}}{r^h_{h,t}}\right),$$

$$r^h_{h,t} = r^h_{w,t}[1 + \psi\{\exp(-b^h_{h,t} + \bar{d}) - 1\}],$$

and

$$r^f_{h,t} = r^f_{w,t}[1 + \psi\{\exp(-b^f_{h,t} + \bar{d}) - 1\}].$$

Similarly, the stochastically de-trended versions of the FONCs of the foreign country, (2) (6), (7), and (8), are

$$\frac{q_t p_{h,t}c_{f,t}}{a_t} - s_t b^f_{h,t} - b^h_{h,t} = -\frac{(1+r^f_{w,t-1})s_t b^f_{h,t-1}}{\gamma^f_{M,t}} - \frac{(1+r^h_{w,t-1})a_t b^h_{h,t-1}}{\gamma^h_{M,t}} + \frac{q_t p_{h,t}y_{f,t}}{a_t},$$

$$\frac{a_t s_t}{q_t p_{h,t}c_{f,t}} = \beta(1+r^f_{w,t})E_t\frac{a_{t+1}s_{t+1}}{\gamma^f_{M,t+1}q_{t+1}p_{h,t+1}c_{f,t+1}},$$

$$s_t(1+r^h_{w,t})E_t\left(\frac{a_{t+1}}{q_{t+1}p_{h,t+1}c_{f,t+1}\gamma^h_{M,t+1}}\right) = (1+r^f_{w,t})E_t\left(\frac{a_{t+1}s_{t+1}}{q_{t+1}p_{h,t+1}c_{f,t+1}\gamma^f_{M,t+1}}\right),$$

and

$$\frac{a_t s_t m_{f,t}}{q_t p_{h,t}} = \phi_{f,t}c_{f,t}\left(\frac{1+r^f_{w,t}}{r^f_{w,t}}\right).$$

Finally The stochastically de-trended PPP condition is $s_t = p_{h,t}q_t/(a_t p_{f,t})$.

If the TFP differential a_t is I(1) as assumed in NR, the above system of stochastic difference equations becomes non-stationary through the home and foreign budget constraints and there is no deterministic steady state to converge. Notice that the cross-country permanent money supply differential $\ln M^\tau_{h,t}/M^\tau_{f,t}$ does not appear in the stochastically de-trended system of the FONCs. In contrast to the TFP differential a_t, the I(1) property of $\ln M^\tau_{h,t}/M^\tau_{f,t}$ in Assumption 2 does not matter for the closing of the model. This might be an obvious result of the model's property that the super-neutrality of money holds in the money-in-utility model: Money growth does not matter for the deterministic steady state.

Notice that at the deterministic steady state, the TFP differential a^* is one. Because of the stationarity of the above system of equations, the deterministic steady state is characterized by constants c_h^*, c_f^*, p_h^*, s^*, b_h^{h*}, b_h^{f*}, r_h^{h*}, r_h^{f*}, r_w^{h*}, and r_w^{f*} that satisfy

$$b_h^{h*} = b_h^{f*} = \bar{d},$$
$$r^* \equiv r_h^{h*} = r_f^{f*} = r_w^{h*} = r_w^{f*} = \gamma_M/\beta - 1,$$
$$s^* = \frac{y_f(\phi\gamma_M)^{-1}r^* + (y_h + y_f)(1 - \beta^{-1})\bar{d}}{y_h(\phi\gamma_M)^{-1}r^* - (y_h + y_f)(1 - \beta^{-1})\bar{d}},$$
$$p_h^* y_h = (1 - \beta^{-1})(1 + s^*)\bar{d} + (\phi\gamma_M)^{-1}r^*,$$
$$p_h^* c_h^* = (\phi\gamma_M)^{-1}r^*,$$
$$c_f^* = s^* c_h^*.$$

Appendix B. Solving a case with two symmetric countries

To understand the equilibrium transitory dynamics of the exchange rate in this model, it is informative to scrutinize a simpler version of the model that includes two symmetric countries. For this purpose, I set the parameter $\bar{d}$ to zero and assume that the transitory output components of the two countries, y_h and y_f, are equal to y. Notice that the deterministic steady state in this case is characterized by $s^* = 1$, $c_h^* = c_f^* = y$, and $p_h^* = (\phi\gamma_M)^{-1}r^*$, where $r^* = \gamma_M/\beta - 1$.

I combine the log-linearized Euler equations of the home and foreign countries, (12) and (16), with those of the home country's interest rates (19) to yield the first-order expectational difference equation of $\hat{s}_t + \hat{c}_{h,t} - \hat{c}_{f,t}$:

$$\hat{s}_t + \hat{c}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = \kappa E_t(\hat{s}_{t+1} + \hat{c}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1} + \hat{a}_{t+1}) + \kappa E_t \hat{\gamma}_{M,t+1} + (1 - \kappa)(\hat{m}_t - \hat{\phi}_t).$$

Since κ takes a value between zero and one, the above expectational difference equation has a forward solution of $\hat{s}_t + \hat{c}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = \kappa\rho_M(1 - \kappa\rho_M)^{-1}\hat{\gamma}_{M,t} + (1 - \kappa)(1 - \kappa\rho_m)^{-1}\hat{m}_t - (1 - \kappa)(1 - \kappa\rho_\phi)^{-1}\hat{\phi}_t$ under a suitable transversality condition. By exploiting this forward solution and the stochastic processes of both countries' TFPs (10), I rewrite the foreign UIP condition (17) as

$$E_t\hat{s}_{t+1} - \hat{s}_t = \psi(1 - \kappa)\tilde{b}_t - \frac{\kappa\rho_M(1 - \rho_M)}{1 - \kappa\rho_M}\hat{\gamma}_{M,t} - \frac{(1 - \kappa)(1 - \rho_m)}{1 - \kappa\rho_m}\hat{m}_t + \frac{(1 - \kappa)(1 - \rho_\phi)}{1 - \kappa\rho_\phi}\hat{\phi}_t, \qquad (B.1)$$

Furthermore, taking a difference between the log-linearized budget constraints of the home and foreign countries, (11) and (15), I find the law of motion of the international bond holdings

$$\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + p_h^* y^* \hat{s}_t - p_h^* y^*(\hat{q}_t - \hat{a}_t) - \frac{p_h^* y^* \kappa\rho_M}{1 - \kappa\rho_M}\hat{\gamma}_{M,t} - \frac{p_h^* y^*(1 - \kappa)}{1 - \kappa\rho_m}\hat{m}_t + \frac{p_h^* y^*(1 - \kappa)}{1 - \kappa\rho_\phi}\hat{\phi}_t + p_h^* y^* \hat{y}_t, \; (B.2)$$

where $y^* = y/4$ and $\hat{y}_t \equiv \hat{y}_{h,t} - \hat{y}_{f,t}$.

Combining equation (B.1) with equation (B.2) then yields the following second-order expectational difference equation with respect to international bond holdings:

$$E_t\tilde{b}_{t+1} - [1 + \beta^{-1} + p_h^* y^* \psi(1 - \kappa)]\tilde{b}_t + \beta^{-1}\tilde{b}_{t-1} = -\lambda p_h^* y^* \hat{a}_t + p_h^* y^*(1 - \rho_q)\hat{q}_t - p_h^* y^*(1 - \rho_y)\hat{y}_t \qquad (B.3)$$

It is straightforward to show that equation (B.3) has two roots, one of which is greater than one and the other of which is less than one.[22] Without losing generality, let η denote the root that is less than one. Solving equation (B.3) by forward iterations then shows that the equilibrium international bond holdings level is determined by the following cross-equation restriction (CER):

$$\tilde{b}_t = \eta\tilde{b}_{t-1} + \beta\eta\lambda p_h^* y^* \sum_{j=0}^{\infty} (\beta\eta)^j E_t \hat{a}_{t+j} + \beta\eta p_h^* y^* (1-\rho_y) \sum_{j=0}^{\infty} (\beta\eta)^j E_t \hat{y}_{t+j} - \beta\eta p_h^* y^* (1-\rho_q) \sum_{j=0}^{\infty} (\beta\eta)^j E_t \hat{q}_{t+j},$$

$$= \eta\tilde{b}_{t-1} + \frac{\beta\eta\lambda p_h^* y^*}{1-\beta\eta(1-\lambda)} \hat{a}_t + \frac{\beta\eta p_h^* y^* (1-\rho_y)}{1-\beta\eta\rho_y} \hat{y}_t - \frac{\beta\eta p_h^* y^* (1-\rho_q)}{1-\beta\eta\rho_q} \hat{q}_t. \tag{B.4}$$

Substituting equation (B.4) back into equation (B.2) provides the CER for the exchange rate (26):

$$\hat{s}_t = \frac{\beta\eta - 1}{\beta p_h^* y^*} \tilde{b}_{t-1} - \frac{1-\beta\eta}{1-\beta\eta(1-\lambda)} \hat{a}_t + \frac{1-\kappa}{1-\kappa\rho_m} \hat{m}_t - \frac{1-\kappa}{1-\kappa\rho_\phi} \hat{\phi}_t$$

$$- \frac{1-\beta\eta}{1-\beta\eta\rho_y} \hat{y}_t + \frac{1-\beta\eta}{1-\beta\eta\rho_q} \hat{q}_t + \frac{\kappa\rho_M}{1-\kappa\rho_M} \hat{\gamma}_{M,t}.$$

Therefore, in this symmetric case, the competitive equilibrium along the balanced growth path is characterized by a lower dimensional dynamic system of $(\hat{s}_t, \tilde{b}_t, \hat{a}_t, \hat{\gamma}_{M,t}, \hat{m}_t, \hat{\phi}_t, \hat{y}_t, \hat{q}_t)$.

Adding the log-linearized home and foreign budget constraints together implies the resource constraint $\hat{c}_{h,t} + \hat{c}_{f,t} = \hat{y}_{h,t} + \hat{y}_{f,t}$. Since the equilibrium dynamics of the consumption differential follow $\hat{c}_{h,t} - \hat{c}_{f,t} = -\hat{s}_t + \hat{q}_t - \hat{a}_t + \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} + (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t - (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$, the home country's consumption obeys $2\hat{c}_{h,t} = (\hat{y}_{h,t} + \hat{y}_{f,t}) - \hat{s}_t + \hat{q}_t - \hat{a}_t + \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} + (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t - (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$, while the foreign country's is $2\hat{c}_{f,t} = (\hat{y}_{h,t} + \hat{y}_{f,t}) + \hat{s}_t - \hat{q}_t + \hat{a}_t - \kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}_{M,t} - (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_t + (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_t$. The home country's price $\hat{p}_{h,t}$ then is determined as follows. The Euler equation and the money demand function of the foreign country, (16) and (18), imply the expectational difference equation of $\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t}$

$$\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = \kappa E_t(\hat{s}_{t+1} - \hat{p}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1} + \hat{a}_{t+1} - \hat{\gamma}^f_{M,t+1}) - (1-\kappa)(\hat{m}_{f,t} - \hat{\phi}_{f,t}).$$

Solving the above equation by forward iterations and imposing a suitable transversality condition yields the CER $\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = -\kappa\rho_M(1-\kappa\rho_M)^{-1}\hat{\gamma}^f_{M,t} - (1-\kappa)(1-\kappa\rho_m)^{-1}\hat{m}_{f,t} + (1-\kappa)(1-\kappa\rho_\phi)^{-1}\hat{\phi}_{f,t}$. This CER characterizes the equilibrium home price

$$2\hat{p}_{h,t} = \hat{s}_t - (\hat{y}_{h,t} + \hat{y}_{f,t}) - \hat{q}_t + \hat{a}_t + \frac{\kappa\rho_M}{1-\kappa\rho_M}(\hat{\gamma}^h_{M,t} + \hat{\gamma}^f_{M,t}) + \frac{1-\kappa}{1-\kappa\rho_m}(\hat{m}_{h,t} + \hat{m}_{f,t}) - \frac{1-\kappa}{1-\kappa\rho_\phi}(\hat{\phi}_{h,t} + \hat{\phi}_{f,t}).$$

The money demand functions of both countries, (14) and (18), imply that the interest rates in the two countries are

$$(1 + \hat{r}^h_{h,t}) = (1-\kappa)\left(\frac{\rho_M}{1-\kappa\rho_M}\hat{\gamma}^h_{M,t} - \frac{1-\rho_m}{1-\kappa\rho_m}\hat{m}_{h,t} + \frac{1-\rho_\phi}{1-\kappa\rho_\phi}\hat{\phi}_{h,t}\right)$$

$$(1 + \hat{r}^f_{w,t}) = (1-\kappa)\left(\frac{\rho_M}{1-\kappa\rho_M}\hat{\gamma}^f_{M,t} - \frac{1-\rho_m}{1-\kappa\rho_m}\hat{m}_{f,t} + \frac{1-\rho_\phi}{1-\kappa\rho_\phi}\hat{\phi}_{f,t}\right).$$

Finally, as the last endogenous variable, the world interest rate of the home bonds then fluctuates in

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[22]To characterize the roots of the second-order expectational difference equation, see, for example, Sargent (1987).

response to the risk premium, following $(1 + \hat{r}_{w,t}^h) = (1 + \hat{r}_{h,t}) + \psi(1 - \kappa)\tilde{b}_t$.

Suppose that $\psi = 0$: There is no debt elastic risk premium in the home country's interest rate. It is easy to show that in this case, the second-order expectational difference equation (B.3) has a unit root, i.e., $\eta = 1$, and the resulting forward solution turns out to be

$$\tilde{b}_t = \tilde{b}_{t-1} + \frac{\beta\lambda p_h^* y^*}{1 - \beta(1 - \lambda)}\hat{a}_t + \frac{\beta p_h^* y^*(1 - \rho_y)}{1 - \beta\rho_y}\hat{y}_t - \frac{\beta p_h^* y^*(1 - \rho_q)}{1 - \beta\rho_q}\hat{q}_t.$$

Hence, the stochastic process of the de-trended international bond holding $\tilde{b}_t$ contains a permanent unit root component and never converges to the steady state. This lack of stationarity of the equilibrium balance growth path motivates this paper to allow for a positive elasticity of the risk premium with respect to the debt level.

Importantly, a permanent stochastic process of the de-trended international bond holding also emerges even when $\kappa = 1$. Because the log-linearized home country's interest rates (19) imply that under $\kappa = 1$, the debt elastic risk premia in play no role in determining the interest rates faced by the home country. As a result, the de-trended international bond holding $\tilde{b}_t$ contains a permanent unit root component, as in the case where $\psi = 0$. Hence, the closing of the two-country DSGE model in this paper requires the market discount factor to be strictly less than one.

Appendix C. Derivation of the error correction representation (23)

Let n_t denote the fundamental of the DSGE-PVM (21): $n_t \equiv \ln M_t - \ln C_t - \psi\kappa\tilde{b}_t - \ln\phi_t + \ln q_t$. Consider the currency return $\Delta\ln S_t$ adjusted by the fundamental $(1 - \kappa)n_{t-1}$: $\Delta\ln S_t + (1 - \kappa)n_{t-1}$. The DSGE-PVM (21) then implies:

$$\begin{aligned}
\Delta\ln S_t + (1 - \kappa)n_{t-1} &= (1 - \kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i} + (1 - \kappa)\sum_{j=0}^{\infty}\kappa^j E_{t-1}n_{t+i} \\
&\quad - (1 - \kappa)\sum_{j=0}^{\infty}\kappa^j E_{t-1}n_{t+i-1} + (1 - \kappa)n_{t-1}, \\
&= (1 - \kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i} + \frac{(1 - \kappa)^2}{\kappa}\sum_{i=0}^{\infty}\kappa^i E_{t-1}n_{t+i-1} - \frac{(1 - \kappa)^2}{\kappa}n_{t-1}, \\
&= (1 - \kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i} + \frac{1 - \kappa}{\kappa}\ln S_{t-1} - \frac{(1 - \kappa)^2}{\kappa}n_{t-1}.
\end{aligned}$$

This result means that the currency return has the following error correction representation, given by equation (23):

$$\Delta\ln S_t = \frac{1 - \kappa}{\kappa}(\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} + \psi\kappa\tilde{b}_{t-1} + \ln\phi_{t-1} - \ln q_{t-1})$$

$$+ (1 - \kappa)\sum_{j=0}^{\infty}\kappa^j(E_t - E_{t-1})n_{t+i}.$$

Appendix D. Data description and construction

All data for the United States are distributed by Federal Reserve Economic Data (FRED), operated by the Federal Reserve Bank of St. Louis (http://http://research.stlouisfed.org/fred2/). The consumption data are constructed as the sum of the real personal consumption expenditures on non-durables and services. FRED, however, distributes only the nominal values of the two categories of personal consumption expenditures as *Personal Consumption Expenditure on Non-Durables (PCND)* and *Personal Consumption Expenditure on Services (PCESV)*. To construct the real total personal consumption expenditure $C_{us,t}$, I first calculate the share of the two nominal consumption categories in the nominal total personal consumption expenditure *Personal Consumption Expenditure* and then multiply the real total personal consumption expenditures, *Real Personal Consumption Expenditures at Chained 2005 Dollars (PCECC96)*, by the calculated share. Following NR, I adopt the M1 money stock, *M1SL*, as the aggregate money supply $M_{us,t}$. The nominal interest rate $r_{us,t}$ is provided by *three-month Treasury Bill (TB3MS)*. All the variables except the nominal interest rate are seasonally adjusted at annual rates and converted to the corresponding per capita terms by *Total Population (POP)*.

All Canadian data are distributed by Statistics Canada (CANSIM) (http://www5.statcan.gc.ca/cansim/). The real consumption data $C_{can,t}$ are constructed as the sum of *Personal Expenditure on Non-Durables at Chained 2002 Dollars*, *Personal Expenditure on Semi-Durables at Chained 2002 Dollars*, and *Personal Expenditure on Services at Chained 2002 Dollars*. I use the M1 money stock as the money supply $M_{can,t}$. The nominal interest rate $r_{can,t}$ is provided by *three-Month Treasury Bills*. All the variables except the nominal interest rate are seasonally adjusted at annual rates and converted to the corresponding per capita terms by *Estimate of Total Population*.

The output measures for Canada and the United States, $Y_{can,t}$ and $Y_{us,t}$, are constructed as in a model-consistent way. In this two-country endowment economy model, a country ' s output is given by the sum of consumption and the trade balance. To measure the bilateral trade balance between Canada and the United States, TB_t, I use the Canadian goods trade balance for the United States included in CANSIM's balance of international payments data (CANSIM Table 376-0005). The Canadian output $Y_{can,t}$ is constructed by $C_{can,t} + TB_t$ and the United States output $Y_{us,t}$ is constructed by $C_{us,t} - TB_t/S_t$, where S_t is the bilateral exchange rate between Canada and the United States.

Table 1: Prior Distributions of Structural Parameters

	Parameters	Distribution	Mean	S.D.	95 % Coverage
β	Household Subjective Discount Factor	Uniform(0,1)	—	—	[0.025 0.975]
γ_M	Deterministic (Gross) Money Growth	Gamma	1.015	0.005	[1.005 1.024]
γ_S	Deterministic EX Trend	Normal	0.000	1.500	[-2.939 2.939]
ψ	Debt Elasticity of Risk Premium	Gamma	0.010	0.001	[0.008 0.012]
λ	Technology Diffusion Speed	Beta	0.010	0.001	[0.008 0.012]
ϕ	Mean Money Demand Shock	Gamma	1.000	0.010	[0.981 1.019]
ρ_M	Permanent Money Growth AR(1) Coef.	Beta	0.100	0.010	[0.081 0.120]
ρ_q	RER AR(1) Coef.	Beta	0.850	0.100	[0.607 0.983]
ρ_ϕ	Money Demand AR(1) Coef.	Beta	0.850	0.100	[0.607 0.983]

Note 1. The AR(1) coefficients of the transitory money and output shocks, ρ_m and ρ_y respectively, have the mass points of zero for identification.

Note 2. The standard deviations of all the structural shocks, σ_M, σ_A, σ_m, σ_y, σ_q, σ_ϕ, have the identical inverse Gamma prior distribution, with a mean of 0.01 and standard deviation of 0.01 for the benchmark information set.

Note 3.: The prior distribution of β is given by the Beta distribution, with a mean of 0.999 and standard deviation of 0.001 for the High Discount Factor model.

Table 2:   Posterior Distributions of Structural Parameters

Parameters	Benchmark			HDF		
	Mean	S.D.	95 % Interval	Mean	S.D.	95 % Interval
κ	0.512	0.028	[0.462 0.569]	0.932	0.000	[0.931 0.934]
β	0.519	0.028	[0.468 0.577]	0.998	0.000	[0.998 0.999]
γ_M	1.015	0.001	[1.013 1.016]	1.071	0.001	[1.069 1.072]
γ_S	-0.000	0.001	[-0.004 0.002]	0.002	0.000	[0.001 0.002]
ψ	0.010	0.001	[0.008 0.012]	0.012	0.001	[0.009 0.013]
λ	0.009	0.001	[0.008 0.011]	0.001	0.001	[0.006 0.008]
ϕ	0.999	0.009	[0.980 1.019]	0.999	0.009	[0.979 1.018]
ρ_M	0.091	0.001	[0.088 0.092]	0.105	0.001	[0.104 0.107]
ρ_q	0.820	0.056	[0.714 0.923]	0.985	0.003	[0.978 0.991]
ρ_ϕ	0.997	0.001	[0.996 0.999]	0.969	0.001	[0.967 0.971]
σ_M	0.017	0.001	[0.016 0.018]	0.029	0.001	[0.026 0.030]
σ_A	0.006	0.000	[0.005 0.007]	0.006	0.000	[0.005 0.007]
σ_m	0.006	0.001	[0.005 0.008]	0.060	0.002	[0.057 0.063]
σ_y	0.003	0.000	[0.002 0.003]	0.003	0.000	[0.002 0.003]
σ_q	0.005	0.001	[0.004 0.007]	0.005	0.000	[0.005 0.006]
σ_ϕ	0.027	0.001	[0.024 0.030]	0.049	0.001	[0.047 0.051]
Marginal Likelihood	2143.269			1873.109		

Note 1: The "Benchmark" represents the Benchmark specification of the two-country model and the "HDF" represents the High Discount Factor specification.

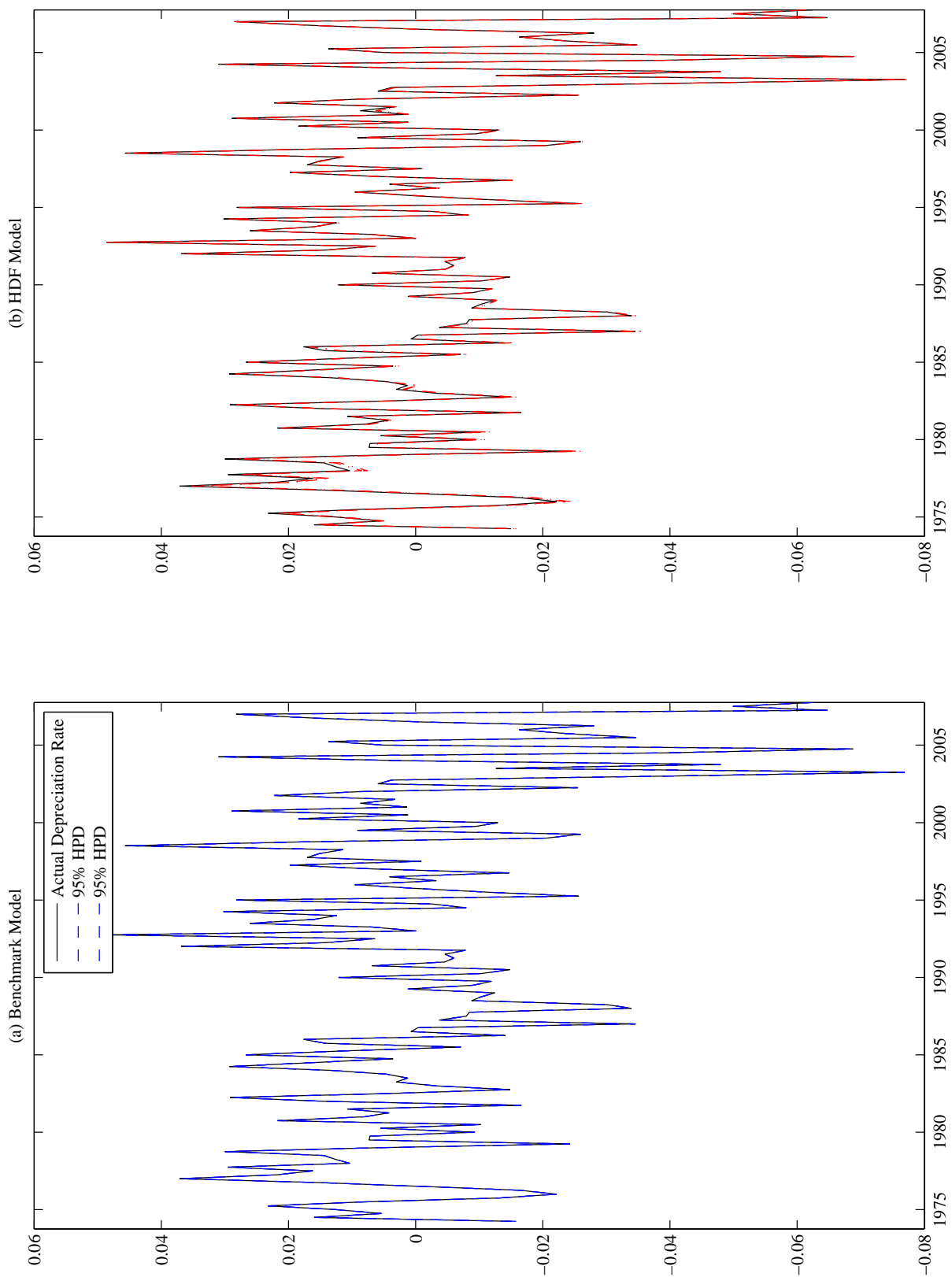Note 2: The marginal likelihoods are estimated based on Geweke's (1999) harmonic mean estimator.

Figure 1: Depreciation rates and in-sample predictions. Note: The solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. The dashed blue and dotted red lines in (a) and (b) respectively indicate the 95 % HPD intervals of the in-sample predictions on the depreciation rate for the Benchmark and HDF models.
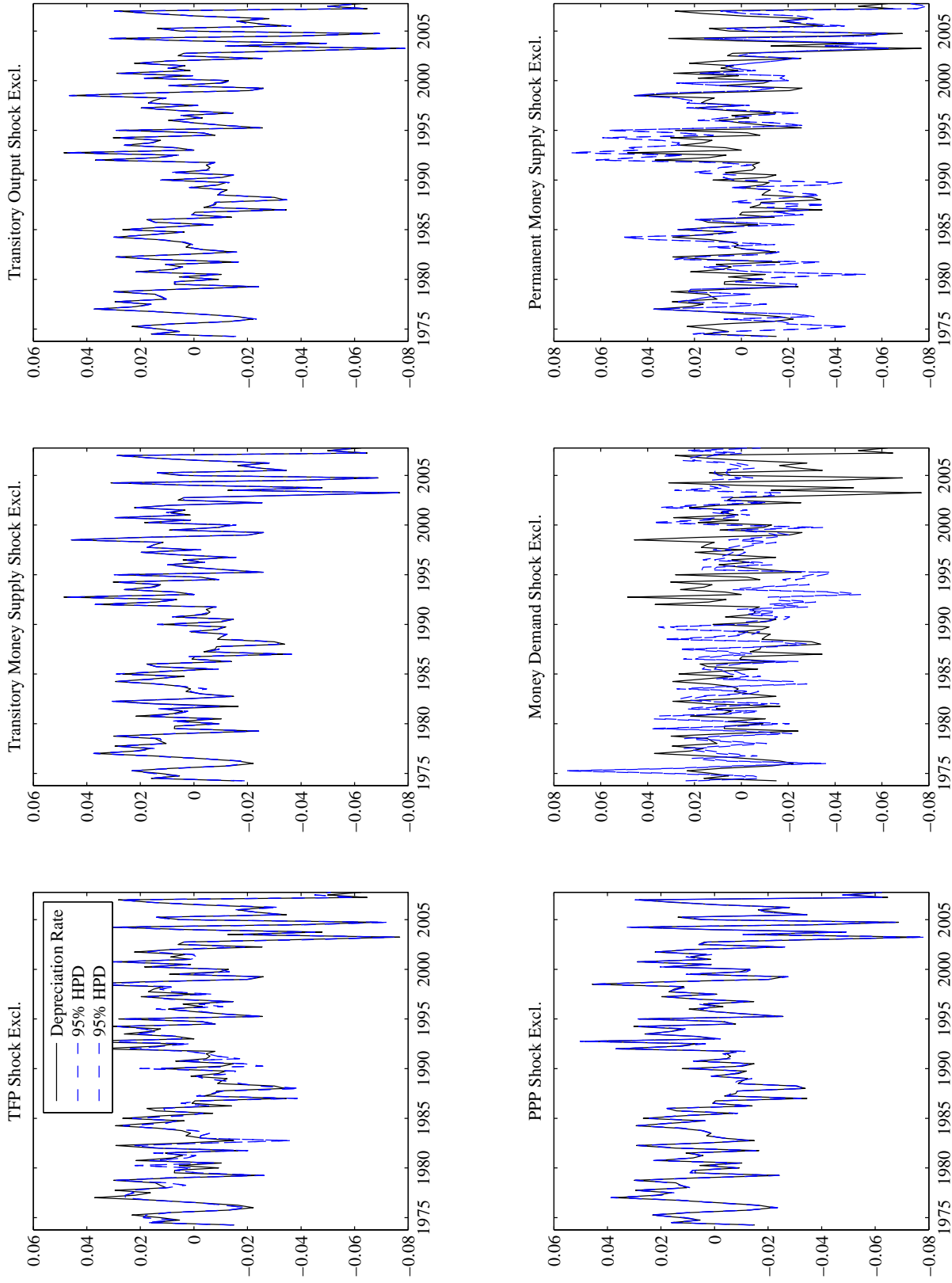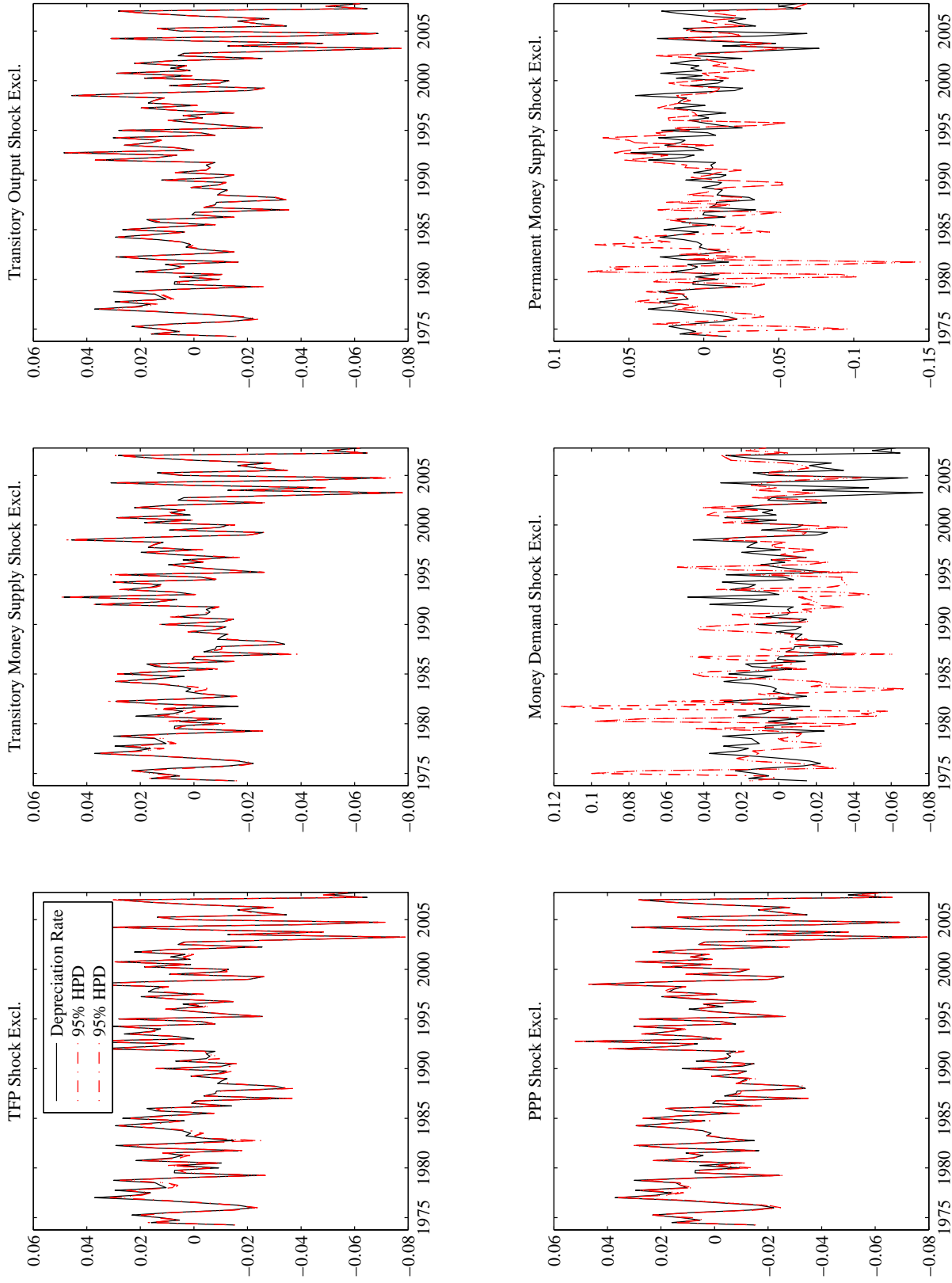
Figure 2: (a) Depreciation rates and historical decomposition of the in-sample prediction of the depreciation rate into structural shocks: the Benchmark model. Note: In each window, the solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. In each window with a particular structural shock, the dashed blue lines represents the 95% HPD interval of the in-sample prediction of the depreciation rate with the Kalman smoother of the corresponding structural shock excluded.
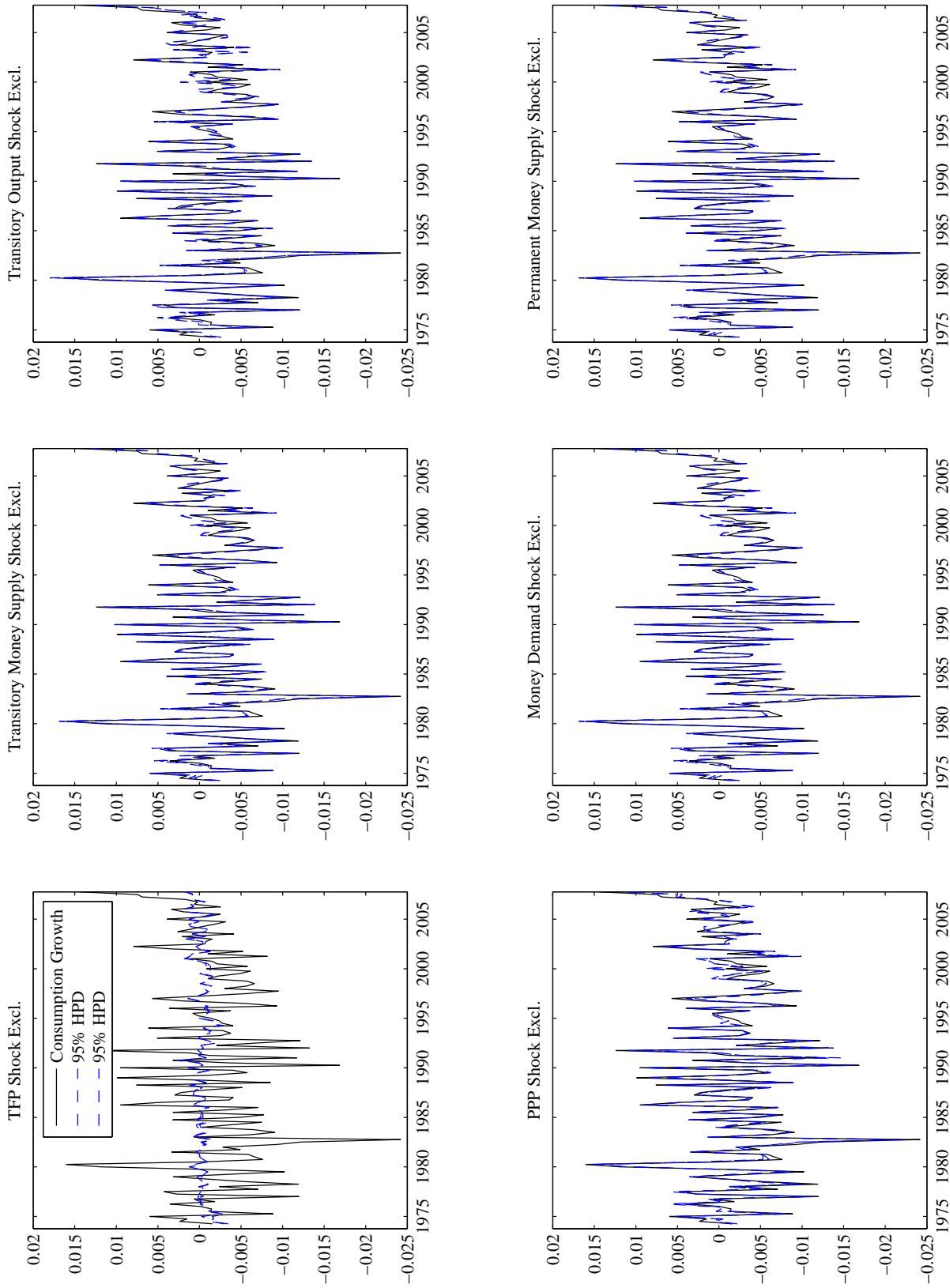
Figure 2: (b) Depreciation rates and historical decomposition of the in-sample prediction of the depreciation rate into structural shocks: the HDF model. Note: In each window, the solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. In each window with a particular structural shock, the dotted red lines represents the 95% HPD interval of the in-sample prediction of the depreciation rate with the Kalman smoother of the corresponding structural shock excluded.

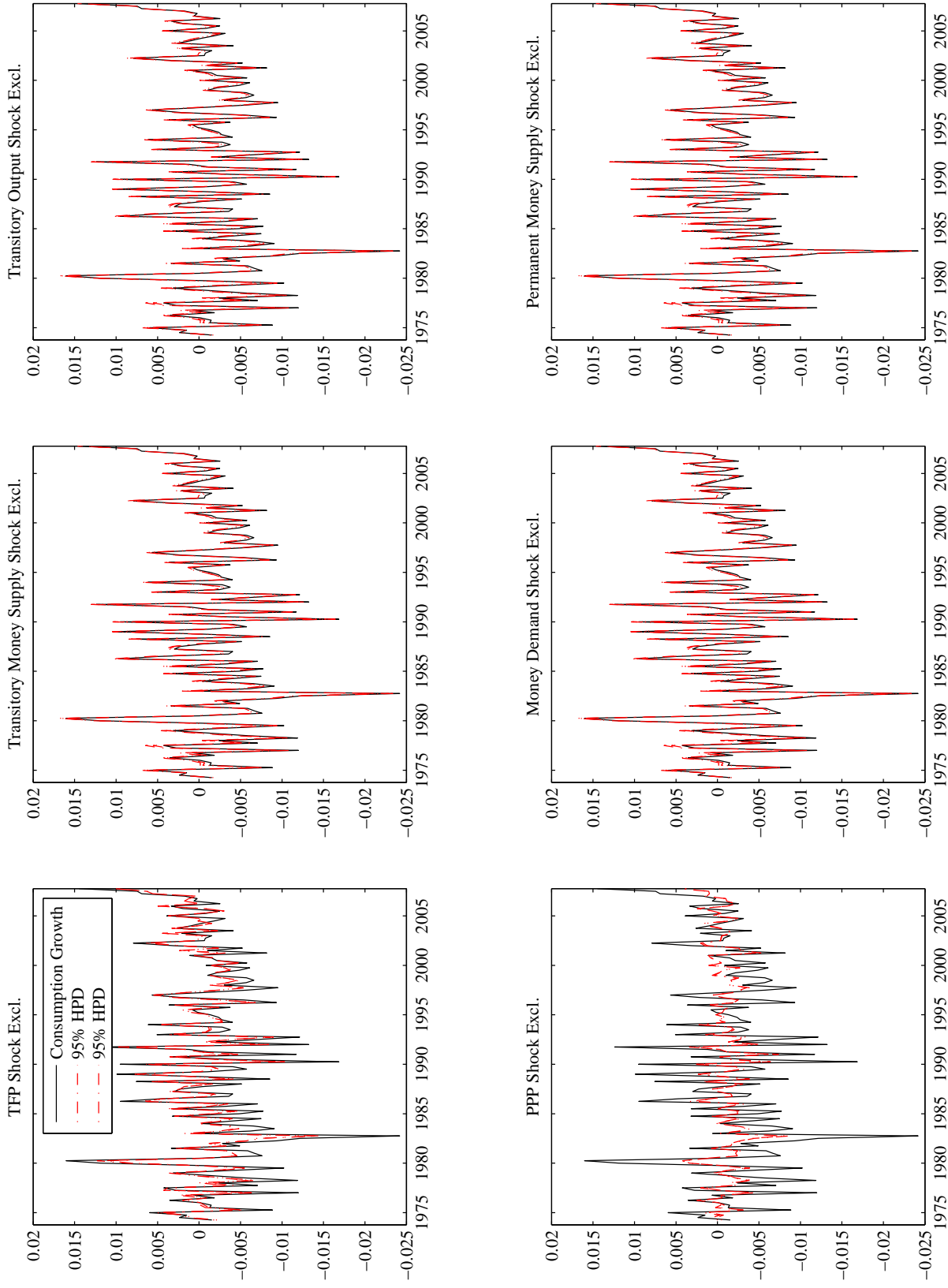Figure 3: (a) Consumption growth differential and historical decomposition of the consumption growth differential into structural shocks: the Benchmark model. Note: In each window, the solid black line represents the actual consumption growth differential. In each window with a particular structural shock, the dashed blue lines represents the 95% HPD interval of the in-sample prediction of the consumption growth differential with the Kalman smoother of the corresponding structural shock excluded.

Figure 3: (b) Consumption growth differential and historical decomposition of the in-sample prediction of the consumption growth differential into structural shocks: the HDF model. Note: In each window, the solid black line represents the actual consumption growth differential. In each window with a particular structural shock, the dotted red lines represents the 95% HPD interval of the in-sample prediction of the consumption growth differential with the Kalman smoother of the corresponding structural shock excluded.
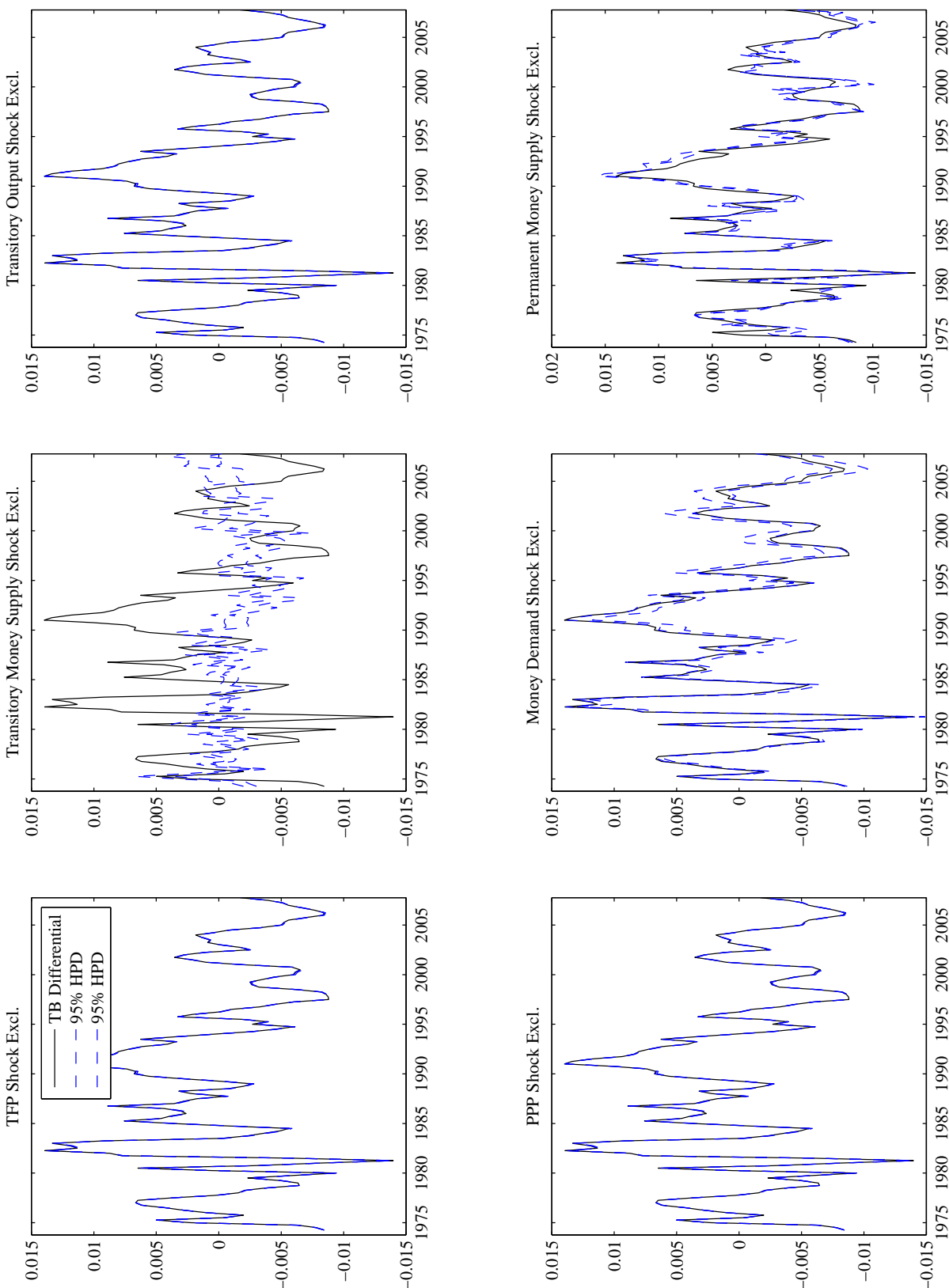
Figure 4: (a) TB differential and historical decomposition of the in-sample prediction of the TB differential into structural shocks: the Benchmark model. Note: In each window, the solid black line represents the actual TB differential. In each window with a particular structural shock, the dashed blue lines represents the 95% HPD interval of the in-sample prediction of the TB differential with the Kalman smoother of the corresponding structural shock excluded.
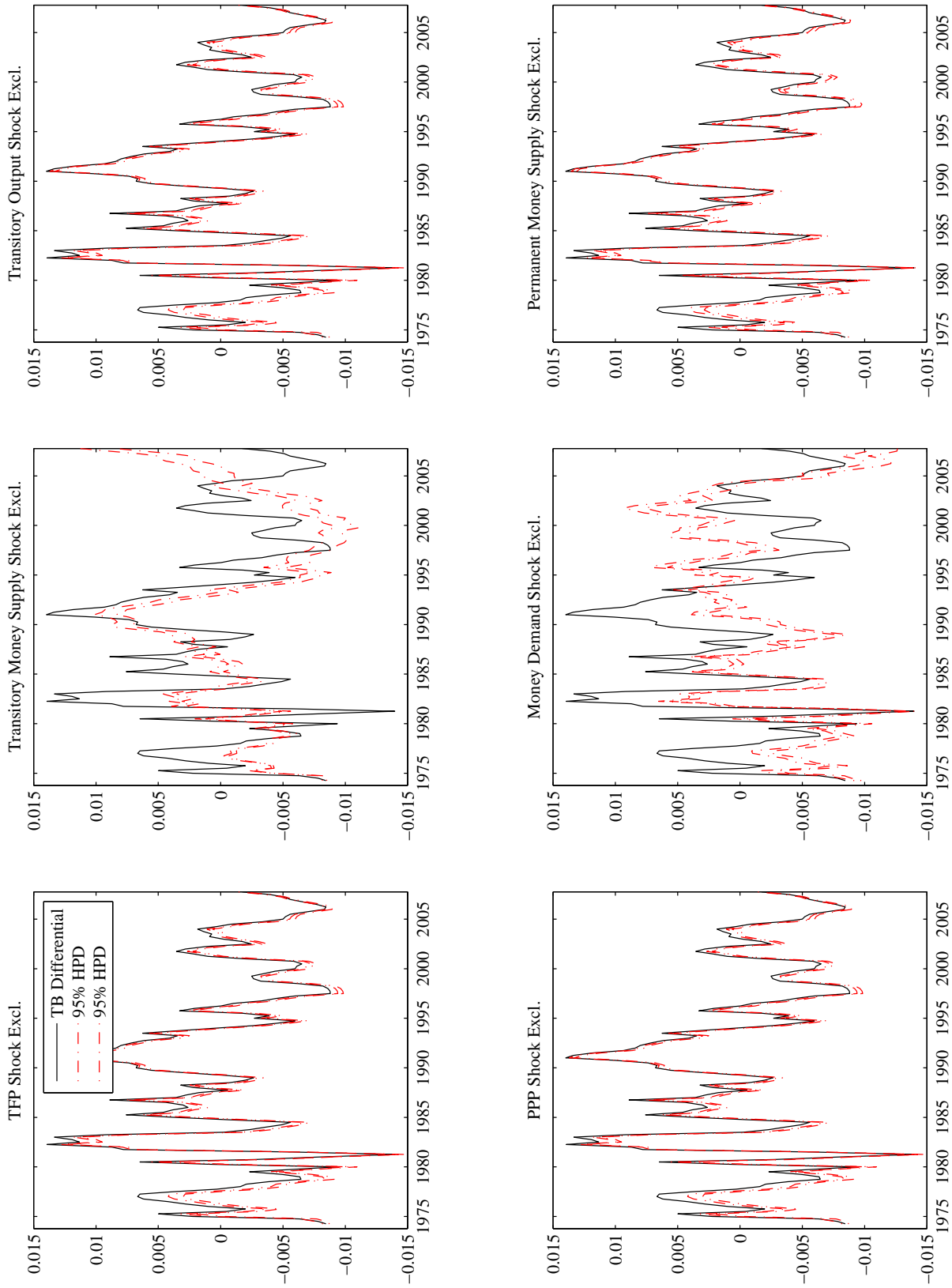
Figure 4: (b) TB differential and historical decomposition of the in-sample prediction of the TB differential into structural shocks: the HDF model.
Note: In each window with a particular structural shock. In each window, the solid black line represents the actual TB differential. The dashed blue lines represents the 95% HPD interval of the in-sample prediction of the TB differential with the Kalman smoother of the corresponding structural shock excluded.
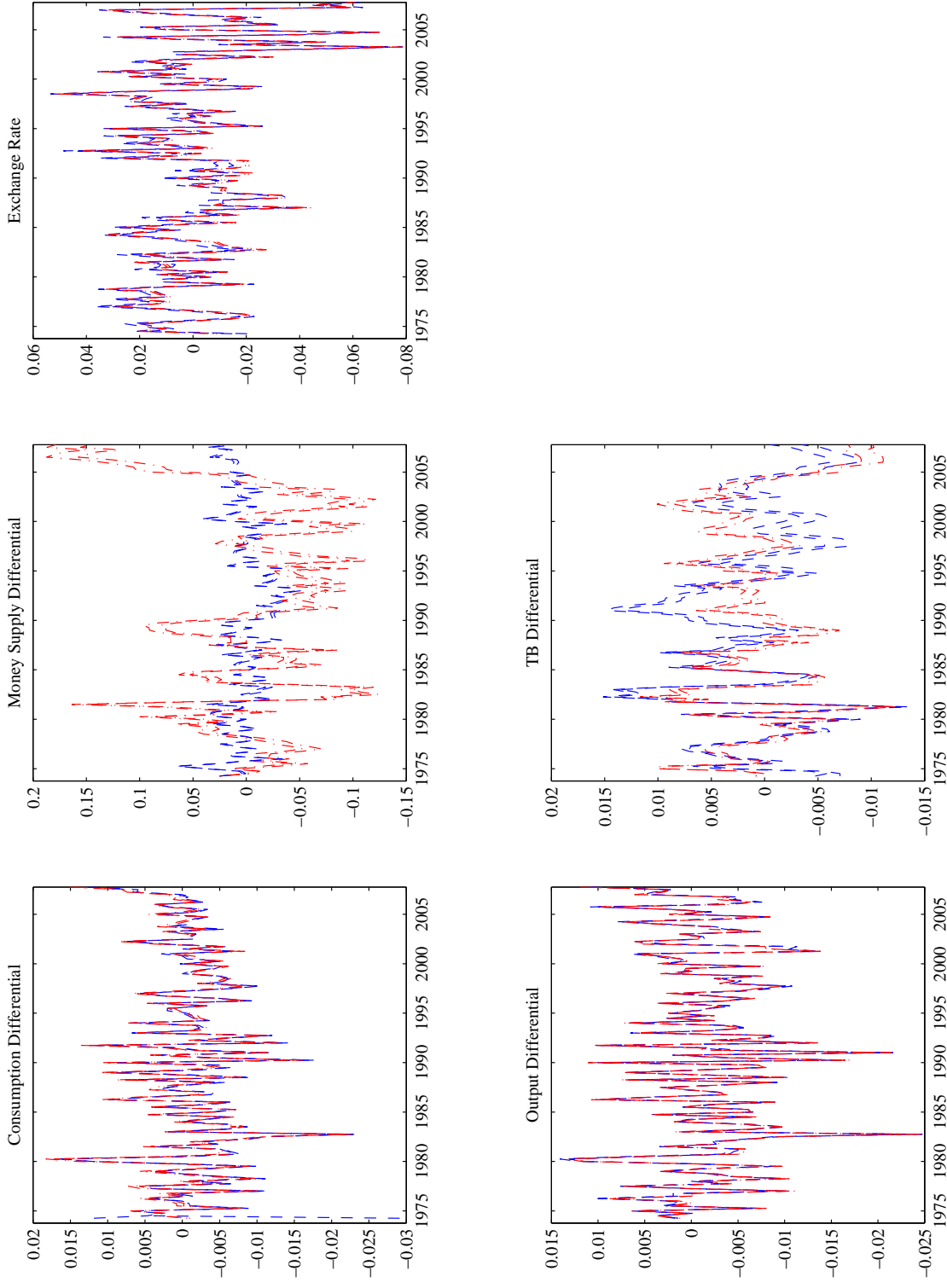
Figure 5: Forecast errors for observations. Note: In each window corresponding to a particular observation, the dashed blue and dotted red lines display the 95 % HPD intervals of the one-period-ahead forecast error calculated through the Kalman forward recursion based of the state space representation of the Benchmark and HDF models, respectively.