Discussion Paper Series A No.579

Exploitation in Economies with Heterogeneous Preferences, Skills and Assets: An Axiomatic Approach

Roberto Veneziani
(School of Economics and Finance,
Queen Mary University of London)
and
Naoki Yoshihara
(Institute of Economic Research, Hitotsubashi University)

November, 2012

Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603 Japan

Exploitation in economies with heterogeneous preferences, skills and assets: An axiomatic approach*

Roberto Veneziani[†] and Naoki Yoshihara[‡]

August 8, 2013

Abstract

This paper provides a novel axiomatic analysis of exploitation as the unequal exchange of labour in economies with heterogeneous optimising agents endowed with unequal amounts of physical and human capital. A definition of exploitation is proposed, which emphasises the relational nature of exploitation and the resulting inequalities in the allocation of labour and income. It is shown that, among all of the major definitions, this is the only one which satisfies two formally weak and normatively salient axioms, and allows one to generalise a number of core insights of exploitation theory.

Keywords: Exploitation, unequal exchange of labour, Profit Exploitation Correspondence Principle.

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*Paper prepared for the symposium on exploitation theory in the occasion of the 30th anniversary of the publication of John Roemer's 'A General Theory of Exploitation and Classes'. We are grateful to Duncan Foley and John Roemer for enlightening discussions that have significantly contributed to shape the main ideas of this paper. We thank Meghnad Desai, Peter Flaschel, Marco Mariotti, Peter Hans Matthews, Simon Mohun, J. Barkley Rosser Jr., Rajiv Sethi, Anwar Shaikh, Gil Skillman, Peter Skott, an anonymous referee, the Editors of this Journal, and participants in the APE 2010 workshop, the EEA 2011 conference, the WAPE 2011 conference, the FMM 2011 conference, the AEA 2012 conference for comments and suggestions. The usual disclaimer applies.

†(Corresponding author) School of Economics and Finance, Queen Mary University of London, Mile End Road, London E1 4NS, UK, *and* Department of Economics, University of Massachusetts, Amherst. E-mail: r.veneziani@qmul.ac.uk

‡The Institute of Economic Research, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-0004, Japan. Phone: (81)-42-580-8354. E-mail: yosihara@ier.hit-u.ac.jp

1 Introduction

What is exploitation? In political philosophy, the most general definition affirms that A exploits B if and only if A takes unfair advantage of B. Despite its intuitive appeal, this definition leaves two major issues in need of a precise specification, namely the kind of unfairness involved and the structure of the relationship between A and B that allows A to take advantage of B. There is considerable debate in the economic and philosophical literature on both issues. Although both aspects of exploitative relations are arguably crucial, the analytical focus of this paper is on the unfairness, or more precisely, on the economic inequalities involved in the concept of exploitation.[1]

To be specific, this paper analyses the theory of *exploitation as an unequal exchange* (hereafter, UE) *of labour*, according to which exploitative relations are characterised by systematic differences between the amount of labour that individuals contribute to the economy, in some relevant sense, and the amount of labour they receive, in some relevant sense, via their income.

There are several reasons to focus on *labour* as the measure of the injustice of exploitative relations. First, in many economic interactions, the notion of exploitation is inextricably linked with some form of labour exchange. Second, as Fleurbaey [7], [8] has argued, the UE definition of exploitation captures some inequalities in the distribution of material well-being and free hours that are - at least *prima facie* - of normative relevance. For instance, they are relevant for *inequalities of well-being freedom*, as discussed by Rawls [19] and Sen [28],[2] because material well-being and free hours are two key determinants of individual well-being freedom. Third, a UE exploitation-free allocation coincides with the so-called *proportional solution*, a well-known fair allocation rule whereby every agent's income is proportional to her contribution to the economy (Roemer and Silvestre [25]). Proportionality is a strongly justified normative principle, whose philosophical foundations can be traced back to Aristotle (Maniquet [14]) and it can be justified in terms

[1] For a discussion of the relevance of power in exploitation theory, see Veneziani [31].

[2] The notion of well-being freedom emphasises an individual's ability to pursue the life she values. In the Rawls-Sen theory, inequalities in the distribution of well-being freedom are formulated as *inequalities of capabilities*, whereas they are formulated as *inequalities of (comprehensive) resources* in Dworkin's theory [4] (see Yoshihara [33]).

of the Kantian categorical imperative (Roemer [23], [24]). Empirical studies have shown that proportionality is indeed a widely held idea of equity (Tornblom [29]). Finally, in a private-ownership economy with positive profits, class and UE exploitation are strictly related, and they reflect an unequal distribution of assets (Roemer [22]; Yoshihara [34]; Yoshihara and Veneziani [35]): in equilibrium the wealthy emerge as exploiters and members of the capitalist class, whereas the poor are exploited and members of the working class. From this perspective, UE exploitative relations are relevant because they reflect unequal opportunities of life options, due to differential ownership of productive assets.

Although the definition of UE exploitation is seemingly intuitive, it has proved surprisingly difficult to provide a fully satisfactory general theory of exploitation. Outside of stylised, two-class economies with a simple linear (Leontief) technology, homogeneous labour, and restrictive assumptions on agents' preferences over consumption and leisure, two problems arise. First, the appropriate definition of the amounts of labour 'contributed to' and 'received by' agents is not obvious, and several approaches have been proposed, which incorporate rather different, and often implicit, normative and positive intuitions.[3] Second, the core insights of exploitation theory do not necessarily hold (Yoshihara and Veneziani [38]).

In his classic work, John Roemer [20], [22] has analysed the normative foundations of exploitation theory and has extended exploitation analysis to include a general convex technology, a complex class structure, and optimising agents. This paper builds on Roemer's seminal work and extends his key insights from both a substantive and a methodological viewpoint.

To be specific, exploitation is analysed in economies with a general convex technology and optimising agents with heterogeneous preferences and with different amounts of both physical and human capital.[4] The formal model is outlined in section 2: it extends Roemer's [20], [22] classic economies, and the related equilibrium notion, to include some key features of advanced economies, such as heterogeneous skills and general preferences over con-

[3] The literature is too vast for a comprehensive list of references. Classic contributions include Morishima [16]; Duménil [1]; Foley [9]; Roemer [22]; and Flaschel [5], [6]. For a brief discussion, see section 3 below. A more thorough analysis is in Yoshihara [34].

[4] This paper focuses on models with optimising agents because they provide a rigorous framework to analyse some key issues in exploitation theory. This does not imply that methodological individualism and rational choice theory are in general the only possible foundations for social theory. For a critical discussion, see Veneziani [31].

sumption and leisure, that are central in debates about the normative and positive relevance of the concept of exploitation.

One substantive contribution of the paper is to provide a novel definition, which extends the core insights of exploitation theory and allows one to characterise the exploitation status of all agents in such general economies. This definition is conceptually related to the 'New Interpretation' (Duménil [1], [2]; Foley [9], [10]; Duménil, Foley, and Lévy [3]), and it states that an agent is exploited (resp., an exploiter) if and only if the labour she contributes is greater (resp., lower) than the share of aggregate social labour that she receives via her income.

This approach defines exploitation as a feature of the competitive allocation of social labour rather than as the result of productive inefficiencies, or imperfections in the labour market. Unlike the main received approaches, it has a clear empirical content, for it is firmly anchored to the actual data of the economy. Perhaps more importantly, it clearly captures the inequalities arising from exploitative relations. First, it identifies exploitation as a social relation: in equilibrium there are some exploited agents if and only if there are some exploiters. As Yoshihara and Veneziani [35] have shown, except for the New Interpretation, none of the main definitions in the literature satisfies this fundamental *relational* property in general. Second, if the New Interpretation is adopted, exploitative relations are characterised by inequalities in individual income/labour ratios, which is an important normative intuition of the UE approach, as Fleurbaey [7], [8] has forcefully argued.

Another contribution of the paper is methodological: UE exploitation is analysed by using a novel, general axiomatic framework.[5] An axiomatic approach was long overdue in exploitation theory. As already noted, a number of alternative definitions with rather different normative and positive implications can be, and have in fact been, proposed: the main approaches are discussed in section 3. By adopting an axiomatic method, this paper suggests to start from first principles, thus explicitly discussing the intuitions underlying UE exploitation.

To be specific, section 4 analyses two axioms. The first is called *Labour Exploitation*, and it restricts the way in which the set of exploited agents is identified. This axiom is interpreted as a minimal necessary condition to capture the core intuitions of exploitation theory, and indeed all of the main approaches satisfy it (see Morishima [16]; Foley [9]; Roemer [22]; Flaschel [5],

[5] For a related approach see Yoshihara and Veneziani [35] and Yoshihara [34].

[6]). Second, the *Profit-Exploitation Correspondence Principle* states that, in equilibrium, propertyless workers are exploited if and only if profits are positive. This axiom incorporates the intuition that in private ownership economies, profits are one of the key mechanisms to transform unequal holdings of scarce productive assets into exploitative relations, unequal exchange of labour, and inequalities in well-being freedom. Given private ownership of productive assets, one should expect profits to allow a transfer of social products and social labour towards wealthy agents. In equilibria with zero profits, the allocation of social labour and income is driven by the wage, and so no UE of labour should occur even if productive assets are unequally owned. Theorem 1 provides the first rigorous characterisation of the class of definitions satisfying Labour Exploitation which meet the Profit-Exploitation Correspondence Principle. Based on this characterisation, Corollary 1 shows that, among all the main definitions, the New Interpretation is the only one that preserves the Profit-Exploitation Correspondence Principle.

The Profit-Exploitation Correspondence Principle captures some important and widely held intuitions in exploitation theory,[6] and so Theorem 1 provides strong support for the New Interpretation as the appropriate formulation of UE exploitation. Yet two objections may be raised at this point. First, a focus on the poorest segment of the working class, namely agents without any physical assets, is appropriate from the axiomatic viewpoint: focusing on a strict subset of agents implies that the axioms impose formally weak and theoretically robust restrictions on the set of admissible definitions. Yet one may argue that this is reductive and some features of capitalist economies should be explicitly considered, which make the issue of exploitation a contentious one today - such as the fact that many workers own some non-labour assets, and even stock in firms, through their pension funds. Second, although exploitation is traditionally analysed by focusing on equilibria (Morishima [16]; Roemer [20]), one may question general equilibrium-type constructions as representations of allocation and distribution in market economies because they depend on the often tacit assumption

[6] The relation between exploitation and profits features prominently, for example, in the literature on the so-called *Fundamental Marxian Theorem* (see, among the many contributions, Morishima [16], [17]; Roemer [20]; Krause [13]; Fleurbaey [7]; Flaschel [5], [6]; Yoshihara and Veneziani [38]). The Fundamental Marxian Theorem and the Profit-Exploitation Correspondence Principle are related from a broad conceptual viewpoint, but as discussed in Section 5.1 below, they are logically and theoretically distinct.

of equal-treatment.[7] A general theory of exploitation should be able to take account of transactions at disequilibrium prices and the resulting inequity in distribution endogenous to market allocation.

Sections 5 and 6 present two extensions of the analysis, which address these objections and provide further support for the New Interpretation. Section 5 shows that the New Interpretation can be extended to analyse the exploitation status of all agents, in economies with heterogeneous preferences, physical assets, and skills (Theorem 2). Further, if the New Interpretation is adopted, positive profits are synonymous with exploitative *social relations*: some agents are exploited if and only if there is someone exploiting them (Corollary 2), a desirable property that is unique to the New Interpretation. Section 6 proves that, under the New Interpretation, there exists a relation between exploitation and profits even out of equilibrium (Theorem 3).

Section 7 concludes and suggests some directions for further research.

2 The model

This section presents a generalisation of Roemer's [20], [22] classic economies and of the related equilibrium notion.

2.1 Production

An economy comprises a set of agents $\mathcal{N} = \{1, ..., N\}$. Let $\mathbb{R}$ ($\mathbb{R}_+$) be the set of (nonnegative) real numbers. Let $\mathbf{0}$ denote the null vector. Production technology is freely available to all agents, who can operate any activity in the production set P, which has elements of the form $\alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha})$ where $\alpha_l \in \mathbb{R}_+$ is the *effective* labour input of the process; $\underline{\alpha} \in \mathbb{R}^n_+$ are the inputs of the produced goods used in the process; and $\overline{\alpha} \in \mathbb{R}^n_+$ are the outputs of the n goods. It is assumed that production displays constant returns to scale, or more precisely that P is a closed convex cone.[8]

The set of production activities feasible with $\alpha_l = k$ units of effective labour can be defined as follows:

$$P(\alpha_l = k) \equiv \{(-\alpha_l, -\underline{\alpha}, \overline{\alpha}) \in P \mid \alpha_l = k\},$$

[7] This issue has been brought to our attention by Duncan Foley in a private exchange. For an analysis of the implications of trading at disequilibrium prices, see Foley [11].

[8] A formal exposition and discussion of the properties of P is in Appendix 8.1.

and $\partial P \equiv \{\alpha \in P \mid \nexists \alpha' \in P \text{ s.t. } \alpha' > \alpha\}$ is the frontier of P.

Let the net output vector arising from α be denoted as $\widehat{\alpha} \equiv \overline{\alpha} - \underline{\alpha}$. For any $c \in \mathbb{R}^n_+$, the set of activities that produce at least c as net output is:[9]

$$\phi(c) \equiv \{\alpha \in P \mid \widehat{\alpha} \geqq c\}.$$

2.2 Agents

This paper investigates exploitation when heterogeneous agents are endowed with unequal amounts of physical and human capital. In the economy, agents produce, consume, and trade labour. On the production side, they can either sell their labour-power or hire workers to work on their capital, or they can be self-employed and work on their own assets. More precisely, for all $\nu \in \mathcal{N}$, let $s^\nu > 0$ be agent ν's skill level and let $\omega^\nu \in \mathbb{R}^n_+$ be the vector of productive assets inherited by ν. Then, $\alpha^\nu = (-\alpha^\nu_l, -\underline{\alpha}^\nu, \overline{\alpha}^\nu) \in P$ is the production process operated by ν as a self-employed producer, with her own capital, where $\alpha^\nu_l = s^\nu a^\nu_l$ and a^ν_l is the labour *time* expended by ν; $\beta^\nu = \left(-\beta^\nu_l, -\underline{\beta}^\nu, \overline{\beta}^\nu\right) \in P$ is the production process that ν operates by hiring (effective) labour β^ν_l; $\gamma^\nu = s^\nu l^\nu$ is ν's effective labour supply, where l^ν is the labour *time* supplied by ν on the market. Thus, let $\lambda^\nu = (a^\nu_l + l^\nu)$ be the total amount of labour time expended by ν, and let $\Lambda^\nu = \alpha^\nu_l + \gamma^\nu = s^\nu \lambda^\nu$ be the total amount of effective labour performed by ν, either as a self-employed producer or working for some other agent.[10]

On the consumption side, let $C \subseteq \mathbb{R}^n_+$ be the consumption space of each agent with generic element c^ν as a consumption vector of agent ν, and assume that total labour hours expended by each agent cannot exceed the total amount of time available, which is normalised to one. Agent ν's welfare is representable by a monotonic function $u^\nu : C \times [0,1] \to \mathbb{R}_+$, which is increasing in consumption and decreasing in labour time. For the sake of simplicity, and with no loss of generality, in what follows, u^ν is assumed to be strictly monotonic on C in at least one argument c^ν_i, for all ν, and the consumption space for any such goods is assumed to be sufficiently large.

[9] For all vectors $x, y \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$.

[10] The model does not include different types of labour to be used in production. This is only for simplicity: this additional source of heterogeneity can be dealt with, albeit at the cost of a substantial increase in technicalities.

Let p denote the $1 \times n$ vector of commodity prices and let w denote the wage rate *per unit of effective labour*. Given (p, w), each agent ν is assumed to choose a plan $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)$ to maximise her welfare subject to the constraint that (1) net income is sufficient for consumption plans; (2) wealth is sufficient to purchase the inputs necessary for production plans; (3) production plans are technically feasible; and (4) the consumption bundle is in the feasible set and working time does not exceed the total amount of time available. Formally, each agent ν solves:[11]

$$MP^\nu : \max_{(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)} u^\nu (c^\nu, \lambda^\nu)$$

subject to

$$\begin{aligned}
[p(\overline{\alpha}^\nu - \underline{\alpha}^\nu)] + \left[p\left(\overline{\beta}^\nu - \underline{\beta}^\nu\right) - w\beta_l^\nu\right] + [w\gamma^\nu] &= pc^\nu, && (1)\\
p\left(\underline{\alpha}^\nu + \underline{\beta}^\nu\right) &\leqq p\omega^\nu, && (2)\\
\alpha^\nu, \beta^\nu &\in P, && (3)\\
c^\nu &\in C,\ \lambda^\nu \in [0,1]. && (4)
\end{aligned}$$

MP^ν is a generalisation of similar optimisation programmes in Roemer [20], [22] and in Yoshihara [34]. As in standard microeconomic theory, agents are not assumed to be simply 'agents of capital' or to produce for production's own sake: they are endowed with general preferences over consumption and leisure, $u^\nu (c^\nu, \lambda^\nu)$. However, following Roemer [20], [22], MP^ν differs from the standard approach in two respects. First, it incorporates the simultaneous role of economic actors as consumers (see, in particular, (1) and (4)) and producers (see, in particular, (2) and (3)), so that no separate consideration of firms is necessary. As shown below, although agents are not assumed to maximise profits, profit maximisation is a corollary of MP^ν. Second, it explicitly takes into account the time structure of the production process. It is thus assumed that, at the beginning of the period, agents need to lay out in advance the capital needed for production and they can do so only by using their own wealth (see (2)).[12] Production then takes place and gross revenues

[11] The first constraint is written as an equality without loss of generality, given the assumptions on the monotonicity of u^ν.

[12] A financial market may be introduced but it would not change the main results. For an interesting analysis, see Roemer ([21], chapter 3). Roemer [22] also shows that the introduction of financial markets does not change the structure of exploitative relations.

(including wages and profits) can be used to finance consumption and the reproduction of initial wealth at the end of the period (see (1)).[13]

2.3 Equilibrium

Let $E\left\langle P, \mathcal{N}, (u^\nu)_{\nu\in\mathcal{N}}, (s^\nu)_{\nu\in\mathcal{N}}, (\omega^\nu)_{\nu\in\mathcal{N}}\right\rangle$, or as a shorthand notation E, denote the economy with technology P, agents $\mathcal{N}$, utility functions $(u^\nu)_{\nu\in\mathcal{N}}$, labour skills $(s^\nu)_{\nu\in\mathcal{N}}$, and productive endowments $(\omega^\nu)_{\nu\in\mathcal{N}}$. Let the set of all such economies be denoted by $\mathcal{E}$. Following Roemer ([20], p.514; [22], pp.64, 114), the equilibrium concept can be defined.

Definition 1: A *reproducible solution* (RS) for $E \in \mathcal{E}$ is a price vector (p, w) and an associated profile of actions $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)_{\nu\in\mathcal{N}}$ such that:

(i) $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)$ solves MP^ν for all ν (optimality);
(ii) $\sum_{\nu\in\mathcal{N}}\left(\widehat{\alpha}^\nu + \widehat{\beta}^\nu\right) \geqq \sum_{\nu\in\mathcal{N}} c^\nu$ (reproducibility);
(iii) $\sum_{\nu\in\mathcal{N}}\left(\underline{\alpha}^\nu + \underline{\beta}^\nu\right) \leqq \sum_{\nu\in\mathcal{N}} \omega^\nu$ (feasibility);
(iv) $\sum_{\nu\in\mathcal{N}} \beta_l^\nu = \sum_{\nu\in\mathcal{N}} \gamma^\nu$ (labour market equilibrium).

In other words, at a reproducible solution (i) every agent optimises; (iii) there are enough resources for production plans in aggregate; and (iv) the labour market clears. Condition (ii) states that aggregate net outputs should at least suffice for aggregate consumption. This is equivalent to requiring that the vector of social endowments does not decrease component-wise, because (ii) is equivalent to $\sum_{\nu\in\mathcal{N}}\left[\omega - \left(\underline{\alpha}^\nu + \underline{\beta}^\nu\right) + \left(\overline{\alpha}^\nu + \overline{\beta}^\nu - c^\nu\right)\right] \geqq \sum_{\nu\in\mathcal{N}} \omega^\nu$, which states that aggregate stocks at the beginning of next period should not be smaller than aggregate stocks at the beginning of the current period. Indeed, although the reproducible solution is defined as a temporary equilibrium in a static general equilibrium framework, it can be seen as a one-shot slice of a *stationary equilibrium* in a dynamic general equilibrium framework.[14]

By the assumptions on u^ν, it immediately follows that both the wage and the prices of all goods must be nonnegative, and at least one good must have

[13] Because of the time structure of production, prices may differ at the beginning and at the end of the period. Given the focus of this paper, however, it is appropriate to analyse stationary state equilibria, and write the individual optimisation programme as MP^ν, where agents rationally expect prices to be constant, as in Roemer [20], [21], [22].

[14] Roemer [20] and Veneziani [30] provide two alternative dynamic frameworks that generalise the one-period reproducible solution. See also Fleurbaey [7].

a strictly positive price, at any non-trivial reproducible solution - i.e. at any equilibrium where *some* production process is activated.

Let $\pi^{\max} = \max_{\alpha \in P} \frac{p\widehat{\alpha} - w\alpha_l}{p\underline{\alpha}}$ denote the maximum profit rate that can be obtained at prices (p, w), and let $P^{\pi}(p, w) = \left\{ \alpha \in P \mid \pi^{\max} = \frac{p\widehat{\alpha} - w\alpha_l}{p\underline{\alpha}} \right\}$ denote the set of production processes that yield the maximum profit rate. Lemma 1 derives some useful properties of the equilibria of the economy.

Lemma 1: *Let (p, w) be a non-trivial reproducible solution for $E \in \mathcal{E}$ such that $\sum_{\nu \in \mathcal{N}} c^{\nu} \geq \mathbf{0}$. Then, (i) $p\widehat{\alpha} - w\alpha_l \geqq 0$ for some $\alpha \in P \setminus \{\mathbf{0}\}$, and (ii) $\alpha^{\nu}, \beta^{\nu} \in P^{\pi}(p, w)$ for all ν.*

The formal proofs of all results are in Appendix 8.2. However, the intuition behind Lemma 1 is simple: by individual optimality (Definition 1(i)), in equilibrium agents will only operate activities that yield the maximum profit rate (Lemma 1(ii)) and will not operate any activities that yield negative profits. Therefore at any reproducible solution where at least *some* production process is activated, profits must be nonnegative (Lemma 1(i)).

3 Defining labour exploitation

In the UE approach, exploitative relations are characterised by systematic differences between the labour that agents contribute to the economy and the labour 'received' by them, which is given by the amount of labour contained, or embodied, in some relevant consumption bundle(s). Therefore, in order to define exploitation status, it is necessary both to select the relevant reference bundle(s) and to identify their labour content. In general economies, neither choice is obvious, and various definitions have, in fact, been proposed. In this section, some of the main definitions - suitably extended to economies with heterogeneous skills - are briefly analysed. The purpose is to illustrate the key issues involved in defining exploitation in general economies, rather than to provide a comprehensive survey of alternative approaches.

As a starting point, consider a simple economy with a standard Leontief technology (A, L), where A is a square $n \times n$ nonnegative and productive matrix and L is a strictly positive $1 \times n$ vector describing, respectively, the amount of each input and the (homogeneous) labour necessary to produce one unit of the n goods. Assume that all agents have equal skills and consume the same subsistence bundle b. Under these assumptions, the definition of UE exploitation is relatively uncontroversial: the reference bundle is b and its labour content is equal to vb, where $v = L(I - A)^{-1}$ is the vector describing the labour embodied in one unit of each good. Then agent ν is exploited (resp., an exploiter) if and only if the labour she contributes to the economy, Λ^{ν}, is greater (resp., lower) than the labour she receives, vb.

As soon as these assumptions are dropped, however, the definition of exploitation is not obvious. If more general technologies are considered, the simple generalisation of the standard approach can yield paradoxical results - such as bundles containing a negative amount of labour - and so various definitions of the labour contained in a *given* bundle have been proposed, focusing either on actual production activities in the economy or on some feasible, possibly counterfactual, technology. Moreover, if agents do *not* consume a given, equal subsistence bundle, then the choice of reference bundle is not obvious: one may focus either on agents' actual choices or on some alternative (affordable) bundle. The former approach takes a subjectivist view by emphasising the actual choices made by agents in the determination of their exploitation status. Scholars adopting the latter perspective argue instead that a subjectivist perspective makes exploitation depend on *consumption* decisions so that agents who consume different bundles but are otherwise identical may end up having a different exploitation status.

In his classic definition, Morishima [16] focuses on the bundles actually consumed by agents, c^{ν}, but adopts a counterfactual definition of labour content, whereby for all bundles $c \in \mathbb{R}^n_+$:

$$l.v.\left(c\right) \equiv \min\left\{\alpha_l \mid \alpha = \left(-\alpha_l, -\underline{\alpha}, \overline{\alpha}\right) \in \phi\left(c\right)\right\}.$$

According to Morishima, agent ν is exploited (resp., an exploiter) if and only if the labour she contributes to the economy, Λ^{ν}, is greater (resp., lower) than $l.v.\left(c^{\nu}\right)$, that is the *minimum* amount of (effective) labour necessary to produce c^{ν} as net output. Formally:

Definition 2 (Morishima [16])**:** Agent $\nu \in \mathcal{N}$, who supplies Λ^{ν} and consumes c^{ν}, is *exploited* if and only if $\Lambda^{\nu} > l.v.\left(c^{\nu}\right)$ and *an exploiter* if and only if $\Lambda^{\nu} < l.v.\left(c^{\nu}\right)$.

Definition 2 has some desirable characteristics, according to Morishima ([16], pp.616-618): the notion of exploitation is well-defined because $l.v.\left(c\right)$ is unique, well-defined and positive whenever $c \neq \mathbf{0}$;[15] and exploitation status is

[15] This follows from assumptions **A0**$\sim$**A2** in Appendix 8.1 (see Roemer [20], Proposition 2.1). The same holds for $l.v.\left(c; p, w\right)$ below at a reproducible solution.

determined prior to and independent of price information, as in the standard Marxian approach, focusing only on production data.

According to Roemer [21], [22], however, Definition 2 is conceptually flawed as it identifies exploitation status (potentially) based on production techniques that will never be used by profit maximising capitalists, and the labour received by agents must be based on equilibrium price information. Like Morishima [16], Roemer [22] focuses on the bundle actually consumed by the agents but argues that its labour content should be given by the minimum amount of (effective) labour necessary to produce it as net output *among profit-rate-maximising activities at given equilibrium prices*, for only the latter production processes will be activated in a capitalist economy.

Formally, for all $c \in \mathbb{R}^n_+$, the labour content of c is defined as follows:

$$l.v.\,(c; p, w) \equiv \min \left\{ \alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha}) \in \phi(c) \cap P^{\pi}(p, w) \right\}.$$

Then, agent ν is exploited (resp., an exploiter) if and only if the labour she contributes to the economy, λ^{ν}, is greater (resp., lower) than the *minimum* amount of (effective) labour necessary to produce c^{ν} as net output with a profit-rate-maximising activity at given equilibrium prices.

Definition 3 (Roemer [22])**:** Consider an economy $E \in \mathcal{E}$. Let (p, w) be a reproducible solution for E. Agent $\nu \in \mathcal{N}$, who supplies Λ^{ν} and consumes c^{ν}, is *exploited* if and only if $\Lambda^{\nu} > l.v.\,(c^{\nu}; p.w)$ and *an exploiter* if and only if $\Lambda^{\nu} < l.v.\,(c^{\nu}; p, w)$.

Although they preserve some important insights of standard exploitation theory,[16] Definitions 2 and 3 have been criticised because exploitation status depends on counterfactual amounts of labour content. For the production activities yielding $l.v.\,(c^{\nu})$ or $l.v.\,(c^{\nu}; p, w)$ may be different from those actually used in equilibrium. According to critics, this use of counterfactuals is theoretically undesirable and it makes exploitation an empirically vacuous notion, since the computation of $l.v.\,(c^{\nu})$ and $l.v.\,(c^{\nu}; p, w)$ requires information that is not available, including, in Morishima's own words, "information about *all* the available techniques of production, actually chosen *or potentially usable*" ([16], pp.617, italics added).[17]

[16] See section 5.1 for a brief discussion. For a more thorough analysis, see Yoshihara and Veneziani [38].

[17] For a thorough discussion, see, for example, Flaschel [5], [6].

An alternative approach has been recently proposed by Yoshihara and Veneziani [35], [37] and Yoshihara [34]. For any $p \in \mathbb{R}^n_+$ and $c \in \mathbb{R}^n_+$, let $\mathcal{B}(p,c) \equiv \left\{x \in \mathbb{R}^n_+ \mid px = pc\right\}$ be the set of bundles that cost exactly as much as c at prices p. Let $\alpha^{p,w} \equiv \sum_{\nu \in \mathcal{N}} (\alpha^\nu + \beta^\nu)$ denote the aggregate equilibrium production activity at a reproducible solution (p,w) for economy E.

Definition 4: Consider an economy $E \in \mathcal{E}$. Let (p,w) be a reproducible solution for E such that $\alpha^{p,w}$ is the aggregate production activity. For each $c \in \mathbb{R}^n_+$ with $pc \leqq p\widehat{\alpha}^{p,w}$, let $\tau^c \in [0,1]$ be such that $\tau^c \widehat{\alpha}^{p,w} \in \mathcal{B}(p,c)$. The *labour content of* c at the aggregate production activity $\alpha^{p,w}$ is $\tau^c \alpha_l^{p,w}$.

According to Definition 4, the total labour content of aggregate net output, $\widehat{\alpha}^{p,w}$, is equal to total social labour, $\alpha_l^{p,w}$. Then, for any bundle c whose value does not exceed national income, the labour contained in c is equal to the fraction τ^c of social labour necessary to produce a fraction of aggregate net output, $\tau^c \widehat{\alpha}^{p,w}$, that has the same value as c.[18]

As in Roemer's [22] approach, in Definition 4 the labour content of a bundle can be identified only if the price vector is known. Yet social relations play a more central role, because the definition of labour content requires a prior knowledge of the social reproduction point and labour content is explicitly linked to the redistribution of total social labour. Then:

Definition 5: Consider any economy $E \in \mathcal{E}$. Let (p,w) be a reproducible solution for E with aggregate production activity $\alpha^{p,w}$. For any $\nu \in \mathcal{N}$, who supplies Λ^ν and consumes c^ν, let τ^{c^ν} be defined as in Definition 4. Agent ν is exploited if and only if $\Lambda^\nu > \tau^{c^\nu} \alpha_l^{p,w}$ and an *exploiter* if and only if $\Lambda^\nu < \tau^{c^\nu} \alpha_l^{p,w}$.

Definition 5 is conceptually related to the 'New Interpretation' (Duménil [1], [2]; Foley [9], [10]). In fact, for any agent $\nu \in \mathcal{N}$, τ^{c^ν} represents ν's share of national income, and so $\tau^{c^\nu} \alpha_l^{p,w}$ is the share of social labour that ν receives by earning income barely sufficient to buy c^ν. Then, as in the New Interpretation, the notion of exploitation is related to the production and distribution of national income and social labour, and it depends on empirically observable magnitudes. Yet, Definition 5 has been criticised because, unlike

[18] At a reproducible solution with $p\widehat{\alpha}^{p,w} = 0$, τ^c is actually undetermined, but in equilibria in which the value of aggregate net output is zero, it seems reasonable to impose that $\tau^c = 0$.

Definitions 2 and 3, the actual consumption choices of the agents are only indirectly relevant to determine exploitation status, and unlike Definition 2, the notion of exploitation depends on price information.

In summary, various definitions with different normative and positive implications can be, and have in fact been, proposed. The question is how to adjudicate alternative approaches. This is the topic of the next section.

4 An axiomatic approach

In this section, a novel, general axiomatic framework is developed in order to analyse exploitation theory. The adoption of an axiomatic method allows us to adjudicate alternative approaches by starting from first principles, thus explicitly discussing the intuitions underlying UE exploitation.

The first step of the analysis is to define a domain condition: an axiom that captures the core insights of UE exploitation shared by all of the main approaches, including those discussed in section 3. In the UE approach, exploitative relations are characterised by systematic differences between the labour that agents contribute to the economy and the labour 'received' by them, which is given by the amount of labour contained, or embodied, in some relevant consumption bundle(s). The domain condition sets some weak restrictions *both* on the choice of the reference bundle(s) *and* on the definition of their labour content that all of the main UE approaches satisfy.

Let $W \equiv \{\nu \in \mathcal{N} \mid \omega^{\nu} = \mathbf{0}\}$ be the set of agents with no initial endowments. The economies analysed in this paper are more general than the polarised, two-class societies usually considered in the literature, and in the next section the exploitation status of *all* agents is derived. Yet the set W is of clear focal interest in exploitation theory: theoretically, if *any* agents are exploited, then one would expect propertyless individuals to be among them, if they work at all. It is therefore opportune, from an axiomatic viewpoint, to focus on W in order to provide a domain condition defining some minimum requirements that all definitions of UE exploitation should satisfy.[19]

Given any definition of exploitation, let $N^{ter} \subseteq \mathcal{N}$ and $N^{ted} \subseteq \mathcal{N}$ denote, respectively, the *set of exploiters* and the *set of exploited agents* at a given

[19] Alternatively, one may focus on the set of agents with no wealth $W' = \{\nu \in \mathcal{N} \mid p\omega^{\nu} = 0\}$. This distinction is relevant only if some goods are free in equilibrium and makes no difference for the results. Indeed, in order to define a weak domain condition to identify exploited agents, it is appropriate to focus on the smaller set $W \subseteq W'$.

allocation, where $N^{ter} \cap N^{ted} = \varnothing$. A basic axiom can now be formally introduced that captures the key intuitions of UE exploitation theory.

Labour Exploitation (LE): *Consider any economy $E \in \mathcal{E}$. Let (p, w) be a reproducible solution for E. Given any definition of exploitation, the set of exploited agents $N^{ted} \subseteq \mathcal{N}$ should have the following property at (p, w). There exists a profile of bundles $(c_e^1, ..., c_e^{|W|})$ such that, for any $\nu \in W$, $c_e^\nu \in \mathbb{R}_+^n$, $pc_e^\nu = w\Lambda^\nu$, and for some $\alpha^{c_e^\nu} \in \phi(c_e^\nu) \cap \partial P$ with $\widehat{\alpha}^{c_e^\nu} \not> c_e^\nu$:*

$$\nu \in N^{ted} \text{ if and only if } \alpha_l^{c_e^\nu} < \Lambda^\nu.$$

Labour Exploitation requires that, at any equilibrium, a definition of exploitation determines whether each propertyless agent $\nu \in W$ is exploited or not by identifying a nonnegative vector c_e^ν - call it an *exploitation reference bundle* (hereafter, ERB) - and its associated labour content, $\alpha_l^{c_e^\nu}$.

The ERB must have two properties:

(I): It must be on ν's budget line, i.e. it must be (*just*) *affordable*, at prices p, by a propertyless agent $\nu \in W$, who supplies Λ^ν units of labour at a wage rate w ($pc_e^\nu = w\Lambda^\nu$).

(II): It must be *technically feasible* with an efficient production process ($\alpha^{c_e^\nu} \in \phi(c_e^\nu) \cap \partial P$).

The labour content of the ERB - and thus the amount of labour that ν receives - is then identified as the labour necessary to produce the ERB efficiently as net output, $\alpha_l^{c_e^\nu}$. Thus, if $\nu \in W$ supplies Λ^ν, and Λ^ν is more than $\alpha_l^{c_e^\nu}$, then ν is regarded as contributing more labour than ν receives. According to **Labour Exploitation**, the definition of exploitation should consider all such agents as exploited, i.e. as members of N^{ted}.

As a domain condition for the admissible class of exploitation-forms, **Labour Exploitation** captures some key insights of UE exploitation theory that are shared by *all* of the main approaches.[20] In the UE theory, the exploitation status of agent ν is determined by the difference between the amount of labour that ν 'contributes' to the economy, and the amount she

[20] **Labour Exploitation** only applies to labour-based definitions of exploitation. It is not relevant, for example, for Roemer's [22] property-relations definition. Related axioms are analysed by Yoshihara and Veneziani [35] and Yoshihara [34].

'receives'. In economies with the type of labour heterogeneity considered here, the former quantity is given by the amount of labour supplied, Λ^{ν}.[21] In contrast, there are many possible UE views concerning the amount of labour that each agent receives. As a domain condition, **Labour Exploitation** provides some minimal, key restrictions on the definition of the amount of labour that a theoretically relevant subset of agents receives.[22]

First, the amount of labour that $\nu \in W$ receives depends on her income, or more precisely, it is determined in equilibrium by some reference bundle that ν can purchase (property **(I)**). In the standard approaches, the ERB is the bundle actually chosen by the agent. **Labour Exploitation** is weaker in that it only requires that the ERB be *potentially* affordable.

Second, the amount of labour associated with the ERB - and thus 'received' by an agent - is related to the production conditions of the economy. More precisely, **Labour Exploitation** states that the ERB be technologically feasible as net output, and defines its labour content as the amount of labour socially necessary to produce it (property **(II)**). Observe that the axiom requires that the amount of labour associated with each ERB be uniquely determined with reference to production conditions, but it does not specify how such amount should be chosen. There may be in principle many (efficient) ways of producing the ERB c_e^{ν}, and thus of determining its labour content $\alpha_l^{c_e^{\nu}}$.

Third, **Labour Exploitation** is weak also because it does not provide comprehensive conditions for the determination of exploitation status: it only focuses on the strict subset of agents who own no physical assets and is silent on the exploitation status of all other agents. Further, it imposes no restrictions on the set of exploiters $N^{ter} \subseteq \mathcal{N}$.

Finally, it is worth noting that **Labour Exploitation** allows the ERB, c_e^{ν}, to be variable and a function of equilibrium prices (p, w).[23]

To verify that **Labour Exploitation** captures the key tenets of UE ex-

[21] This is the standard approach in the literature (see, e.g., Krause [13]; Duménil, Foley, and Lévy [3]). For a slightly different, but related approach based on the notion of 'abstract labour', see Fleurbaey ([7], section 8.5).

[22] The axiom allows for the possibility that *all* agents in the economy be exploited ($N^{ted} = \mathcal{N}$). This is theoretically appropriate, given the nature of **Labour Exploitation** as a *minimum* domain condition. For even some of the classic definitions of exploitation - such as Morishima's [16] - do not exclude this case.

[23] Once the ERB c_e^{ν} is identified, the existence of $\alpha^{c_e^{\nu}}$ is guaranteed by assumptions **A2** and **A3** on the production set P in Appendix 8.1.

ploitation, it is worth checking that all of the main definitions satisfy it.

Consider Definition 2: the ERB is the *actual* consumption bundle of agent $\nu \in W$ - i.e. at any reproducible solution, $c_e^\nu \equiv c^\nu$ - and its labour content is given by choosing $\alpha^{c_e^\nu}$ as the production activity that minimises direct labour among those that produce c_e^ν as net output, so that $\alpha_l^{c_e^\nu} = l.v.\,(c_e^\nu)$.

Consider Definition 3: the ERB is again the *actual* consumption bundle of agent $\nu \in W$ - i.e. at any reproducible solution, $c_e^\nu \equiv c^\nu$ - but its labour content is given by choosing $\alpha^{c_e^\nu}$ as the production activity that minimises direct labour among the profit-rate maximising activities that produce c_e^ν as net output, so that $\alpha_l^{c_e^\nu} = l.v.\,(c_e^\nu; p, w)$.[24]

Consider Definition 5: take any (p, w) and associated aggregate production activity $\alpha^{p,w}$, and let $\tau^{c^\nu} = \frac{pc^\nu}{p\widehat{\alpha}^{p,w}}$, if $p\widehat{\alpha}^{p,w} > 0$ and $\tau^{c^\nu} = 0$, otherwise. In the New Interpretation, the ERB is defined counterfactually by identifying the share of net output that agent $\nu \in W$ could (just) buy - formally, $c_e^\nu \equiv \tau^{c^\nu}\widehat{\alpha}^{p,w}$ - and its labour content is given by choosing $\alpha^{c_e^\nu} \equiv \tau^{c^\nu}\alpha^{p,w}$, so that $\alpha_l^{c_e^\nu} = \tau^{c^\nu}\alpha_l^{p,w}$. In general, for any $\nu \in W$, the ERB is different from the actual consumption bundle chosen - that is, in general $c_e^\nu \neq c^\nu$, unlike in Definitions 2 and 3.

The previous arguments forcefully suggest that **Labour Exploitation** does represent an appropriate domain condition in exploitation theory: it is formally weak and it incorporates some widely shared views on UE exploitation. Thus, although the axiom is not trivial and not all definitions in the literature satisfy it, all of the major approaches do.[25] The next question, then, is how to discriminate among the various definitions satisfying it.

A key tenet of UE exploitation theory is the idea that, in private ownership economies with unequal distribution of productive assets, profits are one of the main determinants of the existence of exploitation, and of inequalities in well-being freedom. Given private ownership of productive assets, one should expect profits to make a transfer of social surplus and social labour from asset-poor agents to wealthy ones possible, and a general correspondence should exist between positive profits and the exploitation of at least the poorest segments of the working class. This is formalised in the next

[24] Formally, for Definition 2, $\alpha^{c_e^\nu} \in \arg\min\{\alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha}) \in \phi(c_e^\nu)\}$, and for Definition 3, $\alpha^{c_e^\nu} \in \arg\min\{\alpha_l \mid \alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha}) \in \phi(c_e^\nu) \cap P^\pi(p, w)\}$.

[25] Based on Flaschel's [5], [6] notion of *actual labour values*, another definition of exploitation can be derived that satisfies **Labour Exploitation**. In contrast, the subjectivist definition of exploitation based on workers' preferences recently proposed by Matsuo [15] does not meet **Labour Exploitation**.

axiom, according to which for any economy, and any equilibrium, aggregate profits are strictly positive if and only if propertyless workers are exploited.

Profit-Exploitation Correspondence Principle (PECP): *Given an economy $E \in \mathcal{E}$ and a reproducible solution for E, (p, w), with aggregate production activity $\alpha^{p,w}$:*

$$\left[p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0 \text{ if and only if } N^{ted} \supseteq W_+\right],$$

whenever $W_+ \equiv \{\nu \in W \mid \Lambda^\nu > 0\} \neq \varnothing$.

A number of points are worth noting about the **Profit-Exploitation Correspondence Principle**. First, it is formulated without specifying *any* definition of exploitation: *whatever* the definition adopted, propertyless workers should be exploited if and only if profits are positive in equilibrium. Second, it is formally weak in that it only focuses on a strict subset of the set of agents $\mathcal{N}$; it is silent on the set of exploiters N^{ter}; it imposes no constraints on the definition of exploitation in economies where $W_+ = \varnothing$ in equilibrium; and when equilibrium profits are zero it only requires that *some* propertyless agents not be exploited. Thus, it establishes a rather weak link between exploitation and profits. Third, it is fairly general, because it both applies to economies with a complex class structure, and allows for the possibility that propertyless workers are a *strict* subset of the set of exploited agents N^{ted}. Note that it focuses only on propertyless agents who perform some labour, W_+. This restriction is theoretically appropriate, since the exploitation status of agents who do not engage in any economic activities is unclear. Fourth, it allows for fairly general assumptions on agents and technology, including heterogeneous preferences and skills, a convex technology, and so on.

The next theorem characterises the class of definitions of exploitation that satisfy **Labour Exploitation** and such that the **Profit-Exploitation Correspondence Principle** holds.

Theorem 1: *For any definition of exploitation satisfying* **Labour Exploitation**, *the following statements are equivalent for any economy $E \in \mathcal{E}$ and any reproducible solution (p, w) with aggregate production activity $\alpha^{p,w}$:*
(1) *the* **Profit-Exploitation Correspondence Principle** *holds*;
(2) *if $\pi^{\max} > 0$ then for all $\nu \in W_+$, there is $\alpha_\pi^\nu \in P\left(\alpha_l = \Lambda^\nu\right) \cap \partial P$ such that $\widehat{\alpha}_\pi^\nu \in \mathbb{R}_+^n$, $p\widehat{\alpha}_\pi^\nu > w\Lambda^\nu$ and $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$, some $\delta^\nu > 1$.*

Theorem 1 provides a demarcation line (condition (2)) which allows one to test which of a potentially infinite number of definitions properly capture a key property of profits and exploitation in capitalist economies. Though its main theoretical implication is drawn in Corollary 1 below, it can be interpreted as follows. The **Profit-Exploitation Correspondence Principle** states that every employed propertyless agent is exploited if and only if equilibrium profits are positive. According to **Labour Exploitation**, the exploitation status of propertyless agents is determined by identifying a profile of exploitation reference bundles that are affordable by the agents and producible with less than Λ^{ν} units of labour for all exploited workers. By Theorem 1, in every convex economy, the **Profit-Exploitation Correspondence Principle** holds if and only if positive equilibrium profits are associated with the existence of a profile of reference bundles - call them the *profit-reference bundles*, $\widehat{\alpha}_{\pi}^{\nu}$. According to condition (2), for all workers $\nu \in W_{+}$, the profit-reference bundles must be producible with a technically efficient process using Λ^{ν} units of labour, yield positive profits, and dominate the ERBs. Theorem 1 thus identifies a general condition for the validity of the relation between exploitation and profits,[26] but, methodologically, it also suggests that different views about exploitation, and the analysis of some key features of exploitation theory, should focus on the choice of the relevant (exploitation and profit) reference bundles.[27]

Theorem 1 does not identify a unique definition of exploitation that meets the **Profit-Exploitation Correspondence Principle**, but rather a class of definitions satisfying condition (2). Yet it has surprising implications for the main approaches in exploitation theory. For there are economies in which for all $\nu \in W_{+}$, condition (2) never holds, if either Definition 2 or Definition 3 is adopted. In contrast, Definition 5 satisfies condition (2), and thus the **Profit-Exploitation Correspondence Principle** holds, in general.

Corollary 1: *There exists an economy $E \in \mathcal{E}$ and a reproducible solution*

[26] Actually, Theorem 1 proves an even stronger result. The **Profit-Exploitation Correspondence Principle** allows for the possibility that, when profits are zero, some agents in W_{+} be exploited. The proof of Theorem 1 shows that for all definitions satisfying **Labour Exploitation**, condition (2) holds if and only if the **Profit-Exploitation Correspondence Principle** holds *and no agent in W_{+} is exploited whenever profits are zero.* The latter property follows directly from **Labour Exploitation**.

[27] It is worth noting that if one restricts the analysis to simple economies with Leontief technologies and homogeneous skills, then any definition meeting **Labour Exploitation** also satisfies the **Profit-Exploitation Correspondence Principle**.

(p, w) *such that neither* Definition 2 *nor* Definition 3 *satisfies the* **Profit-Exploitation Correspondence Principle***. Instead,* Definition 5 *satisfies the* **Profit-Exploitation Correspondence Principle** *for all economies* $E \in \mathcal{E}$ *and all reproducible solutions* (p, w)*.*

5 Exploitation as a social relation

Given the theoretical relevance of the **Profit-Exploitation Correspondence Principle** in exploitation theory, Theorem 1 and Corollary 1 provide strong support for Definition 5 as the appropriate notion of UE exploitation. In this section and in the next, two extensions of the analysis are presented, which provide further support to the New Interpretation. For it is shown that Definition 5 can be extended to analyse, first, the exploitation status of *all* agents and the existence of exploitative *relations*; and then the correspondence between exploitation and profits outside of equilibrium allocations, in economies with heterogeneous preferences and unequal endowments of physical and human capital. This suggests that, if the New Interpretation is adopted, then exploitation theory can be extended to yield interesting insights on unequal relations between agents in advanced capitalist economies.

Theorem 2 proves that, based on Definition 5, it is possible to derive the exploitation status of *all* agents and a more general relation between profits and exploitation beyond the subset of propertyless agents.

Theorem 2: *Consider an economy* $E \in \mathcal{E}$*. Let* (p, w) *be a reproducible solution for* E *and aggregate production activity* $\alpha^{p,w}$*. Under* Definition 5:
(1) *if* $\pi^{\max} > 0$, ν *is* exploited *if and only if* $\frac{p\omega^{\nu}}{p\omega} < \frac{\Lambda^{\nu}}{\alpha_l^{p,w}}$; *and* ν *is an* exploiter *if and only if* $\frac{p\omega^{\nu}}{p\omega} > \frac{\Lambda^{\nu}}{\alpha_l^{p,w}}$.
(2) *if* $\pi^{\max} > 0$, *then all agents* $\nu \in \mathcal{N}$ *such that* $\frac{p\omega}{\alpha_l^{p,w}} < \frac{p\omega^{\nu}}{s^{\nu}}$ *are* exploiters. *Furthermore, if there is a subsistence bundle* $b \in \mathbb{R}_+^n$ *such that* $c^{\nu} \geqq b$*, for all* $\nu \in \mathcal{N}$*, then all agents* $\nu \in \mathcal{N}$ *such that* $\frac{p\omega^{\nu}}{p\omega} < \frac{pb}{p\widehat{\alpha}^{p,w}}$ *are* exploited.
(3) *if* $\pi^{\max} = 0$, *no agents are* exploited *or* exploiters: $N^{ted} = N^{ter} = \varnothing$.

Theorem 2 emphasises the importance of wealth inequalities *and* profits in generating exploitation and in determining the exploitation structure of an economy. Formally, Definition 5 can be interpreted as stating that an agent is exploited (resp., an exploiter) if and only if the share of social labour she contributes is greater (resp., lower) than the share of total income she receives. Given a positive profit rate, and for a given amount of labour performed, Theorem 2 suggests that the key determinant of agents' income - and thus of their exploitation status - is their wealth.

Theorem 2-(1) completely characterises the exploitation structure of an economy in equilibrium: an agent is exploited (resp., an exploiter) if and only if her share of social wealth is lower (resp., higher) than her share of social labour. Theorem 2-(2) shows that at the two extremes of the wealth distribution, exploitation status can be determined independently of individual choices, an intuition of standard exploitation theory that is proved robust. On the one hand, agents with a sufficiently high initial wealth are exploiters regardless of the amount of work they expend in production. On the other hand, if a subsistence bundle exists, the set of agents that are exploited regardless of their individual choices is larger than the set of propertyless workers (those who have 'nothing to lose but their chains'), as it also includes workers with nonnegligible initial wealth. This set can be sizable if b is not just a *physical* subsistence bundle, but it incorporates moral and social elements. Jointly with Theorem 2-(3), this establishes a correspondence between positive profits and the exploitation of a larger set of agents than the set of propertyless workers. Indeed, an important property of the New Interpretation can be immediately derived from Theorems 1 and 2.

Corollary 2: *Consider an economy $E \in \mathcal{E}$. Let (p, w) be a reproducible solution for E with aggregate production activity $\alpha^{p,w}$ such that $W_+ \neq \varnothing$. Under* Definition 5, *the following statements are equivalent*:
(1) *the equilibrium rate of profit is* positive, $\pi^{\max} > 0$;
(2) *some agents are* exploited, $W_+ \subseteq N^{ted} \neq \varnothing$;
(3) *some agents are* exploiters, $N^{ter} \neq \varnothing$.

Corollary 2 implies that in equilibrium positive profits are necessary and sufficient for the existence of *exploitative relations*, where the latter notion can be formalised as requiring that $N^{ted} \neq \varnothing$ if and only if $N^{ter} \neq \varnothing$. This seems a weak and reasonable property in exploitation theory: some agents are exploited if and only if there is someone exploiting them. Yet Yoshihara and Veneziani [35] have proved that none of the main received definitions satisfies it in general. In contrast, Corollary 2 shows that, according to the New Interpretation, exploitation has an inherently relational nature. Further, the New Interpretation captures inequalities between classes of individuals concerning the allocation of *labour*. In fact, it can be proved that, unlike in other approaches, if some other good is used as the exploitation numéraire in Definition 5, neither the **Profit-Exploitation Correspondence Principle** nor Corollary 2 holds (Yoshihara and Veneziani [39]).

5.1 Relation with the literature

Although the main contribution of this paper lies in the discussion of a new definition of exploitation and in a novel axiomatic approach to exploitation theory, it is worth briefly discussing the relation with some traditional strands of the literature. The uninterested reader can safely skip this subsection.

The previous results can be read as a generalisation of the so-called *Fundamental Marxian Theorem* (hereafter, FMT), which states that the existence of exploitation is synonymous with positive aggregate profits.[28] For they prove that there exists a nonempty class of definitions of exploitation such that a correspondence between exploitation and profits exists in general convex economies with heterogeneous agents. Yet, the previous analysis bears a relation to the literature on the FMT only at the broad conceptual level.

The **Profit-Exploitation Correspondence Principle** is logically different from the standard FMT. For unlike in the literature on the FMT, the Principle applies to economies with a complex class structure *and* focuses on the exploitation status of a specific set of agents, namely those without any assets, rather than on the aggregate rate of exploitation in the economy. Indeed, as noted above, the **Profit-Exploitation Correspondence Principle** imposes no constraints on the definition of exploitation at equilibria in which propertyless agents do not work ($W_+ = \varnothing$). Moreover, it does *not* require that when equilibrium profits are zero there be no exploitation in the economy, but only that some propertyless agents are not exploited.

Finally, in the standard Okishio-Morishima approach, the existence of (aggregate) UE exploitation is just a numerical representation of the existence of surplus products in a productive economy. Thus, the FMT establishes the equivalence between positive profits and the productiveness of the economy measured in terms of the labour numéraire. Yet, analogous results can be proved when productiveness is measured in terms of any other good (this is the *Commodity Exploitation Theorem*; Roemer [22]; see also Fujimoto and Fujita [12] and Yoshihara and Veneziani [36]), which raises doubts on the significance of the FMT. Instead, if the New Interpretation is adopted, no

[28] See the contributions cited in footnote 6 above.

equivalence between profits and exploitation holds if another commodity is used to define exploitation, as Yoshihara and Veneziani [38] have shown.

6 Exploitation, profits and disequilibrium

Theorem 2 and Corollary 2 complete the analysis of the relation between exploitation and profits, and extend the main insights of UE exploitation theory to all agents in the general economies considered in this paper, under Definition 5, in equilibrium. In this section, an extension of Definition 5 is proposed, and a general relation between exploitation and profits is derived, at *any feasible allocation*.

The key point to note is that there are various possible ways of conceptualising exploitation at general disequilibrium allocations and, consequently, there is no trivial way of extending Definition 5. For outside of a reproducible solution, it is unclear whether exploitation status should be determined relative to the *actual* features of the allocation. On the one hand, if individual plans are not realised, coordination failures arise, and perhaps even sheer mistakes are made, then by focusing on *actual* data one may be capturing purely transient and ephemeral phenomena unrelated to the structural features of the economy. On the other hand, one may insist that only the information contained in the *actual* allocation is relevant. For, ultimately, the actual features of the allocation are what matters to the agents.

In the extension of Definition 5 proposed here, the actual features of the allocation, including the price vector, the aggregate production activity, and the individual work and consumption choices remain central, but the effects of sheer individual mistakes in technical choices, or of temporary market imbalances leading to productive inefficiency are discounted.

For any $\alpha \in P$ with $\widehat{\alpha} \in \mathbb{R}_{+}^{n} \backslash \{\mathbf{0}\}$, let $\mu^{\alpha} \equiv \min \{\mu \mid (-\alpha_l, -\underline{\alpha}, \underline{\alpha} + \mu\widehat{\alpha}) \in \partial P \text{ and } \mu \geqq 1\}$. Note that μ^{α} is well-defined for all $\alpha \in P$. Further, if $\alpha \notin \partial P$, then $\mu^{\alpha} > 1$, while if $\alpha \in \partial P$, then $\mu = 1$. Then, for all $c \in \mathbb{R}_{+}^{n}$, $c \leqq \widehat{\alpha}$, define

$$\phi(c;\alpha) \equiv \{\alpha' \in \phi(c) \mid \exists t \in \mathbb{R}_{+} : \alpha' = t(-\alpha_l, -\underline{\alpha}, \underline{\alpha} + \mu^{\alpha}\widehat{\alpha})\}.$$

$\phi(c;\alpha)$ denotes the set of *efficient* production activities which are along the ray defined by $(-\alpha_l, -\underline{\alpha}, \underline{\alpha} + \mu^{\alpha}\widehat{\alpha})$ and produce at least c as net output.

Then:[29]

$$l.v.\left(c;\alpha\right)\equiv\min\left\{\alpha_{l}\mid\alpha\in\phi\left(c;\alpha\right)\right\}.$$

For a given price vector (p,w) and associated aggregate production activity $\alpha^{p,w}$, the labour content of a bundle c is:

Definition 6: Consider an economy $E\in\mathcal{E}$. Let (p,w) be a price vector for E with aggregate production activity $\alpha^{p,w}$ with $\widehat{\alpha}^{p,w}\in\mathbb{R}_{+}^{n}$ and $p\widehat{\alpha}^{p,w}>0$. For each $c\in\mathbb{R}_{+}^{n}$ with $pc\leqq p\widehat{\alpha}^{p,w}$, let $\tau^{c}\in[0,1]$ be such that $\tau^{c}\widehat{\alpha}^{p,w}\in\mathcal{B}\left(p,c\right)$. The *labour content of* c at $\alpha^{p,w}$ is $l.v.\left(\tau^{c}\widehat{\alpha}^{p,w};\alpha^{p,w}\right)$.

The following definition identifies the set of propertyless agents who are exploited at any given allocation.

Definition 7: Consider an economy $E\in\mathcal{E}$. Let (p,w) be a price vector for E with associated aggregate production activity $\alpha^{p,w}$ with $\widehat{\alpha}^{p,w}\in\mathbb{R}_{+}^{n}$ and $p\widehat{\alpha}^{p,w}>0$. For any $\nu\in W$, who supplies Λ^{ν} and consumes c^{ν}, let $\tau^{c^{\nu}}$ be defined as in Definition 6. Then, $\nu\in W$ is *exploited* if and only if $\Lambda^{\nu}>l.v.\left(\tau^{c^{\nu}}\widehat{\alpha}^{p,w};\alpha^{p,w}\right)$.

Formally, Definitions 6 and 7 generalise Definitions 4 and 5 and they reduce to the latter in equilibrium: if (p,w) is a reproducible solution for E, then $\alpha^{p,w}\in\partial P$ and $l.v.\left(\tau^{c}\widehat{\alpha}^{p,w};\alpha^{p,w}\right)=\tau^{c}\alpha_{l}^{p,w}$ holds. Further, note that in section 4 a weak formulation of **Labour Exploitation** is adopted, which focuses on reproducible solutions. It is straightforward, however, to extend the axiom to all price vectors (p,w) with associated aggregate production activity $\alpha^{p,w}$ and, from a theoretical viewpoint, none of the arguments used to defend **Labour Exploitation** depends on the equilibrium assumption. Therefore one may argue that it remains an appropriate domain condition to define UE exploitation even at disequilibrium allocations. From this perspective, it is worth noting that Definition 7 satisfies **Labour Exploitation**, at any (p,w) with associated aggregate production activity $\alpha^{p,w}$.[30]

Theoretically, in Definitions 6 and 7, the actual allocation of the economy plays a pivotal role. In order to define the exploitation status of propertyless agents, the *actual* price vector and the *actual* individual choices on work and

[29] Assumptions **A0**$\sim$**A3** in Appendix 8.1 guarantee that $l.v.\left(c;\alpha\right)$ is well-defined and bounded below by 0.

[30] To see this, let the ERB be $c_{e}^{\nu}\equiv\tau^{c^{\nu}}\widehat{\alpha}^{p,w}$ and let $\alpha^{c_{e}^{\nu}}\equiv\arg\min\left\{\alpha_{l}\mid\alpha\in\phi\left(c_{e}^{\nu};\alpha^{p,w}\right)\right\}$, so that $\alpha_{l}^{c_{e}^{\nu}}=l.v.\left(\tau^{c^{\nu}}\widehat{\alpha}^{p,w};\alpha^{p,w}\right)$.

consumption are central. The only possible deviation from actual data concerns the focus on technically efficient production activities in the definition of labour content: activities in the interior of the production possibility set are the product of transient contingencies and do not reveal much about the structural features of the economy.[31] Yet the set of admissible efficient activities in Definitions 6 and 7 is significantly constrained by the *actual* social production point $\alpha^{p,w}$ (unlike in Roemer's or Morishima's definitions).

Let $\Lambda^W = \sum_{\nu \in W} \Lambda^\nu$ be the total labour expended by propertyless agents at a given allocation. For all $\nu \in \mathcal{N}$, let $(\alpha^\nu,\ \beta^\nu,\ \gamma^\nu, c^\nu)$ be *individually feasible* if it satisfies constraints (1)-(4) of MP^ν. Based on Definition 7, Theorem 3 establishes a general relation between exploitation and profits at any feasible allocations.

Theorem 3: *Consider any economy $E \in \mathcal{E}$, any nonnegative price vector $(p, w) \in \mathbb{R}_+^{n+1}$ with a positive wage $w > 0$, and any allocation $(\alpha^\nu,\ \beta^\nu,\ \gamma^\nu, c^\nu)_{\nu \in \mathcal{N}}$ that is individually feasible for all $\nu \in N$ and such that some propertyless agents work, $W_+ \neq \varnothing$.*

Then, the following statements are equivalent for any $\alpha^* \in P\left(\alpha_l = \Lambda^W\right) \cap \partial P$ with $\widehat{\alpha}^* \in \mathbb{R}_+^n$:
(1) $p\widehat{\alpha}^* - w\alpha_l^* > 0$ *holds*;
(2) *for any $\nu \in W_+$, $\Lambda^\nu > l.v.\left(\tau^{c^\nu}\widehat{\alpha}^*; \alpha^*\right)$, where $l.v.\left(\tau^{c^\nu}\widehat{\alpha}^*; \alpha^*\right) = \tau^{c^\nu}\alpha_l^*$ for $\tau^{c^\nu} \in [0, 1)$ with $\tau^{c^\nu}\widehat{\alpha}^* \in \mathcal{B}(p, c^\nu)$.*

Theorem 3 states that a general relation between exploitation and profits holds, at any price vector and corresponding allocation, provided productive inefficiencies and temporary disequilibrium phenomena are ruled out: at every technically efficient production vector α^* (which is feasible using actual, effective labour Λ^W), society realises positive profits if and only if every propertyless worker is exploited. This result is fairly general: no significant restriction is imposed on individual behaviour (except that income should not be wasted: the budget constraint holds for all agents) and on the actual allocation. As a result, Theorem 3 does not establish necessary and sufficient conditions for the existence of positive profits and the exploitation of propertyless workers at the *actual* allocation, and the social production point $\alpha^{p,w}$ may, or may not, coincide with one of the vectors α^*. For given the rather

[31] Indeed, Marx's own notion of *Socially Necessary Labour Time* may be interpreted as ruling out inefficient technologies and involving a counterfactual analysis. See Sen [27]. For an alternative interpretation, see Flaschel [5], [6].

large set of admissible allocations, the link between profits and exploitation may be somewhat weakened.[32] However, Theorem 3 derives the general conditions under which propertyless workers are exploited if and only if the economy *can* generate positive profits, starting from the actual individual consumption/leisure choices, price system, and aggregate production activity, even if exchanges do not take place at equilibrium prices. Indeed, if the actual social production point $\alpha^{p,w}$ under the presumption of Theorem 3 is technically *in*efficient with $\widehat{\alpha}^{p,w} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, there is some $\alpha^* \in P\left(\alpha_l = \Lambda^W\right) \cap \partial P$ such that $\alpha^* = \alpha^{\mu^{\alpha^{p,w}}}$, and then $l.v.\left(\tau^{c^\nu}\widehat{\alpha}^*; \alpha^*\right) = l.v.\left(\tau^{c^\nu}\widehat{\alpha}^{p,w}; \alpha^{p,w}\right)$ holds. Therefore, Theorem 3 implies that the exploitation status of employed propertyless agents in this disequilibrium feasible allocation can be verified by examining the profitability of the counterfactual production point $\alpha^* = \alpha^{\mu^{\alpha^{p,w}}}$ at (p, w).

7 Conclusions

What is exploitation? The analysis developed in this paper provides two important, albeit partial answers to the question in the opening paragraph, focusing on the theory of exploitation as the unequal exchange of labour.

First, from a methodological viewpoint, an axiomatic approach provides key insights into UE exploitation in general convex economies with agents endowed with heterogeneous preferences and different amounts of physical and human capital. In these economies, the definition of exploitation is inherently ambiguous and controversial. An axiomatic framework allows for a general analysis of the main approaches, and indeed of *all* conceivable definitions, that starts from first principles, thus explicitly discussing the positive and normative foundations of UE exploitation.

The axiomatic results in section 4 are quite striking. A weak domain condition called **Labour Exploitation** is presented which captures the key insights of UE exploitation and which is satisfied by all of the main approaches. Then, Theorem 1 identifies a demarcation line that partitions the set of (infinitely many, conceivable) definitions satisfying **Labour Exploitation** into those that preserve the **Profit-Exploitation Correspondence Principle**, and those that do not. Based on Theorem 1, Corollary 1 shows

[32] For example, it is possible to have allocations $\alpha^{p,w} \notin \partial P$ such that profits are non-positive but propertyless workers are exploited.

that among all of the main approaches, only a definition of exploitation related to the 'New Interpretation' satisfies the principle in general. Given the theoretical relevance of the relation between exploitation and profits in exploitation theory, the axiomatic analysis provides significant support to the New Interpretation, as opposed to alternative views.

Both **Labour Exploitation** and the **Profit-Exploitation Correspondence Principle** are theoretically robust and formally weak properties. But even if they are rejected on theoretical or exegetical grounds, the methodological conclusion of the paper stands: significant progress can be made in exploitation theory by adopting an axiomatic method.

Second, from a substantive viewpoint, the New Interpretation has several desirable properties and preserves some of the key intuitions of exploitation theory. This is important because the difficulty of extending the normative and positive insights of UE exploitation outside of stylised, two-class economies has produced significant scepticism in the literature about the relevance of exploitation theory in advanced capitalist economies.

According to the New Interpretation, an agent is exploited (resp., an exploiter) if and only if the labour she contributes is greater (resp., lower) than the share of social labour that she receives via her income. This definition conceives of exploitation as the result of the competitive allocation of social labour (rather than productive inefficiencies, or market imperfections) and it has a clear empirical content. Moreover, it preserves the fundamental link between the appropriation of surplus and the exploitation of (at least some) workers in general economies both in equilibrium (Corollary 1) and - remarkably - outside of equilibrium allocations (Theorem 3).

Furthermore, under the New Interpretation, it possible to derive the exploitation structure of general economies with heterogeneous preferences, physical assets, and skills (Theorem 2). From a normative perspective, the exploitation status of agents is in general related to inequalities in the allocation of labour and income, as well as to wealth inequalities. And for both very wealthy and very poor agents, exploitation status is independent of individual choices, as in the classical-Marxian approach. Perhaps even more interestingly, under the New Interpretation, UE exploitation is an inherently relational phenomenon, whereby some agents are exploited if and only if some agents are exploiting them (Corollary 2).

To be sure, the results obtained in this paper are not sufficient to conclude that a logically coherent and normatively relevant definition exists which preserves all of the key insights of UE exploitation theory. Although the economies analysed in this paper are significantly more general than those usually considered in exploitation theory, they remain fairly stylised.

There are several reasons, however, to believe that the main conclusions of the paper can be significantly extended, and the New Interpretation may provide the foundations for a *general* theoretical framework that can deal with many unresolved issues in exploitation theory. For example, it can be shown that all of the key analytical results of the paper remain valid if, in addition to heterogeneous labour skills, one also allows for *different types of labour inputs* (provided each agent's effective labour contribution per unit of time is measured by her marginal productivity).

The New Interpretation may also provide interesting insights on another tenet of exploitation theory, namely the correspondence between class and exploitation status (Roemer [22]). For example, Yoshihara and Veneziani [35] and Yoshihara [34] have proved that, unlike in the standard approaches, if the New Interpretation is adopted, it is possible to derive the full class and exploitation structure, and a correspondence between class and exploitation in economies with a general convex technology and agents endowed with identical preferences and skills. To extend these results to economies with heterogeneous agents is an interesting direction for further research.

8 Appendix

8.1 Assumptions on the production set P

Let $\mathbb{R}_-$ be the set of nonpositive real numbers. The following assumptions on $P \subset \mathbb{R}^{2n+1}$ hold throughout the paper.

Assumption 0 (A0). P is a closed convex cone in $\mathbb{R}^{2n+1}$ and $\mathbf{0} \in P$.

Assumption 1 (A1). For all $\alpha \in P$, if $\overline{\alpha} \geq \mathbf{0}$ then $\alpha_l > 0$.

Assumption 2 (A2). For all $c \in \mathbb{R}^n_+$, there is a $\alpha \in P$ such that $\widehat{\alpha} \geqq c$.

Assumption 3 (A3). For all $\alpha \in P$ and all $\alpha' \in \mathbb{R}_- \times \mathbb{R}^n_- \times \mathbb{R}^n_+$, if $\alpha' \leqq \alpha$ then $\alpha' \in P$.

A1 implies that labour is indispensable to produce any non-negative output vector. **A2** states that any non-negative commodity vector is producible as a net output. **A3** is a standard *free disposal* condition.

8.2 Proofs of formal results

Proof of Lemma 1: Straightforward and therefore omitted. ■

Proof of Theorem 1: First, note that at any $E \in \mathcal{E}$ and any RS (p, w) with $\alpha^{p,w}$ such that either $W = \varnothing$ or $\Lambda^\nu = 0$ for all $\nu \in W$, the equivalence is immediately established, for both **PECP** and condition (2) are vacuously satisfied. Therefore, in the rest of the proof, suppose that $\Lambda^\nu > 0$ for at least some $\nu \in W$, and $W \neq \varnothing$.

(2)⇒(1): Consider any $E \in \mathcal{E}$ and any RS (p, w) with $\alpha^{p,w}$. Suppose that if $\pi^{\max} > 0$, then for each $\nu \in W_+$, there exists $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu) \cap \partial P$ such that $p\widehat{\alpha}_\pi^\nu > w\Lambda^\nu$ and $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$ for some $\delta^\nu > 1$.

Let $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} = 0$. Then by Lemma 1, $\pi^{\max} = 0$ and condition (2) is vacuously satisfied. Moreover, at any RS, $p_i > 0$ for at least some good i and so by **A2**, $\pi^{\max} = 0$ implies $w > 0$. (If $w = 0$ then $p_i > 0$, some i, and **A2** together imply that there exist $\alpha \in P$ such that $p\widehat{\alpha} - w\alpha_l = p\widehat{\alpha} > 0$, contradicting $\pi^{\max} = 0$.) By **LE**, for each $\nu \in W_+$, $c_e^\nu \in \mathbb{R}_+^n$, $pc_e^\nu = w\Lambda^\nu$ and $\alpha^{c_e^\nu} \in \phi(c_e^\nu)$. Therefore, noting that $p\widehat{\alpha}^{c_e^\nu} \geqq pc_e^\nu = w\Lambda^\nu > 0$, $\pi^{\max} = 0$ implies that $\alpha_l^{c_e^\nu} \geqq \Lambda^\nu$. Hence, by **LE**, $\nu \notin N^{ted}$ holds for all $\nu \in W_+$.

Let $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0$ so that $\pi^{\max} > 0$. For any $\nu \in W_+$, $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu)$ and $\delta^\nu > 1$ imply $\alpha_l^{c_e^\nu} < \Lambda^\nu$. Thus, by **LE**, $\nu \in N^{ted}$ holds for any $\nu \in W_+$.

In sum, (2) implies that **PECP** holds under any definition of exploitation satisfying **LE**.

(1)⇒(2): Consider any $E \in \mathcal{E}$ and any RS (p, w) with $\alpha^{p,w}$. Suppose that $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0 \Leftrightarrow N^{ted} \supseteq W_+$.

Suppose that $\pi^{\max} > 0$. By Lemma 1, $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0$ holds, and by **LE** and **PECP**, for each $\nu \in W_+$, there exist $c_e^\nu \in \mathbb{R}_+^n$ and $\alpha^{c_e^\nu} \in \phi(c_e^\nu) \cap \partial P$ with $\widehat{\alpha}^{c_e^\nu} \not> c_e^\nu$ such that $pc_e^\nu = w\Lambda^\nu$ and $\alpha_l^{c_e^\nu} < \Lambda^\nu$.

Suppose first that $\alpha_l^{c_e^\nu} = 0$ for some $\nu \in W_+$. By **A1** and **LE**, this implies that $\overline{\alpha}^{c_e^\nu} = \mathbf{0}$ holds. Moreover, since $\alpha^{c_e^\nu} \in \phi(c_e^\nu)$ and $c_e^\nu \geqq \mathbf{0}$, $\overline{\alpha}^{c_e^\nu} = \mathbf{0}$ implies that $\underline{\alpha}^{c_e^\nu} = \mathbf{0}$. Hence, we have $\widehat{\alpha}^{c_e^\nu} = \mathbf{0}$ and therefore $c_e^\nu = \mathbf{0}$, which implies $pc_e^\nu = 0$ and $w = 0$. Hence $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} = p\widehat{\alpha}^{p,w} > 0$ and $p \geq \mathbf{0}$ imply $\widehat{\alpha}^{p,w} \geq \mathbf{0}$, which in turn implies $\alpha_l^{p,w} > 0$, by **A1**. Therefore let $\alpha_\pi^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}}\alpha^{p,w}$: at a RS $\alpha^{p,w} \in \partial P$ by Lemma 1 and so by **A0**, $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu) \cap \partial P$. Moreover $\widehat{\alpha}_\pi^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}}\widehat{\alpha}^{p,w}$, and so $\widehat{\alpha}_\pi^\nu \in \mathbb{R}_+^n$ and $p\widehat{\alpha}_\pi^\nu > w\Lambda^\nu = 0$. Finally, $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$ for all $\delta^\nu > 1$, as sought.

Suppose that $\alpha_l^{c_e^\nu} > 0$ for all $\nu \in W_+$. Then, for all $\nu \in W_+$, let δ^ν be such that $\delta^\nu \alpha_l^{c_e^\nu} = \Lambda^\nu$. Suppose $w > 0$. For each $\nu \in W_+$, let $\alpha_\pi^\nu \equiv \delta^\nu \alpha^{c_e^\nu}$. By definition, $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$ for $\delta^\nu = \frac{\Lambda^\nu}{\alpha_l^{c_e^\nu}} > 1$. Further, by **A0**, since $\alpha^{c_e^\nu} \in \partial P$, $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu) \cap \partial P$. Finally, by construction, $\widehat{\alpha}_\pi^\nu \in \mathbb{R}_+^n$ and $p\widehat{\alpha}_\pi^\nu > w\Lambda^\nu$, as sought.

Next, suppose $w = 0$. If $p\widehat{\alpha}^{c_e^\nu} > 0$, the result follows in a similar manner. Let $p\widehat{\alpha}^{c_e^\nu} = 0$. Then, let $x^\nu \equiv t\frac{\Lambda^\nu}{\alpha_l^{c_e^\nu}}\alpha^{c_e^\nu} + (1-t)\frac{\Lambda^\nu}{\alpha_l^{p,w}}\alpha^{p,w}$ where $t \in (0,1)$ is sufficiently close to 1. If $x^\nu \in \partial P$, then let $\alpha_\pi^\nu \equiv x^\nu$. If $x^\nu \notin \partial P$, then by **A0-A3** there is $y^\nu \in \partial P$ such that $\left(-y_l^\nu, -\underline{y}^\nu\right) = \left(-x_l^\nu, -\underline{x}^\nu\right)$ and $\overline{y}^\nu > \overline{x}^\nu$. Then, let $\alpha_\pi^\nu \equiv y^\nu$. In either case, by definition, $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu) \cap \partial P$. Moreover, $\widehat{\alpha}_\pi^\nu \in \mathbb{R}_+^n$ and since $p\widehat{\alpha}^{p,w} > 0$ by $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0$, $p\widehat{\alpha}_\pi^\nu > 0 = w\Lambda^\nu$ holds. Finally, since t is sufficiently close to 1, there exists $\delta^\nu \in \left(1, \frac{\Lambda^\nu}{\alpha_l^{c_e^\nu}}\right)$ which is sufficiently close to 1, such that $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$.

In sum, if **PECP** holds, then (2) holds under any definition of exploitation satisfying **LE**. ■

Proof of Corollary 1: For a proof that neither Definition 2 nor Definition 3 satisfies **PECP**, see Lemma A2.1 in [32]. We need to prove that Definition 5 satisfies condition (2) of Theorem 1. Consider any $E \in \mathcal{E}$ and any RS (p, w) with $W_+ \neq \varnothing$. We show that if $\pi^{\max} > 0$, then for each $\nu \in W_+$, there exists $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu) \cap \partial P$ such that $p\widehat{\alpha}_\pi^\nu > w\Lambda^\nu$ and $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$ for some $\delta^\nu > 1$.

Suppose $\pi^{\max} > 0$. By Lemma 1, this implies $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0$. Then, for all $\nu \in W_+$, let $\alpha_\pi^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}}\alpha^{p,w}$: at a RS $\alpha^{p,w} \in \partial P$ by Lemma 1 and so by **A0** $\alpha_\pi^\nu \in P(\alpha_l = \Lambda^\nu) \cap \partial P$. Moreover $\widehat{\alpha}_\pi^\nu = \frac{\Lambda^\nu}{\alpha_l^{p,w}}\widehat{\alpha}^{p,w}$, and so $\widehat{\alpha}_\pi^\nu \in \mathbb{R}_+^n$, and since $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0$ it follows that $p\widehat{\alpha}_\pi^\nu > w\Lambda^\nu$.

Finally, under Definition 5 $\alpha^{c_e^\nu} = \tau^{c^\nu}\alpha^{p,w}$, where $\tau^{c^\nu} = \frac{pc^\nu}{p\widehat{\alpha}^{p,w}}$ for all $\nu \in W_+$. Hence $(\alpha_{\pi l}^\nu, \underline{\alpha}_\pi^\nu, \overline{\alpha}_\pi^\nu) \geqq \delta^\nu \left(\alpha_l^{c_e^\nu}, \underline{\alpha}^{c_e^\nu}, \overline{\alpha}^{c_e^\nu}\right)$ for some $\delta^\nu > 1$ if and only if $\frac{\Lambda^\nu}{\alpha_l^{p,w}}\left(\alpha_l^{p,w}, \underline{\alpha}^{p.w}, \overline{\alpha}^{p,w}\right) \geqq \delta^\nu \frac{pc^\nu}{p\widehat{\alpha}^{p,w}}\left(\alpha_l^{p,w}, \underline{\alpha}^{p.w}, \overline{\alpha}^{p,w}\right)$ for some $\delta^\nu > 1$, and the latter inequality holds for all $\nu \in W_+$ whenever $p\widehat{\alpha}^{p,w} - w\alpha_l^{p,w} > 0$.

In summary, condition (2) of Theorem 1 holds in any $E \in \mathcal{E}$ and at any RS (p, w). ■

Remark: A similar argument to Lemma A2.1 in [32] is used in Yoshihara

([34]; Corollaries 1 and 2) to prove that the *Class-Exploitation Correspondence Principle* [22] does not hold under Definition 2 or Definition 3.

Proof of Theorem 2: 1. Consider the case $p\widehat{\alpha}^{p,w} > 0$.

Part (1). Let (p, w) be a RS for $E \in \mathcal{E}$. Then by Definition 1-(i), it follows that $p\widehat{\alpha}^{\nu} + \left[p\widehat{\beta}^{\nu} - w\beta_l^{\nu}\right] + w\gamma^{\nu} = pc^{\nu}$ for all $\nu \in \mathcal{N}$. Since $p\left(\underline{\alpha}^{\nu} + \underline{\beta}^{\nu}\right) = p\omega^{\nu}$ for all $\nu \in \mathcal{N}$, and noting that only processes yielding the maximum rate of profit are going to be activated, the latter expression can be written as $\pi^{\max}p\omega^{\nu} + w\Lambda^{\nu} = pc^{\nu}$. Then, by Definition 1-(ii) and Definition 1-(iv), it follows that $\pi^{\max}p\omega + w\alpha_l^{p,w} = p\widehat{\alpha}^{p,w}$. Therefore $\Lambda^{\nu} \gtrless \tau^{c^{\nu}}\alpha_l^{p,w}$ if and only if $\Lambda^{\nu} \gtrless \frac{\pi^{\max}p\omega^{\nu}+w\Lambda^{\nu}}{\pi^{\max}p\omega+w\alpha_l^{p,w}}\alpha_l^{p,w}$, which yields the desired result.

Part (2). Let (p, w) be a RS for $E \in \mathcal{E}$. The first part of the statement follows immediately from part 1, noting that $\lambda^{\nu} \leqq 1$. In order to prove the second part of the statement, note, as in the proof of Part (1), that by Definition 1-(i), $\pi^{\max}p\omega^{\nu} + w\Lambda^{\nu} = pc^{\nu}$, for all $\nu \in \mathcal{N}$. If $w > 0$, then it follows that $\Lambda^{\nu} > \tau^{c^{\nu}}\alpha_l^{p,w}$ if and only if $\left[\frac{pc^{\nu}-\pi^{\max}p\omega^{\nu}}{w}\right] > \frac{pc^{\nu}}{p\widehat{\alpha}^{p,w}}\alpha_l^{p,w}$, which is in turn equivalent to $pc^{\nu}\left[1 - \frac{w\alpha_l^{p,w}}{p\widehat{\alpha}^{p,w}}\right] > \pi^{\max}p\omega^{\nu}$. Then, setting $c^{\nu} \geqq b$, for all $\nu \in \mathcal{N}$, gives the desired result.

If $w = 0$, there is no agent $\nu \in \mathcal{N}$ such that $\frac{p\omega^{\nu}}{p\omega} < \frac{pb}{p\widehat{\alpha}^{p,w}}$, and so the second part of Theorem 2(2) vacuously holds. To see this, note that if $w = 0$ then $pc^{\nu} = \pi^{\max}p\omega^{\nu}$ for all $\nu \in \mathcal{N}$ and $p\widehat{\alpha}^{p,w} = \pi^{\max}p\omega$. Therefore $\frac{p\omega^{\nu}}{p\omega} = \frac{pc^{\nu}}{p\widehat{\alpha}^{p,w}}$ and for all $\nu \in \mathcal{N}$, $c^{\nu} \geqq b$ implies $pc^{\nu} \geqq pb$ and so $\frac{p\omega^{\nu}}{p\omega} \geqq \frac{pb}{p\widehat{\alpha}^{p,w}}$ for all $\nu \in \mathcal{N}$.

Part (3). If $\pi^{\max} = 0$, then it follows that $w\Lambda^{\nu} = pc^{\nu}$, for all $\nu \in \mathcal{N}$, and $w\alpha_l^{p,w} = p\widehat{\alpha}^{p,w}$, which yields the desired result.

2. Consider the case $p\widehat{\alpha}^{p,w} = 0$. Then, since $p\widehat{\alpha}^{p,w} = w\alpha_l^{p,w} + \pi^{\max}p\underline{\alpha}^{p,w}$ holds in the RS, $w\alpha_l^{p,w} = 0$ holds. Then, $p\widehat{\alpha}^{p,w} = 0 = w\alpha_l^{p,w}$ implies that $\pi^{\max} = 0$, thus we only examine Part (3).

By the same argument used in the proof of Theorem 1, $\pi^{\max} = 0$ implies $w > 0$. Therefore, at the RS $\alpha_l^{p,w} = 0$, and so $\Lambda^{\nu} = 0$ holds for all $\nu \in \mathcal{N}$. Thus, $\Lambda^{\nu} = \tau^{c^{\nu}}\alpha_l^{p,w}$ holds for any $\nu \in \mathcal{N}$, which implies $N^{ted} = N^{ter} = \varnothing$. ■

Proof of Theorem 3: Consider any $\alpha^* \in P\left(\alpha_l = \Lambda^W\right) \cap \partial P$. Let $C^W = \sum_{\nu \in W} c^{\nu}$.

Suppose (1) holds. Then, $p\widehat{\alpha}^* - w\Lambda^W = p\left(\widehat{\alpha}^* - C^W\right) > 0$, since constraints (1)-(2) hold for all agents. Note that, for any $\nu \in W_+$, $pc^{\nu} = w\Lambda^{\nu} = w\Lambda^W\frac{\Lambda^{\nu}}{\Lambda^W} = pC^W\frac{\Lambda^{\nu}}{\Lambda^W}$. Then, let $\tau^{c^{\nu}} = \frac{pc^{\nu}}{p\widehat{\alpha}^*}$ for any $\nu \in W_+$. Clearly $\tau^{c^{\nu}} \in [0, 1)$ with $\tau^{c^{\nu}}\widehat{\alpha}^* \in \mathcal{B}(p, c^{\nu})$. Moreover, for any $\nu \in W_+$, $\tau^{c^{\nu}}\alpha_l^* = \frac{pc^{\nu}}{p\widehat{\alpha}^*}\Lambda^W =$

$\Lambda^{\nu}\frac{pC^{W}}{p\widehat{\alpha}^{*}} < \Lambda^{\nu}$, where the latter inequality follows from $p\left(\widehat{\alpha}^{*} - C^{W}\right) > 0$. Finally, since $\alpha^{*} \in P\left(\alpha_{l} = \Lambda^{W}\right) \cap \partial P$, $l.v.\left(\tau^{c^{\nu}}\widehat{\alpha}^{*};\alpha^{*}\right) = \tau^{c^{\nu}}\alpha_{l}^{*}$ holds. Thus, (2) is obtained.

Suppose (2) holds. Then, for any $\nu \in W_{+}$, $\Lambda^{\nu} > l.v.\left(\tau^{c^{\nu}}\widehat{\alpha}^{*};\alpha^{*}\right)$, where $l.v.\left(\tau^{c^{\nu}}\widehat{\alpha}^{*};\alpha^{*}\right) = \tau^{c^{\nu}}\alpha_{l}^{*}$ holds for $\tau^{c^{\nu}} \in [0,1)$ with $\tau^{c^{\nu}}\widehat{\alpha}^{*} \in \mathcal{B}\left(p,c^{\nu}\right)$. Thus, $\Lambda^{W} > \sum_{\nu\in W_{+}}\tau^{c^{\nu}}\alpha_{l}^{*}$ holds. Note that for any $\nu \in W_{+}$, $w\Lambda^{\nu} = pc^{\nu} > 0$ by $w > 0$. Then, $\tau^{c^{\nu}}\widehat{\alpha}^{*} \in \mathcal{B}\left(p,c^{\nu}\right)$ implies $\tau^{c^{\nu}} > 0$ and $p\widehat{\alpha}^{*} > 0$. Since $\tau^{c^{\nu}} = \frac{pc^{\nu}}{p\widehat{\alpha}^{*}}$ for any $\nu \in W_{+}$, $\Lambda^{W} > \sum_{\nu\in W_{+}}\tau^{c^{\nu}}\alpha_{l}^{*}$ implies that $\Lambda^{W} > \frac{pC^{W}}{p\widehat{\alpha}^{*}}\Lambda^{W}$, thus $p\left(\widehat{\alpha}^{*} - C^{W}\right) > 0$ holds. Since $pC^{W} = w\Lambda^{W} = w\alpha_{l}^{*}$ by the budget constraint, $p\widehat{\alpha}^{*} - w\alpha_{l}^{*} > 0$ holds. ■

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1 Addendum: The existence of a RS

This addendum proves the existence of equilibrium for the economies analysed in this paper. This completes the analysis by showing the consistency of the economic framework.

To be precise, the existence of an equilibrium is proved for a theoretically relevant subset of the set of economies $\mathcal{E}$. It focuses on the polar case where $C = \mathbb{R}^n_+$ and it generalises the proofs of existence in Roemer [2], [3]. Yoshihara and Veneziani [5] prove the existence of a RS for another polar case where $C = \left\{c \in \mathbb{R}^n_+ \mid c \geqq b\right\}$ for some subsistence vector $b \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, u^ν is not strictly increasing on C, and agents minimise labour.

Theorem A1.1 below generalises Theorem 2.5 and Corollary 2.8 of Roemer [2] in two respects: first, although the assumptions on technology are the same, a more general model of individuals is considered by allowing for unequal skills and general, heterogeneous preferences over consumption and leisure. Second, the subsistence wage hypothesis is dropped in favour of a more general endogenous determination of the equilibrium wage rate (see Definition 1(iv) above). Despite these generalisations, the existence of a *non-trivial* RS is proved for essentially the same set of initial physical endowments as in Roemer [2]. Note that Theorem 2.5 and Corollary 2.8 of Roemer [2] do not show the existence of a non-trivial RS: the non-triviality of a RS is guaranteed only by referring to the Fundamental Marxian Theorem.

It is assumed that u^ν is continuous, quasi-concave, and strictly increasing on C for all $\nu \in \mathcal{N}$. Further, the following standard *boundary condition of utility functions* is assumed: $u^\nu(c, \lambda) > u^\nu(\mathbf{0}, \lambda')$ for any $c \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, and any $\lambda, \lambda' \in [0,1]$. This assumption implies that any propertyless agent $\nu \in W$ would rather participate in the labour market to earn some revenue and purchase *some* consumption goods, than drop out of the labour market consuming nothing. Finally, **A1** is slightly strengthened to require that *some* produced inputs be used in the production of commodities:

Assumption 1′ (A1′). For all $\alpha \in P$, $\overline{\alpha} \geq \mathbf{0} \Rightarrow [\alpha_l > 0$ and $\underline{\alpha} \geq \mathbf{0}]$.

A1′ is an essential property of a capitalist economy in the sense that if it is not satisfied, anyone - including propertyless agents - can in principle hire workers. Given the twin role of agents as consumers and producers, **A1**$'$ guarantees the boundedness of the aggregate demand correspondences.

Let a profile $(c^\nu, \gamma^\nu, \beta^\nu)_{\nu\in\mathcal{N}}$ be a *feasible allocation* for $E \in \mathcal{E}$ if and only if $(c^\nu, \gamma^\nu, \beta^\nu)_{\nu\in\mathcal{N}}$ satisfies Definition 1-(ii), 1-(iii), and 1-(iv), and $(c^\nu, \gamma^\nu, \beta^\nu) \in C \times [0, s^\nu] \times P$ holds for all $\nu \in \mathcal{N}$. If the social endowment of capital ω of an economy $E \in \mathcal{E}$ only allows for feasible allocations with $\sum_{\nu\in\mathcal{N}} c^\nu = \mathbf{0}$, then if a RS exists for this economy, it can only be a trivial RS. However, by **A2**, it is always possible to have a non-trivial feasible allocation with $\sum_{\nu\in\mathcal{N}} c^\nu \neq \mathbf{0}$ if ω is placed appropriately. Thus, in order to guarantee the existence of non-trivial feasible allocations, the following assumption is made:

Assumption 4 (A4). $E(P, \mathcal{N}, \mathbf{u}, \mathbf{s}, \Omega)$ has the following property:

$$\omega \in \left\{ \underline{\alpha} \in \mathbb{R}^n_+ \mid \exists \alpha \in P \text{ s.t. } \alpha_l \leqq \sum_{\nu\in\mathcal{N}} s^\nu \text{ and } \widehat{\alpha} \geq \mathbf{0} \right\}.$$

By **A4**, there exists $\alpha' \in P$ with $\alpha'_l \leqq \sum_{\nu\in\mathcal{N}} s^\nu$ and $\underline{\alpha}' = \omega$ such that for any $p > \mathbf{0}$, $p\,(\overline{\alpha}' - \omega) > 0$. Thus, for a sufficiently small $w' > 0$, $p\,(\overline{\alpha}' - \omega) - w\alpha'_l \geqq 0$ holds for any $w \leqq w'$. This implies that for any $p > \mathbf{0}$, there is $w' > 0$ such that for any $w \leqq w'$, $\max_{\alpha\in P:\, p\underline{\alpha}=p\omega} p\widehat{\alpha} - w\alpha_l$ is non-negative.

Let $\mathcal{O}^\nu(p, w)$ be the set of plans $(\alpha^\nu, \beta^\nu, \gamma^\nu, c^\nu)$ that solve MP^ν at prices (p, w). For any vector (p, w), let $\Pi^\nu(p, w) \equiv p\widehat{\alpha}^\nu + \left[p\widehat{\beta}^\nu - w\beta^\nu_l\right] + w\gamma^\nu$ denote agent ν's net revenue. Note that, for any (p, w), $\mathcal{O}^\nu(p, w)$ always contains vectors of the form $(\mathbf{0},\ \beta^\nu,\ \gamma^\nu, c^\nu)$ such that $\Pi^\nu(p, w) = \pi^{\max} p\underline{\beta}^\nu + w\gamma^\nu = pc^\nu$ with $p\underline{\beta}^\nu = p\omega^\nu$ for all ν. Let $\Delta \equiv \left\{(p, w) \in \mathbb{R}^{n+1}_+ \mid \sum_{i=1}^n p_i + w = 1\right\}$ and $\Delta_+ \equiv \{(p, w) \in \Delta \mid p > \mathbf{0}\}$.

In order to analyse the existence of a RS, for all $(p, w) \in \Delta_+$, and for all $\nu \in \mathcal{N}$, define the feasibility correspondence

$$B^\nu(p, w) \equiv \left\{(c^\nu, \beta^\nu, \gamma^\nu) \in C \times P \times [0, s^\nu] \mid pc^\nu \leqq \Pi^\nu(p, w)\,;\ p\underline{\beta}^\nu \leqq p\omega^\nu\right\}.$$

The next result establishes some basic properties of $B^\nu(p, w)$.

Lemma A1.1: *For each $\nu \in \mathcal{N}$, the correspondence B^ν is non-empty, closed-valued and convex-valued, and continuous on Δ_+. Moreover, every (c^ν, γ^ν) in $B^\nu(p, w)$ is bounded for each $(p, w) \in \Delta_+$.*

Proof. It is obvious that B^ν is non-empty, closed-valued, and convex-valued. Since $pc^\nu \leqq \Pi^\nu(p, w) \leqq \pi^{\max} p\omega^\nu + ws^\nu$, the boundedness of (c^ν, γ^ν) in $B^\nu(p, w)$ follows from **A1**$'$, for all $(p, w) \in \Delta_+$.

Finally, we prove the continuity of B^ν. First, we show that B^ν is lower hemi-continuous. Let $\left\{\left(p^k, w^k\right)\right\} \subseteq \Delta_+$ be a sequence such that $\left(p^k, w^k\right) \to (p, w)$ and $(c^\nu, \beta^\nu, \gamma^\nu) \in B^\nu(p, w)$.

Case 1: Suppose $\left(p\widehat{\beta}^\nu - w\beta_l^\nu\right) + w\gamma^\nu > 0$. Then, for each $\left(p^k, w^k\right)$, let $\beta^{k\nu} \equiv \mu^{k\nu}\beta^\nu$ where if $\beta^\nu = \mathbf{0}$ then $\mu^{k\nu} = 1$ and if $\beta^\nu \neq \mathbf{0}$, then

$$\mu^{k\nu} \equiv \min\left\{\frac{\min\left\{\max\left\{p^k\widehat{\beta}^\nu - w^k\beta_l^\nu + w^k\gamma^\nu, 0\right\}, p\widehat{\beta}^\nu - w\beta_l^\nu + w\gamma^\nu\right\}}{p\widehat{\beta}^\nu - w\beta_l^\nu + w\gamma^\nu}, \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}\right\},$$

and let $\gamma^{k\nu} = \gamma^\nu$. Note that if $\beta^\nu \neq \mathbf{0}$, then by $\mathbf{A1'}$ and $\left(p^k, w^k\right) \in\Delta_+$, $p^k\underline{\beta}^\nu > 0$. Moreover, if $c^\nu \neq \mathbf{0}$, then let $\sigma^{k\nu} \equiv \min\left\{\frac{\mu^{k\nu}\left(p^k\widehat{\beta}^\nu - w^k\beta_l^\nu\right) + w^k\gamma^{k\nu}}{p^k c^\nu}, 1\right\}$ and $c^{k\nu} \equiv \sigma^{k\nu}c^\nu$, whereas if $c^\nu = \mathbf{0}$, then let $c^{k\nu} \equiv c^\nu$. Then, since $\mu^{k\nu} \leqq \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}$, $p^k\underline{\beta}^{k\nu} \leqq p^k\omega^\nu$, and since $\mu^{k\nu} = 0$ for $p^k\widehat{\beta}^\nu - w^k\beta_l^\nu + w^k\gamma^\nu \leqq 0$, $\Pi^\nu\left(p^k, w^k\right) \geqq 0$ holds. Therefore, $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \in B^\nu(p^k, w^k)$ with $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \to (c^\nu, \beta^\nu, \gamma^\nu)$ as $\left(p^k, w^k\right) \to (p, w)$. The last convergence property follows from $\mu^{k\nu} \to 1$ as $\left(p^k, w^k\right) \to (p, w)$.

Case 2: Suppose $\left(p\widehat{\beta}^\nu - w\beta_l^\nu\right) + w\gamma^\nu = 0$. In this case, $c^\nu = \mathbf{0}$ holds. Then, for each $\left(p^k, w^k\right)$, let $\gamma^{k\nu} = \gamma^\nu$, $c^{k\nu} = \mathbf{0}$, and $\beta^{k\nu} \equiv \mu^{k\nu}\beta^\nu$ where if $\beta^\nu = \mathbf{0}$ then $\mu^{k\nu} = 1$ and if $\beta^\nu \neq \mathbf{0}$, then

$$\mu^{k\nu} \equiv \begin{cases} \min\left\{1, \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}\right\} & \text{if } \left(p^k\widehat{\beta}^\nu - w^k\beta_l^\nu\right) + w^k\gamma^\nu \geqq 0, \\ \min\left\{\frac{w^k\gamma^\nu}{\left|p^k\widehat{\beta}^\nu - w^k\beta_l^\nu\right|}, \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}\right\} & \text{if } \left(p^k\widehat{\beta}^\nu - w^k\beta_l^\nu\right) + w^k\gamma^\nu < 0. \end{cases}$$

Then, since $\mu^{k\nu} \leqq \frac{p^k\omega^\nu}{p^k\underline{\beta}^\nu}$, $p^k\underline{\beta}^{k\nu} \leqq p^k\omega^\nu$. Also, since $\mu^{k\nu} \leqq \frac{w^k\gamma^\nu}{\left|p^k\widehat{\beta}^\nu - w^k\beta_l^\nu\right|}$ for $\left(p^k\widehat{\beta}^\nu - w^k\beta_l^\nu\right) + w^k\gamma^\nu < 0$, $\Pi^\nu\left(p^k, w^k\right) \geqq 0$ holds. Therefore, $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \in B^\nu(p^k, w^k)$ with $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \to (c^\nu, \beta^\nu, \gamma^\nu)$ as $\left(p^k, w^k\right) \to (p, w)$. The last convergence property follows from $\mu^{k\nu} \to 1$ as $\left(p^k, w^k\right) \to (p, w)$.

The previous arguments show that B^ν is lower hemi-continuous.

To prove that B^ν is upper hemi-continuous, suppose that $\left\{\left(p^k, w^k\right)\right\} \subseteq \Delta_+$ is a sequence such that $\left(p^k, w^k\right) \to (p, w)$ and $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \in B^\nu(p^k, w^k)$ with $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \to (c^\nu, \beta^\nu, \gamma^\nu)$ as $\left(p^k, w^k\right) \to (p, w)$, and $(c^\nu, \beta^\nu, \gamma^\nu) \notin B^\nu(p, w)$. Then, either $(c^\nu, \beta^\nu, \gamma^\nu) \notin C \times P \times [0, s^\nu]$, or $pc^\nu > \Pi^\nu(p, w)$, or $p\underline{\beta}^{\nu} > p\omega^{\nu}$. Since $C \times P \times [0, s^{\nu}]$ is closed, $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \to (c^{\nu}, \beta^{\nu}, \gamma^{\nu})$ implies that $(c^{\nu}, \beta^{\nu}, \gamma^{\nu}) \in C \times P \times [0, s^{\nu}]$. Thus, either $pc^{\nu} > \Pi^{\nu}(p, w)$ or $p\underline{\beta}^{\nu} > p\omega^{\nu}$. Suppose $p\underline{\beta}^{\nu} > p\omega^{\nu}$. Then, for some $\left(p^{k}, w^{k}\right)$ close enough to (p, w), its corresponding $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right)$ is also sufficiently close to $(c^{\nu}, \beta^{\nu}, \gamma^{\nu})$, which implies $p^{k}\underline{\beta}^{k\nu} > p^{k}\omega^{\nu}$, which yields a contradiction. This implies that $(c^{\nu}, \beta^{\nu}, \gamma^{\nu}) \in B^{\nu}(p, w)$. A similar argument holds if $pc^{\nu} > \Pi^{\nu}(p, w)$ and therefore B^{ν} is upper hemi-continuous. ■

Lemma A1.2 analyses optimal choice correspondences.

Lemma A1.2: *For each ν, the correspondence $\mathcal{O}^{\nu}$ is non-empty, closed-valued, convex-valued, and upper hemi-continuous on Δ_{+}. Moreover, every (c^{ν}, γ^{ν}) in $\mathcal{O}^{\nu}(p, w)$ is bounded for each $(p, w) \in \Delta_{+}$.*

Proof. Non-emptiness, closed-valuedness, and convexity can be proved in the standard manner. Since every (c^{ν}, γ^{ν}) in $B^{\nu}(p, w)$ is bounded by Lemma A1.1, every (c^{ν}, γ^{ν}) in $\mathcal{O}^{\nu}(p, w)$ is bounded for any $(p, w) \in \Delta_{+}$.

We only need to show that $\mathcal{O}^{\nu}$ is upper hemi-continuous. Let $\left\{\left(p^{k}, w^{k}\right)\right\} \subseteq \Delta_{+}$ be a sequence such that $\left(p^{k}, w^{k}\right) \to (p, w)$ and $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \in \mathcal{O}^{\nu}(p^{k}, w^{k})$ with $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \to (c^{\nu}, \beta^{\nu}, \gamma^{\nu})$ as $\left(p^{k}, w^{k}\right) \to (p, w)$. Suppose $(c^{\nu}, \beta^{\nu}, \gamma^{\nu}) \notin \mathcal{O}^{\nu}(p, w)$. This implies that (c^{ν}, γ^{ν}) is not a maximizer of u^{ν} over $B^{\nu}(p, w)$ and $(c^{\nu}, \beta^{\nu}, \gamma^{\nu}) \in B^{\nu}(p, w)$ by the upper hemi-continuity of B^{ν}. Then, there exists $(c'^{\nu}, \beta'^{\nu}, \gamma'^{\nu}) \in B^{\nu}(p, w)$ such that $u^{\nu}\left(c'^{\nu}, \frac{\gamma'^{\nu}}{s^{\nu}}\right) > u^{\nu}\left(c^{\nu}, \frac{\gamma^{\nu}}{s^{\nu}}\right)$. Since B^{ν} is lower hemi-continuous, there exists a sequence $\left\{\left(c'^{k\nu}, \beta'^{k\nu}, \gamma'^{k\nu}\right)\right\}$ such that for each $\left(p^{k}, w^{k}\right) \in \Delta_{+}$, $\left(c'^{k\nu}, \beta'^{k\nu}, \gamma'^{k\nu}\right) \in B^{\nu}(p^{k}, w^{k})$ with $\left(c'^{k\nu}, \beta'^{k\nu}, \gamma'^{k\nu}\right) \to (c'^{\nu}, \beta'^{\nu}, \gamma'^{\nu})$ as $\left(p^{k}, w^{k}\right) \to (p, w)$. Then, for $\left(p^{k}, w^{k}\right)$ which is sufficiently close to (p, w), $u^{\nu}\left(c'^{k\nu}, \frac{\gamma'^{k\nu}}{s^{\nu}}\right) > u^{\nu}\left(c^{k\nu}, \frac{\gamma^{k\nu}}{s^{\nu}}\right)$ holds. However, since $\left(c^{k\nu}, \beta^{k\nu}, \gamma^{k\nu}\right) \in \mathcal{O}^{\nu}(p^{k}, w^{k})$, this is a contradiction. Thus, $(c^{\nu}, \beta^{\nu}, \gamma^{\nu}) \in \mathcal{O}^{\nu}(p, w)$, and so $\mathcal{O}^{\nu}$ is upper hemi-continuous. ■

For any $\nu \in \mathcal{N}$, if $(p, w) \in \Delta_{+}$ is associated with $p\widehat{\alpha} - w\alpha_{l} < 0$ for all $\alpha \in P \setminus \{\mathbf{0}\}$, then $(c^{\nu}, \beta^{\nu}, \gamma^{\nu}) \in \mathcal{O}^{\nu}(p, w)$ implies $\beta^{\nu} = \mathbf{0}$. However, by A4, for any $p > \mathbf{0}$, there is $w' > 0$ such that for any $w \leqq w'$, $\max_{\alpha \in P:\, p\underline{\alpha} = p\omega} p\widehat{\alpha} - w\alpha_{l}$ is non-negative, so that there is $(c^{\nu}, \beta^{\nu}, \gamma^{\nu})_{\nu \in \mathcal{N}} \in \times_{\nu \in \mathcal{N}} \mathcal{O}^{\nu}(p, w)$ with $\sum_{\nu \in \mathcal{N}} \beta^{\nu} \neq \mathbf{0}$.

For each $(p, w) \in \Delta_{+}$, let $P(p, w; \omega) \equiv \left\{\alpha \in \arg\max_{\alpha' \in P:\, p\underline{\alpha} \leqq p\omega} p\widehat{\alpha}' - w\alpha'_{l}\right\}$. Let $\alpha_{l}^{m} \equiv \max_{(p', w') \in \Delta} \min\left\{\alpha_{l} \mid \alpha \in P(p', w'; \omega)\right\}$. Then, let $P^{*}(p, w; \omega) \equiv$

$\left\{\alpha \in P(p,w;\omega) \mid \alpha_l \leqq \max\left\{\alpha_l^m, \sum_{\nu\in\mathcal{N}} s^\nu\right\}\right\}$ for each $(p,w) \in\Delta_+$. By this definition, $P^*(p,w;\omega)$ is non-empty, convex, compact, and upper hemi-continuous at every $(p,w) \in\Delta_+$.

For each $(p,w) \in\Delta_+$, define the aggregate excess demand correspondence:

$$\begin{aligned} Z(p,w) \equiv & \left\{\left(\sum_{\nu\in\mathcal{N}} c^\nu - \sum_{\nu\in\mathcal{N}} \widehat{\beta}^\nu, \sum_{\nu\in\mathcal{N}} \beta_l^\nu - \sum_{\nu\in\mathcal{N}} \gamma^\nu\right) \mid \sum_{\nu\in\mathcal{N}} \beta^\nu \in P^*(p,w;\omega)\right. \\ & \left. \& \ \ (c^\nu, \beta^\nu, \gamma^\nu) \in \mathcal{O}^\nu(p,w) \ \ (\forall \nu \in \mathcal{N})\right\}. \end{aligned}$$

Given the above Lemmas and the definition of $P^*(p,w;\omega)$, it follows that Z is compact-valued, convex-valued, and upper hemi-continuous on Δ_+. To see that it is non-empty, firstly suppose that $(p,w) \in\Delta_+$ is such that $p\widehat{\alpha} - w\alpha_l < 0$ for all $\alpha \in P\backslash\{\mathbf{0}\}$. Then, $P(p,w;\omega) = \{\mathbf{0}\} = P^*(p,w;\omega)$, so that there exists $(\beta^\nu)_{\nu\in\mathcal{N}}$ such that $\beta^\nu = \mathbf{0}$ for all ν. Next, if $p\widehat{\alpha} - w\alpha_l \geqq 0$ for some $\alpha \in P\backslash\{\mathbf{0}\}$, $P(p,w;\omega) \supseteq \left\{\alpha \in \arg\max_{\alpha' \in P: p\underline{\alpha}' = p\omega} p\widehat{\alpha}' - w\alpha_l'\right\}$ holds by **A1′**, so that $P^*(p,w;\omega)\backslash\{\mathbf{0}\} \neq \varnothing$, and so if $\alpha \in P^*(p,w;\omega)\backslash\{\mathbf{0}\}$ then there is $(\beta^\nu)_{\nu\in\mathcal{N}}$ such that $\sum_{\nu\in\mathcal{N}} \beta^\nu = \alpha$, and $p\underline{\beta}^\nu = p\omega^\nu$ for all ν. In either case, for (c^ν, γ^ν) in $\mathcal{O}^\nu(p,w)$, it follows that $(c^\nu, \beta^\nu, \gamma^\nu) \in \mathcal{O}^\nu(p,w)$ for each $\nu \in \mathcal{N}$. By definition, since $\sum_{\nu\in\mathcal{N}} \beta^\nu \in P^*(p,w;\omega)$, $Z(p,w)$ is non-empty. Then:

Lemma A1.3: *There exists a price vector* $(\overline{p}, \overline{w}) \in\Delta_+$ *such that* $\mathbf{0} \in Z(\overline{p}, \overline{w})$.

Proof. 1. First, we prove that Z satisfies the Strong Walras Law (SWL), namely for each $(p,w) \in\Delta_+$, and each $(z_1, z_2) \in Z(p,w)$, $pz_1 + wz_2 = 0$. In fact, for each $(p,w) \in\Delta_+$, and each $(z_1,z_2) \in Z(p,w)$,

$$\begin{aligned} pz_1 + wz_2 &= p\left(\sum_{\nu\in\mathcal{N}} c^\nu - \sum_{\nu\in\mathcal{N}} \widehat{\beta}^\nu\right) + w\left(\sum_{\nu\in\mathcal{N}} \beta_l^\nu - \sum_{\nu\in\mathcal{N}} \gamma^\nu\right) \\ &= \sum_{\nu\in\mathcal{N}} \left[pc^\nu - \left\{\left(p\widehat{\beta}^\nu - w\beta_l^\nu\right) + w\gamma^\nu\right\}\right] = 0, \end{aligned}$$

since $pc^\nu = \left(p\widehat{\beta}^\nu - w\beta_l^\nu\right) + w\gamma^\nu$ for every ν, by the strict monotonicity of u^ν.

2. Next, we prove that Z satisfies the following Boundary condition: there is a $(\widetilde{p}, \widetilde{w}) \in\Delta_+$ such that for every sequence $\left\{\left(p^k, w^k\right)\right\} \subseteq\Delta_+$ with $\left(p^k, w^k\right) \to (p,w) \in\Delta \setminus \Delta_+$, there is an M such that for every $k \geqq M$, $(\widetilde{p}, \widetilde{w}) \cdot \left(z_1^k, z_2^k\right) > 0$ holds for every $\left(z_1^k, z_2^k\right) \in Z\left(p^k, w^k\right)$. Take a sufficiently

small but positive real number ε, and define $(\widetilde{p},\widetilde{w}) \in \Delta_+$ as $\widetilde{w}=\varepsilon>0$, and for all j, $\widetilde{p}_j=\frac{1-\varepsilon}{n}>0$. Then, consider any price vector $(p,w) \in \Delta \setminus \Delta_+$, such that $p_i=0$ for one i. Firstly, note that because $\left\{\left(p^k,w^k\right)\right\} \subseteq \Delta_+$, it is possible that $w^k=0$ for sufficiently large k. Thus, in this case, $c^{k\nu}=\mathbf{0}$ for any $\nu \in W$. However, in this case, the corresponding $\pi^{\max k}$ is strictly positive by **A4**, and so $\Pi^\nu\left(p^k,w^k\right)>0$ for any $\nu \in \mathcal{N} \backslash W$. Hence, by the strict monotonicity of utility functions, $c^{k\nu} \geq \mathbf{0}$ for any $\nu \in \mathcal{N} \backslash W$, and in particular, $c_i^{k\nu}$ is sufficiently large at p^k for sufficiently large k. Secondly, $\left\{\left(p^k,w^k\right)\right\} \subseteq \Delta_+$ may also contain the case that $w^k>0$ but $\pi^{\max k}$ is zero for sufficiently large k. In this case, because of the boundary condition for utility functions, any $\nu \in \mathcal{N}$ optimally supplies a positive amount of labour, so that $\Pi^\nu\left(p^k,w^k\right)>0$. Thus, by the strict monotonicity of utility functions, $c^{k\nu} \geq \mathbf{0}$ for any $\nu \in \mathcal{N}$, and in particular, $c_i^{k\nu}$ is sufficiently large at p^k for sufficiently large k. In sum, noting that $\beta^k \in P^*(p^k,w^k;\omega)$ is bounded above, it follows that $z_{1i}^k>0$ is sufficiently large for p^k sufficiently close to p. Then, even if $\widetilde{w}>0$, $\widetilde{w}z_2^k$ will never compensate for $\widetilde{p}z_1^k>0$, since z_2^k is bounded below by $-\sum_{\nu \in \mathcal{N}} s^\nu$ whereas $\widetilde{p}z_1^k$ grows infinitely large due to a sufficiently large $z_{1i}^k>0$. Thus, there is a neighbourhood $\mathcal{H}\left((p,w),\delta\right)$ of (p,w) such that $(\widetilde{p},\widetilde{w}) \cdot \left(z_1^k,z_2^k\right)>0$ for all $\left(p^k,w^k\right) \in \mathcal{H}\left((p,w),\delta\right) \cap \Delta_+$. A similar argument holds if $(p,w) \in \Delta \setminus \Delta_+$, with $p_i=0$, for more than one i.

3. Set $K_m \equiv co\left\{(q,w) \in \Delta_+ \middle|\, dist\left((q,w),\Delta \setminus \Delta_+\right) \geqq \frac{1}{m}\right\}$. Then, $\{K_m\}$ is an increasing family of compact convex sets and $\Delta_+ = \cup_m K_m$. Then, as in Border ([1], Theorem 18.13, p. 85), it follows that there exist $(\overline{p},\overline{w}) \in \Delta_+$ and $\overline{z} \in Z\left(\overline{p},\overline{w}\right)$ such that $\overline{z} \leqq \mathbf{0}$. This fact together with (SWL) imply that $\overline{z}=\mathbf{0}$. In fact, since $\overline{p}>\mathbf{0}$, (SWL) and $\overline{z} \leqq \mathbf{0}$ imply that $\overline{z}_1=\mathbf{0}$. Second, if $\overline{w}>0$, then $\overline{z}_2=0$ holds by (SWL) and $\overline{z} \leqq \mathbf{0}$. Thus, suppose $\overline{w}=0$ and $\overline{z}_2 \equiv \sum_{\nu \in \mathcal{N}} \beta_l^{*\nu} - \sum_{\nu \in \mathcal{N}} \gamma^{*\nu}<0$. Given that every agent's utility function u^ν is strictly monotonic on C, the real-valued function $V^\nu\left(\Pi^\nu\left(p,w\right),\gamma^\nu\right) \equiv \max_{(c^\nu,\beta^\nu,\gamma^\nu) \in B^\nu(p,w)} u^\nu\left(c^\nu,\gamma^\nu\right)$ is strictly monotonic on $\Pi^\nu\left(p,w\right)$, for all ν.

Since $\Pi^\nu\left(\overline{p},\overline{w}\right)=\pi^{\max}\overline{p}\underline{\beta}^{*\nu}+\overline{w}\gamma^{*\nu}=\pi^{\max}\overline{p}\underline{\beta}^{*\nu}$, then $V^\nu\left(\Pi^\nu\left(\overline{p},\overline{w}\right),\gamma^{*\nu}\right)= V^\nu\left(\Pi^\nu\left(\overline{p},\overline{w}\right),0\right)$ because u^ν is (weakly) decreasing in γ^ν on $[0,1]$. Thus, whenever $(c^{*\nu},\beta^{*\nu},\gamma^{*\nu}) \in \mathcal{O}^\nu(\overline{p},\overline{w})$ for all $\nu \in \mathcal{N}$, then for any $\gamma^{**\nu} \in [0,\gamma^{*\nu}]$, we have $(c^{*\nu},\beta^{*\nu},\gamma^{**\nu}) \in \mathcal{O}^\nu(\overline{p},\overline{w})$, which implies that, for any $\left(\gamma^{**\nu}\right)_{\nu \in \mathcal{N}} \in \times_{\nu \in \mathcal{N}}\left[0,\gamma^{*\nu}\right]$ with $\sum_{\nu \in \mathcal{N}} \gamma^{**\nu}=\sum_{\nu \in \mathcal{N}} \beta_l^{*\nu}$, $(c^{*\nu},\beta^{*\nu},\gamma^{**\nu}) \in \mathcal{O}^\nu(\overline{p},\overline{w})$ holds for any $\nu \in \mathcal{N}$. Let $\overline{z}_2' \equiv \sum_{\nu \in \mathcal{N}} \beta_l^{*\nu} - \sum_{\nu \in \mathcal{N}} \gamma^{**\nu}=0$. Then, $(\overline{z}_1,\overline{z}_2') \in Z\left(\overline{p},\overline{w}\right)$, which yields the desired result. ■

Lemma A1.3 proves the existence of a fixed point for the aggregate excess demand correspondences: there exists a price vector $(\overline{p}, \overline{w}) \in \Delta_+$ such that conditions (i), (ii) and (iv) of Definition 1 are satisfied. In order to complete the proof of existence of a RS, it is necessary to show that condition (iii) also holds. Theorem A1.1 provides a condition on aggregate social endowments under which the capital constraint (iii) is satisfied.

Theorem A1.1: *Let* **A0, A1**$'$ $\sim$**A3** *hold and let* u^ν *be continuous, quasi-concave, strictly increasing on* C *, and satisfying the boundary condition for all* $\nu \in \mathcal{N}$*. For any profile* $\Omega = (\omega^\nu)_{\nu \in \mathcal{N}}$ *with* $\sum_{\nu \in \mathcal{N}} \omega^\nu = \omega \geq \mathbf{0}$ *which satisfies* **A4***, there exist a distribution* $\Omega' = (\omega'^\nu)_{\nu \in \mathcal{N}}$ *with* $\sum_{\nu \in \mathcal{N}} \omega'^\nu = \omega'$ *and a RS* $(p, w) \in \Delta_+$ *for the economy* $E(P, \mathcal{N}, \mathbf{u}, \mathbf{s}, \Omega')$ *with* $p\omega'^\nu = p\omega^\nu$ *for all* $\nu \in \mathcal{N}$*.*

Proof. Let $P, \mathcal{N}, \mathbf{s}$, and $\Omega = (\omega^\nu)_{\nu \in \mathcal{N}}$ satisfy **A0, A1**$'$ $\sim$**A4**, and let $\mathbf{u}$ be such that for all $\nu \in \mathcal{N}$, u^ν is continuous, quasi-concave, strictly increasing on C , and it satisfies the boundary condition. Then, we can apply Lemmas A1.1-A1.3, to prove that there exists $(p^*, w^*) \in \Delta_+$ such that $\left(\sum_{\nu \in \mathcal{N}} c^{*\nu} - \sum_{\nu \in \mathcal{N}} \widehat{\beta}^{*\nu}\right) = \mathbf{0}$ and $\left(\sum_{\nu \in \mathcal{N}} \beta_l^{*\nu} - \sum_{\nu \in \mathcal{N}} \gamma^{*\nu}\right) = 0$.

Thus, (p^*, w^*) is associated with $p^*\widehat{\alpha} - w^*\alpha_l \geqq 0$ for some $\alpha \in P \backslash \{\mathbf{0}\}$. In fact, if (p^*, w^*) is such that $p^*\widehat{\alpha} - w^*\alpha_l < 0$ for all $\alpha \in P \backslash \{\mathbf{0}\}$, then $\beta^{*\nu} = \mathbf{0}$ for all $\nu \in \mathcal{N}$, but $\gamma^{*\nu} > 0$ and $c^{*\nu} \neq \mathbf{0}$ follow from $w^* > 0$ and the boundary condition for utility functions. (Note that if $p^*\widehat{\alpha} - w^*\alpha_l < 0$ for all $\alpha \in P \backslash \{\mathbf{0}\}$, then $w^* > 0$.) Hence, $\left(\sum_{\nu \in \mathcal{N}} c^{*\nu} - \sum_{\nu \in \mathcal{N}} \widehat{\beta}^{*\nu}\right) \geq \mathbf{0}$ and $\left(\sum_{\nu \in \mathcal{N}} \beta_l^{*\nu} - \sum_{\nu \in \mathcal{N}} \gamma^{*\nu}\right) < 0$ follow if $p^*\widehat{\alpha} - w^*\alpha_l < 0$ for all $\alpha \in P \backslash \{\mathbf{0}\}$, which is a contradiction. Thus, $p^*\widehat{\alpha} - w^*\alpha_l \geqq 0$ for some $\alpha \in P \backslash \{\mathbf{0}\}$.

Since $p^*\widehat{\alpha} - w^*\alpha_l \geqq 0$ for some $\alpha \in P \backslash \{\mathbf{0}\}$, $(\mathbf{0}, \beta^{*\nu}, \gamma^{*\nu}, c^{*\nu})_{\nu \in \mathcal{N}}$ is a profile of optimal solutions of all MP^ν with $p^*\underline{\beta}^{*\nu} = p^*\omega^\nu$ for all $\nu \in \mathcal{N}$, thus $p^*\underline{\beta}^* = p^*\omega$ at (p^*, w^*). By **A4**, the existence of such profile is guaranteed.

Let us define $\Omega' = (\omega'^\nu)_{\nu \in \mathcal{N}}$ as $\omega'^\nu = \underline{\beta}^{*\nu}$ for each $\nu \in \mathcal{N}$. Then, since $p^*\omega'^\nu = p^*\omega^\nu$ holds for each $\nu \in \mathcal{N}$, it follows that $(\mathbf{0}, \beta^{*\nu}, \gamma^{*\nu}, c^{*\nu})_{\nu \in \mathcal{N}}$ remains a profile of optimal solutions of all MP^ν such that $\left(\sum_{\nu \in \mathcal{N}} c^{*\nu} - \sum_{\nu \in \mathcal{N}} \widehat{\beta}^{*\nu}\right) = \mathbf{0}$ and $\left(\sum_{\nu \in \mathcal{N}} \beta_l^{*\nu} - \sum_{\nu \in \mathcal{N}} \gamma^{*\nu}\right) = 0$. Moreover $\underline{\beta}^* = \omega'$, and so condition (iii) of Definition 1 is also satisfied. Hence, for the economy $E(P, \mathcal{N}, \mathbf{u}, \mathbf{s}, \Omega')$, (p^*, w^*) is a RS with associated profile $(\mathbf{0}, \beta^{*\nu}, \gamma^{*\nu}, c^{*\nu})_{\nu \in \mathcal{N}}$. ∎

As shown in Roemer ([2]; Appendix II) and Yoshihara ([4]; Proposition 1), the proof of existence of a RS is based on the identification of a specific domain of capital stocks, ω, and of an appropriate price vector (p^*, w^*), as in the existence proof of the stationary-state dynamic competitive equilibrium. This is because a RS is a one-shot slice of a *stationary equilibrium* in a dynamic general equilibrium framework.

In the classical economies analysed in [2] and [4], since the aggregate consumption vector is exogenously given, it is easy to identify the set of endowment vectors such that there exists a feasible allocation in terms of Definition 1(ii)-(iii). Given any such vector, to prove the existence of a RS one only needs to find an *efficient* production activity which is feasible in terms of Definition 1(ii)-(iii): if such an activity is found, then the existence of a RS can be proved by applying the supporting hyperplane theorem. Because of this simple structure, Yoshihara ([4]; Proposition 1) provides a full characterisation of the set of endowment vectors such that a RS exists.

In contrast, this paper considers an economy with heterogeneous agents, in which the aggregate consumption vector is endogenously determined, the set of endowment vectors such that there exists a feasible allocation in terms of Definition 1(ii)-(iii) cannot be identified prior to the determination of the price vector. As a result, Theorem A1.1 does not provide a complete characterisation of the domain of endowments such that a RS exists. It does prove, however, that starting from *any* aggregate endowment vector satisfying **A4**, there exists an equilibrium price vector such that the economy can 'purchase' another suitable aggregate endowment vector at those prices, which makes the afore-mentioned stationary-state feasible.

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