Hitotsubashi University

Graduate School of Economics

Discussion Paper No. 2013-13

Goods Revenue Monotonicity in Combinatorial Auctions

by

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October 2013

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October 10, 2013

Abstract

We study a new monotonicity problem in combinatorial auctions called *goods revenue monotonicity*, which requires that the auctioneer earn no more revenue by dropping goods from the endowments. Although no mechanism satisfies goods revenue monotonicity together with strategy-proofness, efficiency, and participation even in the domain of substitute valuations, we find a restricted domain called *per-capita goods-bidder submodular domain* in which there exists a goods revenue monotone mechanism satisfying the above three conditions. The restriction is likely to be met when bidders' valuations are similar. Finally, we provide a relation to the monopoly theory, and argue that per-capita goods-bidder submodularily is independent of the standard elasticity argument.

1 Introduction

A combinatorial auction is an auction in which the auctioneer attempts to sell combinations of multiple objects. In combinatorial auctions, large possibilities of combinations of objects aggravate a difficulty in designing a suitable mechanism. In the literature, the

*We are very grateful to Professor Akira Okada for his guidance and encouragement. We also thank Shigehiro Serizawa, Taiki Todo, Takuro Yamashita, and participants in seminars at Hitotsubashi University, SWET at Hokkaido University, 11th Meeting of Society for Social Choice and Welfare, the 2012 Asian meeting of the Econometric Society, and the 19th Decentralization Conference in Japan for their helpful comments. Muto gratefully acknowledges support from the Spanish Ministry of Science and Innovation through grant "Consolidated Group-C" ECO2008-04756 and FEDER. Shirata gratefully acknowledges support from the MEXT of Japan through the grant Global COE Hi-Stat and JSPS KAKENHI 24830004.

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Vickrey-Clarke-Groves (VCG) mechanism is one of the most widely accepted as a desirable candidate possessing nice properties. In the VCG mechanism, no bidder obtains a negative payoff, no bidder has an incentive to misreport his true preference, and the outcome is always efficient in terms of the reported valuations.

On the other hand, Milgrom (2004, Chapter 2) points out that the VCG mechanism has several weaknesses. One of the most important is a *monotonicity* problem that the auctioneer's revenue is non-monotone with respect to the set of bidders. It may cause the auctioneer to disqualify bidders for an increase of the revenue. Rastegari et al. (2011) show that this bidder revenue monotonicity problem arises in the domain of single-minded valuations.[1] In their seminal paper, Ausubel and Milgrom (2002) show that the above monotonicity problem disappears when goods are substitutes for all bidders.[2]

Our finding is that, even if goods are substitutes, the VCG mechanism suffers another monotonicity problem, *goods revenue monotonicity*. In words, goods revenue monotonicity requires that the auctioneer earn no more revenue by dropping goods. This property is desirable since if violated, an incentive would arise for the auctioneer to drop goods, leading to misallocation. This problem is generally serious as the price theory predicts that a monopolist may under-supply. Indeed, implementing goods revenue monotone outcomes together with strategy-proofness, efficiency, and participation is impossible in the substitutes domain.

Nevertheless, we find a new restricted domain, *per-capita goods-bidder submodular domain*. This condition requires that the social welfare per capita be submodular with respect to goods and bidders. In this domain, we show that the VCG mechanism is goods revenue monotone. The per-capita goods-bidder submodular domain is nonempty, and likely to hold when all bidders' valuations are similar. We also demonstrate by examples that if this condition holds, the valuations are relatively close to the additive. This contrasts bidder revenue monotonicity which is always satisfied in the substitutes domain.

[1] A bidder's valuation is called single-minded if he demands only a target bundle of goods. See Section 4 for more details.

[2] Furthermore, they show that if goods are substitutes for all bidders, the VCG mechanism satisfies the false-name proofness, and the VCG outcome is always in the core.

Our goods revenue monotonicity is related to the standard monopoly theory. To discuss this relation, we investigate another domain in which all goods are homogeneous for all bidders, and thus the auction can be viewed as a multi-unit auction. We show, thanks to the homogeneity, that the VCG mechanism in the multi-unit auction is goods revenue monotone if the marginal value elasticity of demand is higher than or equal to one. This parallels the fact in the monopoly theory that the monopolist's revenue is monotonically increasing with respect to the quantity if the price elasticity of demand is higher than or equal to one.

Per-capita goods-bidder submodularity also works in this environment. We argue however that it is logically independent of the above elasticity in the market with homogeneous goods. Since cross elasticity matters in the market with heterogeneous goods, it is hard to generalize the elasticity argument to the combinatorial auctions. Hence our per-capita goods-bidder submodularity is a new condition that guarantees no under-supply in the context of combinatorial auctions.

The rest of the paper is organized as follows. Section 2.1 defines a combinatorial auction mechanism, and introduces goods revenue monotonicity. In Section 2.2 we show the impossibility. In Section 3.1 we show a possibility in the per-capita goods-bidder submodular domain. Section 3.2 studies a multi-unit auction with homogeneous goods. Section 4 discusses relations to the literature and bidder revenue monotonicity. Section 5 concludes.

2 Preliminaries

2.1 The model

An auctioneer faces a problem of selling multiple goods to bidders. Let $\mathcal{G}$ be a *universal* set of indivisible goods which are potentially to be sold. We assume that $\mathcal{G}$ contains at least two goods. We analyze problems for multiple sets of goods contained in $\mathcal{G}$, as an auction mechanism generally works with distinct sets of goods. We denote the set of goods actually sold in the auction by $G \subseteq \mathcal{G}$. We also denote the finite set of bidders who actually participate in the auction by N with $|N| \geq 2$. Let $X_G = \{(x_1, \ldots, x_{|N|}) \subseteq G^{|N|} \mid x_i \cap x_j = \emptyset$ for all $i, j \in N$ $(i \neq j)\}$ be the set of feasible allocations when the set of goods to be sold is $G \subseteq \mathcal{G}$. An auction mechanism allocates goods to bidders while we allow the auctioneer to retain some goods unsold.

Given a set of goods G, each bidder i has a private valuation function $v_i : x_i \mapsto v_i(x_i) \in \mathbb{R}$ over all bundles of goods $x_i \subseteq G$. Let V_i be the set of valuation functions of bidder i, and $\mathcal{V} = \prod_{i \in N} V_i$ be the set of valuation profiles of all bidders.[3] We always assume *free-disposal*; $x_i \subseteq x_i'$ implies $v_i(x_i) \leq v_i(x_i')$ for all $i \in N$ and all $v_i \in V_i$. For normalization, each bidder i values no goods at zero, i.e. $v_i(\emptyset) = 0$ for all $i \in N$ and all $v_i \in V_i$. All bidders have quasi-linear payoff functions. If bidder i obtains a bundle of goods $x_i \subseteq G$ in exchange of payment $t_i \in \mathbb{R}$, his payoff is $v_i(x_i) - t_i$.

Let us introduce a standard notion of substitutes. Suppose that G is finite and each good $g \in G$ is sold separately at price p_g. Then, the demand correspondence for each bidder i at price vector $p = (p_g)_{g \in G}$ is defined by

$$D_i(p) = \underset{x_i \subseteq G}{\operatorname{argmax}} \Big(v_i(x_i) - \sum_{g \in x_i} p_g \Big).$$

Definition 1. Goods are *substitutes* for bidder i if for any p, p' with $p \leq p'$ and any $x_i \in D_i(p)$, there exists an $x_i' \in D_i(p')$ such that $\{g \mid g \in x_i, p_g = p_g'\} \subseteq x'$.[4]

Note that since any bidder has a quasi-linear payoff function with no budget constraints, this condition is equivalent to the gross substitutes condition defined by Kelso and Crawford (1982). We denote the set of all substitute valuations by V_{Sub}.

We consider a *deterministic direct combinatorial auction mechanism* (*CA mechanism* for short) $M = (M_G)_{G \subseteq \mathcal{G}}$.By the revelation principle, we can focus on CA mechanisms without loss of generality. Each $M_G = (x(G), t(G))$ is a CA mechanism for G in which each

[3] We do not write dependency of V_i on G explicitly. We note that V_i can be regarded as of a set of valuation functions on $\mathcal{G}$ restricted to G.

[4] We can generalize the definition for the infinitely many goods straightforwardly.

bidder i simultaneously bids a valuation function $\hat{v}_i \in V_i$, and the goods and monetary transfers are allocated according to the reported valuation profile $\hat{v} = (\hat{v}_1, \ldots \hat{v}_{|N|})$. For each $\hat{v} \in \mathcal{V}$, $x(G)(\hat{v}) = \big(x_1(G)(\hat{v}), \ldots, x_{|N|}(G)(\hat{v})\big) \in X_G$ is the allocation function, and $t(G)(\hat{v}) = \big(t_1(G)(\hat{v}), \ldots, t_{|N|}(G)(\hat{v})\big) \in \mathbb{R}^{|N|}$ is the payment function. In what follows, we denote $x(G)(\hat{v})$ and $t(G)(\hat{v})$ as $x(\hat{v}; G)$ and $t(\hat{v}; G)$, respectively. If bidders report $\hat{v} \in \mathcal{V}$, then each bidder i with true valuation v_i obtains payoff $v_i(x_i(\hat{v}; G)) - t_i(\hat{v}; G)$.

We next define properties of CA mechanisms.

Definition 2. (i) CA mechanism M satisfies *participation* if a payment is zero for any bidder obtaining payoff zero. That is, for all $(\hat{v}_i, \hat{v}_{-i}) \in \mathcal{V}$, and all $G \subseteq \mathcal{G}$,

$$\text{if } \hat{v}_i(x_i(\hat{v}_i, \hat{v}_{-i}; G)) = 0 \text{ then } t_i(\hat{v}_i, \hat{v}_{-i}; G) = 0.$$

(ii) CA mechanism M is *strategy-proof* if no bidder has an incentive to misreport his valuations in $M_{G,N} = (x, t)$. That is, for all $v_i \in V_i$, $(\hat{v}_i, \hat{v}_{-i}) \in \mathcal{V}$, and all $G \subseteq \mathcal{G}$,

$$v_i(x_i(v_i, \hat{v}_{-i}; G)) - t_i(v_i, \hat{v}_{-i}; G) \geq v_i(x_i(\hat{v}_i, \hat{v}_{-i}; G)) - t_i(\hat{v}_i, \hat{v}_{-i}; G).$$

(iii) CA mechanism M is *efficient* if for all $v \in \mathcal{V}$ and all $G \subseteq \mathcal{G}$,

$$x(v; G) \in \operatorname*{argmax}_{y \in X_G} \sum_{i \in N} v_i(y_i).$$

We assume that this is well-defined for all $v \in \mathcal{V}$.[5]

Now, we introduce a notion of *goods revenue monotonicity*, which will play the central role in this paper. A CA mechanism is goods revenue monotone if the auctioneer earns no more revenue by dropping any goods. Formally, we define this concept as follows:

Definition 3. A strategy-proof CA mechanism M is *goods revenue monotone* if for all $\hat{v} \in \mathcal{V}$

[5] This maximum exists under a weak condition. A sufficient condition is that each v_i is upper semi-continuous and the domain is compact in a suitable topology (see Holmström (1979, footnote 6)).

and all sets of goods G, G' with $G' \subseteq G \subseteq \mathcal{G}$,

$$\sum_{i \in N} t_i(\hat{v}; G) \geq \sum_{i \in N} t'_i(\hat{v}; G').$$

We denote welfare for coalition $S \subseteq N$ with set of goods $G \subseteq \mathcal{G}$ at valuation profile $v \in \mathcal{V}$ by $w(v; G, S) = \max_{x \in X_G} \sum_{i \in S} v_i(x_i)$.

A natural candidate satisfying the desirable properties defined in Definition 2 is the Vickrey-Clarke-Groves (VCG) mechanism. The VCG mechanism $(M_G^{\text{VCG}})_{G \subseteq \mathcal{G}}$ is a CA mechanism in which $x^{\text{VCG}}(\hat{v}; G) \in \text{argmax}_{y \in X_G} \sum_{i \in N} \hat{v}_i(y_i)$ and

$$t_i^{\text{VCG}}(\hat{v}; G) = w(\hat{v}; G, N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} \hat{v}_j(x_j^{\text{VCG}}(\hat{v}; G))$$

for each $i \in N$ and $G \subseteq \mathcal{G}$. The VCG mechanism obviously satisfies participation, strategy-proofness, and efficiency for any environment. By strategy-proofness, the revenue of the auctioneer $\sum_{i \in N} t_i^{\text{VCG}}(v; G)$ for each G is computed with respect to the bidders' true valuations. The following example demonstrates that the VCG mechanism is not goods revenue monotone in some environment.

Example 1. Let $\mathcal{G} = \{a, b\}$ be the set of two goods. There are two bidders 1, 2. For each bidder $i = 1, 2$, the valuation v_i is given as follows; $v_1(\{a\}) = 7$, $v_1(\{b\}) = 3$, $v_1(\{a, b\}) = 8$, $v_2(\{a\}) = 3$, $v_2(\{b\}) = 7$, and $v_2(\{a, b\}) = 8$. The outcome of the VCG mechanism is allocating a to 1 and b to 2, and payment 1 by both, i.e. $x^{\text{VCG}}(v; \{a, b\}) = (\{a\}, \{b\})$ and $t^{\text{VCG}}(v; \{a, b\}) = (1, 1)$. Hence, the revenue is $1 + 1 = 2$.

On the other hand, if the auctioneer sells only good b, the VCG outcome is allocating b to 1 with payment 3. The auctioneer's revenue is 3, which exceeds the revenue obtained from selling both goods a and b. Hence the VCG mechanism is not goods revenue monotone if $\mathcal{V}$ contains the above valuation profile. $\Diamond$

2.2 An impossibility

Goods revenue monotonicity, as one may suppose, is a strong requirement in general. In this section, we prove an impossibility in any domain including all "single-unit demand" valuations. We say that a valuation function v_i is *single-unit demand* if for all $x_i \subseteq G$, $v_i(x_i) = \sup_{g \in x_i} v_i(\{g\})$. Let V_{SUD} be the set of valuation functions with a single-unit demand. This is an extreme case of substitutes since obtaining any combination of two bundles of goods causes no increase in valuations.

Proposition 1. *Suppose that $V_i \supseteq V_{\text{SUD}}$ for all $i \in N$. Then, any CA mechanism M that satisfies efficiency, strategy-proofness, and participation is not goods revenue monotone.*

The proof is given in the Appendix. Since $V_{\text{Sub}} \supseteq V_{\text{SUD}}$, we obtain the following corollary:

Corollary 2. *Suppose that $V_i \supseteq V_{\text{Sub}}$ is for all $i \in N$. Then, any CA mechanism M that satisfies efficiency, strategy-proofness, and participation is not goods revenue monotone.*

3 Main results

3.1 A possibility result: Per-capita goods-bidder submodular domain

We showed an impossibility in Proposition 1 that if $V_i \supseteq V_{\text{SUD}}$ for all i, then no mechanism satisfies goods revenue monotonicity together with efficiency, strategy-proofness, and participation. This section examines the existence of such mechanisms on a restricted domain of valuations.

We consider a restricted domain satisfying the following property: Let $\tilde{w}(v; G, S) = \frac{1}{|S|} w(v; G, S)$ be the welfare per capita for coalition S with $\emptyset \neq S \subseteq N$.

Definition 4. A profile of valuation functions v is *per-capita goods-bidder submodular* if for all $G' \subseteq G \subseteq \mathcal{G}$ and all $i \in N$,

$$\tilde{w}(v; G, N) - \tilde{w}(v_{-i}; G, N \setminus \{i\}) \leq \tilde{w}(v; G', N) - \tilde{w}(v_{-i}; G', N \setminus \{i\}).$$

A set of all per-capita goods-bidder submodular valuation profiles is called the *per-capita goods-bidder submodular domain*.[6]

This domain has a nonempty interior; for example, suppose that $v = (v_1, \dots, v_n)$ is such that v_i is additive (i.e., satisfies $v_i(G) = \sum_{g \in G} v_i(\{g\})$) for all $i \in N$, and $v_i = v_j$ for all $i, j \in N$. Then the total welfare $w(v; G, N) = v_i(G)$ is independent of N, and thus $\tilde{w}(v; G, N) - \tilde{w}(v_{-i}; G, N \setminus \{i\}) = -\frac{1}{|N|(|N|-1)} w(v; G, N)$ is nonincreasing in G. This monotonicity directly implies per-capita goods-bidder submodularity. If v_i is strictly increasing in G, the above monotonicity is also strict. In such a case, per-capita goods-bidder submodularity holds with strict inequalities for this v, and also for any valuation profiles close to v.

Theorem 3. *Suppose that $\mathcal{V}$ is the per-capita goods-bidder submodular domain. Then, the VCG mechanism satisfies goods revenue monotonicity.*

Proof. Fix G. In the VCG mechanism, the payment of each bidder $i \in N$ is $t_i^{\mathrm{VCG}}(v; G) = w(v_{-i}; G, N \setminus \{i\}) - [w(v; G, N) - v_i(x_i^{\mathrm{VCG}}(v; G))]$.

Then, the revenue is

$$\begin{aligned}
\sum_{i \in N} t_i^{\mathrm{VCG}}(v; G) &= \sum_{i \in N} w(v_{-i}; G, N \setminus \{i\}) - |N| w(v; G, N) + \sum_{i \in N} v_i(x_i^{\mathrm{VCG}}(v; G)) \\
&= \sum_{i \in N} w(v_{-i}; G, N \setminus \{i\}) - (|N| - 1) w(v; G, N) \\
&= \sum_{i \in N} (|N| - 1) \tilde{w}(v_{-i}; G, N \setminus \{i\}) - (|N| - 1)|N| \tilde{w}(v; G, N) \\
&= \sum_{i \in N} (|N| - 1) \big(\tilde{w}(v_{-i}; G, N \setminus \{i\}) - \tilde{w}(v; G, N) \big).
\end{aligned}$$

This is nondecreasing in G since, by per-capita goods-bidder submodularity, $\tilde{w}(v; G, N \setminus \{i\}) - \tilde{w}(v; G, N)$ is nondecreasing in G for each $i \in N$.

It is obvious that this argument holds for all $G \subseteq \mathcal{G}$. Hence, the auctioneer cannot earn more revenue by dropping goods. $\square$

[6]This terminology follows "bidder submodularity" (Ausubel and Milgrom (2002)), submodularity of the welfare function with respect to the set of bidders. We will discuss the relation in Section 4.

We now discuss properties of the per-capita goods-bidder submodular domain.

For a positive integer K, a set of bidders N, and $v \in \mathcal{V}$, let $N(K) = \{(i,k) \mid i \in N, k = 1, \ldots, K\}$ be a set of $K|N|$ bidders, and v^K be the K-replica valuation profile with $v_{(i,k)} = v_i$ for each $(i,k) \in N(K)$.

Proposition 4. *Suppose that $|G|$ is finite.*[7] *For any valuation profile $v \in \mathcal{V}$, there exists $\bar{K}$ such that for all $K \geq \bar{K}$, the K-replica valuation profile v^K satisfies per-capita goods-bidder submodularity.*

Proof. Let $\bar{K} = |G| + 1$. Since for any $K \geq \bar{K}$ and any $i \in N$, there exists $k = 1, \ldots, K$ such that bidder (i,k) obtains no goods in an efficient allocation in the K-replica environment, dropping any bidder (i,k) does not influence the resulting efficient allocation. Therefore, for any $K \geq \bar{K}$, any $(i,k) \in N(K)$, and any $G' \subseteq G$, we have $w(v; G', N(K)) = w(v_{-(i,k)}; G', N(K) \setminus \{(i,k)\})$. Since $w(v; G', N(K)) \geq 0$ is nondecreasing in $G' \subseteq G$, $\tilde{w}(v; G', N(K)) - \tilde{w}(v_{-(i,k)}; G', N(K) \setminus \{(i,k)\}) = -\frac{1}{(K|N|-1)K|N|} w(v; G', N(K))$ must be nonincreasing in G'. Hence per-capita goods-bidder submodularity is satisfied. $\square$

Proposition 4 implies that if the set of goods is finite, for any valuation function v_i, a profile $(v_i, v_i, \ldots, v_i)$ satisfies per-capita goods-bidder submodularity whenever the set of bidders is sufficiently large. This roughly suggests that per-capita goods-bidder submodularily holds if every bidder's valuation is close to each other. Conversely, we can show that if some bidder's valuation is very far from the others', per-capita goods-bidder submodularily may fail.

Proposition 5. *Let $\mathcal{V}$ be any domain of valuations for a set of bidders N. For any $v \in \mathcal{V}$ and any $i \in N$, there exists a valuation function $\tilde{v}_i \in \mathbb{R}_+^{2^G}$ such that $(\tilde{v}_i, v_{-i})$ is not per-capita goods-bidder submodular.*

Proof. Let us fix $v \in \mathcal{V}$ and $i \in N$. For a good $a \in G$, take a valuation $\tilde{v}_i$ such that $\tilde{v}_i(x) \geq \frac{|N|}{|N|-1} w(v_{-i}; G, N \setminus \{i\}) + 1$ if $a \in x$, and $\tilde{v}_i(x) = 0$ otherwise. Since $\tilde{v}_i(\{a\}) >$

[7]For infinite goods, there is the following counter-example. Suppose that G is countably infinite and take any positive integer K. Then profile $(v_1, \ldots, v_{K|N|})$ with $v_i(x) = 1$ for any $x \neq \emptyset$ and any i satisfies $\tilde{w}(v; G) - \tilde{w}(v; G, N \setminus \{i\}) = 0$, but if we take G' with $|G'| < |N|$, $\tilde{w}(v; G, N', N) - \tilde{w}(v; G, N', N \setminus \{i\}) < 0$.

$w(v_{-i}; G, N \setminus \{i\})$, bidder i must obtain a in any efficient allocation under $(\tilde{v}_i, v_{-i})$. Then,

$$\begin{aligned}
&\tilde{w}(\tilde{v}_i, v_{-i}; G, N) - \tilde{w}(\tilde{v}_i, v_{-i}; G \setminus \{a\}, N) - \tilde{w}(v_{-i}; G, N \setminus \{i\}) + \tilde{w}(v_{-i}; G \setminus \{a\}, N \setminus \{i\}) \\
&\quad = \frac{\tilde{v}_i(\{a\}) + w(v_{-i}; G \setminus \{a\}, N \setminus \{i\})}{|N|} - \frac{w(v_{-i}; G \setminus \{a\}, N \setminus \{i\})}{|N|} \\
&\qquad\qquad - \frac{w(v_{-i}; G, N \setminus \{i\})}{|N| - 1} + \frac{w(v_{-i}; G \setminus \{a\}, N \setminus \{i\})}{|N| - 1} \\
&\quad \geq \frac{\tilde{v}_i(\{a\})}{|N|} - \frac{w(v_{-i}; G, N \setminus \{i\})}{|N| - 1} \\
&\quad = \frac{1}{|N|}\Big(\tilde{v}_i(\{a\}) - \frac{|N|}{|N| - 1} w(v_{-i}; G, N \setminus \{i\})\Big) \\
&\quad \geq \frac{1}{|N|} > 0.
\end{aligned}$$

Hence per-capita goods-bidder submodularity fails. □

Per-capita goods-bidder submodularily is a condition of a profile of valuations, not of an individual valuation function. These two propositions suggest that it is not fit to discuss per-capita goods-bidder submodularily with respect to a valuation of an individual bidder, since closeness among valuations is important.

We conclude this section by presenting a simple example to show that the per-capita goods-bidder submodular domain has a nonempty intersection with the domain of substitutes.

Example 2. Consider an environment with two bidders and two goods. Let $N = \{1, 2\}$ and $\mathcal{G} = \{a, b\}$. We assume substitutes, namely, $v_i(\{a\}) + v_i(\{b\}) \geq v_i(\{a, b\})$ for all $v_i \in V_i$ and $i \in N$. The valuation v_i is additive if the equality holds. By the definition, the free-disposal holds if and only if $v_i(\{a, b\}) \geq \max\big\{v_i(\{a\}), v_i(\{b\})\big\}$ for all $v_i \in V_i$ and $i \in N$. The valuation v_i is single-unit demand if the equality holds.

For simplicity we focus on the domain $\mathcal{V}$ in which bidder 1 values $\{a\}$ no lower than the opponent, while bidder 2 values $\{b\}$ no lower than the opponent, i.e. $v_1(\{a\}) \geq v_2(\{a\})$ and $v_2(\{b\}) \geq v_1(\{b\})$. In this case, $(x_1, x_2) = (\{a\}, \{b\})$ is an efficient allocation for any $v \in \mathcal{V}$. Then a straight-forward computation proves that per-capita goods-bidder submodularity holds true if $\tilde{v}_i \leq v_i(\{a,b\}) \leq v_i(\{a\}) + v_i(\{b\})$ for all $i \in N$ where

$$\tilde{v}_1 = \max\Big\{v_1(\{b\}) + \frac{1}{2}v_1(\{a\}), v_1(\{a\}) + \frac{1}{2}v_2(\{b\})\Big\},$$
$$\tilde{v}_2 = \max\Big\{v_2(\{b\}) + \frac{1}{2}v_1(\{b\}), v_2(\{a\}) + \frac{1}{2}v_2(\{b\})\Big\}.$$

Such a domain actually exists whenever $2v_2(\{a\}) \geq v_1(\{a\})$ and $2v_1(\{b\}) \geq v_2(\{b\})$. Since one can easily show that $\tilde{v}_i > \max\{v_i(\{a\}), v_i(\{b\})\}$ if each bidder has a positive valuation to either good, the single-unit demand domain cannot be included as we showed in Proposition 1. To sum up, the valuations are not close to the single-unit demand domain, i.e., close to the additive if per-capita goods-bidder submodularity is satisfied. $\Diamond$

3.2 Multi-unit auctions—A relation to the monopoly theory

To connect goods revenue monotonicity with the standard monopoly theory, this section studies a multi-unit auction where all goods are homogeneous. Of course, per-capita goods-bidder submodularity works also in this environment. We furthermore introduce another condition that ensures goods revenue monotonicity. This condition is related to a well-known argument in the monopoly theory that the monopolist's revenue is increasing in the quantity if the price elasticity of demand is larger than one. We argue that the above condition is independent of per-capita goods-bidder submodularity becuase the elasticity argument crucially relies on the homogeneity assumption. Therefore per-capita goods-bidder submodularity is a new requirement of the domain of valuation functions.

Let $\mathcal{Q}$ be the potential amount of the homogeneous goods, and Q be the total quantity of the homogeneous goods to be sold ($Q \leq \mathcal{Q}$). To convey an intuition we assume divisible goods, and discuss whether the auctioneer's revenue is nondecreasing with respect to the quantity Q of the goods.

Every bidder i's valuation function depends only on the quantity of goods allocated to i, denoted by q_i. Let $v_i(q_i)$ be the valuation when i obtains $q_i \in [0, \mathcal{Q}]$.[8] We assume

[8]One can accommodate this notion to the original model by letting $\mathcal{G} = [0, \mathcal{Q}]$, and the domain V_i satisfy

that for all $i \in N$ and all $v_i \in V_i$, valuation v_i is twice continuously differentiable, $v_i' \geq 0$, $v_i'' < 0$, and $\lim_{q_i \to 0} v_i'(q_i) = \infty$. Let $q^*(v; Q, S)$ be an efficient allocation in a coalition S with $\emptyset \neq S \subseteq N$ for a valuation profile $v \in \mathcal{V}$ such that $q_i^{\mathrm{VCG}}(v; Q) = q_i^*(v; Q, N)$ for all $v \in \mathcal{V}$ and all $Q \leq \mathcal{Q}$. By efficiency, $\sum_{i \in S} q_i^*(v; Q, S) = Q$, and there is a common marginal value $p(v; Q, S)$ such that $p(v; Q, S) = v_i'(q_i^*(v; Q, S))$ for all $i \in S$.

Denote by $e(v; Q, S) = -\frac{p/Q}{\partial p / \partial Q} \geq 0$, the marginal value elasticity of aggregated demand of coalition S with $\emptyset \neq S \subseteq N$. Then, we obtain the following result:

Proposition 6. *In the multi-unit auction, if $e(v_{-i}; Q, N \setminus \{i\}) \geq 1$ for all $i \in N$, $v \in \mathcal{V}$, and all $Q \in [0, \mathcal{Q}]$, then the VCG mechanism satisfies goods revenue monotonicity.*

Proof. Since $w(v; G, S) = \sum_{i \in S} v_i(q_i^*(v; Q, S))$, the VCG payment of bidder i is $t_i^{\mathrm{VCG}}(v; Q) = \sum_{j \neq i} \big(v_j(q_j^*(v_{-i}; Q, N \setminus \{i\})) - v_j(q_j^*(v_i; Q, N)) \big)$. Then, for all $i \in N$,

$$
\begin{aligned}
\frac{\partial}{\partial Q} t_i^{\mathrm{VCG}}(v; Q) &= \frac{\partial}{\partial Q} \sum_{j \neq i} v_j(q_j^*(v_{-i}; Q, N \setminus \{i\})) - \frac{\partial}{\partial Q} \sum_{j \neq i} v_j(q_j^*(v; Q, N)) \\
&= \sum_{j \neq i} p(v_{-i}; Q, N \setminus \{i\}) \frac{\partial}{\partial Q} q_j^*(v_{-i}; Q, N \setminus \{i\}) - \sum_{j \neq i} p(v; Q, N) \frac{\partial}{\partial Q} q_j^*(v; Q, N) \\
&= p(v_{-i}; Q, N \setminus \{i\}) \frac{\partial}{\partial Q} Q - p(v; Q, N) \frac{\partial}{\partial Q} (Q - q_i^*(v; Q, N)) \\
&= p(v_{-i}; Q, N \setminus \{i\}) - p(v; Q, N) \Big(1 - \frac{\partial q_i^*(v; Q, N)}{\partial Q} \Big).
\end{aligned}
$$

Summing these up with respect to $i \in N$ yields

$$
\begin{aligned}
\frac{\partial}{\partial Q} \sum_{i \in N} t_i^{\mathrm{VCG}}(v; Q) &= \sum_{i \in N} p(v_{-i}; Q, N \setminus \{i\}) - p(v; Q, N) \sum_{i \in N} \Big(1 - \frac{\partial q_i^*(v; Q, N)}{\partial Q} \Big) \\
&= \sum_{i \in N} p(v_{-i}; Q, N \setminus \{i\}) - (|N| - 1) p(v; Q, N). \qquad (1)
\end{aligned}
$$

Since $e(v_{-i}; Q, N \setminus \{i\}) \geq 1$, the function pQ satisfies that for all j and all Q,

$$
\frac{\partial}{\partial Q} p(v_{-i}; Q, N \setminus \{i\}) Q = p(v_{-i}; Q, N \setminus \{i\}) + (\partial p / \partial Q) \cdot Q
$$

$\tilde{v}_i(x_i) = \tilde{v}_i(x_i')$ for all $\tilde{v}_i \in V_i$ and all $x_i, x_i' \subseteq G \subseteq \mathcal{G}$ with an equal Lebesgue measure q_i.

$$= p(v_{-i}; Q, N \setminus \{i\})\Big(1 - \frac{1}{e(v_{-i}; Q, N \setminus \{i\})}\Big) \geq 0.$$

Fix i. Since $q_j^*(v; Q, N) = q_j^*(v_{-i}; Q - q_i^*(v; Q, N), N \setminus \{i\})$ for all $j \neq i$,

$$\begin{aligned} p(v; Q, N) &= v_j'(q_j^*(v; Q, N)) = v_j'(q_j^*(Q - q_i^*(v_{-i}; Q, N), N \setminus \{i\})) \\ &= p(Q - q_i^*(v; Q, N), N \setminus \{i\}). \end{aligned}$$

This implies that by $\partial(pQ)/\partial Q \geq 0$,

$$\begin{aligned} p(v; Q, N)(Q - q_i^*(v; Q, N)) &= p(v_{-i}; Q - q_i^*(v; Q, N), N \setminus \{i\})(Q - q_i^*(v; Q, N)) \\ &\leq p(v_{-i}; Q, N \setminus \{i\})Q. \end{aligned} \tag{2}$$

Summing these up with respect to all $i \in N$ yields

$$(|N| - 1)p(v; Q, N) \leq \sum_{i \in N} p(v_{-i}; Q, N \setminus \{i\}). \tag{3}$$

By (1) and (3), $\frac{\partial}{\partial Q} \sum_{i \in N} t_i^{\mathrm{VCG}}(v; Q) \geq 0$. This implies goods revenue monotonicity. □

An immediate sufficient condition for the above is that each bidder's elasticity is larger than or equal to one. Formally, let $e_i(v_i; q_i) = -\frac{v_i'(q_i)/q_i}{v_i''(q_i)}$ be the marginal value elasticity of demand of buyer i. If $e_i(v_i; q_i) \geq 1$ for all $i \in N$ and all $q_i \in [0, \mathcal{Q}]$, then $e(v; Q, S) \geq 1$ for any coalition S. Then we obtain the following:

Corollary 7. *In the multi-unit auction, if $e_i(v_i; q_i) \geq 1$ for all $i \in N$, all $v_i \in V_i$, and all $q_i \in [0, \mathcal{Q}]$, then the VCG mechanism satisfies goods revenue monotonicity.*

This corollary claims that if the valuation function of each bidder is a concave function which is close to linear, then goods revenue monotonicity is satisfied. In contrast to the per-capita goods-bidder submodular valuations, the elasticity of aggregated demand can be decomposed to the each individual bidder's elasticity.

To provide an economic intuition of Proposition 6, suppose that there are many bidders, and that the price remains almost the same if a single bidder i drops out of the auction. Then, we can interpret $p(v; Q, N)$ as the inverse aggregated demand function that the monopolist (auctioneer) faces, and $p(v; Q, N)Q$ as the monopolist's revenue function. The monopoly theory shows that if the aggregated demand is elastic such that the price elasticity of aggregated demand $e(v; Q, N) \geq 1$, then the revenue is monotonically non-decreasing in Q, i.e. $\frac{\partial p(v;Q,N)Q}{\partial Q} \geq 0$. Hence, the monopolist has no incentive to undersupply, and the goods revenue monotonicity is satisfied.

Remark 1. While they are related, the domains of per-capita goods-bidder submodular valuations and of the elastic demands are logically independent. In this environment with homogeneous goods, a similar computation to that in the proof of Proposition 6 shows that per-capita goods-bidder submodularity holds if and only if

$$p(v; Q, N)\frac{|N| - 1}{|N|}Q \leq p(v; Q, N \setminus \{i\})Q \tag{4}$$

for all Q and all $i \in N$. Obviously this inequality (4) is logically independent of (2). Therefore per-capita goods-bidder submodularity is not directly related to the standard argument of elasticity. We note that the VCG price, which may not be uniform among bidders, is different from the monopoly price, and elasticity of the demand is not implied by goods revenue monotonicity. $\Diamond$

Remark 2. One might wonder if it is possible to generalize the elasticity argument to the general environment as in Sections 2 and 3. There is, however, no natural generalization of Proposition 6 in the environment with heterogeneous goods because cross price elasticity of demand matters when discussing the revenue monotonicity in quantity. $\Diamond$

4 Relations to the literature

We review existing papers and discuss properties related to goods revenue monotonicity.

Goods revenue monotonicity is related to a well-known property of *bidder revenue monotonicity*: A CA mechanism is bidder revenue monotone if the auctioneer earns no more revenue by excluding some bidders. Ausubel and Milgrom (2002) show that the VCG mechanism is bidder revenue monotone if and only if goods are substitutes, and goods are substitutes if and only if the welfare function is submodular with respect to the set of bidders whenever there are four or more bidders. By Theorem 3, both goods revenue monotonicity and bidder revenue monotonicity are ensured in the intersection of the per-capita goods-bidder submodular domain and the bidder submodular domain with four or more bidders, and this intersection is nonempty as we saw in Section 3.1. In contrast to bidder revenue monotonicity, goods revenue monotonicity often fails for substitute valuations by Proposition 1. This implies that in the substitutes domain, goods revenue monotonicity is stronger than bidder revenue monotonicity.

There are other papers that discuss bidder revenue monoconicity: Rastegari et al. (2011) show that in the single-minded domain[9] no mechanism satisfies bidder revenue monotonicity together with participation, consumer sovereignty, and a property that any good should be allocated to a bidder who positively values it. They provide inefficient mechanisms that satisfy bidder revenue monotonicity.[10] Todo et al. (2009) characterize strategy-proof and bidder revenue monotone auction mechanisms in a general domain of valuations. They also discuss relations between bidder revenue monotonicity and false-name-proofness. Lamy (2010) shows that there is no bidder-optimal core selecting auctions which satisfy bidder revenue monotonicity if there are more than two goods for sale, while there exists one if there are only two goods.

Beck and Ott (2009) introduce a condition stronger than both bidder and goods revenue monotonicities; the revenue should not decrease if bidders report weakly higher valuations for all bundles. They show a necessary condition of this stronger monotonicity, and propose core-selecting mechanisms satisfying their monotonicity condition. Their

[9]A valuation function v_i is *single-minded* if there is a particular bundle of goods $x_i \subseteq G$ such that i wants only x_i. That is, $v_i(y_i) = v_i(x_i)$ if $x_i \subseteq y_i \subseteq G$, and $v_i(y_i) = 0$ otherwise. Goods are not substitutes if the targeted bundle contains two or more goods.

[10]Rastegari et al. (2011, Section 4.2) consider goods revenue monotonicity in the single-minded domain. Their impossibility is, however, immediately followed from that with bidder revenue monotonicity since dropping a good g is equivalent to disqualifying every single-minded bidder with a target bundle including g.

monotonicity is much stronger than goods revenue monotonicity. In fact, their necessary condition is so strong that it rules out most non-additive valuations. In contrast, our Theorem 3 provides a sufficient condition which is satisfied in a domain containing non-additive valuations with a nonempty interior.

5 Conclusion

We found a new monotonicity problem—a problem of *goods revenue monotonicity* in combinatorial auctions. The monotonicity requires that the auctioneer earn no higher revenue by dropping goods from the set of endowments. We first demonstrated that in the domain containing valuations with a single-unit demand, there exists no mechanism satisfying strategy-proofness, efficiency, participation, and goods revenue monotonicity. This suggests that combinatorial auction design can be seriously affected by seller's manipulation of the set of objects for sale, even if goods are supposed to be substitutes for all bidders.

Nevertheless, we found a possibility in a restricted domain of valuations satisfying per-capita goods-bidder submodularity. This condition means that the welfare per capita is submodular with respect to the set of bidders and goods, and is likely to hold when bidders' valuations are similar to each other and close to the additive.

We further investigate a relation to the monopoly theory by discussing the multi-unit auction with homogeneous goods. In the multi-unit auction, the VCG mechanism is goods revenue monotone if the marginal value elasticity of the aggregated demand is larger than or equal to one. Such elasticity is meaningful thanks to the homogeneity. We showed, however, that this elasticity is independent of per-capita goods-bidder submodularity. This suggests that in the context of combinatorial auctions, per-capita goods-bidder submodularity is a new condition under which the monopolist may not under-supply.

Given our results, further investigations will be necessary to implement desirable allocations in a larger domain; e.g. a domain containing all subsutitute valuations. Since a combinatorial auction which is not goods revenue monotone would mis-allocate endowments of the seller, a social planner should design goods revenue monotone auctions to achieve her purpose, weakening some other desirable properties.

We propose two directions that would be interesting. One is to design a second-best goods revenue monotone auction in terms of efficiency, while maintaining strategy-proofness and participation. The other is to design a goods revenue monotone auction mechanism satisfying efficiency, participation, and Bayesian Nash incentive compatibility instead of strategy-proofness. These issues are left for future research.

Appendix: Proof of Proposition 1

The appendix proves Proposition 1. When $\mathcal{G}$ is finite or countably infinite, the revenue equivalence shown by Chung and Olszewski (2007) enables us to skip Step 1 of the proof below. In a general environment with a possibly uncountable cardinality of the set of goods, we adopt the graph theoretic method developed by Heydenreich et al. (2009).

Fix a bidder $i \in N$ and a profile v_{-i}, and let $f(v_i) = x_i^{\mathrm{VCG}}(v_i, v_{-i}; G)$ for all v_i. We define $G_f = (X, l)$ as the weighted complete directed allocation graph, where X is the set of nodes with $X = f(V_i)$, and $l(x,y) = \inf_{v_i \in f^{-1}(y)} \big(v_i(y) - v_i(x)\big)$ is the length function for $x, y \in X$. A path from node x to node y is defined as $P = (x = a^0, a^1, ..., a^k = y)$ such that $a^j \in X$ for $j = 0, ..., k$. Let $\mathcal{P}(x,y)$ be the set of all paths from x to y. Define the distance of (x,y) as $d(x,y) = \inf_{P \in \mathcal{P}(x,y)} \sum_{j=0}^{k-1} l(a^j, a^{j+1})$.

Proof of Proposition 1. **Step 1**: Prove the revenue equivalence.

Suppose that $V_i = V_{\mathrm{SUD}}$ for all $i \in N$. Take any bidder $i \in N$ and any profile $v_{-i} \in V_{\mathrm{SUD}}^{N-1}$, and consider the allocation graph G_f.

First we prove the revenue equivalence of f. Heydenreich et al. (2009, Theorem 1) show that a necessary and sufficient condition is $d(x,y) + d(y,x) = 0$ for all $x, y \subseteq G$. Since strategy-proofness generally ensures $d(x,y) + d(y,x) \geq 0$ (Heydenreich et al. (2009, Observation 2)), it suffices to show that $d(x,y) + d(y,x) \leq \varepsilon$ for all $\varepsilon > 0$.

Take any $v_i^0 \in f^{-1}(x)$ and $v_i^3 \in f^{-1}(y)$. For any $\delta \in (0, \varepsilon/4]$, let $\bar{x}(\delta) = \{g \in x \,|\, v_i^0(\{g\}) \geq v_i^0(x) - \delta\}$, and $\bar{y}(\delta) = \{g \in y \,|\, v_i^3(\{g\}) \geq v_i^3(y) - \delta\}$. Note that these sets are nonempty since we consider single-unit demand valuations. First suppose that $\bar{x}(\delta) \cap \bar{y}(\delta) \neq \emptyset$. Then we have $v_i^0(y) \geq v_i^0(x) - \delta$ and $v_i^3(x) \geq v_i^3(y) - \delta$, which imply $d(x,y), d(y,x) \leq \delta$. Therefore $d(x,y) + d(y,x) \leq 2\delta \leq \varepsilon$.

Next suppose that $\bar{x}(\delta) \cap \bar{y}(\delta) = \emptyset$. Let us fix $g \in \bar{x}(\delta)$ and $g' \in \bar{y}(\delta)$ ($g \neq g'$). For $\alpha, \beta \geq 0$, we denote a single-unit demand valuation function by $\bar{v}_i^{\alpha,\beta} \in V_i$ satisfying $\bar{v}_i^{\alpha,\beta}(\{g\}) = \alpha$, $\bar{v}_i^{\alpha,\beta}(\{g'\}) = \beta$, and $\bar{v}_i^{\alpha,\beta}(\{g''\}) = 0$ for all $g'' \neq g, g'$. We consider two cases:

Case 1: Suppose that there do not exist α, β such that $f(\bar{v}_i^{\alpha,\beta}) \supseteq \{g, g'\}$. Since the efficient allocation assigns a good to bidder i who values the good very highly, there exists a large value C such that $f(\bar{v}_i^{\alpha,\beta}) \ni g$ for all $\alpha, \beta \geq C$ with $\alpha \geq \beta + C$, and $f(\bar{v}_i^{\alpha,\beta}) \ni g'$ for all $\alpha, \beta \geq C$ with $\beta \geq \alpha + C$. Therefore there exist $\tilde{\alpha}, \tilde{\beta} \geq C$ and $\delta', \delta'' \in (0, \delta]$ such that for $v_i^1 := \bar{v}_i^{\tilde{\alpha}+\delta', \tilde{\beta}}$ and $v_i^2 := \bar{v}_i^{\tilde{\alpha}, \tilde{\beta}+\delta''}$, the allocations $f(v_i^1) \ni g$ and $f(v_i^2) \ni g'$. Let $\tilde{x} := f(v_i^1)$ and $\tilde{y} := f(v_i^2)$. Since the assumption implies $\tilde{x} \not\ni g'$ and $\tilde{y} \not\ni g$, each length is bounded as follows:

$$
\begin{aligned}
l(x, \tilde{x}) &\leq v_i^1(\tilde{x}) - v_i^1(x) = 0 \\
l(\tilde{x}, \tilde{y}) &\leq v_i^2(\tilde{y}) - v_i^2(\tilde{x}) = \tilde{\beta} - \tilde{\alpha} + \delta'' \\
l(\tilde{y}, y) &\leq v_i^3(y) - v_i^3(\tilde{y}) \leq \delta \\
l(y, \tilde{y}) &\leq v_i^2(\tilde{y}) - v_i^2(y) = 0 \\
l(\tilde{y}, \tilde{x}) &\leq v_i^1(\tilde{x}) - v_i^1(\tilde{y}) = \tilde{\alpha} - \tilde{\beta} + \delta' \\
l(\tilde{x}, x) &\leq v_i^0(x) - v_i^0(\tilde{x}) \leq \delta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d(x,y) + d(y,x) &\leq \big(l(x,\tilde{x}) + l(\tilde{x},\tilde{y}) + l(\tilde{y},y)\big) + \big(l(y,\tilde{y}) + l(\tilde{y},\tilde{x}) + l(\tilde{x},x)\big) \\
&\leq 4\delta \leq \varepsilon.
\end{aligned}
$$

Case 2: Suppose that there exist α, β such that $f(\bar{v}_i^{\alpha,\beta}) \supseteq \{g, g'\}$. By efficiency, this means that no other bidders $-i$ positively value $\{g, g'\}$ since $\bar{v}_i^{\alpha,\beta}$ is a unit-demand valuation. Then for $v_i^1 := \bar{v}_i^{\delta,0}$ and $v_i^2 := v_i^{0,\delta}$, we have $\tilde{x} := f(v_i^1) \ni g$ and $\tilde{y} := f(v_i^2) \ni g'$. Therefore, applying the same computation as in Case 1 for $\tilde{\alpha} = \tilde{\beta} = 0$, $\delta' = \delta'' = \delta$, we have $d(x,y) + d(y,x) \le 4\delta \le \varepsilon$.

This completes the proof of revenue equivalence.

Step 2: Prove impossibility.

Recall that $V_i = V_{\mathrm{SUD}}$ for all $i \in N$. Then, V_i is connected, i.e., for any $v_i, \tilde{v}_i \in V_i$ there is a path in V_i connecting v_i and $\tilde{v}_i$. Suppose that a mechanism $M_G = (x(\hat{v}; G), t(\hat{v}; G))$ for G satisfies efficiency and strategy-proofness in the domain $\mathcal{V} = V_i \times \cdots \times V_n$. By the revenue equivalence, the revenue $\sum t_i(v; G) = \sum t_i^{\mathrm{VCG}}(v; G) + c$, where c is a constant.

Let $\underline{v}_i \in V_i$ be the zero valuation function with $\underline{v}_i(x_i) = 0$ for all bundles $x_i \subseteq G$. Since $\underline{v}_i(x_i(\underline{v}_i, v_{-i}; G)) = 0$, participation implies $t_i(v_i, v_{-i}; G) = t_i^{\mathrm{VCG}}(v_i, v_{-i}; G)$ for any $v_i \in V_i$ and any $v_{-i} \in V_{-i}$. Therefore, the constant c is zero if M_G satisfies efficiency, strategy-proofness, and participation.

Consider the following valuation functions with a single-unit demand: $v_i(x_i) = 1$ for any bundle $x_i \neq \emptyset$ and any bidder $1 \le i \le \min\{|G|, |N|\}$, and $v_i(x_i) = 0$ for any bundle $x_i \subseteq G$ and any bidder $i > \min\{|G|, |N|\}$. Then, $w(G, N) = \min\{|G|, |N|\}$, and $w(G, N \setminus \{i\}) = w(G) - 1$ for all $1 \le i \le \min\{|G|, |N|\}$ and $w(G, N \setminus \{i\}) = w(G, N)$ otherwise. Thus, the payment $t_i^{\mathrm{VCG}}(v; G) = 1$ for all $1 \le i \le \min\{|G|, |N|\}$ and 0 otherwise. Hence the revenue is $\min\{|G|, |N|\} - \sum_{i \in N} t_i^{\mathrm{VCG}}(v; G) = 0$.

Suppose that the auctioneer drops some goods, and that a set of goods $G' \subsetneq G$ with $|G'| = \min\{|G|, |N|\} - 1$ (≥ 1) remains to be sold. Since $w(G', N) - w(G', N \setminus \{i\}) = 0$ for all $i \in N$, the payment $t_i^{\mathrm{VCG}}(v; G) = 1$ for any bidder $1 \le i \le |G'|$, and 0 otherwise. Therefore, the revenue increases; $|G'| > 0$.

This implies that the VCG mechanism is not goods revenue monotone. By revenue equivalence, M is not goods revenue monotone in the domain $\mathcal{V}$ if M satisfies efficiency, strategy-proofness, and participation. Hence M cannot be goods revenue monotone in any larger domains if M satisfies efficiency, strategy-proofness, and participation. $\square$

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