A note on the existence of Walras equilibrium in irreducible economies with satiable and non-ordered preferences[*]

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Abstract. Irreducible exchange economies in which consumers' preferences are satiable and non-ordered are considered. A general existence theorem of dividend quasi-equilibrium is proved and by the theorem the existence of Walras equilibrium is proved under weaker assumptions of non-satiation.

Keywords: dividend equilibrium, Walras equilibrium, irreducibility, satiation

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The purpose of this note is to prove the existence of Walras equilibrium in an economy with possibly satiated consumers under a set of weaker assumptions. We consider a bounded model of exchange economy where consumers' preferences are non-ordered.[*1]

Our main generalization has two new features. First, we do not necessarily assume that the initial endowment of each consumer belongs to the interior of his consumption set. We shall show that only the irreducibility is required even in the case that there are some satiable consumers. Second, our assumptions of non-satiation are weaker and admit that satiation may unexceptionally occur in feasible allocations.

We consider an exchange economy with L commodities and N consumers. The set of consumers is denoted by $I = \{1, \cdots, N\}$. The commodity space is an L-dimensional Euclidean space $\mathbb{R}^L$. The consumption set of each consumer $i \in I$ is denoted by $X_i \subset \mathbb{R}^L$ and the initial endowment is by $e_i \in X_i$. The preference of each consumer $i \in I$ is denoted by a mapping by $P_i : X_i \to 2^{X_i}$.

An *allocation* is an N-tuple of vectors, $\mathbf{x} = (x_1, \cdots, x_N)$, where $x_i \in X_i$ is an amount of commodities allotted to consumer $i \in I$. An allocation $\mathbf{x} = (x_1, \cdots, x_N)$ is said to be *feasible* if $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} e_i$. The set of all feasible allocations is denoted by $\mathbf{A}$. Since $e_i \in X_i$ for each $i \in I$, set $\mathbf{A}$ is non-empty.

A *dividend vector* is a non-negative vector, $d = (d_1, \cdots, d_N) \in \mathbb{R}_+^N$, where d_i is an extra income given to consumer $i \in I$.

A *dividend quasi-equilibrium* is a triplet $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ of a feasible allocation $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_N) \in \mathbf{A}$, a price vector $\hat{p} \in \mathbb{R}^L \setminus \{\mathbf{0}\}$, and a dividend $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_N) \in \mathbb{R}_+^N$ such that for each $i \in I$,

(1) $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + \hat{d}_i$,
(2) $\hat{p} \cdot y_i \geq \hat{p} \cdot e_i + \hat{d}_i$ for all $y_i \in P_i(\hat{x}_i)$.

A *dividend equilibrium* is a dividend quasi-equilibrium $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ such that, instead of condition (2), $\hat{p} \cdot y > \hat{p} \cdot e_i + \hat{d}_i$ for all $y \in P_i(\hat{x}_i)$.

Moreover, a *Walras equilibrium* (resp. *quasi-equilibrium*) is a pair $\{\hat{\mathbf{x}}, \hat{p}\}$ of a dividend equilibrium (resp. dividend quasi-equilibrium) $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ such that $\hat{d} = \mathbf{0}$.

Throughout this paper, for each consumer $i \in I$, we assume the following:

(A.1) X_i is a closed and convex subset of $\mathbb{R}^L$.
(A.2) $P_i : X_i \to 2^{X_i}$ is lower hemi-continuous, i.e., if $y^0 \in P_i(x^0)$ and a sequence $\{x^n\}$ converges to x^0, then there is a sequence $\{y^n\}$ converging to y^0 such that $y^n \in P_i(x^n)$ for all n sufficiently large.
(A.3) For every $x_i \in X_i$, $P_i(x_i)$ is convex and $x_i \notin P_i(x_i)$.

In addition, we assume the boundedness of the economy:

(A.4) The set $\mathbf{A}$ is bounded, i.e., there is a number $\bar{b} > 0$ such that for any $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$, $\|x_i\| \leq \bar{b}$ for all $i \in I$.

For each $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$, we define two sets of consumers as follows:

$$I^S(\mathbf{x}) := \{i \in I | P_i(x_i) = \emptyset\} \quad \text{and} \quad I^{NS}(\mathbf{x}) := I \setminus I^S(\mathbf{x}).$$

To exclude trivial cases in which every consumer is simultaneously satiated and a dividend equilibrium always exists, we assume the following condition which is the weakest non-satiation assumption.

(A5) For any $\mathbf{x} \in \mathbf{A}$, $I^{NS}(\mathbf{x}) \neq \emptyset$.

Step 1. We shall prove an existence theorem of dividend quasi-equilibrium under a most general setting.

Let us modify consumers' preference relations. For each $i \in I$ and each $x_i \in X_i$, we define a convex cone $K_i(x_i)$ by

$$K_i(x_i) := \{\lambda(y_i - x_i) | y_i \in P_i(x_i), \lambda > 0\}.$$

Set $K_i(x_i)$ indicates the desirable directions from $x_i \in X_i$ for consumer $i \in I$. Moreover, for each $i \in I$, a set $\bar{X}_i$ and a mapping $\bar{P}_i : \bar{X}_i \to 2^{\bar{X}_i}$ are defined in the following way:

$$\bar{X}_i := \left\{x_i \in X_i | \|x_i\| \leq \bar{b} + 1\right\}.$$
$$\bar{P}_i(x_i) := \{z_i + x_i | z_i \in K_i(x_i)\} \cap \bar{X}_i \text{ for each } x_i \in \bar{X}_i.$$

Consider a modified economy in which the consumption set and the preference relation of each consumer $i \in I$ are replaced by $\bar{X}_i$ and $\bar{P}_i : \bar{X}_i \to 2^{\bar{X}_i}$. It should be noted that for any $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$, $\bar{P}_i(x_i) = \emptyset$ if and only if $P_i(x_i) = \emptyset$. Therefore, we can easily show that any dividend quasi-equilibrium for the modified economy is a dividend quasi-equilibrium for the original economy. Thus, in order to prove the existence of a dividend quasi-equilibrium, it suffices only to prove the existence of a dividend quasi-equilibrium for the modified economy.

In addition, it is easy to show that mapping $\bar{P}_i$ has the same properties as mapping P_i has, that is, it is lower hemi-continuous, convex-valued, and $x_i \notin \bar{P}_i(x_i)$ for every $x_i \in \bar{X}_i$. Thus, in what follows, we shall identify X_i with $\bar{X}_i$ and $P_i : X_i \to 2^{X_i}$ with $\bar{P}_i : \bar{X}_i \to 2^{\bar{X}_i}$.

The following theorem on the existence of dividend quasi-equilibrium is a basic and key theorem for our argument.

Theorem 1. *Under assumptions (A.1)-(A.5), there exists a dividend quasi-equilibrium. More precisely, there exists a dividend quasi-equilibrium $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$ and $\hat{d}_1 = \cdots = \hat{d}_N$.*

To prove the above theorem, we shall follow a standard process that was innovated by Gale and Mas-Colell (1975). Let us confine prices to the closed unit ball, $\mathbf{B} = \{p \in \mathbb{R}^n | \|p\| \le 1\}$. For each $i \in I$, define a mapping $\beta_i : \mathbf{B} \to 2^{X_i}$ by:

$$\beta_i(p) := \{y_i \in X_i | p \cdot y_i < p \cdot e_i + 1 - \|p\|\} \text{ for each } p \in \mathbf{B}.$$

Moreover, define mappings $F_0 : \mathbf{B} \times \mathbb{R}^L \to 2^{\mathbf{B}}$ and $F_i : \mathbf{B} \times X_i \to 2^{X_i}$ $(i \in I)$ by:

$$F_0(p, z) := \{q \in \mathbf{B} | q \cdot z > p \cdot z\} \text{ for each } (p, z) \in \mathbf{B} \times \mathbb{R}^L,$$

$$F_i(p, x_i) := \begin{cases} \{y_i \in X_i | p \cdot y_i < p \cdot x_i\} & \text{when } p \cdot x_i > p \cdot e_i + 1 - \|p\| \\ \beta_i(p) \cap P_i(x_i) & \text{otherwise} \end{cases}$$

for each $(p, x_i) \in \mathbf{B} \times X_i$. This mapping F_i is a modification of the mapping originally constructed by Gale and Mas-Colell (1975). The modification is slight, but crucial since we do not assume that $e_i \in \text{int} X_i$ for each $i \in I$.[*2] Our mapping can be applied to cases where consumers' budget sets do not always have interior. We can check easily that mappings $F_0 : \mathbf{B} \times \mathbb{R}^L \to 2^{\mathbf{B}}$ and $F_i : \mathbf{B} \times X_i \to 2^{X_i}$ for each $i \in I$ are convex-valued and lower hemi-continuous.

Then, applying the fixed point theorem in Gale=Mas-Colell (1975, 1979), there exist $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_N)$ and $\hat{p}$ such that

$$F_0\left(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i\right) = \emptyset \text{ and } F_i(\hat{p}, \hat{x}_i) = \emptyset \text{ for each } i \in I.$$

Since $F_i(\hat{p}, \hat{x}_i) = \emptyset$ for each $i \in I$, it follows from the definition of F_i that

$$\hat{p} \cdot \hat{x}_i \le \hat{p} \cdot e_i + 1 - \|\hat{p}\| \text{ and } \beta_i(\hat{p}) \cap P_i(\hat{x}_i) = \emptyset \text{ for each } i \in I \qquad (1)$$

Furthermore, suppose that $\sum_{i \in I} \hat{x}_i \ne \sum_{i \in I} e_i$. Since $F_0\left(\hat{p}, \sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i\right) = \emptyset$, by the definition of F_0, $q \cdot \left(\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i\right) \le \hat{p} \cdot \left(\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i\right)$ for any $q \in \mathbf{B}$. Therefore, $\|\hat{p}\| = 1$

[*2] For any set $X \subset \mathbb{R}^L$, $\text{int} X$ denotes the interior of set X in $\mathbb{R}^L$.

and $\hat{p} \cdot \left(\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i \right) > 0$. Thus, (1) implies that $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i$ for each $i \in I$, and that $\hat{p} \cdot \left(\sum_{i \in I} \hat{x}_i - \sum_{i \in I} e_i \right) \leq 0$, which is a contradiction. Hence, we can conclude that $\sum_{i \in I} \hat{x}_i = \sum_{i \in I} e_i$.

Suppose that $\|\hat{p}\| = 0$. Then, $\beta(\hat{p}) = X_i$ for all $i \in I$, and, by (1), $P_i(\hat{x}_i) = \emptyset$ for all $i \in I$, which contradicts assumption (A.5). Thus, $\|\hat{p}\| \neq 0$.

Now, let $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_N)$ be a vector such that $\hat{d}_1 = \cdots = \hat{d}_N := 1 - \|\hat{p}\|$. Then, by (1), for each $i \in I$
,
$$\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + \hat{d}_i \text{ and } \hat{p} \cdot y_i \geq \hat{p} \cdot e_i + \hat{d}_i \text{ for all } y_i \in P_i(\hat{x}_i). \tag{2}$$

Thus, we have shown that $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend quasi-equilibrium.

Finally, since we identify P_i with $\bar{P}_i$, for $i \in I^{NS}(\hat{\mathbf{x}})$ there is a point $y \in P_i(\hat{x}_i)$ which is arbitrarily close to $\hat{x}_i$. Therefore, (2) implies that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$. This completes the proof of Theorem 1.

Step 2. We shall define the irreducibility condition for an economy with possibly satiated consumers and prove the existence of dividend equilibrium in the irreducible economy.

A well-known sufficient condition under which any quasi-equilibrium is an equilibrium is the irreducibility assumption that originates with McKenzie (1956). And a weaker condition of irreducibility was considered by Bergstrom (1976). The condition can be defined for economies with possibly satiated consumers in the following fashion:[*3]

(A.6) Let $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbf{A}$ and $j \in I^{NS}(\mathbf{x})$. If $I^{NS}(\mathbf{x}) \setminus \{j\} \neq \emptyset$ then there exist an allocation $\mathbf{y} = (y_1, \cdots, y_N) \in X_1 \times \cdots \times X_N$ and a scalar $\theta > 0$ such that
(1) $\theta(y_j - e_j) + \sum_{i \in I^S(\mathbf{x})} (x_i - e_i) + \sum_{i \in I^{NS}(\mathbf{x}) \setminus \{j\}} (y_i - e_i) = 0$,
(2) $y_i \in P_i(x_i)$ for each $i \in I^{NS}(\mathbf{x}) \setminus \{j\}$.

The meaning of irreducibility is that the initial endowment of any non-satiated consumer is desired by some other non-satiated consumers. Evidently, when $I^S(\mathbf{x}) = \emptyset$, the above condition is equivalent to the condition of Bergstrom (1976).

In order to prove that any quasi-equilibrium is an equilibrium, we need the following assumptions:

(A.7) For each $i \in I$, $P_i(x_i)$ is open in X_i for every $x_i \in X_i$.[*4]
(A.8) For any $\mathbf{x} \in \mathbf{A}$, $\sum_{i \in I^{NS}(\mathbf{x})} e_i \in \text{int} \sum_{i \in I^{NS}(\mathbf{x})} X_i$.

Condition (A.8) is weaker than the assumption that $e_i \in \text{int} X_i$ for all $i \in I$. For example, if there is at least one consumer who is never satiated and if his initial endowment belongs to the interior of his consumption set, then the condition holds.

The following is a fundamental lemma which is a modification of the lemma of Debreu (1962).

Lemma 1. *Under assumptions (A.6) and (A.7), for any dividend quasi-equilibrium $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$, if $\hat{p} \cdot e_i + \hat{d}_i >$*

[*3] In Bergstrom (1976), the irreducibility condition is assumed only on individually rational allocations in which no consumers are worse off than in their initial endowments. Since in a quasi-equilibrium some consumers might be worse off than in their endowments, we need to take account of all feasible allocations in defining irreducibility.

[*4] As Bergstrom (1976) and Won & Yannelis (2011) showed, we can replace this assumption with the following condition: If $y_i \in P_i(x_i)$ and $z_i \in X_i$, then there exists a number $\theta > 0$ such that $(1 - \theta)y_i + \theta z_i \in P_i(x_i)$.

$inf\ \hat{p} \cdot X_i$ *occurs for some* $i \in I^{NS}(\hat{\mathbf{x}})$, *then it occurs for every* $i \in I^{NS}(\hat{\mathbf{x}})$.

Proof. Let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend quasi-equilibrium. Define two sets in the following way:

$$I_1 := \{i \in I^{NS}(\hat{\mathbf{x}}) \mid \hat{p} \cdot e_i + \hat{d}_i = \inf \hat{p} \cdot X_i\}.$$
$$I_2 := \{i \in I^{NS}(\hat{\mathbf{x}}) \mid \hat{p} \cdot e_i + \hat{d}_i > \inf \hat{p} \cdot X_i\}.$$

Assume that I_2 is non-empty. Choose $j \in I_1$. Then, (A.6) implies that there are an allocation $\mathbf{y} = (y_1, \cdots, y_N) \in X_1 \times \cdots \times X_N$ and a scalar $\theta > 0$ such that

$$\theta(y_j - e_j) + \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} (y_i - e_i) + \sum_{i \in I^S(\mathbf{x})} (\hat{x}_i - e_i) = 0 \tag{3}$$

and $y_i \in P_i(\hat{x}_i)$ for each $i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}$. Since $\sum_{i \in I}(\hat{x}_i - e_i) = 0$, by (3) we have

$$\theta(y_j - e_j) + (e_j - \hat{x}_j) + \sum_{i \in I^{NS}(\mathbf{x}) \setminus \{j\}} (y_i - \hat{x}_i) = 0. \tag{4}$$

On the other hand, from the definition of quasi-equilibrium, it follows that $\hat{p} \cdot y_i \geq \hat{p} \cdot e_i + \hat{d}_i \geq \hat{p} \cdot \hat{x}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$ and in particular, by (A.7), $\hat{p} \cdot y_i > \hat{p} \cdot e_i + \hat{d}_i$ for each $i \in I_2$. In addition, as for j, since $e_j \in X_j$, we have $\hat{d}_j = 0$ and $\hat{p} \cdot e_j = \hat{p} \cdot \hat{x}_j$. Hence, $\hat{p} \cdot (y_j - e_j) \geq 0$. Thus, we have

$$\theta \hat{p} \cdot (y_j - e_j) + \hat{p} \cdot (e_j - \hat{x}_j) + \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} \hat{p} \cdot (y_i - \hat{x}_i) \geq \sum_{i \in I^{NS}(\hat{\mathbf{x}}) \setminus \{j\}} \hat{p} \cdot (y_i - \hat{x}_i) > 0.$$

This is a contradiction to (4). This shows that $I_2 \neq \emptyset$ implies that $I_1 = \emptyset$. $\square$

Now, let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend quasi-equilibrium. From (A.8), it follows that $\hat{p} \cdot e_i > \inf \hat{p} \cdot X_i$ occurs for some $i \in I^{NS}(\hat{\mathbf{x}})$, i.e., $\hat{p} \cdot e_i + \hat{d}_i > \inf \hat{p} \cdot X_i$ occurs for some $i \in I^{NS}(\hat{\mathbf{x}})$. Therefore, by Lemma 1, it occurs for all $i \in I^{NS}(\hat{\mathbf{x}})$, i.e., $\hat{p} \cdot e_i + \hat{d}_i > \inf \hat{p} \cdot X_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$, and by (A.7) we can easily show that $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ is a dividend equilibrium. Thus, by Theorem 1 we have the following theorem:

Theorem 2. *Under assumptions (A.1)-(A.8), there exists a dividend equilibrium. More precisely, there exists a dividend equilibrium* $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ *such that* $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ *for all* $i \in I^{NS}(\hat{\mathbf{x}})$ *and* $\hat{d}_1 = \cdots = \hat{d}_N$.

Step 3. We shall prove the existence of Walras equilibrium under weaker non-satiation assumptions.

The irreducibility condition (A.6) is a relation among non-satiated consumers. In order to prove the existence of Walras equilibrium, we need an additional assumption which relates satiated consumers to non-satiated ones. In what follows, we shall consider two types of conditions which are weaker than that of Won & Yannelis (2011).[*5]

For $\mathbf{x} \in \mathbf{A}$, we define a cone by

$$K(\mathbf{x}) := \sum_{i \in I^{NS}(\mathbf{x})} K_i(x_i).$$

First we consider the following condition which is a straightforward extension of that of Won & Yannelis (2011).

[*5] Won & Yannelis (2011) assumed the following: For any $\mathbf{x} \in \mathbf{A}$, $x_i - e_i \in \text{cl}[\sum_{j \in I^{NS}(\mathbf{x})}(P_j(x_j) - \{x_j\})]$ for all $i \in I^S(\mathbf{x})$.

(R.1) For any $\mathbf{x} \in \mathbf{A}$, $\sum_{i \in I^S(\mathbf{x})} (x_i - e_i) \in \mathrm{cl}\, K(\mathbf{x})$.

Set $K(\mathbf{x})$ indicates the desirable directions for the non-satiated consumers in allocation $\mathbf{x}$. Therefore, roughly speaking, condition (R.1) means that in any feasible allocation the sum of the directions of satiation points from initial endowments for satiated consumers is one of the directions which are desirable for non-satiated consumers.

Now, let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend equilibrium such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$ and $\hat{d}_1 = \cdots = \hat{d}_N$. From the definition of dividend equilibrium, it follows that for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot y_i > \hat{p} \cdot e_i + \hat{d}_i = p \cdot \hat{x}_i$ for any $y_i \in P_i(\hat{x}_i)$, and therefore, $\hat{p} \cdot z > 0$ for any $z \in K_i(\hat{x}_i)$. Hence, by the definition of $K(\hat{\mathbf{x}})$, $\hat{p} \cdot z \geq 0$ for any $z \in \mathrm{cl}\, K(\hat{\mathbf{x}})$. Thus, by (R.1), we have $\hat{p} \cdot \sum_{i \in I^S(\hat{\mathbf{x}})} (\hat{x}_i - e_i) \geq 0$.

Since $\sum_{i \in I} (\hat{x}_i - e_i) = 0$, it follows that $\hat{p} \cdot \sum_{i \in I^{NS}(\hat{\mathbf{x}})} (\hat{x}_i - e_i) \leq 0$. Since $\hat{p} \cdot (\hat{x}_i - e_i) = \hat{d}_i \geq 0$ for all $i \in I^{NS}(\hat{\mathbf{x}})$, we conclude that $\hat{p} \cdot (\hat{x}_i - e_i) = \hat{d}_i = 0$ for all $i \in I^{NS}(\hat{\mathbf{x}})$, and that $\hat{d}_i = 0$ for all $i \in I$. Therefore, $\{\hat{\mathbf{x}}, \hat{p}\}$ is a Walras equilibrium. Thus, by virtue of Theorem 2 we have proved the following theorem.

Theorem 3. *Under assumptions (A.1)-(A.8) and (R.1), there exists a Walras equilibrium $\{\hat{\mathbf{x}}, \hat{p}\}$ such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ for all $i \in I$.*

It should be noted that the existence of a dividend equilibrium with equal dividends is essential in proving Theorem 3.

Next, we consider another weaker assumption of non-satiation. For $\mathbf{x} \in \mathbf{A}$, we define the following set:

$$L_i(x_i) := \begin{cases} \{\lambda(z + x_i - e_i) \,|\, z \in K_i(x_i), \lambda > 0\} & \text{for each } i \in I^{NS}(\mathbf{x}) \\ \{\lambda(x_i - e_i) \,|\, \lambda > 0\} & \text{for each } i \in I^S(\mathbf{x}). \end{cases}$$

(R.2) For any $\mathbf{x} \in \mathbf{A}$, if $\mathbf{0} \notin \mathrm{int} \sum_{i \in I^{NS}(\mathbf{x})} L_i(x_i)$, then $\mathbf{0} \notin \mathrm{int} \sum_{i \in I} L_i(x_i)$.[*6]

For consumer $i \in I^{NS}(\mathbf{x})$, set $L_i(x_i)$ indicates the desirable directions from initial endowment e_i and for consumer $i \in I^S(\mathbf{x})$, the directions of satiation point x_i from initial endowment e_i. Therefore, roughly speaking, condition (R.2) means that in any feasible allocation the directions of satiation points of satiated consumers are almost the same as the desirable directions for non-satiated consumers in that the desirable directions for both satiated and non-satiated consumers are contained in a common half space of $\mathbb{R}^L$.

Now, let $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ be a dividend equilibrium such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for all $i \in I^{NS}(\hat{\mathbf{x}})$. From the definition of dividend equilibrium, it follows that for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot y_i > p \cdot \hat{x}_i = \hat{p} \cdot e_i + \hat{d}_i$ for any $y_i \in P_i(\hat{x}_i)$, which implies that

$$\hat{p} \cdot (y_i - \hat{x}_i) > 0 \text{ and } \hat{p} \cdot (\hat{x}_i - e_i) \geq 0 \text{ for any } y_i \in P_i(\hat{x}_i),$$
$$\text{accordingly, } \hat{p} \cdot \lambda(y_i - \hat{x}_i) + \hat{p} \cdot (\hat{x}_i - e_i) > 0 \text{ for any } y_i \in P_i(\hat{x}_i) \text{ and } \lambda > 0,$$
$$\text{accordingly, } \hat{p} \cdot (z_i + \hat{x}_i - e_i) > 0 \text{ for any } z_i \in K_i(\hat{x}_i).$$

[*6] This condition can be proved to be weaker than that of Won and Yannelis (2011). Also, we can show that this condition neither weaker nor stronger than condition (R.1). For details, see Miyazaki and Takekuma (2012).

This implies that for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\hat{p} \cdot w_i > 0$ for any $w_i \in L_i(\hat{x}_i)$, and that $\mathbf{0} \notin \text{int} \sum_{i \in I^{NS}(\hat{\mathbf{x}})} L_i(\hat{x}_i)$.

Under assumption (R.2), by Minkowski's separation theorem we have a vector $\bar{p} \in \mathbb{R}^L$ with $\bar{p} \neq \mathbf{0}$ such that $\bar{p} \cdot z \geq 0$ for all $z \in \sum_{i \in I} L_i(\hat{x}_i)$. Namely,

$$\bar{p} \cdot \sum_{i \in I^{NS}(\hat{\mathbf{x}})} \lambda_i(z_i + \hat{x}_i - e_i) + \bar{p} \cdot \sum_{i \in I^S(\hat{\mathbf{x}})} \lambda_i(\hat{x}_i - e_i) \geq 0$$

for any $z_i \in K_i(\hat{x}_i)$ and $\lambda_i > 0$. Hence, for each $i \in I^{NS}(\hat{\mathbf{x}})$, $\bar{p} \cdot (\hat{x}_i - e_i) \geq 0$ and $\bar{p} \cdot (y_i - e_i) \geq 0$ for any $y_i \in P_i(\hat{x}_i)$. Moreover, for each $i \in I^S(\hat{\mathbf{x}})$, $\bar{p} \cdot (\hat{x}_i - e_i) \geq 0$. Since $\sum_{i \in I}(\hat{x}_i - e_i) = 0$, $\bar{p} \cdot (\hat{x}_i - e_i) = 0$ for all $i \in I$. Thus, $\{\hat{\mathbf{x}}, \bar{p}\}$ is a Walras quasi-equilibrium such that $\bar{p} \cdot \hat{x}_i = \bar{p} \cdot e_i$ for all $i \in I$. Under assumptions (A.6), (A.7), and (A.8), by the same argument in proving Theorem 2, we can show that $\{\hat{\mathbf{x}}, \bar{p}\}$ is a Walras equilibrium.[*7] Thus, again by virtue of Theorem 2, we have proved the following theorem:

Theorem 4. *Under assumptions (A.1)-(A.8) and (R.2), there exists a Walras equilibrium $\{\hat{\mathbf{x}}, \hat{p}\}$ such that $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ for all $i \in I$.*

Concluding Remarks. As for the existence of dividend quasi-equilibrium, Theorem 1 is more general than the result by Allouch & Le Van (2009, Prop.1, p.321) since we consider economies with consumers whose preferences are non-ordered. As for the existence of dividend equilibrium, Theorem 2 is more general than the results by Mas-Colell (1992, Thm.1, p.205) and Kajii (1996, Prop.1, p.79) since we consider irreducible economies in which consumers have not always positive incomes.

As for the existence of Walras quasi-equilibrium, Theorems 3 and 4 are neither more special nor more general compared with the results by Allouch & Le Van (2009, Thm.2, p.323) and by Sato (2010a, Thm.2, p541, Thm.3, p.543). While our assumption on consumers' preferences is weaker than their assumptions in the sense that consumers' preferences are non-ordered, the assumptions of non-satiation are different and cannot be compared directly with each other. However, our assumption admits that satiation generally occurs in the set of feasible allocations, and our theorem applies to a broader set of economies. In fact, the example of economy shown by Sato (2010a, Eg.1, p.537) satisfies our assumptions.

In comparison with the work of Won & Yannelis (2011, Thm.4.1, p.249) in which economies with non-ordered preferences are considered, our results are an extension of their result, since we use the assumptions of irreducibility and weaker assumptions of non-satiation.

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[*7] We can construct an example of economy in which there is a dividend equilibrium $\{\hat{\mathbf{x}}, \hat{p}, \hat{d}\}$ that is not a Walras equilibrium and, if we choose appropriately a different price vector $\bar{p}$, $\{\hat{\mathbf{x}}, \bar{p}\}$ becomes a Walras equilibrium. See Miyazaki and Takekuma (2012, Eg.1).

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