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# Strong $L^p$ Convergence Associated with Rellich-type Discrete Compactness for Discontinuous Galerkin FEM

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**Abstract**

In a preceding paper, we proved the discrete compactness properties of Rellich type for some 2D discontinuous Galerkin finite element methods (DGFEM), that is, the strong $L^2$ convergence of some subfamily of finite element functions bounded in an $H^1$-like mesh-dependent norm. In this note, we will show the strong $L^p$ convergence of the above subfamily for $1 \le p < \infty$. To this end, we will utilize the duality mappings and special auxiliary problems. The results are applicable to numerical analysis of various semi-linear problems.
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## 1 Introduction

Various discontinuous Galerkin finite element methods (DGFEM) have been developed and analyzed in recent years[1, 2]. Since they use discontinuous approximation functions, some important results in the conventional functional analysis are not directly available, so that we are obliged to establish their discrete analogs.

In [3], we proved discrete compactness properties of Rellich type for some 2D DGFEM. That is, from a mesh-dependent family of functions bounded in a broken $H^1$-like Sobolev norm, we can choose a subfamily which is strongly convergent and whose approximate first-order derivatives are weakly convergent in the $L^2$ sense. The obtained results can be applied to justification of numerical approximations to various linear problems. However, in the 2D cases, the original Rellich theorem also assures the strong $L^p$ convergence for $1 \le p < \infty$, and this property is effective to analysis of some semi-linear problems. So we will derive such property for some DGFEM by making use of the duality maps and regularity results of special auxiliary boundary value problems.

## 2 Preliminaries

### 2.1 Function spaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. We assume that its maximum interior angle is strictly less than $2\pi$. For $\Omega$, we can define the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{s,p}(\Omega)$ ( $s \ge 0$,

$1 \le p \le \infty$, $L^p(\Omega) = W^{0,p}(\Omega)$ ), where fractional cases ($s \notin \mathbb{N} \cup \{0\}$) are included[2, 4]. We will also use $H^s(\Omega) := W^{s,2}(\Omega)$. The inner products of both $L^2(\Omega)$ and $L^2(\Omega)^2$ are designated by $(\cdot,\cdot)_\Omega$, with the associated norms by $\|\cdot\|_\Omega$, and the norm and the standard semi-norm of $W^{s,p}(\Omega)$, as well as those of $W^{s,p}(\Omega)^2$, are denoted by $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$, respectively. For domains other than $\Omega$, notations of the above spaces, norms etc. will be used with $\Omega$ replaced appropriately.

Let us consider a subset $\partial\Omega_D$ of $\partial\Omega$, which either is empty or consists of finitely many closed segments. Then we introduce a closed subspace $H^1_D(\Omega)$ of $H^1(\Omega)$ by

$$H^1_D(\Omega) = \{v \in H^1(\Omega)\,;\ v = 0 \text{ on } \partial\Omega_D\}\,. \tag{1}$$

## 2.2 Definitions and notations for triangulations

We first construct a family of triangulations $\{\mathscr{T}^h\}_{h>0}$ of $\Omega$ by polygonal finite elements (or shortly elements): each $\mathscr{T}^h$ consists of a finite number of elements, and each element $K \in \mathscr{T}^h$ is a bounded $m$-polygonal (open) domain, where $m$ is an integer which can differ with $K$ such that $3 \le m \le M$ for an integer $M \ge 3$ common to the considered family $\{\mathscr{T}^h\}_{h>0}$. Thus the boundary $\partial K$ of $K \in \mathscr{T}^h$ is a closed simple polygonal curve composed of $m$ edges. We do not avoid non-convex cases for $K$ unlike in the classical quadrilateral elements, cf.[5].

We use the notation $e$ to denote an edge of $K$, which is assumed here to be an open segment. The sets of edges of $K \in \mathscr{T}^h$ and of $\mathscr{T}^h$ are respectively denoted by $\mathscr{E}^K$and $\mathscr{E}^h$. For each triangulation $\mathscr{T}^h$, we define its "skeleton" $\Gamma^h$ as $\Gamma^h = \cup_{e\in\mathscr{E}^h}\bar{e}$. We assume that the triangulations are so constructed that any edge $e \in \mathscr{E}^h$ such that $e \cap \partial\Omega_D \neq \emptyset$ is entirely contained in $\partial\Omega_D$.

The diameter of $K$ is denoted by $h_K$, and the length of $e \in \mathscr{E}^K$ by $|e|$. Moreover, $h = \max_{K\in\mathscr{T}^h} h_K$. We will designate the inner products of $L^2(\partial K)$ and $L^2(\partial K)^2$ by $[\cdot,\cdot]_{\partial K}$, and the associated norms by $|\cdot|_{\partial K}$. For $e \in \mathscr{E}^K$, $[\cdot,\cdot]_e$ and $|\cdot|_e$ are defined similarly, and the norm of $L^p(e)$ $(1 \le p \le \infty)$ is denoted by $|\cdot|_{p,e}$ ($|\cdot|_e = |\cdot|_{2,e}$).

We will also impose the "regularity" conditions on $\{\mathscr{T}^h\}_{h>0}$ presented in [3], cf. also [5]. In particular, we adopt the chunkiness condition[2], the triangle condition, and the local quasi-uniformity of edge lengths.

## 2.3 Function spaces associated to triangulations

Over $\mathscr{T}^h$, we consider the broken Sobolev spaces[1, 2]:

$$W^{s,p}(\mathscr{T}^h) = \{v \in L^p(\Omega); v|_K \in W^{s,p}(K)\,(\forall K \in \mathscr{T}^h)\},\ \ H^s(\mathscr{T}^h) = W^{s,2}(\mathscr{T}^h) \quad (s \ge 0, 1 \le p \le \infty). \tag{2}$$

Here, $W^{s,p}(\mathscr{T}^h)$ can be identified with $\Pi_{K\in\mathscr{T}^h} W^{s,p}(K)$. For $v \in H^{\frac{1}{2}+\sigma}(\mathscr{T}^h)$ $(\sigma > 0)$ and $K \in \mathscr{T}^h$, the trace of $v|_K$ to $\partial K$ is well defined as an element of $L^2(\partial K)$ and denoted by $v|_{\partial K}$ or simply $v$, which can be double-valued on edges shared by two elements [1, 2].

On $\Gamma^h$, we consider a kind of flux $\hat{v} \in L^2(\Gamma^h)$, which is single-valued on each edge shared by two elements [1, 2]. To deal with the boundary condition in (1), define

$$L^2_D(\Gamma^h) = \{\hat{v} \in L^2(\Gamma^h)\,; \hat{v} = 0 \text{ on } \partial\Omega_D\}\,. \tag{3}$$

In the hybrid(ized) DGFEM, the flux $\hat{v}$ is independent of $v$, and they are used as a pair. On the other hand, in some genuine (non-hybridized) DGFEM like IP and LDG methods[1, 2], we make $\hat{v}$ to be subject to $v$ by introducing appropriate constraints between them. A typical approach is: first define $\{\{v\}\} \in L^2(\Gamma^h)$ for $v \in H^1(\mathscr{T}^h)$ by: for an edge $e \in \mathscr{E}^h$, we set $\{\{v\}\}|_e = v|_e$ if $e \subset \partial\Omega$, while we take as follows (simple averaging) if $e$ is shared by two elements $K_1, K_2 \in \mathscr{T}^h$;

$$\{\{v\}\}|_e = (v_1 + v_2)/2\,, \tag{4}$$

where $v_1$ ($v_2$ resp.) = trace of $v|_{K_1}$ ($v|_{K_2}$ resp.) to $e$. Then we can use such $\{\{v\}\}|_e$ as $\hat{v}|_e$ when $e \not\subset \partial\Omega_D$.

For each $\mathscr{T}^h$, let us define a mesh-dependent semi-norm for $\{v, \hat{v}\} \in H^1(\mathscr{T}^h) \times L^2(\Gamma^h)$ by

$$|\{v,\hat{v}\}|_h^2 = \|\nabla_h v\|_\Omega^2 + \sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K} |e|^{-1}|v - \hat{v}|_e^2\,, \tag{5}$$

where $v$ on $e \in \mathscr{E}^K$ implies the trace of $v|_K$ to $e$, and $\nabla_h : H^1(\mathscr{T}^h) \to L^2(\Omega)^2$ is characterized by $(\nabla_h v)|_K = \nabla(v|_K)$ for $v \in H^1(\mathscr{T}^h)$ and $K \in \mathscr{T}^h$.

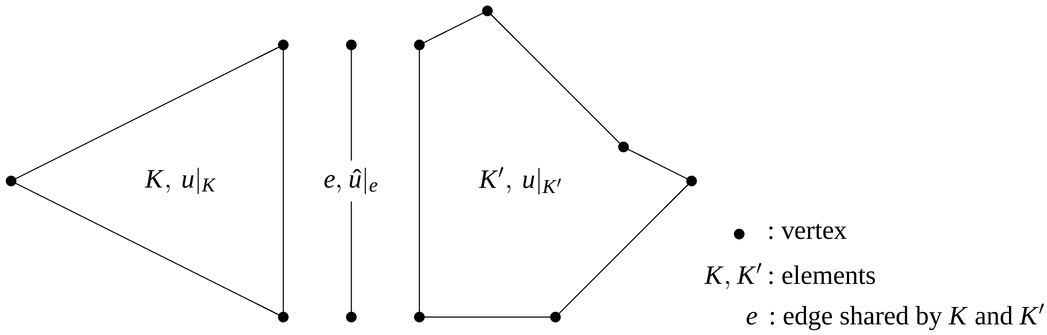


Fig. 1 interior function $u$ and flux function $\hat{u}$

## 2.4 Finite element spaces

To approximate $\{v,\hat{v}\}\in H^{\frac{3}{2}+\sigma}(\mathscr{T}^h)\times L^2(\Gamma^h)$ $(0<\sigma\le\frac{1}{2})$ associated to $\mathscr{T}^h$, let us prepare two concrete finite dimensional spaces for a specified $k\in\mathbb{N}$:

$$U^h=\Pi_{K\in\mathscr{T}^h}P_k(K)\subset W^{2,\infty}(\mathscr{T}^h)\subset H^{\frac{3}{2}+\sigma}(\mathscr{T}^h), \tag{6}$$

$$\hat{U}^h=\Pi_{e\in\mathscr{E}^h}P_k(e)\subset L^\infty(\Gamma^h) \text{ or } \Pi_{e\in\mathscr{E}^h}P_k(e)\cap C(\Gamma^h), \tag{7}$$

where $P_k(K)$ and $P_k(e)$ are the spaces of polynomials of degree $\le k$ on $K$ and $e$, respectively, and $C(\Gamma^h)$ is the space of continuous functions on $\Gamma^h$.

To deal with the Dirichlet condition in (1), define also

$$\hat{U}^h_D=\{\hat{v}_h\in\hat{U}^h;\hat{v}_h=0 \text{ on } \partial\Omega_D\}=\hat{U}^h\cap L^2_D(\Gamma^h). \tag{8}$$

We will employ the finite element spaces given by

$$V^h=U^h\times\hat{U}^h,\ \ V^h_D=U^h\times\hat{U}^h_D. \tag{9}$$

## 2.5 Lifting operators

First let us introduce, for the same $k\in\mathbb{N}$ as in 2.4,

$$Q^K=P_k(K) \text{ or } P_{k-1}(K). \tag{10}$$

Then the local lifting operator $R_K:g\in L^2(\partial K)\mapsto\xi\in(Q^K)^2$ is defined as: given $g\in L^2(\partial K)$, find $\xi=\{\xi_1,\xi_2\}\in(Q^K)^2$ such that, $\forall\eta=\{\eta_1,\eta_2\}\in(Q^K)^2$,

$$(\xi,\eta)_K=[g,\eta\cdot n]_{\partial K}\ \ (\eta\cdot n=\eta_1n_1+\eta_2n_2), \tag{11}$$

where $n=\{n_1,n_2\}$ is the outward unit normal on $\partial K$.

Identifying $Q^h:=\Pi_{K\in\mathscr{T}^h}Q^K$ with a subspace of $L^2(\Omega)$ and further $\Pi_{K\in\mathscr{T}^h}(Q^K)^2$ with $(Q^h)^2$, the global lifting operator $R_h$ is defined by

$$R_h:\tilde{g}=\{g_{\partial K}\}_{K\in\mathscr{T}^h}\in\Pi_{K\in\mathscr{T}^h}L^2(\partial K)\mapsto\{R_Kg_{\partial K}\}_{K\in\mathscr{T}^h}\in(Q^h)^2\subset L^2(\Omega)^2. \tag{12}$$

Since $\hat{v}\in L^2(\Gamma^h)$ is single-valued on $e\in\mathscr{E}^h$, it can be naturally identified with an element of $\Pi_{K\in\mathscr{T}^h}L^2(\partial K)$. On the other hand, the trace of $v\in H^1(\mathscr{T}^h)$ to $e\not\subset\partial\Omega$ may be double-valued. To use $R_h$ for such $v$, we define

$$S_h:v\in H^1(\mathscr{T}^h)\mapsto\{v|_{\partial K}\}_{K\in\mathscr{T}^h}\in\Pi_{K\in\mathscr{T}^h}L^2(\partial K). \tag{13}$$

For the present choice of the discrete spaces, we can show that $R_h$ in (12) satisfies [1, 3]

$$\|R_h\tilde{g}\|_\Omega\le C\Big(\sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K}|e|^{-1}|g_{\partial K}|_e^2\Big)^{\frac{1}{2}}. \tag{14}$$

Here $C>0$ is independent of $h>0$ and $\tilde{g}$, and, along with $C^*$ and $c$, will denote generic positive constants[2].

## 3 Rellich type discrete compactness

In [3], we showed the following results.

**Theorem 1.** *We employ the above finite element spaces and assume the regularity conditions in* [3]. *Let* $\{\{u_h,\hat{u}_h\} \in V_D^h\}_{h>0}$ *be a family associated to* $\{\mathscr{T}^h\}_{h>0}$ *such that* $|\{u_h,\hat{u}_h\}|_h^2 + \|u_h\|_\Omega^2 \leq 1$. *Then there exist a function* $u_0 \in H_D^1(\Omega)$ *and a subfamily, denoted again by* $\{\{u_h,\hat{u}_h\}\}_{h>0}$ *for convenience, such that, as* $h \downarrow 0$,

$$u_h \to u_0 \ \text{ in } L^2(\Omega), \tag{15}$$

$$u_h|_{\partial\Omega_D} \to u_0|_{\partial\Omega_D} = 0 \ \text{ in } L^2(\partial\Omega_D) \text{ if } \partial\Omega_D \neq \emptyset, \tag{16}$$

$$\nabla_h u_h + R_h(\hat{u}_h - S_h u_h) \rightharpoonup \nabla u_0 \ \text{ in } L^2(\Omega)^2, \tag{17}$$

*where* $\to$ *and* $\rightharpoonup$ *respectively denote the strong and weak convergences.*

To prove the above, we used some assumptions on the family of finite element spaces, which can be assured for the present types of triangulations and piecewise polynomial spaces[3], cf. also [1]. However, we should supplement the techniques used there. In the former proof[3], we utilized an $h-$family $(h > 0)$ of problems $-\Delta u^h + u^h = u_h$ $(u_h \in U^h)$ under the mixed Dirichlet-Neumann boundary conditions associated to $H_D^1(\Omega)$. But this choice yields so severe regularity results for $\Omega$ of general shape[4], that some arguments employed there may loose the validity, unless we put additional restrictions on the interior angles of the polygonal domain $\Omega$. Instead, we can use the pure Neumann condition without any essential changes of the proof.

**Remark 1.** *Another approach of showing the discrete compactness for some genuine DGFEM is to use the reconstruction operators, see e.g.*[6]. *Probably, we can also apply such techniques to various hybrid(ized) DGFEM.*

## 4 Strong $L^p$ convergence for $1 \leq p < \infty$

Let us show that the subfamily $\{u_h\}_{h>0}$ in Theorem 1 also converges strongly to $u_0$ in $L^p(\Omega)$ for $p \in [1,\infty[$. Since $\Omega$ is bounded, the conclusion is obvious for $p \in [1,2[$, so that we will consider only for $p \in ]2,\infty[$. We will use the notation $q \in ]1,2[$ characterized by $\frac{1}{p} + \frac{1}{q} = 1$.

Notice here the following lemma[7].

**Lemma 1.** *If* $f \in L^2(\Omega) \cap L^p(\Omega)$ $(p \in ]2,\infty[)$, *it also belongs to* $L^{p^*}(\Omega)$ *where* $1/p^* = (1-\alpha)/2 + \alpha/p$ *for* $\alpha \in ]0,1[$, *and the following "interpolation inequality" holds*:

$$\|f\|_{0,p^*,\Omega} \leq \|f\|_\Omega^{1-\alpha} \|f\|_{0,p,\Omega}^\alpha. \tag{18}$$

Thus we can conclude the strong convergence of $\{u_h\}_{h>0}$ in $L^{p^*}(\Omega)$ for all $p^* \in ]2,p[$ by deriving the $L^p$ boundedness for $p > 2$. Moreover, to our aim, it suffices to show such boundedness for each sufficiently large $p$.

Let $J_p : L^p(\Omega) \to L^q(\Omega)$ be the duality map characterized for each $v \in L^p(\Omega)$ by

$$\int_\Omega (J_p v)\, v\, dx = \|v\|_{0,p,\Omega}^2\ , \quad \|J_p v\|_{0,q,\Omega} = \|v\|_{0,p,\Omega}, \tag{19}$$

where $x = \{x_1,x_2\}$ denotes the variable in $\mathbb{R}^2$, and $J_p v$ is uniquely given by $J_p v = v \cdot |v|^{p-2}/\|v\|_{0,p,\Omega}^{p-2}$[7].

For each $\{u_h,\hat{u}_h\} \in V^h$ with $|\{u_h,\hat{u}_h\}|_h^2 + \|u_h\|_\Omega^2 \leq 1$, define $u^{h,p} \in W^{1,q}(\Omega)$ $(p \in ]2,\infty[,\ \frac{1}{p} + \frac{1}{q} = 1)$ such that

$$\int_\Omega \Big(\sum_{i=1}^2 \frac{\partial u^{h,p}}{\partial x_i} \frac{\partial v}{\partial x_i} + u^{h,p} v\Big) dx = \int_\Omega (J_p u_h)\, v\, dx\ ; \ \ \forall v \in W^{1,p}(\Omega). \tag{20}$$

This is a variational formulation to $-\Delta u^{h,p} + u^{h,p} = J_p u_h$ with the homogeneous Neumann condition. For $p = 2$, it reduces to the auxiliary problem in Sec. 3.

Since we are dealing with a bounded polygonal domain $\Omega$ whose maximum interior angle is strictly smaller than $2\pi$, we have the following existence and regularity results for elliptic problems with possible corner singularities (cf. Theorems 1.4.5.3 and 4.4.3.7 in [4]).

**Lemma 2.** *For the present domain $\Omega$ and any sufficiently large $p < \infty$, there exists a unique solution $u^{h,p} \in W^{1,q}(\Omega)$ of (20), which also satisfies*

$$u^{h,p} \in W^{s,q}(\Omega) \quad (s = \min\{2, \tfrac{1}{2} + \tfrac{2}{q} + \delta\}) , \quad \|u^{h,p}\|_{s,q,\Omega} \leq C_{p,\Omega} \|J_p u_h\|_{0,q,\Omega} = C_{p,\Omega} \|u_h\|_{0,p,\Omega} . \tag{21}$$

*Here, $\delta > 0$ depends only on $p$ and the maximum interior angle of $\Omega$, and $C_{p,\Omega} > 0$ does only on $p$ and $\Omega$.*

**Remark 2.** *The present results may not hold for some finite $p$[4]. Moreover, for $2 < p < \infty$, the number $\frac{1}{2} + \frac{2}{q}$ in the definition of $s$ is evaluated as $\frac{3}{2} < \frac{1}{2} + \frac{2}{q} < \frac{5}{2}$.*

Let us integrate $-\Delta u^{h,p} + u^{h,p} = J_p u_h$ over $\Omega$ after multiplying $u_h$ to the both sides, and then apply the Green formula with carefully handling the singularities around the vertices[2]. To justify such calculations, we should notice that $u^{h,p} \in W^{s,q}(\Omega)$, $u_h \in W^{1,\infty}(\mathscr{T}^h)$ and $\hat{u}^h \in L^\infty(\Gamma^h)$ for any $h > 0$, and in addition, for any $K \in \mathscr{T}^h$ and $e \in \mathscr{E}^K$, $(\nabla u^{h,p})|_e \in W^{s-1-\frac{1}{q},q}(e)^2$, and $u_h|_e \in L^\infty(e)$, where $u_h|_e$ for example denotes the trace of $u_h|_K$ to $e$ and the trace theorem from $W^{s-1,q}(K)$ to $W^{s-1-\frac{1}{q},q}(e)$ is used by taking account that $s - 1 - \frac{1}{q} = \min\{1 - \frac{1}{q}, -\frac{1}{2} + \frac{1}{q} + \delta\} > 0$[4]. Using also (19), we finally obtain

$$\|u_h\|_{0,p,\Omega}^2 = \int_\Omega (J_p u_h)\, u_h \, dx = I_1 + I_2 \ ; \quad \begin{cases} I_1 = \displaystyle\sum_{K \in \mathscr{T}^h} \sum_{i=1}^{2} \int_K \frac{\partial u^{h,p}}{\partial x_i} \frac{\partial u_h}{\partial x_i} \, dx + \int_\Omega u^{h,p}\, u_h \, dx , \\ I_2 = \displaystyle\sum_{K \in \mathscr{T}^h} \int_{\partial K} [(\nabla u^{h,p}) \cdot n]\, (\hat{u}_h - u_h) \, ds , \end{cases} \tag{22}$$

where $ds$ denotes the infinitesimal line element.

By the Sobolev imbedding theorem (Theorem 1.4.4.1 in [4]), we have the continuous inclusions

$$W^{s,q}(\Omega) \subseteq H^{s+1-\frac{2}{q}}(\Omega) \subseteq H^1(\Omega), \tag{23}$$

since $s + 1 - \frac{2}{q} = \min\{3 - \frac{2}{q}, \frac{3}{2} + \delta\} > 1$. Thus $I_1$ in (22) can be expressed by $I_1 = (\nabla u^{h,p}, \nabla_h u_h)_\Omega + (u^{h,p}, u_h)_\Omega$, and is estimated as, for a generic constant $C > 0$,

$$|I_1| \leq C \|u^{h,p}\|_{s,q,\Omega} (\|\nabla_h u_h\|_\Omega^2 + \|u_h\|_\Omega^2)^{\frac{1}{2}} . \tag{24}$$

To estimate $I_2$ in (22), we need some inverse inequalities and trace theorems to $e$ and $K$ along with Lemma 2. To this end, recall the triangle condition in [3]: *Let $T_0$ be a fixed isosceles triangle with unit base length. For each $h > 0$ and each edge $e$ of any $K \in \mathscr{T}^h$, there exists a isosceles triangle $T_e \subset K$ that is similar to $T_0$ with the similarity ratio $|e|$ and whose base coincides with $e$.*

Let us first show a trace theorem related to $K \in \mathscr{T}^h$.

**Lemma 3.** *Let $e$ be an arbitrary edge of $K \in \mathscr{T}^h$, and $v$ be an arbitrary element of $W^{t,r}(K)$ with*

$$1 < r < \infty, \quad 1/r < t \leq 1 . \tag{25}$$

*Then the trace of $v$ to $e$ exists as an element of $L^r(e)$ and satisfies, with $C > 0$ independent of $h > 0$ and $v$,*

$$|v|_{r,e} \leq C (|e|^{-\frac{1}{r}} \|v\|_{0,r,K} + |e|^{t-\frac{1}{r}} |v|_{t,r,K}). \tag{26}$$

*Proof.* For a reference triangle $T_0$ in the triangle condition, whose base $e_0$ has unit length. By the trace theorem for $T_0$[1, 4], any $\tilde{v} \in W^{t,r}(T_0)$ has a trace to $e_0$ as an element of $L^r(e_0)$, and satisfies for an appropriate $C > 0$

$$|\tilde{v}|_{r,e_0} \leq C \left( \|\tilde{v}\|_{0,r,T_0} + |\tilde{v}|_{t,r,T_0} \right) .$$

Let us introduce a suitable similarity transformation from $T_0$ to $T_e \subset K$ in the triangle condition. By relating $v$ to an appropriate $\tilde{v}$ and using the scaling arguments, we can derive the desired results. □

We also need the following inverse inequalities.

**Lemma 4.** *Let $e$ be an arbitrary edge of $K \in \mathscr{T}^h$, and $\{v_h, \hat{v}_h\}$ be an arbitrary element of $V^h$. For any $p$ with $2 < p < \infty$, $(v_h|_K)|_e$ and $\hat{v}_h|_e$ can be regarded as elements of $L^p(e)$, and satisfy*

$$|v_h - \hat{v}_h|_{p,e} \le C|e|^{\frac{1}{p}-\frac{1}{2}}|v_h - \hat{v}_h|_e, \tag{27}$$

*where $C > 0$ depends on $p$ and the polynomial degree $k$ but is independent of $h > 0$, $e$ and $\{v_h, \hat{v}_h\}$.*

*Proof.* Using some notations in the preceding proof, we find for $\tilde{u} \in P^k(T_0)$ and $\tilde{v} \in P^k(e_0)$ $(k \in \mathbb{N})$

$$|\tilde{u} - \tilde{v}|_{p,e_0} \le C|\tilde{u} - \tilde{v}|_{e_0} \quad (\tilde{u} = \tilde{u}|_{e_0}),$$

since $\tilde{u}|_{e_0} - \tilde{v}$ belongs to the finite-dimensional space $P^k(e_0)$. Here, $C > 0$ depends on $k$ but does not on $\tilde{u}$ and $\tilde{v}$. Connecting $\tilde{u}$ and $\tilde{v}$ respectively with $v_h$ and $\hat{v}_h$ by an appropriate similarity transformation between $T_0$ and $T_e$, we have the desired estimation. □

In addition, let us define $\xi_h \in (Q^h)^2$ such that $\xi_K = \xi_h|_K$ for each $K \in \mathscr{T}^h$ is given by

$$\xi_K \in P_0(K)^2,\ \xi_K(x) = \frac{1}{|K|}\int_K \nabla u^{h,p} dx\ (x \in K), \tag{28}$$

where $|K|$ is the measure of $K$. Then we find that[2]

$$\|\xi_h\|_\Omega \le \|\nabla u^{h,p}\|_\Omega\,, \tag{29}$$

$$\|\nabla u^{h,p} - \xi_K\|_{0,q,K} \le C h_K^{s-1}|u^{h,p}|_{s,q,K}\,, \tag{30}$$

where $C > 0$ is independent of $h$ and $K$. Using the above $\xi_h$, we split $I_2$ into $I_3 + I_4$ with

$$I_3 = \sum_{K\in\mathscr{T}^h}\int_{\partial K}(\xi_h\cdot n)\,(\hat{u}_h - u_h)\,ds = (R_h(\hat{u}_h - S_h u_h), \xi_h)_\Omega \qquad (\text{by } (11))\,,$$

$$I_4 = \sum_{K\in\mathscr{T}^h}\int_{\partial K}[(\nabla u^{h,p} - \xi_h)\cdot n]\,(\hat{u}_h - u_h)\,ds\,. \tag{31}$$

By (5), (14), (23) and (29), $I_3$ is estimated as

$$|I_3| \le C\|\nabla u^{h,p}\|_\Omega \cdot |\{u_h, \hat{u}_h\}|_h \le C^*\|u^{h,p}\|_{s,q,\Omega}\,. \tag{32}$$

Finally, let us estimate $I_4$.

**Lemma 5.** *For $\{u_h, \hat{u}_h\} \in V^h$ and $u^{h,p}$ in* (20), *we have*

$$|I_4| \le C\|u^{h,p}\|_{s,q,\Omega}\,, \tag{33}$$

*where $C > 0$ is independent of $h > 0$ and $\{u_h, \hat{u}_h\}$.*

*Proof.* For $\alpha, \beta, \gamma \ge 1$ and the present $\{u_h, \hat{u}_h\}$, define

$$J_h(\alpha,\beta,\gamma) = \sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K}|e|^{-\frac{\alpha}{2}}|\hat{u}_h - u_h|^\gamma_{\beta,e}\,.$$

By the Hölder inequality, we have, with $\eta^h = \nabla u^{h,p} - \xi_h$,

$$|I_4| \le \sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K}\int_e|\eta^h\cdot n|\cdot|\hat{u}_h - u_h|\,ds \le \Big(\sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K}|e|^{\frac{q}{p}}|\eta^h|^q_{q,e}\Big)^{\frac{1}{q}}J_h(2,p,p)^{\frac{1}{p}}\,, \tag{a}$$

where $|\eta^h|_{q,e} = (\sum_{i=1}^2|\eta_i^h|^q_{q,e})^{1/q}$ $(\eta^h = \{\eta_1^h, \eta_2^h\})$. By Lemma 4, $|\hat{u}_h - u_h|^p_{p,e} \le C^p|e|^{1-\frac{p}{2}}|\hat{u}_h - u_h|^p_e$, so that

$$J_h(2,p,p) \le C^p J_h(p,2,p)\,. \tag{b}$$

It follows from $|\{u_h,\hat{u}_h\}|_h^2+\|u_h\|_\Omega^2 \le 1$ and (5) that

$$J_h(2,2,2)=\sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K}|e|^{-1}|\hat{u}_h-u_h|_e^2\le 1\,.$$

Then we have $|e|^{-\frac{1}{2}}|\hat{u}_h-u_h|_e\le 1$, and hence, for $p>2$, $|e|^{-\frac{p}{2}}|\hat{u}_h-u_h|_e^p\le|e|^{-1}|\hat{u}_h-u_h|_e^2$, which means that

$$J_h(p,2,p)\le J_h(2,2,2)\le 1\,. \tag{c}$$

On the other hand, by noting $\frac{1}{q}<s-1\le 1$ and applying Lemma 3 to $\nabla u^{h,p}\in W^{s-1,q}(\Omega)^2$, we find that

$$|\eta^h|_{q,e}=|\nabla u^{h,p}-\xi_K|_{q,e}\le C(|e|^{-\frac{1}{q}}\|\nabla u^{h,p}-\xi_K\|_{0,q,K}+|e|^{s-1-\frac{1}{q}}|u^{h,p}|_{s,q,K}).$$

Then, by (30) and $|e|/h_K\ge c$ for some $c>0$, we obtain

$$\sum_{K\in\mathscr{T}^h}\sum_{e\in\mathscr{E}^K}|e|^{\frac{q}{p}}|\eta^h|_{q,e}^q\le C^*\sum_{K\in\mathscr{T}^h}h_K^{\frac{q}{p}-1+q(s-1)}|u^{h,p}|_{s,q,K}^q\le C^*h^{qs-2}\|u^{h,p}\|_{s,q,\Omega}^q \tag{d}$$

since $h_K\le h$ and $qs>2$, where $C^*>0$ is independent of $h$. By (a), (b), (c) and (d), we have (33). □

By (21), (24), (32), (33) and $|\{u_h,\hat{u}_h\}|_h^2+\|u_h\|_\Omega^2\le 1$, $\|u_h\|_{0,p,\Omega}^2=I_1+I_3+I_4$ in (22) is bounded from above by $C\|u^{h,p}\|_{s,q,\Omega}\le C^*\|u_h\|_{0,p,\Omega}$ $(C,C^*>0)$. Thus $\|u_h\|_{0,p,\Omega}$ for each sufficiently large $p<\infty$ is uniformly bounded for $h>0$, so that we obtain the theorem below.

**Theorem 2.** *The subfamily $\{u_h\}_{h>0}$ in Theorem 1 also converges strongly to $u_0$ in $L^p(\Omega)$ $(\forall p\in[1,\infty[)$ as $h\downarrow 0$.*

# 5 Concluding remarks

We have proved the strong $L^p$ convergence associated with the Rellich type discrete compactness for some discontinuous Galerkin FEM. The results can be applied to justification of numerical computations of various semilinear problems by DGFEM. To give a firm foundation to DGFEM, we are also planning to show the discrete Korn inequalities, which play essential roles in applications to solid mechanics and fluid dynamics [2, 8]. Moreover, our results on $L^p$ boundedness are only “qualitative” since we have not shown, for example, the dependence of $\|u_h\|_{0,p,\Omega}$ on $p$. Such refined results may be required in certain cases, and we will continue our studies.

# References


[1] D. N. Arnold, F. Brezzi, B. Cockburn and L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal., **39** (2002), 1749-1779.

[2] S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, 3rd ed., Springer, New York, 2008.

[3] F. Kikuchi, Rellich-type discrete compactness for some discontinuous Galerkin FEM, Japan J. Indust. Appl. Math., **29** (2012), 269-288.

[4] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985, and SIAM, Philadelphia, 2011.

[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, 2nd ed., SIAM, Philadelphia, 2002.

[6] A. Buffa and C. Ortner, Compact embeddings of broken Sobolev spaces and applications, IMA J. Numer. Anal., **29** (2009), 827-855.

[7] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.

[8] S. C. Brenner, Korn’s inequalities for piecewise $H^1$ vector fields, Math. Comp., **73** (2003), 1067-1087.