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A Hybridized Discontinuous Galerkin FEM with Lifting Operator for Plane Elasticity Problems

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Abstract

We present a formulation of the hybridized DGFEM with lifting operator for the plane elasticity problem. To validate the formulation, we establish an inequality of the Korn type by following the method of proof due to Brenner [Math. Comp., **73** (2004), 1067–1087]. Using the inequality, we can demonstrate the well-posedness of a discrete problem arising in the formulation, and derive a priori error estimates for solutions of the discrete problem.

1. Introduction

Discontinuous Galerkin Finite Element Methods (DGFEMs) have been applied to various problems arising in scientific and industrial fields. There are many kinds of formulations in the DGFEM [2, 5, 7, 10, 12, 13]. We present a formulation of the hybridized DGFEM for the plane elasticity problem. This formulation is obtained by adding an interior penalty term using a lifting operator, which is called the *lifting term*, to a formulation proposed by Kikuchi–Ishii–Oikawa [10]. To validate our formulation, we establish an inequality of the Korn type. To do so, we follow the method of proof devised by Brenner [4], who proved Korn's inequality for piecewise H^1 vector functions. In our proof, we need to take account of the lifting term and the *numerical flux* which is contained in our formulation as an unknown variable. To estimate the lifting term, we have essentially used an estimation for the lifting operator (Lemma 1) derived by Kikuchi [9]. Using the inequality of the Korn type, we can show that a discrete problem arising in our formulation is well-posed. Furthermore we can derive a priori error estimates for solutions of the discrete problem by a standard method.

The formulation of Kikuchi–Ishii–Oikawa [10], where the lifting term is not employed, is also applied to the Poisson equation by Oikawa–Kikuchi [13]. When we use this formulation in practical computations, we need to carefully choose an interior penalty parameter to get appropriate numerical solutions. To overcome this shortcoming, Oikawa [12] introduced the lifting term in the formulation for the Poisson equation.

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His work inspires us to consider the formulation using the lifting term for the plane elasticity problem. Thanks to the lifting term, discrete problems in the formulation equipped with the lifting term is well-posed for an arbitrary positive interior penalty parameter. This suggests that the lifting term liberates us from the inconvenience of properly choosing the interior penalty parameter in practical computations.

This paper is organized as follows. In Section 2, we introduce the plane stress problem and formulate its weak formulation. In Section 3, we present our formulation of the hybridized DGFEM. In Section 4, we prove the inequality of the Korn type. In Section 5, using this inequality, we establish the well-posedness of the discrete problem. In Section 6, we derive the a priori error estimates. In Appendix A, we introduce the formulation due to Kikuchi–Ishii–Oikawa [10], and moreover we show that a discrete problem in their formulation is well-posed if the interior penalty parameter is large sufficiently.

We close this section with the introduction of several notations. For every open set $\Omega \subset \mathbb{R}^2$, we can define the Hilbetian Sobolev spaces $L^2(\Omega)$ and $H^\kappa(\Omega)$ $(\kappa > 0)$, where fractional cases $(\kappa \notin \mathbb{N})$ are included [1, 11, 8, 5]. The inner product of $L^2(\Omega)^j$ $(j \in \mathbb{N})$ is designated by $(\cdot,\, \cdot)_\Omega$, with the associated norm done by $\|\cdot\|_\Omega$. Furthermore, the norms and the standard semi-norm of $H^\kappa(\Omega)$ are denoted by $\|\cdot\|_{\kappa,\Omega}$ and $|\cdot|_{\kappa,\Omega}$, respectively, where $|v|^2_{\kappa,\Omega} = \|v\|^2_{\kappa,\Omega} - \|v\|^2_{\kappa^*,\Omega}$ for $v \in H^\kappa(\Omega)$ $(\kappa^* = [\kappa]$ for $\kappa \notin \mathbb{N}$ and $\kappa^* = \kappa - 1$ for $\kappa \in \mathbb{N})$. For these spaces associated to domains other than Ω, the same notations of spaces, norms etc. will be used with Ω replaced appropriately. In addition, C denotes a generic positive constant, and can be a different value at each of different places.

2. Linear plane stress problem

We consider a homogeneous isotropic elastic body occupying a reference configuration $\overline{\Omega} \times [-t/2,\, t/2] \subset \mathbb{R}^3$ with $\Omega \subset \mathbb{R}^2$. The body corresponds to a thin elastic plate with middle surface Ω and thickness t. We introduce orthogonal Cartesian coordinates $(x_1,\, x_2,\, x_3)$ such that the x_1x_2 plane contains the middle surface. We consider the linear static plane stress problem. For the two-dimensional displacement $\boldsymbol{u} = \{u_1,\, u_2\}$ of the elastic plate, the (linearized) strain tensor $\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{u})$ is given by $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$ $(1 \le i,\, j \le 2)$. Instead of the strain tensor, we use its engineering expression of vector form:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) \equiv \{\varepsilon_1(\boldsymbol{u}),\, \varepsilon_2(\boldsymbol{u}),\, \gamma_{12}(\boldsymbol{u})\} := \left\{\frac{\partial u_1}{\partial x_1},\, \frac{\partial u_2}{\partial x_2},\, \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right\}.$$

We also use the engineering stress components in vector expression:

$$\boldsymbol{\sigma}(\boldsymbol{u}) \equiv \{\sigma_1(\boldsymbol{u}),\, \sigma_2(\boldsymbol{u}),\, \tau_{12}(\boldsymbol{u})\} := \{\sigma_{11}(\boldsymbol{u}),\, \sigma_{22}(\boldsymbol{u}),\, \sigma_{12}(\boldsymbol{u}) = \sigma_{21}(\boldsymbol{u})\},$$

where σ_{ij} $(1 \le i,\, j \le 2)$ denote the usual tensor expressions for the stress.

To describe the isotropic linear elastic stress-strain relation, we introduce the matrix $[D]$:

$$[D] := \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix},$$

where E is Young's modulus and ν Poisson's ratio. Usually, we assume that $E > 0$ and $0 < \nu < 1/2$. Note that $[D]$ is a symmetric and positive-definite matrix. Then the stress-strain relation is written by

$$\boldsymbol{\sigma} = [D]\boldsymbol{\varepsilon}.$$

Remark 1 We can argue in exactly the same way for the plane strain problem by defining $[D]$ as follows:

$$[D] := \frac{E}{(1+\nu)(1-\nu^2)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}.$$

We assume that Ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Then the outward unit normal is well-defined almost everywhere on $\partial\Omega$, and is denoted by $\boldsymbol{n} = \{n_1, n_2\}$. The surface force $\boldsymbol{\sigma_n}$ on $\partial\Omega$ is given by

$$\boldsymbol{\sigma_n} := \{\sigma_1 n_1 + \tau_{12} n_2, \tau_{12} n_1 + \sigma_2 n_2\}, \tag{1}$$

where $\boldsymbol{n} = \{n_1, n_2\}$ is the outward unit normal.

An equilibrium equation and boundary conditions for the elastic plate are given as follows:

$$-\frac{\partial \sigma_1}{\partial x_1} - \frac{\partial \tau_{12}}{\partial x_2} = f_1, \quad -\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \sigma_2}{\partial x_2} = f_2 \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega_D, \qquad \boldsymbol{\sigma_n}(\boldsymbol{u}) = \overline{\boldsymbol{\sigma}}_{\boldsymbol{n}} \quad \text{on } \partial\Omega_N, \tag{3}$$

where $\boldsymbol{f} = \{f_1, f_2\}$ is the distributed external body force per unit in-plane area, $\partial\Omega$ is decomposed into disjoint two parts: $\partial\Omega_D$ and $\partial\Omega_N$, and $\overline{\boldsymbol{\sigma}}_{\boldsymbol{n}}$ is the surface traction force per unit length on $\partial\Omega_N$. We suppose that the measure $|\partial\Omega_D|$ of $\partial\Omega_D$ is positive. A weak formulation of (2) and (3) is given as follows: find $\boldsymbol{u} \in H_D^1(\Omega)^2$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in H_D^1(\Omega)^2. \tag{4}$$

Here $H_D^1(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega_D\}$,

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega, \quad F(\boldsymbol{v}) := (\boldsymbol{f}, \boldsymbol{v})_\Omega + \int_{\partial\Omega_N} \overline{\boldsymbol{\sigma}}_{\boldsymbol{n}} \cdot \boldsymbol{v}\, ds,$$

where ds is the infinitesimal line element on $\partial\Omega_N$. Bilinear form a is bounded on $H^1(\Omega)^2$, i.e.,

$$|a(\boldsymbol{u}, \boldsymbol{v})| \le C \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)^2, \tag{5}$$

where C is a positive constant independent of $\boldsymbol{u}$ and $\boldsymbol{v}$. We define for $\boldsymbol{p}, \boldsymbol{q} \in [L^2(\Omega)]^3$,

$$\begin{aligned} d(\boldsymbol{p}, \boldsymbol{q}) &:= ([D]\boldsymbol{p}, \boldsymbol{q})_\Omega, \\ d(\boldsymbol{q}) &:= d(\boldsymbol{q}, \boldsymbol{q})^{1/2}. \end{aligned}$$

Since $[D]$ is a positive definite symmetric matrix, $d(\cdot)$ is a norm in $[L^2(\Omega)]^3$, and moreover we have

$$\underline{\alpha} \|\boldsymbol{q}\|_\Omega^2 \le d(\boldsymbol{q})^2 \le \overline{\alpha} \|\boldsymbol{q}\|_\Omega^2 \quad \forall \boldsymbol{q} \in \left[L^2(\Omega)\right]^3, \tag{6}$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are the minimum and maximum eigenvalues of $[D]$, respectively.

It follows from Corollary 11.2.22 in [5] that there exists a positive constant C such that for all $\boldsymbol{v} \in H^1_D(\Omega)^2$

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_\Omega \geq C\|\boldsymbol{v}\|_{1,\Omega} \quad \text{(Korn's inequality in } H^1_D(\Omega)^2\text{)}. \tag{7}$$

Combining (6) and (7), we obtain the coreciveness of $a(\cdot,\, \cdot)$ on $H^1_D(\Omega)^2$, that is, there exists a positive constant C such that for all $\boldsymbol{v} \in H^1_D(\Omega)^2$

$$a(\boldsymbol{v},\, \boldsymbol{v}) \geq C\|\boldsymbol{v}\|^2_{1,\Omega}. \tag{8}$$

We see from (5) and (8) that for every $\boldsymbol{f} \in L^2(\Omega)^2$ and $\overline{\boldsymbol{\sigma}\boldsymbol{n}} \in L^2(\partial\Omega_N)^2$, problem (4) has a unique solution $\boldsymbol{u} \in H^1_D(\Omega)^2$.

3. Hybridized DGFEM with lifting operator

3.1. Partition $\mathcal{T}^h$ of Ω

Hereafter we assume that Ω is a bounded polygonal domain in $\mathbb{R}^2$. We first construct a partition $\mathcal{T}^h$ of Ω, which consists of a finite number of elements K such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}^h} \overline{K}$. Here each element $K \in \mathcal{T}^h$ is a bounded m-polygonal (open) domain, where m is an integer ≥ 3 and can differ with K. Notice here that non-convex elements are available for $m \geq 4$. For different $K,\, K' \in \mathcal{T}^h$, $K \cap K' = \emptyset$ and $\overline{K} \cap \overline{K'}$ is exclusively one of the three sets: (i) $\emptyset$, (ii) one vertex, and (iii) one closed edge. We define the set of nodes of $\mathcal{T}^h$ by

$$\mathcal{V}^h := \left\{p \in \mathbb{R}^2 \mid \exists K \in \mathcal{T}^h \text{ such that } p \text{ is a vertex of polygon } K\right\}.$$

Note that $\mathcal{V}^h$ may include the "hanging" nodes [5], that is, vertices of a polygon $K \in \mathcal{T}^h$ which lie in the interior of an edge of another polygon $K' \in \mathcal{T}^h$ (see Figure 1). We call

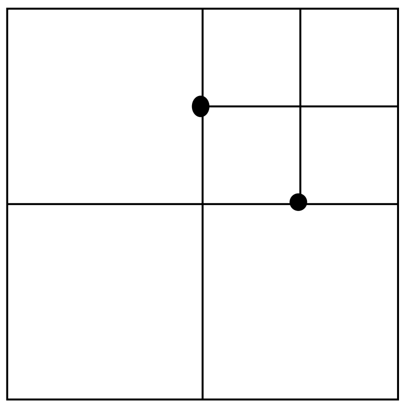

Figure 1: An example of partition of Ω which includes hanging nodes, which are depicted by dots.

an element of $\mathcal{V}^h$ a node of $\mathcal{T}^h$. An open line segment between two different nodes of $\mathcal{T}^h$ is said to be an edge of $\mathcal{T}^h$. The set of edges of $\mathcal{T}^h$ is denoted by $\mathcal{E}^h$. We use notation e to denote an edge of $\mathcal{T}^h$. For each $K \in \mathcal{T}^h$, we define $\mathcal{E}^K := \left\{e \in \mathcal{E}^h \mid e \subset \partial K\right\}$. We call an element of $\mathcal{E}^K$ an edge of K. Note that an element of $\mathcal{E}^K$ can be different from an edge of polygon K. We further define the "skeleton" Γ^h of $\mathcal{T}^h$ as the union of closed edges in $\mathcal{E}^h$: $\Gamma^h := \bigcup_{e\in\mathcal{E}^h} \overline{e}$.

The diameter and measure of K are denoted by h_K and $|K|$, respectively, while the length of an edge $e \in \mathcal{E}^K$ by $|e|$. Furthermore, $h = \max_{K\in\mathcal{T}^h} h_K$. We will use $(\cdot,\,\cdot)_K$ and $\|\cdot\|_K$ for both $L^2(K)^j$ ($j \in \mathbb{N}$), and also define, for $\hat{\boldsymbol{u}},\, \hat{\boldsymbol{v}} \in L^2(\partial K)^j$ ($j \in \mathbb{N}$),

$$\langle \hat{\boldsymbol{u}},\, \hat{\boldsymbol{v}}\rangle_{\partial K} = \int_{\partial K} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}\, ds, \quad |\hat{\boldsymbol{v}}|_{\partial K} = \langle \hat{\boldsymbol{v}},\, \hat{\boldsymbol{v}}\rangle_{\partial K}^{1/2},$$

where ds is the infinitesimal line element on ∂K. For each edge $e \in \mathcal{E}^K$, $\langle\cdot,\,\cdot\rangle_e$ and $|\cdot|_e$ are similarly defined.

3.2. Function spaces associated to partitions

Over $\mathcal{T}^h$, we consider the "broken" or piecewise Sobolev spaces ($\kappa > 0$):

$$H^\kappa(\mathcal{T}^h) := \left\{ v \in L^2(\Omega);\ v|_K \in H^\kappa(K)\ (\forall K \in \mathcal{T}^h) \right\},$$

which can be identified with $\prod_{K\in\mathcal{T}^h} H^\kappa(K)$, where, as was already noted, $H^\kappa(K)$ is the Sobolev space of (possibly fractional) order κ over K. For $v \in H^{\frac{1}{2}+\gamma}(\mathcal{T}^h)$ ($\gamma > 0$) and $K \in \mathcal{T}^h$, the trace of $v|_K$ to ∂K is well-defined as an element of $L^2(\partial K)$ and denoted by $v|_{\partial K}$ or simply v, which can be double-valued on edges shared by two elements [2, 5]. For $v \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)$ ($\gamma > 0$), we can define the trace of $\nabla(v|_K)$ to ∂K and the normal derivative $\partial v/\partial \boldsymbol{n} = (\nabla v)\cdot \boldsymbol{n}$ there in the L^2 sense.

On Γ^h of $\mathcal{T}^h$, we consider a kind of *flux* $\hat{v} \in L^2(\Gamma^h)$, which is single-valued on each edge shared by two elements, unlike various double-valued ones [2, 5].

For each $\mathcal{T}^h$, we define bilinear form I_h: for every $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\ \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^1(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$,

$$I_h(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) := \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \frac{1}{|e|} \langle \hat{\boldsymbol{u}} - \boldsymbol{u},\, \hat{\boldsymbol{v}} - \boldsymbol{v}\rangle_e,$$

and the associated semi-norm: for every $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^1(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$,

$$I_h(\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) := I_h(\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\})^{1/2}.$$

Further, let us define the following semi-norms and norms: for $\boldsymbol{v} = \{v_1,\, v_2\} \in H^1(\mathcal{T}^h)^2$,

$$\begin{aligned} |\boldsymbol{v}|^2_{H^1(\mathcal{T}^h)} &:= \sum_{i=1}^{2} \|\nabla_h v_i\|^2_\Omega, \\ \|\boldsymbol{v}\|^2_{H^1(\mathcal{T}^h)} &:= |\boldsymbol{v}|^2_{H^1(\mathcal{T}^h)} + \|\boldsymbol{v}\|^2_\Omega, \end{aligned}$$

and for $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^1(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$,

$$\begin{aligned} |\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}|^2_h &:= \sum_{i=1}^{2} \|\nabla_h v_i\|^2_\Omega + I_h(\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}), \\ \|\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}\|^2_{1,h} &:= |\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}|^2_h + \|\boldsymbol{v}\|^2_\Omega, \end{aligned}$$

where $\nabla_h : H^1(\mathcal{T}^h) \longrightarrow L^2(\Omega)^2$ is characterized by $(\nabla_h v)|_K = \nabla(v|_K)$ for $v \in H^1(\mathcal{T}^h)$ and $K \in \mathcal{T}^h$. Notice again that v and ∇v can be double-valued on e but $\hat{v}$ is not so, and, in addition, that all the above (semi-)norms are mesh-dependent.

3.3. Finite element spaces

To approximate $\{\boldsymbol{v},\,\hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ $(0 < \gamma \le \frac{1}{2})$ associated to $\mathcal{T}^h$, let us prepare two finite dimensional spaces: for $k \in \mathbb{N}$

$$\begin{aligned} U_k^h &:= \prod_{K\in\mathcal{T}^h} P_k(K)\ \left(\subset H^1(\mathcal{T}^h)\right), \\ \widehat{U}_k^h &:= \prod_{e\in\mathcal{E}^h} P_k(e)\ \left(\subset L^2(\Gamma^h)\right). \end{aligned}$$

Then the finite element spaces are given by

$$V_k^h := \left(U_k^h\right)^2 \times \left(\widehat{U}_k^h\right)^2 \subset H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2 \quad (\gamma > 0).$$

3.4. Lifting operators

To consider a local lifting operator [2, 12] for each $K \in \mathcal{T}^h$, let us introduce

$$Q_k^K := P_{k-1}(K).$$

We can define the local lifting operator $R_K:\ L^2(\partial K)^2 \longrightarrow \left(Q_k^K\right)^3$ by

$$([D]R_K\boldsymbol{g},\,\boldsymbol{q})_K = \langle \boldsymbol{g},\,([D]\boldsymbol{q})_{\boldsymbol{n}}\rangle_{\partial K} \qquad \forall \boldsymbol{g} \in L^2(\partial K)^2, \quad \forall \boldsymbol{q} \in \left(Q_k^K\right)^3, \tag{9}$$

where $\boldsymbol{n} = \{n_1,\,n_2\}$ is the outward unit normal on ∂K, and for every $\boldsymbol{p} = \{p_1,\,p_2,\,p_{12}\} \in \left(Q_k^K\right)^3$,

$$\boldsymbol{p}_{\boldsymbol{n}} := \begin{pmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_{12} \end{pmatrix}. \tag{10}$$

Note that replacing $\boldsymbol{p}$ by $\boldsymbol{\sigma} = \{\sigma_1,\,\sigma_2,\,\tau_{12}\}$ in (10), we get (1).

Identifying $Q_k^h := \prod_{K\in\mathcal{T}^h} Q_k^K$ with a subspace of $L^2(\Omega)$ and making the identification $\prod_{K\in\mathcal{T}^h} \left(Q_k^K\right)^3 = \left(Q_k^h\right)^3$, the global lifting operator is defined by

$$R_h:\ \tilde{\boldsymbol{g}} = \{\boldsymbol{g}_{\partial K}\}_{K\in\mathcal{T}^h} \in \prod_{K\in\mathcal{T}^h} L^2(\partial K)^2 \longrightarrow \{R_K\boldsymbol{g}_{\partial K}\}_{K\in\mathcal{T}^h} \in \left(Q_k^h\right)^3 \subset L^2(\Omega)^3.$$

3.5. Discrete problem

Let $\boldsymbol{u} \in H^{3/2+\gamma}(\Omega)$ $(\gamma > 0)$ satisfy (2) and (3), and let $\hat{\boldsymbol{u}} := \boldsymbol{u}|_{\Gamma^h} \in L^2(\Gamma^h)$. To formulate a discrete problem of (4), we derive a weak formulation which $\{\boldsymbol{u},\,\hat{\boldsymbol{u}}\}$ satisfies. From the Green formula in each $K \in \mathcal{T}^h$, we have, for all $\boldsymbol{v} \in H^1(K)^2$,

$$\int_K \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,d\boldsymbol{x} - \int_{\partial K} \boldsymbol{\sigma}_{\boldsymbol{n}}(\boldsymbol{u}) \cdot \boldsymbol{v}\,ds = \int_K \boldsymbol{f} \cdot \boldsymbol{v}\,d\boldsymbol{x}.$$

Summing up this identity over all $K \in \mathcal{T}^h$, we get for all $\boldsymbol{v} \in H^1(\mathcal{T}^h)^2$,

$$\left(\boldsymbol{\sigma}^h(\boldsymbol{u}),\,\boldsymbol{\varepsilon}^h(\boldsymbol{v})\right)_\Omega - \sum_{K\in\mathcal{T}^h} \langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\,\boldsymbol{v}\rangle_{\partial K} = (\boldsymbol{f},\,\boldsymbol{v})_\Omega\,, \tag{11}$$

where for $\boldsymbol{v} \in H^1(\mathcal{T}^h)$,

$$\boldsymbol{\varepsilon}^h(\boldsymbol{v})|_K := \boldsymbol{\varepsilon}(\boldsymbol{v}|_K) \quad \forall K \in \mathcal{T}^h,$$

$$\boldsymbol{\sigma}^h(\boldsymbol{v}) \equiv \{\sigma_1^h(\boldsymbol{v}),\, \sigma_2^h(\boldsymbol{v}),\, \tau_{12}^h(\boldsymbol{v})\} := [D]\boldsymbol{\varepsilon}^h(\boldsymbol{v}),$$

and for $\boldsymbol{v} \in H^{3/2+\gamma}(\mathcal{T}^h)$ $(\gamma > 0)$,

$$\boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{v}) := \{\sigma_1^h(\boldsymbol{v})n_1 + \tau_{12}^h(\boldsymbol{v})n_2,\, \tau_{12}^h(\boldsymbol{v})n_1 + \sigma_2^h(\boldsymbol{v})n_2\}.$$

On the other hand, we have, for all $\hat{\boldsymbol{v}} \in L^2(\Gamma^h)^2$,

$$\sum_{K\in\mathcal{T}^h} \left\langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\, \hat{\boldsymbol{v}}\right\rangle_{\partial K} = \sum_{e\in\mathcal{E}_\partial^h} \left\langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\, \hat{\boldsymbol{v}}\right\rangle_e, \tag{12}$$

where $\mathcal{E}_\partial^h := \{e \in \mathcal{E}^h \mid e \subset \partial\Omega\}$. Adding (12) to (11) leads to

$$\left(\boldsymbol{\sigma}^h(\boldsymbol{u}),\, \boldsymbol{\varepsilon}^h(\boldsymbol{v})\right)_\Omega + \sum_{K\in\mathcal{T}^h} \left\langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\, \hat{\boldsymbol{v}} - \boldsymbol{v}\right\rangle_{\partial K} = (\boldsymbol{f},\, \boldsymbol{v})_\Omega + \sum_{e\in\mathcal{E}_\partial^h} \left\langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\, \hat{\boldsymbol{v}}\right\rangle_e. \tag{13}$$

Since $\boldsymbol{u} - \hat{\boldsymbol{u}} = \boldsymbol{0}$ on Γ^h, we have

$$a_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) = (\boldsymbol{f},\, \boldsymbol{v})_\Omega + \sum_{e\in\mathcal{E}_\partial^h} \left\langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\, \hat{\boldsymbol{v}}\right\rangle_e$$

for all $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ $(\gamma > 0)$, where a_h^η is defined by

$$\begin{aligned} a_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) \;:=\; & \left(\boldsymbol{\sigma}^h(\boldsymbol{u}),\, \boldsymbol{\varepsilon}^h(\boldsymbol{v})\right)_\Omega \\ & + \sum_{K\in\mathcal{T}^h} \left[\left\langle \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{u}),\, \hat{\boldsymbol{v}} - \boldsymbol{v}\right\rangle_{\partial K} + \left\langle \hat{\boldsymbol{u}} - \boldsymbol{u},\, \boldsymbol{\sigma}_{\boldsymbol{n}}^h(\boldsymbol{v})\right\rangle_{\partial K}\right] \\ & + d\left(R_h(\hat{\boldsymbol{u}} - \boldsymbol{u}),\, R_h(\hat{\boldsymbol{v}} - \boldsymbol{v})\right) \\ & + \eta I_h(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) \end{aligned} \tag{14}$$

for every $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$, $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ $(\gamma > 0)$. Here $\eta \geq 0$ is an interior penalty parameter and the third and fourth terms in the right-hand side of (14) are called the *lifting* and *interior penalty* terms, respectively. Bilinear form a_h^η is symmetric. Moreover, taking account of the boundary condition (3), we have

$$a_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) = F_h(\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) \tag{15}$$

for all $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L_D^2(\Gamma^h)^2$ $(\gamma > 0)$, where

$$\begin{aligned} L_D^2(\Gamma^h) \;&:=\; \left\{\hat{v} \in L^2(\Gamma^h) \mid \hat{v} = 0 \text{ on } \partial\Omega_D\right\}, \\ F_h(\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) \;&:=\; (\boldsymbol{f},\, \boldsymbol{v})_\Omega + \int_{\partial\Omega_N} \overline{\boldsymbol{\sigma}}_{\boldsymbol{n}} \cdot \hat{\boldsymbol{v}}\, ds. \end{aligned}$$

We present a discrete problem of (4) as follows: find $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\} \in V_{k,D}^h$ such that

$$a_h^\eta(\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) = F_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) \quad \forall\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_{k,D}^h, \tag{16}$$

where

$$\widehat{U}_{k,D}^h := \widehat{U}_k^h \cap L_D^2(\Gamma^h), \qquad V_{k,D}^h := \left(U_k^h\right)^2 \times \left(\widehat{U}_{k,D}^h\right)^2.$$

Since the exact solution $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$ satisfies (15), discrete problem (16) is said to be consistent [2].

We will describe the unique solvability of discrete problem (16) for every $\eta > 0$ in Section 5. To do so, we first establish an inequality of Korn type in Section 4.

3.6. Some properties of the bilinear forms

Since $\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)|_K \in \left(Q^K\right)^3 \equiv P_{k-1}(K)^3$ for each $K \in \mathcal{T}^h$ and for all $\boldsymbol{v}_h \in \left(U_k^h\right)^2$, it follows from the definition (9) of R_K that

$$([D]R_K\boldsymbol{g},\, \boldsymbol{\varepsilon}^h(\boldsymbol{v}_h))_K = \langle \boldsymbol{g},\, \boldsymbol{\sigma_n}(\boldsymbol{v}_h)\rangle_{\partial K} \qquad \forall \boldsymbol{g} \in L^2(\partial K)^2, \quad \forall \boldsymbol{v}_h \in \left(U_k^h\right)^2.$$

Thus we find from (14) that for all $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\ \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h$,

$$a_h^0(\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) = d\left(\boldsymbol{\varepsilon}^h(\boldsymbol{u}_h) + R_h(\hat{\boldsymbol{u}}_h - \boldsymbol{u}_h),\, \boldsymbol{\varepsilon}^h(\boldsymbol{v}_h) + R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)\right). \tag{17}$$

This implies

$$a_h^0(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) \geq 0 \quad \forall \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h.$$

So we can define the associated semi-norm: for $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h$,

$$a_h^0(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) := a_h^0(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^{1/2}.$$

Note that for every $\boldsymbol{u}_h,\, \boldsymbol{v}_h \in \left[U_k^h \cap H^1(\Omega)\right]^2$, we have $\boldsymbol{u}_h|_{\Gamma^h},\, \boldsymbol{v}_h|_{\Gamma^h} \in L^2(\Gamma^h)^2$, and moreover

$$a_h^0\left(\{\boldsymbol{u}_h,\, \boldsymbol{u}_h|_{\Gamma^h}\},\, \{\boldsymbol{v}_h,\, \boldsymbol{v}_h|_{\Gamma^h}\}\right) = d(\boldsymbol{\varepsilon}^h(\boldsymbol{u}_h),\, \boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)). \tag{18}$$

Remark 2 If $k = 1$, then we have for every $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\ \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_1^h$,

$$a_h^0(\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) = d\left(R_h\hat{\boldsymbol{u}}_h,\, R_h\hat{\boldsymbol{v}}_h\right). \tag{19}$$

Indeed, it follows from the Green formula that

$$([D]\boldsymbol{\varepsilon}(\boldsymbol{v}),\, \boldsymbol{q})_K = \langle \boldsymbol{v},\, ([D]\boldsymbol{q})_{\boldsymbol{n}}\rangle_{\partial K} \qquad \forall \boldsymbol{v} \in H^1(K)^2, \quad \forall \boldsymbol{q} \in \left(Q_1^K\right)^3 \equiv P_0(K)^3, \tag{20}$$

where we used the fact that all first order partial derivatives of $[D]\boldsymbol{q} \in P_0(K)^3$ vanish. From (9) and (20), we can get

$$([D]\boldsymbol{\varepsilon}(\boldsymbol{v}),\, \boldsymbol{q})_K = ([D]R_K\boldsymbol{v},\, \boldsymbol{q})_K \qquad \forall \boldsymbol{v} \in H^1(K)^2, \quad \forall \boldsymbol{q} \in \left(Q_1^K\right)^3 \equiv P_0(K)^3. \tag{21}$$

Using (21), we can reduce (17) to (19).

4. An inequality of the Korn type

Let K be a star-shaped bounded domain with respect to a open disk $D \subset K$ of positive radius, that is, for every $x \in \overline{K}$ the closed convex hull of $\{x\} \cup D$ is included in $\overline{K}$ [5]. For every star-shaped bounded domain K with respect to a open disk $D \subset K$ of positive radius, we can define ρ_K as the maximum of radii of such possible D's, i.e.,

$$\rho_K := \max\left\{ r(D) \;\middle|\; \begin{array}{l} \text{open disk } D \subset K \text{ such that} \\ K \text{ is a star-shaped with respect to } D \end{array} \right\},$$

where $r(D)$ denotes the radius of D.

Then by using ρ_K and $h_K := \operatorname{diam} K$, the chunkiness parameter ζ_K for K is defined as:

$$\zeta_K := h_K/\rho_K.$$

We consider a family of partitions $\{\mathcal{T}^h\}_{0<h\leq\bar{h}}$ that satisfies the following conditions (cf. [9]):

(H1) For all $h \in (0, \bar{h}]$ ($\bar{h} \leq 1$), each $K \in \mathcal{T}^h$ is star-shaped with respect to a open disk of positive radius.

(H2) There exists a positive integer M (≥ 3) such that for all $h \in (0, \bar{h}]$ and for all $K \in \mathcal{T}^h$, the number m of elements in $\mathcal{E}^K$ is less than or equal to M.

(H3) (Chunkiness condition) There exists a positive constant γ_C such that

$$\sup_{0<h\leq\bar{h}} \max_{K\in\mathcal{T}^h} \zeta_K \leq \gamma_C.$$

(H4) (Local quasi-uniformity of edge sizes) There exists a positive constant γ_U such that for all $h \in (0, \bar{h}]$ and for all $K \in \mathcal{T}^h$

$$\max_{e\in\mathcal{E}^K} |e| \leq \gamma_U \min_{e\in\mathcal{E}^K} |e|.$$

Theorem 1 *Let $k \in \mathbb{N}$. Assume that a family $\left\{\mathcal{T}^h\right\}_{0<h\leq\bar{h}}$ of partitions of Ω satisfies conditions (H1)–(H4). Then there exists a positive constant C such that for all $h \in (0, \bar{h}]$ and for all $\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} \in V_k^h$,*

$$\|\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}\|_{1,h} \leq C\left[a_h^1(\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}) + \|\boldsymbol{v}_h\|_\Omega\right], \tag{22}$$

where C is independent of h and $\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}$.

Hereafter we will assume that $k \in \mathbb{N}$ and that a family $\left\{\mathcal{T}^h\right\}_{0<h\leq\bar{h}}$ of partitions of Ω satisfies conditions (H1)–(H4) unless otherwise stated.

To prove this theorem, we first show three lemmas.

Lemma 1 *There exists a positive constant C_r such that for all $h \in (0, \bar{h}]$, for all $K \in \mathcal{T}^h$, and for all $\boldsymbol{g} \in L^2(\partial K)^2$,*

$$\|R_K\boldsymbol{g}\|_K \leq C_r\left[\sum_{e\in\mathcal{E}^K} \frac{1}{|e|}\left|\boldsymbol{g}|_e\right|_e^2\right]^{1/2}, \tag{23}$$

where C_r is independent of h, K, and $\boldsymbol{g}$.

Proof. Lemma 1 is proved in [9, (42)]. ■

Lemma 2 *There exists a positive constant C such that for all $h \in (0, \bar{h}]$ and for all $\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} \in V_k^h$,*

$$\left\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\right\|_\Omega \leq C a_h^1(\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}), \tag{24}$$

where C is independent of h and $\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}$.

Proof. From (6), we have, for all $\boldsymbol{v}_h \in \left(U_k^h\right)^2$,

$$\left\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\right\|_\Omega \leq \underline{\alpha}^{-1/2} d(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)). \tag{25}$$

The triangle inequality yields that for every $\hat{\boldsymbol{v}}_h \in \left(\widehat{U}_k^h\right)^2$,

$$d(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)) \leq d(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h) + R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)) + d(R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)). \tag{26}$$

Using (23), we have

$$
\begin{aligned}
(27) \qquad d(R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h))^2 &= \sum_{K\in\mathcal{T}^h} ([D]R_K(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h),\, R_K(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h))_K \\
&\le \bar{\alpha} \sum_{K\in\mathcal{T}^h} \|R_K(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)\|_K^2 \quad \text{(by (6))} \\
&\le \bar{\alpha} C_r^2 I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2 \quad \text{(by (23))}.
\end{aligned}
$$

We see from (26) and (27) that

$$
(28) \qquad d(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)) \le d(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h) + R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)) + \bar{\alpha}^{1/2} C_r I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}).
$$

From (28) and (17), we find

$$
(29) \qquad d(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)) \le a_h^0(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) + \bar{\alpha}^{1/2} C_r I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}).
$$

From (25) and (29), we can get (24). ■

Let $\mathcal{E}_0^h$ be the set of interior edges of $\mathcal{T}^h$. For each $e \in \mathcal{E}^h$ let $\mathcal{V}^e$ be the set of two endpoints of $\bar{e}$, and $\mathcal{T}^e$ the set of polygons K such that $e \subset \partial K$. For $\boldsymbol{v}_h \in \left(U_k^h\right)^2$ and $e \in \mathcal{E}_0^h$, let $[\boldsymbol{v}_h]_e$ denote the jump of $\boldsymbol{v}_h$ across the edge e, i.e.,

$$
[\boldsymbol{v}_h]_e := \boldsymbol{v}_h|_{K_1^e} - \boldsymbol{v}_h|_{K_2^e} \quad \text{on} \quad e \qquad (\mathcal{T}^e = \{K_1^e,\, K_2^e\}).
$$

Lemma 3 *There exists a positive constant C such that for all $h \in (0,\, \bar{h}]$ and for all $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h$,*

$$
(30) \qquad \sum_{e\in\mathcal{E}_0^h} \sum_{p\in\mathcal{V}^e} \left|[\boldsymbol{v}_h]_e(p)\right|^2 \le C I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2,
$$

where C is independent of h and $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}$.

Proof. By a standard inverse estimate (cf. (3.10) in [4]), we have, for every $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h$,

$$
\begin{aligned}
\sum_{e\in\mathcal{E}_0^h} \sum_{p\in\mathcal{V}^e} \left|[\boldsymbol{v}_h]_e(p)\right|^2 &= \sum_{e\in\mathcal{E}_0^h} \sum_{p\in\mathcal{V}^e} \left| \boldsymbol{v}_h|_{K_1^e}(p) - \boldsymbol{v}_h|_{K_2^e}(p)\right|^2 \quad (\mathcal{T}^e = \{K_1^e,\, K_2^e\}) \\
&\le 2 \sum_{e\in\mathcal{E}_0^h} \sum_{p\in\mathcal{V}^e} \sum_{K\in\mathcal{T}^e} \left| \boldsymbol{v}_h|_K(p) - \hat{\boldsymbol{v}}_h(p)\right|^2 \\
&\le C \sum_{e\in\mathcal{E}_0^h} |e|^{-1} \sum_{K\in\mathcal{T}^e} \left| \boldsymbol{v}_h|_K - \hat{\boldsymbol{v}}_h\right|_e^2 \\
&\qquad \text{(by a standard inverse estimate)} \\
&\le C I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2. \quad ■
\end{aligned}
$$

4.1. Proof of Theorem 1 in the case when $\left\{\mathcal{T}^h\right\}_{0<h\le\bar{h}}$ is a family of triangulations

We first prove Theorem 1 in the case when each $\mathcal{T}^h$ is a triangulation without hanging nodes, i.e., a partition consisting of triangles having a property that no vertex of any triangle lies in the interior of an edge of another triangle. We then note that a family of triangulations without hanging nodes satisfies conditions (H1)–(H4) if and only if it is regular in the sense of Ciarlet [6].

In Section 4.1 we assume that $\left\{\mathcal{T}^h\right\}_{0<h\le\bar{h}}$ is a family of triangulations which is regular.

We now define

$$W^h := U_1^h \cap H^1(\Omega) = U_1^h \cap C^0(\overline{\Omega}).$$

Let $E:\, U_1^h \longrightarrow W^h$ be the *reconstruction operator* defined in [4], i.e., for every $v_h \in U_1^h$, we can define an element $Ev_h \in W^h$ by

$$(Ev_h)(p) = \frac{1}{|\mathcal{T}^p|}\sum_{K\in\mathcal{T}^p} \left(v_h|_K\right)(p) \quad \forall p \in \mathcal{V}^h,$$

where $\mathcal{T}^p := \left\{K \in \mathcal{T}^h;\, p \in \partial K\right\}$ is the set of triangles sharing p as a common vertex and $|\mathcal{T}^p|$ is the number of triangles in $\mathcal{T}^p$. Note that the restriction of E to W^h is the identity operator of W^h.

Since the family $\{\mathcal{T}^h\}_{0<h\le\bar{h}}$ of triangulations is regular, we have

$$\sup_{0<h\le\bar{h}} \max_{p\in\mathcal{V}^h} |\mathcal{T}^p| < +\infty. \tag{31}$$

We can also regard E as an operator from $\left(U_1^h\right)^2$ onto $\left(W^h\right)^2$ as $E\boldsymbol{v}_h := \{Ev_{h1},\, Ev_{h2}\} \in \left(W^h\right)^2$ for all $\boldsymbol{v}_h = \{v_{h1},\, v_{h2}\} \in \left(U_1^h\right)^2$.

Lemma 4 *Suppose that a family $\{\mathcal{T}^h\}_{0<h\le\bar{h}}$ of triangulations is regular. Then there exists a positive constant C such that for all $h \in (0,\, \bar{h}]$ and for all $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_1^h$,*

$$|\boldsymbol{v}_h - E\boldsymbol{v}_h|_{H^1(\mathcal{T}^h)} \le CI_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}), \tag{32}$$

$$\|\boldsymbol{v}_h - E\boldsymbol{v}_h\|_{\Omega} \le ChI_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}), \tag{33}$$

where C is independent of h and $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}$.

Proof. From (2.10) in [4] and the estimation in Example 2.3 of [4], we have

$$|\boldsymbol{v}_h - E\boldsymbol{v}_h|^2_{H^1(\mathcal{T}^h)} \le C\sum_{e\in\mathcal{E}_0^h}\sum_{p\in\mathcal{V}^e}\left|[\boldsymbol{v}_h]_e(p)\right|^2,$$

$$\|\boldsymbol{v}_h - E\boldsymbol{v}_h\|^2_{\Omega} \le C\sum_{e\in\mathcal{E}_0^h}\sum_{p\in\mathcal{V}^e}|e|^2\left|[\boldsymbol{v}_h]_e(p)\right|^2.$$

Combining these inequalities and (30), we get (32) and (33). ■

Remark 3 Assume that a family $\{\mathcal{T}^h\}_{0<h\le\bar{h}}$ of triangulations is regular, and that $\left\{\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_1^h\right\}_{0<h\le\bar{h}}$ satisfies

$$\|\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\|_{1,h} \le 1. \tag{34}$$

Then there exist a $\boldsymbol{v} \in H^1(\Omega)^2$ and a subsequence of $\left\{\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right\}_{0<h\le\bar{h}}$, which is also denoted by the same notation for convenience, such that $\boldsymbol{v}_h$ and $E\boldsymbol{v}_h$ converge strongly to $\boldsymbol{v}$ in $L^2(\Omega)$ as h tends to zero.

Indeed, it follows from Theorem 1 in [9] that there exist a $\boldsymbol{v} \in H^1(\Omega)^2$ and a subsequence of $\Big\{\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\Big\}_{0<h\leq\bar{h}}$ such that

$$\lim_{h\longrightarrow 0} \|\boldsymbol{v}_h - \boldsymbol{v}\|_{\Omega} = 0. \tag{35}$$

Moreover

$$\begin{aligned}
\|E\boldsymbol{v}_h - \boldsymbol{v}\|_{\Omega} &\leq \|E\boldsymbol{v}_h - \boldsymbol{v}_h\|_{\Omega} + \|\boldsymbol{v}_h - \boldsymbol{v}\|_{\Omega} \\
&\leq ChI_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right) + \|\boldsymbol{v}_h - \boldsymbol{v}\|_{\Omega} \quad \text{(by (33))} \\
&\longrightarrow 0 \quad (h \longrightarrow 0) \quad \text{(by (34) and (35))}.
\end{aligned}$$

Lemma 5 *Suppose that a family $\{\mathcal{T}^h\}_{0<h\leq\bar{h}}$ of triangulations is regular. Then there exists a positive constant C such that for all $h \in (0,\, \bar{h}]$ and for all $\boldsymbol{v}_h \in \left(U_1^h\right)^2$,*

$$I_h\left(\{\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}\right)^2 \leq C \sum_{e\in\mathcal{E}_0^h} \sum_{p\in\mathcal{V}^e} \Big|[\boldsymbol{v}_h]_e(p)\Big|^2. \tag{36}$$

where C is independent of h and $\boldsymbol{v}_h$.

Proof. We have, for every $\boldsymbol{v}_h \in \left(U_1^h\right)^2$,

$$I_h\left(\{\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}\right)^2 \leq \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \|\boldsymbol{v}_h - E\boldsymbol{v}_h\|^2_{C^0(\bar{e})^2}, \tag{37}$$

where $\|\boldsymbol{v}\|^2_{C^0(\bar{e})^2} := \sum_{j=1}^2 \left(\max_{x\in\bar{e}} |v_j(x)|\right)^2$. Since each element of $E\boldsymbol{v}_h - \boldsymbol{v}_h$ is a linear function on $\bar{e}$, it takes a maximum of its absolute value at one of the endpoints of $\bar{e}$. Thus we have

$$\sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \|\boldsymbol{v}_h - E\boldsymbol{v}_h\|^2_{C^0(\bar{e})^2} \leq 2 \sum_{K\in\mathcal{T}^h} \sum_{p\in\mathcal{V}^K} |(\boldsymbol{v}_h - E\boldsymbol{v}_h)(p)|^2, \tag{38}$$

where $\mathcal{V}^K := \big\{p \in \mathcal{V}^h \,|\, p \in \partial K\big\}$. From (31) and Lemma 2.1 in [4], we see that for all $\boldsymbol{v}_h \in \left(U_1^h\right)^2$,

$$\begin{aligned}
\sum_{K\in\mathcal{T}^h} \sum_{p\in\mathcal{V}^K} |(\boldsymbol{v}_h - E\boldsymbol{v}_h)(p)|^2 &\leq C \sum_{p\in\mathcal{V}^h} |(\boldsymbol{v}_h - E\boldsymbol{v}_h)(p)|^2 \quad \text{(by (31))} \\
&\leq C \sum_{p\in\mathcal{V}^h} \sum_{e\in\mathcal{E}_0^p} \Big|[\boldsymbol{v}_h]_e(p)\Big|^2 \quad \text{(by Lemma 2.1 in [4])},
\end{aligned} \tag{39}$$

where $\mathcal{E}_0^p := \big\{e \in \mathcal{E}_0^h;\, p \in \partial e\big\}$. Here noticing

$$\sum_{p\in\mathcal{V}^h} \sum_{e\in\mathcal{E}_0^p} \Big|[\boldsymbol{v}_h]_e(p)\Big|^2 = \sum_{e\in\mathcal{E}_0^h} \sum_{p\in\mathcal{V}^e} \Big|[\boldsymbol{v}_h]_e(p)\Big|^2, \tag{40}$$

we conclude from (37)–(40) that (36) holds. ∎

We will prove (22) for each of the cases when $k = 1$ and $k > 1$ in Sections 4.1.1 and 4.1.2, respectively.

4.1.1. Proof of (22) in the case when $k = 1$

Since $E\boldsymbol{v}_h \in H^1(\Omega)^2$ for every $\boldsymbol{v}_h \in \left(U_1^h\right)^2$, we have $E\boldsymbol{v}_h|_{\Gamma^h} \in \left(\widehat{U}_1^h\right)^2$. For every $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_1^h$, we have

$$
\begin{aligned}
(41)\quad \|\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\|_{1,h} &\leq \|\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} - \{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}\|_{1,h} + \|\{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}\|_{1,h} \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

1° We have

$$
\begin{aligned}
\mathrm{I}^2 &= \|\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} - \{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}\|_{1,h}^2 \\
&= |\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} - \{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}|_h^2 + \|\boldsymbol{v}_h - E\boldsymbol{v}_h\|_\Omega^2 \\
&= \|\boldsymbol{v}_h - E\boldsymbol{v}_h\|_{H^1(\mathcal{T}^h)}^2 + I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} - \{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}\right)^2 \\
&= \|\boldsymbol{v}_h - E\boldsymbol{v}_h\|_{H^1(\mathcal{T}^h)}^2 + I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right)^2.
\end{aligned}
$$

Thus we see from (32) and (33) that

$$(42)\qquad \mathrm{I} \leq C I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}).$$

2° Since $E\boldsymbol{v}_h \in H^1(\Omega)^2$, it follows from Korn's inequality for H^1-functions (see Theorem 11.2.16 in [5]) that

$$(43)\qquad \mathrm{II} = \|E\boldsymbol{v}_h\|_{1,\Omega} \leq C\left[\|\boldsymbol{\varepsilon}(E\boldsymbol{v}_h)\|_\Omega + \|E\boldsymbol{v}_h\|_\Omega\right].$$

Using (6) and (18), we have

$$(44)\qquad \|\boldsymbol{\varepsilon}(E\boldsymbol{v}_h)\|_\Omega \leq \underline{\alpha}^{-1/2} d(\boldsymbol{\varepsilon}^h(E\boldsymbol{v}_h)) = \underline{\alpha}^{-1/2} a_h^0(\{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}).$$

Substituting (44) into (43) yields

$$(45)\qquad \mathrm{II} \leq C\left[a_h^0(\{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}) + \|E\boldsymbol{v}_h\|_\Omega\right].$$

First let us estimate the first term in the right-hand side of (45). We have, for all $\hat{\boldsymbol{v}}_h \in \left(\widehat{U}_1^h\right)^2$,

$$(46)\qquad a_h^0(\{E\boldsymbol{v}_h,\, E\boldsymbol{v}_h|_{\Gamma^h}\}) \leq a_h^0(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) + a_h^0(\{\boldsymbol{v}_h - E\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h - E\boldsymbol{v}_h|_{\Gamma^h}\}).$$

It follows from (17) that

$$
\begin{aligned}
(47)\quad a_h^0(\{\boldsymbol{v}_h - E\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h - E\boldsymbol{v}_h|_{\Gamma^h}\}) &= d\left(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h - E\boldsymbol{v}_h) + R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)\right) \\
&\leq d\left(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h - E\boldsymbol{v}_h)\right) + d\left(R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h)\right).
\end{aligned}
$$

From (6) and (32), we have

$$
\begin{aligned}
(48)\qquad d\left(\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h - E\boldsymbol{v}_h)\right) &\leq \overline{\alpha}^{1/2}\left\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h - E\boldsymbol{v}_h)\right\|_\Omega \\
&\leq C\left|\boldsymbol{v}_h - E\boldsymbol{v}_h\right|_{H^1(\mathcal{T}^h)} \\
&\leq C I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}).
\end{aligned}
$$

Combining (47), (48), and (27), we get

$$(49)\qquad a_h^0(\{\boldsymbol{v}_h - E\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h - E\boldsymbol{v}_h|_{\Gamma^h}\}) \leq C I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}).$$

It follows directly from (46) and (49) that

$$a_h^0(\{E\boldsymbol{v}_h,\ E\boldsymbol{v}_h|_{\Gamma^h}\}) \leq C a_h^1(\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\}). \tag{50}$$

Next let us estimate the second term in the right-hand side of (45). From (33) we get

$$\|E\boldsymbol{v}_h\|_\Omega \leq \|\boldsymbol{v}_h\|_\Omega + \|E\boldsymbol{v}_h - \boldsymbol{v}_h\|_\Omega \leq \|\boldsymbol{v}_h\|_\Omega + C I_h(\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\}). \tag{51}$$

From (45), (50), and (51), we obtain

$$\mathrm{II} \leq C\left[a_h^1(\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\}) + \|\boldsymbol{v}_h\|_\Omega\right]. \tag{52}$$

We conclude from (41), (42), and (52) that (22) holds true for $k = 1$. ■

4.1.2. Proof of (22) in the case when $k > 1$

Assume $k > 1$ in this section.

Let $\mathbf{RM}(\Omega)$ be the space of (infinitesimal) rigid motions on Ω defined by

$$\mathbf{RM}(\Omega) := \left\{\boldsymbol{a} + \boldsymbol{\eta}\boldsymbol{x};\ \boldsymbol{a} \in \mathbb{R}^2 \text{ and } \boldsymbol{\eta} \in \mathfrak{so}(2)\right\},$$

where $\boldsymbol{x} = (x_1,\ x_2)^T$ is the position vector function on Ω and $\mathfrak{so}(2)$ is the Lie algebra of anti-symmetric 2×2 matrices. The spaces $\mathbf{RM}(\Omega)$ is precisely the kernel of the strain tensor [5], i.e., for $\boldsymbol{v} \in H^1(\Omega)^2$,

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = 0 \Longleftrightarrow \boldsymbol{v} \in \mathbf{RM}(\Omega). \tag{53}$$

We define on each $K \in \mathcal{T}^h$ an interpolation operator Π_K from $H^1(K)^2$ onto $\mathbf{RM}(K)$ by the following conditions:

$$\begin{aligned}
\left|\int_K (\boldsymbol{v} - \Pi_K\boldsymbol{v}) d\boldsymbol{x}\right| &= 0 \quad \forall \boldsymbol{v} \in H^1(K)^2, \\
\left|\int_K \nabla \times (\boldsymbol{v} - \Pi_K\boldsymbol{v}) d\boldsymbol{x}\right| &= 0 \quad \forall \boldsymbol{v} \in H^1(K)^2,
\end{aligned}$$

where for $\boldsymbol{v} = \{v_1,\ v_2\} \in H^1(K)^2$,

$$\nabla \times \boldsymbol{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

Let $\Pi :\ H^1(\mathcal{T}^h)^2 \longrightarrow \left(U_1^h\right)^2$ be defined by

$$(\Pi\boldsymbol{v})|_K := \Pi_K\left(\boldsymbol{v}|_K\right) \quad \forall K \in \mathcal{T}^h.$$

We have, for every $\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\} \in V_k^h$,

$$\begin{aligned}
&\|\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\}\|_{1,h} \\
&\leq\ \|\,\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\} - \{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\,\|_{1,h} + \|\,\{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\,\|_{1,h} \\
&=:\ \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{54}$$

1° We have

$$\mathrm{I}^2 = |\boldsymbol{v}_h - \Pi\boldsymbol{v}_h|^2_{H^1(\mathcal{T}^h)} + \|\boldsymbol{v}_h - \Pi\boldsymbol{v}_h\|^2_\Omega + I_h\left(\{\boldsymbol{v}_h,\ \hat{\boldsymbol{v}}_h\} - \{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right)^2. \tag{55}$$

It follows from (3.3) in [4] that

$$|\boldsymbol{v}_h - \Pi\boldsymbol{v}_h|_{H^1(\mathcal{T}^h)} \le C\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega. \tag{56}$$

We see from (3.4) in [4] and (56) that

$$\begin{aligned}\|\boldsymbol{v}_h - \Pi\boldsymbol{v}_h\|_\Omega^2 &= \sum_{K\in\mathcal{T}^h} \|\boldsymbol{v}_h - \Pi\boldsymbol{v}_h\|_K^2 \\ &\le C\sum_{K\in\mathcal{T}^h} h_K^2 \,|\boldsymbol{v}_h - \Pi\boldsymbol{v}_h|_{H^1(K)}^2 \quad \text{(by (3.4) in [4])} \\ &\le Ch^2\,|\boldsymbol{v}_h - \Pi\boldsymbol{v}_h|_{H^1(\mathcal{T}^h)}^2 \\ &\le C\bar{h}^2\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega^2 \quad \text{(by (56))}.\end{aligned} \tag{57}$$

The triangle inequality yields

$$I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} - \{\Pi\boldsymbol{v}_h,\, E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right) \le I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right) + I_h\left(\{\Pi\boldsymbol{v}_h,\, E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right). \tag{58}$$

Using (36) where $\boldsymbol{v}_h$ is replaced by $\Pi\boldsymbol{v}_h$ and (3.12) in [4], we have

$$\begin{aligned}&I_h\left(\{\Pi\boldsymbol{v}_h,\, E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right)^2 \\ &\le C\sum_{e\in\mathcal{E}_0^h}\sum_{p\in\mathcal{V}^e}\Big|\left[\Pi\boldsymbol{v}_h\right]_e(p)\Big|^2 \quad \text{(by (36))} \\ &\le C\left[\left\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\right\|_\Omega^2 + \sum_{e\in\mathcal{E}_0^h}\frac{1}{|e|}\Big|\pi_e[\boldsymbol{v}_h]_e\Big|_e^2\right] \quad \text{(by (3.12) in [4])},\end{aligned} \tag{59}$$

where for $e \in \mathcal{E}^h$, π_e is the orthogonal projection operator from $L^2(e)^2$ onto $P_1(e)^2$. We have

$$\begin{aligned}&\sum_{e\in\mathcal{E}_0^h}\frac{1}{|e|}\Big|\pi_e[\boldsymbol{v}_h]_e\Big|_e^2 \\ &= \sum_{e\in\mathcal{E}_0^h}\frac{1}{|e|}\Big|\pi_e\left(\boldsymbol{v}_h|_{K_1^e} - \boldsymbol{v}_h|_{K_2^e}\right)\Big|_e^2 \quad (\mathcal{T}^e = \{K_1^e,\, K_2^e\}) \\ &\le 2\sum_{e\in\mathcal{E}_0^h}\frac{1}{|e|}\left\{\Big|\boldsymbol{v}_h|_{K_1^e} - \hat{\boldsymbol{v}}_h\Big|_e^2 + \Big|\boldsymbol{v}_h|_{K_2^e} - \hat{\boldsymbol{v}}_h\Big|_e^2\right\} \quad \left(\forall\hat{\boldsymbol{v}}_h \in \left(\widehat{U}_k^h\right)^2\right) \\ &\le 2I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2.\end{aligned} \tag{60}$$

From (59) and (60), we get

$$I_h\left(\{\Pi\boldsymbol{v}_h,\, E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right) \le C\left[\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega + I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})\right]. \tag{61}$$

Note that we may delete π_e in (59) and (60) to derive (61).

From (55)–(58), and (61), we get

$$\mathrm{I} \le C\left[\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega + I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})\right]. \tag{62}$$

2° Since $\{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\} \in V_1^h$, it follows from (22) in the case when $k = 1$ that

$$\begin{aligned}\mathrm{II} &= \|\{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\|_{1,h} \\ &\le C\left[a_h^0\left(\{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right) + I_h\left(\{\Pi\boldsymbol{v}_h,\ E\Pi\boldsymbol{v}_h|_{\Gamma^h}\}\right) + \|\Pi\boldsymbol{v}_h\|_\Omega\right].\end{aligned} \tag{63}$$

Let us first estimate the first term in the right-hand side of (63). Since $\Pi \boldsymbol{v}_h|_K \in \mathbf{RM}(K)$ for all $K \in \mathcal{T}^h$, it follows from (53) that $\boldsymbol{\varepsilon}^h(\Pi \boldsymbol{v}_h) = \mathbf{0}$. Hence we see from (17) that

$$a_h^0(\{\Pi \boldsymbol{v}_h,\ E\Pi \boldsymbol{v}_h|_{\Gamma^h}\}) = d\left(R_h\left(\Pi \boldsymbol{v}_h - E\Pi \boldsymbol{v}_h\right)\right). \tag{64}$$

Further it follows from (6) and (23) that

$$\begin{aligned} d\left(R_h\left(\Pi \boldsymbol{v}_h - E\Pi \boldsymbol{v}_h\right)\right) &\leq \overline{\alpha}^{1/2} \left\|R_h\left(\Pi \boldsymbol{v}_h - E\Pi \boldsymbol{v}_h\right)\right\|_\Omega \quad \text{(by (6))} \\ &\leq C I_h(\{\Pi \boldsymbol{v}_h,\, E\Pi \boldsymbol{v}_h|_{\Gamma^h}\}) \quad \text{(by (23))}. \end{aligned} \tag{65}$$

Let us next estimate the third term in the right-hand side of (63). From (57), we have

$$\|\Pi \boldsymbol{v}_h\|_\Omega \leq \|\boldsymbol{v}_h\|_\Omega + \|\boldsymbol{v}_h - \Pi \boldsymbol{v}_h\|_\Omega \leq \|\boldsymbol{v}_h\|_\Omega + C\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega. \tag{66}$$

From (63)–(66), and (61), we get

$$\mathrm{II} \leq C\left[\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega + I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) + \|\boldsymbol{v}_h\|_\Omega\right]. \tag{67}$$

We finally combine (54), (62), (67), and (24) to get (22) for $k > 1$. ■

4.2. Proof of Theorem 1 in the case when $\left\{\mathcal{T}^h\right\}_{0<h\leq\bar{h}}$ is a family of general polygonal partitions

We first demonstrate that we can construct a regular family of triangulations from a family of partitions satisfying conditions (H1)–(H4).

For each $K \in \mathcal{T}^h$, there exists an open disk $D_K \subset K$ such that the radius of D_K equals ρ_K. Let c be the center of D_K. We subdivide K into m triangles by connecting c with each of nodes in $\mathcal{V}^K$ (see Figure 2), where m is the number of nodes in $\mathcal{V}^K$. We can then obtain a triangulation of Ω. We show that a family of the triangulations constructed in this way from a given family $\{\mathcal{T}^h\}_{0<h\leq\bar{h}}$ satisfies the minimum angle condition [6, 5], that is, the family of triangulations is regular.

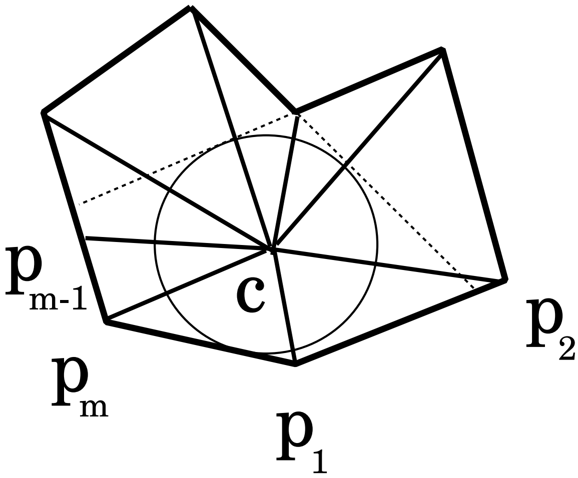

Figure 2: A triangulation of polygonal element K.

According to Kikuchi [9, Remark 4], it follows from (H2) and (H4) that for each $K \in \mathcal{T}^h$

$$\min_{e\in\mathcal{E}^K} |e| \geq \frac{2}{M\gamma_U} h_K. \tag{68}$$

We number the nodes in $\mathcal{V}^K$ from 1 to m anticlockwise as in Figure 2. For each $i = 1, 2 \ldots, m$, let s_i be the line segment between p_i and c, θ_i the angle between s_i and s_{i+1}, e_i the line segment between p_i and p_{i+1}, and T_i the triangle with vertices c, p_i, and p_{i+1}, where $s_{m+1} = s_1$ and $p_{m+1} = p_1$ (see Figure 3).

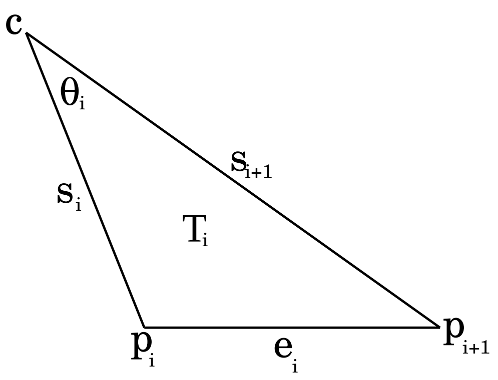

Figure 3: Triangle T_i, line segments s_i, edges e_i, and angle θ_i.

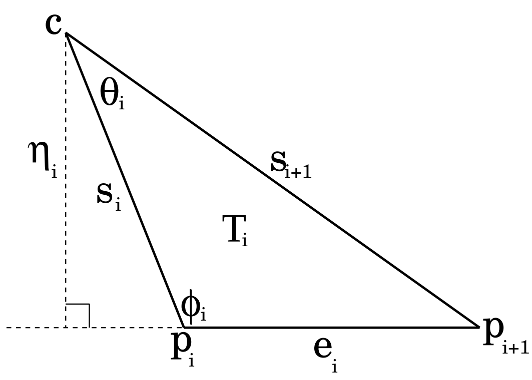

Figure 4: Angle ϕ_i and the distance η_i from c to the line including e_i.

For every $i = 1, 2 \ldots, m$, let η_i be the distance from c to the line including e_i (see Figure 4). Since $|T_i| = \frac{1}{2}\eta_i|e_i|$ and $\eta_i \geq \rho_K$, where $|T_i|$ denotes the area of T_i, we can get

$$|T_i| \geq \frac{1}{2}\rho_K|e_i|. \tag{69}$$

On the other hand, we have $|T_i| = \frac{1}{2}|s_i||s_{i+1}|\sin\theta_i$, and hence, by (69), (68), and (H3),

$$\sin\theta_i = \frac{2|T_i|}{|s_i||s_{i+1}|} \geq \frac{\rho_K|e_i|}{|s_i||s_{i+1}|} \geq \frac{\rho_K}{h_K^2}\min_{1\leq i\leq m}|e_i| \geq \frac{2}{M\gamma_U\gamma_C}. \tag{70}$$

Let ϕ_i be the angle between e_i and s_i (see Figure 4). It follows from the law of sines, (H3), and (70) that

$$\sin\phi_i = \frac{|s_{i+1}|}{|e_i|}\sin\theta_i \geq \frac{\rho_K}{h_K}\sin\theta_i \geq \gamma_C^{-1}\sin\theta_i \geq \frac{2}{M\gamma_U\gamma_C^2}. \tag{71}$$

This estimate holds for the angle between e_i and s_{i+1} as well. Therefore we can conclude from (70) and (71) that the family of triangulations satisfies the minimum angle condition.

We are now in a position to prove (22). Let $\{\widetilde{\mathcal{T}}^h\}_{0<h\leq\bar{h}}$ be the family of triangulations constructed from a given family $\{\mathcal{T}^h\}_{0<h\leq\bar{h}}$ of polygonal partitions of Ω in the way as described above. We will denote $\mathcal{E}^h$, V_k^h, a_h^η, I_h, and R_h corresponding to the triangulation $\widetilde{\mathcal{T}}^h$ by $\widetilde{\mathcal{E}}^h$, $\widetilde{V}_k^h$, $\widetilde{a}_h^\eta$, $\widetilde{I}_h$, and $\widetilde{R}_h$, respectively. For every $\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} \in V_k^h$, we can define $\{\widetilde{\boldsymbol{v}}_h, \widetilde{\hat{\boldsymbol{v}}}_h\} \in \widetilde{V}_k^h$ as follows:

$$\widetilde{\boldsymbol{v}}_h|_{\widetilde{K}} := \boldsymbol{v}_h|_{\widetilde{K}} \quad \text{for } \widetilde{K} \in \widetilde{\mathcal{T}}^h,$$

$$\widetilde{\hat{\boldsymbol{v}}}_h\Big|_{\widetilde{e}} := \begin{cases} \boldsymbol{v}_h|_{\widetilde{e}} & \text{if } \widetilde{e} \in \widetilde{\mathcal{E}}^h \setminus \mathcal{E}^h, \\ \hat{\boldsymbol{v}}_h|_{\widetilde{e}} & \text{if } \widetilde{e} \in \mathcal{E}^h. \end{cases}$$

Then for every $\widetilde{e} \in \widetilde{\mathcal{E}}^h \setminus \mathcal{E}^h$, $\widetilde{\boldsymbol{v}}_h - \widetilde{\hat{\boldsymbol{v}}}_h = 0$ on $\widetilde{e}$, and hence

$$\widetilde{I}_h\left(\left\{\widetilde{\boldsymbol{v}}_h,\, \widetilde{\hat{\boldsymbol{v}}}_h\right\}\right) = I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right), \tag{72}$$

$$\|\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\|_{1,h} = \left\|\left\{\widetilde{\boldsymbol{v}}_h,\, \widetilde{\hat{\boldsymbol{v}}}_h\right\}\right\|_{1,h}. \tag{73}$$

Using (22) in the case of triangulations and (72), we have

$$\left\|\left\{\widetilde{\boldsymbol{v}}_h,\, \widetilde{\hat{\boldsymbol{v}}}_h\right\}\right\|_{1,h} \le C\left[\widetilde{a}_h^0\left(\left\{\widetilde{\boldsymbol{v}}_h,\, \widetilde{\hat{\boldsymbol{v}}}_h\right\}\right) + I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right) + \|\boldsymbol{v}_h\|_\Omega\right], \tag{74}$$

where C is independent of h and $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}$. It follows from (17) that

$$\begin{aligned} \widetilde{a}_h^0\left(\left\{\widetilde{\boldsymbol{v}}_h,\, \widetilde{\hat{\boldsymbol{v}}}_h\right\}\right) &= d\left(\widetilde{\varepsilon}^h(\widetilde{\boldsymbol{v}}_h) + \widetilde{R}_h\left(\widetilde{\hat{\boldsymbol{v}}}_h - \widetilde{\boldsymbol{v}}_h\right)\right) \\ &\le d\left(\varepsilon^h(\boldsymbol{v}_h)\right) + d\left(\widetilde{R}_h\left(\widetilde{\hat{\boldsymbol{v}}}_h - \widetilde{\boldsymbol{v}}_h\right)\right). \end{aligned} \tag{75}$$

It follows from (23) and (72) that

$$\left\|\widetilde{R}_h\left(\widetilde{\hat{\boldsymbol{v}}}_h - \widetilde{\boldsymbol{v}}_h\right)\right\|_\Omega \le C I_h\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right). \tag{76}$$

We see from (75), (6), (24), and (76) that

$$\widetilde{a}_h^0\left(\left\{\widetilde{\boldsymbol{v}}_h,\, \widetilde{\hat{\boldsymbol{v}}}_h\right\}\right) \le C a_h^1\left(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}\right). \tag{77}$$

Combining (73), (74), and (77) leads to (22). ■

5. Unique solvability of discrete problem (16)

We will show that for each $\eta > 0$, discrete problem (16) has a unique solution $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\} \in V_{k,D}^h$ for every $\boldsymbol{f} \in L^2(\Omega)^2$ and $\overline{\boldsymbol{\sigma}}\boldsymbol{n} \in L^2(\partial\Omega_N)^2$.

We introduce the Poincaré–Friedrichs inequality and the Korn inequality of another type which are established by Brenner [3, 4]: there exists a positive constant C such that for all $\boldsymbol{v} \in H^1(\mathcal{T}^h)$,

$$\|\boldsymbol{v}\|_\Omega^2 \le C\left[|\boldsymbol{v}|_{H^1(\mathcal{T}^h)}^2 + \left|\int_{\partial\Omega_D} \boldsymbol{v}\, ds\right|^2 + \sum_{e\in\mathcal{E}_0^h} |e|^{-1}\Big|[\boldsymbol{v}]_e\Big|_e^2\right] \quad \text{(see [3, Remark 1.1])}, \tag{78}$$

$$|\boldsymbol{v}|_{H^1(\mathcal{T}^h)}^2 \le C\left[\|\boldsymbol{\varepsilon}^h(\boldsymbol{v})\|_\Omega^2 + |\boldsymbol{v}|_{\partial\Omega_D}^2 + \sum_{e\in\mathcal{E}_0^h} |e|^{-1}\Big|[\boldsymbol{v}]_e\Big|_e^2\right] \quad \text{(see [4, (1.19)])}, \tag{79}$$

where C is independent of h and $\boldsymbol{v}$. It is necessary to be careful when we use these inequalities under assumptions (H1)–(H4). Because Brenner [3, 4] assumed that there exists a fixed finite set of reference polygons such that every polygonal element in $\bigcup_{0<h\le\bar{h}} \mathcal{T}^h$ is affine homeomorphic to a reference polygon in the set, and this assumption is not included in assumptions (H1)–(H4). However, this assumption is automatically satisfied for an arbitrary family of triangulations. Moreover, according to [3, 4], we see that (78) and (79) are true for a regular family of triangulations. This implies that (78) and (79) hold good for the family $\{\widetilde{\mathcal{T}}^h\}_{0<h\le\bar{h}}$ of triangulations constructed as

in Section 4.2 from a family $\{\mathcal{T}^h\}_{0<h\le\bar{h}}$ of polygonal partitions satisfying (H1)–(H4). Furthermore we have, for all $\boldsymbol{v} \in H^1(\mathcal{T}^h) \subset H^1(\widetilde{\mathcal{T}}^h)$,

$$|\boldsymbol{v}|_{H^1(\mathcal{T}^h)} = |\boldsymbol{v}|_{H^1(\widetilde{\mathcal{T}}^h)} \quad \text{and} \quad \sum_{e\in\mathcal{E}_0^h} |e|^{-1}\Big|[\boldsymbol{v}]_e\Big|_e^2 = \sum_{e\in\widetilde{\mathcal{E}}_0^h} |e|^{-1}\Big|[\boldsymbol{v}]_e\Big|_e^2,$$

where $\widetilde{\mathcal{E}}_0^h$ is the set of interior edges of $\widetilde{\mathcal{T}}^h$. Therefore we find that (78) and (79) hold for a family $\{\mathcal{T}^h\}_{0<h\le\bar{h}}$ of polygonal partitions satisfying (H1)–(H4) as well.

From these inequalities and Theorem 1, we can establish the coreciveness of $a_h^\eta(\cdot,\,\cdot)$ on $V_{k,D}^h$. Note that $|\cdot|_h$ becomes a norm of $V_{k,D}^h$ if $|\partial\Omega_D| > 0$.

Proposition 1 *There exists a positive constant C such that for all $h \in (0,\,\bar{h}]$, for all $\eta > 0$, and for all $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_{k,D}^h$,*

$$a_h^\eta(\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}) \ge C\min\{1,\,\eta\}|\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}|_h, \tag{80}$$

where C is independent of h, η, and $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}$.

Proof. We will show (80) only in the case when $\eta = 1$. Because it follows easily from this result that (80) also holds for an arbitrary positive η. For this purpose, because of Theorem 1, it is sufficient to show that there exists a positive constant C such that for all $h \in (0,\,\bar{h}]$ and for all $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_{k,D}^h$,

$$\|\boldsymbol{v}_h\|_\Omega \le C a_h^1(\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}). \tag{81}$$

It follows from (78) and (79) that for all $\boldsymbol{v}_h \in \left(U_k^h\right)^2$,

$$\|\boldsymbol{v}_h\|_\Omega^2 \le C\left[\|\boldsymbol{\varepsilon}^h(\boldsymbol{v}_h)\|_\Omega^2 + \left|\int_{\partial\Omega_D} \boldsymbol{v}_h\,ds\right|^2 + \sum_{e\in\mathcal{E}_0^h} |e|^{-1}\Big|[\boldsymbol{v}_h]_e\Big|_e^2 + |\boldsymbol{v}_h|_{\partial\Omega_D}^2\right]. \tag{82}$$

Let us estimate each term in the right-hand side of (82). We have, for all $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_{k,D}^h$,

$$\begin{aligned} |\boldsymbol{v}_h|_{\partial\Omega_D}^2 &= \sum_{e\in\mathcal{E}_D^h} \frac{|e|}{|e|}\,|\boldsymbol{v}_h|_e^2 \\ &\le |\partial\Omega_D| \sum_{e\in\mathcal{E}_D^h} \frac{1}{|e|}\,|\boldsymbol{v}_h|_e^2 \quad (\text{by } |e| \le |\partial\Omega_D|) \\ &\le |\partial\Omega_D|\, I_h(\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\})^2 \quad (\text{by } \hat{\boldsymbol{v}}_h = 0 \text{ on } \partial\Omega_D)\,, \end{aligned} \tag{83}$$

where $\mathcal{E}_D^h := \left\{e \in \mathcal{E}^h \,|\, e \subset \partial\Omega_D\right\}$. This implies that for all $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_{k,D}^h$,

$$\left|\int_{\partial\Omega_D} \boldsymbol{v}_h\,ds\right| \le |\partial\Omega_D|\, I_h(\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}). \tag{84}$$

In a similar manner to (60) we get, for all $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_k^h$,

$$\sum_{e\in\mathcal{E}_0^h} |e|^{-1}\Big|[\boldsymbol{v}_h]_e\Big|_e^2 \le 2I_h(\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\})^2. \tag{85}$$

From (82)–(85), and (24), we can obtain (81). ■

We now define another semi-norm: for $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ $(\gamma > 0)$,

$$|||\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}|||_h^2 := |\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}|_h^2 + \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} |e| \Big|\nabla(\boldsymbol{v}|_K)\Big|_e^2. \tag{86}$$

Proposition 2 (Boundedness) *There exists a positive constant C such that for all $h \in (0,\, \bar{h}]$, for all $\eta > 0$, and for all $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\ \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ $(\gamma > 0)$,*

$$a_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\ \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) \le C \max\{1,\, \eta\} |||\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}|||_h |||\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}|||_h, \tag{87}$$

where C is independent of h, η, $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$, and $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}$.

Proof. Let us estimate each term in the right-hand side of (14). Let $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$ and $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}$ be arbitrary elements in $H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$. The Schwarz inequality yields the following estimates:

$$\big|(\boldsymbol{\sigma}^h(\boldsymbol{u}),\, \boldsymbol{\varepsilon}^h(\boldsymbol{v}))_\Omega\big| \le \overline{\alpha} \left\|\boldsymbol{\varepsilon}^h(\boldsymbol{u})\right\|_\Omega \left\|\boldsymbol{\varepsilon}^h(\boldsymbol{v})\right\|_\Omega \le C \left\|\nabla_h \boldsymbol{u}\right\|_\Omega \left\|\nabla_h \boldsymbol{v}\right\|_\Omega, \tag{88}$$

$$\begin{aligned} \sum_{K\in\mathcal{T}^h} \big|\langle \hat{\boldsymbol{u}} - \boldsymbol{u},\, \boldsymbol{\sigma}^h_{\boldsymbol{n}}(\boldsymbol{v})\rangle_{\partial K}\big| &\le \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \big|\langle \hat{\boldsymbol{u}} - \boldsymbol{u},\, \boldsymbol{\sigma}^h_{\boldsymbol{n}}(\boldsymbol{v})\rangle_e\big| \\ &\le \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \left|\hat{\boldsymbol{u}} - \boldsymbol{u}\right|_e \left|\boldsymbol{\sigma}^h(\boldsymbol{v})\right|_e \\ &= \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \frac{1}{|e|^{1/2}} \left|\hat{\boldsymbol{u}} - \boldsymbol{u}\right|_e |e|^{1/2} \left|\boldsymbol{\sigma}^h(\boldsymbol{v})\right|_e. \end{aligned} \tag{89}$$

Here we have, for each $e \in \mathcal{E}^h$,

$$\begin{aligned} \left|\boldsymbol{\sigma}^h_{\boldsymbol{n}}(\boldsymbol{v})\right|_e^2 &\le \int_e \left|\sigma_1^h n_1 + \tau_{12}^h n_2\right|^2 + \left|\tau_{12}^h n_1 + \sigma_2^h n_2\right|^2 ds \\ &\le \int_e \left|\sigma_1^h\right|^2 + 2\left|\tau_{12}^h\right|^2 + \left|\sigma_2^h\right|^2 ds \\ &\le \overline{\alpha}^2 \int_e \left|\boldsymbol{\varepsilon}^h\right|^2 ds \\ &\le C \int_e |\nabla \boldsymbol{v}|^2 ds. \end{aligned} \tag{90}$$

Substituting (90) into (89) and applying the Schwarz inequality, we get

$$\begin{aligned} &\sum_{K\in\mathcal{T}^h} \big|\langle \hat{\boldsymbol{u}} - \boldsymbol{u},\, \boldsymbol{\sigma}^h_{\boldsymbol{n}}(\boldsymbol{v})\rangle_{\partial K}\big| \\ &\le C \sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \frac{1}{|e|^{1/2}} \left|\hat{\boldsymbol{u}} - \boldsymbol{u}\right|_e |e|^{1/2} \Big|\nabla(\boldsymbol{v}|_K)\Big|_e \\ &\le C \left(\sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} \frac{1}{|e|} \left|\hat{\boldsymbol{u}} - \boldsymbol{u}\right|_e^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}^h} \sum_{e\in\mathcal{E}^K} |e| \Big|\nabla(\boldsymbol{v}|_K)\Big|_e^2\right)^{1/2}. \end{aligned} \tag{91}$$

Using (23), we obtain

$$
\begin{aligned}
(92)\qquad & \sum_{K\in\mathcal{T}^h} |([D]R_K(\hat{\boldsymbol{u}}-\boldsymbol{u}),\, R_K(\hat{\boldsymbol{v}}-\boldsymbol{v}))_K| \\
&\le \overline{\alpha} \sum_{K\in\mathcal{T}^h} \|R_K(\hat{\boldsymbol{u}}-\boldsymbol{u})\|_K \, \|R_K(\hat{\boldsymbol{v}}-\boldsymbol{v})\|_K \\
&\le C \sum_{K\in\mathcal{T}^h} \left(\sum_{e\in\mathcal{E}^K} \frac{1}{|e|}\,|\hat{\boldsymbol{u}}-\boldsymbol{u}|_e^2\right)^{1/2} \left(\sum_{e\in\mathcal{E}^K} \frac{1}{|e|}\,|\hat{\boldsymbol{v}}-\boldsymbol{v}|_e^2\right)^{1/2} \\
&\le C \left(\sum_{K\in\mathcal{T}^h}\sum_{e\in\mathcal{E}^K} \frac{1}{|e|}\,|\hat{\boldsymbol{u}}-\boldsymbol{u}|_e^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}^h}\sum_{e\in\mathcal{E}^K} \frac{1}{|e|}\,|\hat{\boldsymbol{v}}-\boldsymbol{v}|_e^2\right)^{1/2}.
\end{aligned}
$$

In addition, the Schwarz inequality gives us

$$
(93)\qquad |I_h(\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})| \le I_h(\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\}) I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}).
$$

From (14), (88), (91), (92), (93), and (86), we obtain (87). ■

Lemma 6 (Local inverse inequality) *There exists a positive constant C such that for all $h \in (0, \bar{h}]$, $K \in \mathcal{T}^h$, $e \in \mathcal{E}^K$, and $v \in P_k(K)$,*

$$
(94)\qquad |e|^{1/2}|v|_e \le C\|v\|_K,
$$

where C is independent of h, K, e, and v.

Proof. Let $\left\{\widetilde{\mathcal{T}}^h\right\}_{0<h\le\bar{h}}$ be the family of triangulations that is constructed from $\{\mathcal{T}^h\}_{0<h\le\bar{h}}$ as in Section 4.2. For each $K \in \mathcal{T}^h$ and $e \in \mathcal{E}^K$, there exists a $\widetilde{K} \in \widetilde{\mathcal{T}}^h$ such that $\widetilde{K} \subset K$ and $e \in \mathcal{E}^{\widetilde{K}}$. We choose a reference triangle $\widehat{K}$. Then there exists an affine map $F_{\widetilde{K}} : \overline{\widehat{K}} \longrightarrow \overline{\widetilde{K}}$ such that $F_{\widetilde{K}}\left(\overline{\widehat{K}}\right) = \overline{\widetilde{K}}$. For every $\widetilde{K} \in \widetilde{\mathcal{T}}^h$, $e \in \mathcal{E}^{\widetilde{K}}$, and $v \in P_k(\widetilde{K})$, we have

$$
(95)\qquad |v|_e^2 = \int_e |v|^2\, ds \le \|v\|^2_{C^0(\overline{e})}|e| \le \|v\|^2_{C^0(\overline{\widetilde{K}})}|e| = \|v \circ F_{\widetilde{K}}\|^2_{C^0(\overline{\widehat{K}})}|e|.
$$

Since norms $\|\cdot\|_{C^0(\overline{\widehat{K}})}$ and $\|\cdot\|_{\widehat{K}}$ are equivalent on $P_k(\widehat{K})$, there exists a positive constant C such that

$$
(96)\qquad \|v \circ F_{\widetilde{K}}\|_{C^0(\overline{\widehat{K}})} \le C\|v \circ F_{\widetilde{K}}\|_{\widehat{K}} = C\left(\left|\widehat{K}\right| / \left|\widetilde{K}\right|\right)^{1/2}\|v\|_{\widetilde{K}}.
$$

Further, since $\left\{\widetilde{\mathcal{T}}^h\right\}_{0<h\le\bar{h}}$ is satisfies the minimum angle condition, there exists a positive constant C such that for all $h \in (0, \bar{h}]$, $\widetilde{K} \in \widetilde{\mathcal{T}}^h$, and $e \in \mathcal{E}^{\widetilde{K}}$,

$$
(97)\qquad |e|^2|\widetilde{K}|^{-1} \le C
$$

where C is independent of h, $\widetilde{K}$, and e. From (95)–(97),

$$
|e|^{1/2}|v|_e \le C\left(\left|\widehat{K}\right| / \left|\widetilde{K}\right|\right)^{1/2}|e|\|v\|_{\widetilde{K}} \le C\|v\|_K.
$$

This shows that (94) holds. ■

Lemma 7 *The semi-norm $|||\cdot|||_h$ is equivalent to the semi-norm $|\cdot|_h$ on V_k^h.*

Proof. It is trivial that

$$|\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}|_h \le |||\,\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}\,|||_h \quad \forall\,\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_k^h.$$

So we will show that there exists a positive constant C such that

$$|||\,\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}\,|||_h \le C\,|\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}|_h \quad \forall\,\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_k^h. \tag{98}$$

For this purpose it is sufficient to estimate the term $\displaystyle\sum_{K\in\mathcal{T}^h}\sum_{e\in\mathcal{E}^K}|e|\Big|\nabla(\boldsymbol{v}_h|_K)\Big|_e^2$. For each $h \in (0,\,\bar{h}]$, $K \in \mathcal{T}^h$, $e \in \mathcal{E}^K$, and $\boldsymbol{v}_h \in \left(U_k^h\right)^2$, we have, by Lemma 6,

$$\begin{aligned}
|e|\Big|\nabla(\boldsymbol{v}_h|_K)\Big|_e^2 &= |e|\sum_{i,j=1}^{2}\left|\frac{\partial v_{hj}}{\partial x_i}\right|_e^2 \\
&\le C\sum_{i,j=1}^{2}\left\|\frac{\partial v_{hj}}{\partial x_i}\right\|_K^2 \qquad \text{(by Lemma 6)} \\
&= C\Big\|\nabla(\boldsymbol{v}_h|_K)\Big\|_K^2.
\end{aligned}$$

Thus it follows from (H2) that

$$\sum_{e\in\mathcal{E}^K}|e|\Big|\nabla(\boldsymbol{v}_h|_K)\Big|_e^2 \le CM\Big\|\nabla(\boldsymbol{v}_h|_K)\Big\|_K^2,$$

and hence

$$\sum_{K\in\mathcal{T}^h}\sum_{e\in\mathcal{E}^K}|e|\Big|\nabla(\boldsymbol{v}_h|_K)\Big|_e^2 \le CM\,\|\nabla_h\boldsymbol{v}_h\|_\Omega^2\,.$$

Thus we see that (98) holds. ■

Proposition 3 (Coreciveness) *There exists a positive constant C such that for all $h \in (0,\,\bar{h}]$, for all $\eta > 0$, and for all $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_k^h$,*

$$a_h^\eta(\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}) \ge C\min\{1,\,\eta\}|||\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}|||_h, \tag{99}$$

where C is independent of h, η, and $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}$.

Proof. Proposition 3 follows directly from Proposition 1 and Lemma 7. ■

We find from the boundedness (87) and coreciveness (99) of a_h^η that for each $\eta > 0$, discrete problem (16) has a unique solution $\{\boldsymbol{u}_h,\,\hat{\boldsymbol{u}}_h\} \in V_{k,D}^h$ for every $\boldsymbol{f} \in L^2(\Omega)^2$ and $\overline{\boldsymbol{\sigma}}\boldsymbol{n} \in L^2(\partial\Omega_N)^2$.

6. A priori error estimates

According to Kikuchi [9], if a family $\left\{\mathcal{T}^h\right\}_{0<h\le\bar{h}}$ of partitions of Ω satisfies conditions (H1)–(H4), then there exists a positive constant C such that for every $\boldsymbol{v} \in \left[H_D^1(\Omega)\cap H^{3/2+\gamma}(\Omega)\right]^2$ $(0<\gamma\le 1/2)$ we have a sequence $\{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\} \in V_{k,D}^h$ $(0<h\le\bar{h})$ which satisfies

$$|||\{\boldsymbol{v},\,\hat{\boldsymbol{v}}\} - \{\boldsymbol{v}_h,\,\hat{\boldsymbol{v}}_h\}|||_h \le Ch^{\frac{1}{2}+\gamma}\|\boldsymbol{v}\|_{\frac{3}{2}+\gamma,\Omega}, \tag{100}$$

where $\hat{\boldsymbol{v}} := \boldsymbol{v}|_{\Gamma^h}$ and C is independent of $\boldsymbol{v}$ and h.

As mentioned above, we have the consistency (15) of discrete problem (16) and the boundedness (87) and coreciveness (99) of a_h^η. Hence we can obtain the following a priori error estimates by a standard method [5, 12, 13].

Theorem 2 *Let $\boldsymbol{u}$ and $\{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}$ be the solutions of problems (4) and (16), respectively. Assume that $\boldsymbol{u} \in H^{3/2+\gamma}(\Omega)^2$ $(0 < \gamma \leq 1/2)$. Then we have*

$$|||\{\boldsymbol{u}, \hat{\boldsymbol{u}}\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}|||_h \leq C \max\{\eta^{-1}, \eta\} h^{\frac{1}{2}+\gamma} \|\boldsymbol{u}\|_{\frac{3}{2}+\gamma,\Omega}, \tag{101}$$

where $\hat{\boldsymbol{u}} := \boldsymbol{u}|_{\Gamma^h}$, and C is a positive constant independent of h, η, $\boldsymbol{u}$, and $\{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}$.

Proof. Let $\boldsymbol{e}_h := \boldsymbol{u} - \boldsymbol{u}_h$ and $\hat{\boldsymbol{e}}_h := \hat{\boldsymbol{u}} - \hat{\boldsymbol{u}}_h$. The consistency (15) implies the Galerkin orthogonality:

$$a_h^\eta(\{\boldsymbol{e}_h, \hat{\boldsymbol{e}}_h\}, \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}) = 0 \quad \forall \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} \in V_{k,D}^h. \tag{102}$$

Let $\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}$ be an arbitrary element of $V_{k,D}^h$. The triangle inequality yields

$$|||\{\boldsymbol{e}_h, \hat{\boldsymbol{e}}_h\}|||_h \leq |||\{\boldsymbol{u}, \hat{\boldsymbol{u}}\} - \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}|||_h + |||\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}|||_h. \tag{103}$$

We have

$$\begin{aligned} &|||\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}|||_h^2 \\ &\leq C \max\{1, \eta^{-1}\} a_h^\eta(\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}, \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}) \\ &\qquad \text{(by the coreciveness (99))} \\ &\leq C \max\{1, \eta^{-1}\} a_h^\eta(\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}, \hat{\boldsymbol{u}}\}, \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}) \\ &\qquad \text{(by the Galerkin orthogonality (102))} \\ &\leq C \max\{\eta^{-1}, \eta\} |||\{\boldsymbol{u}, \hat{\boldsymbol{u}}\} - \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}|||_h |||\{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\} - \{\boldsymbol{u}_h, \hat{\boldsymbol{u}}_h\}|||_h \\ &\qquad \text{(by the boundedness (87))}. \end{aligned} \tag{104}$$

It follows from (103) and (104) that

$$|||\{\boldsymbol{e}_h, \hat{\boldsymbol{e}}_h\}|||_h \leq C \max\{\eta^{-1}, \eta\} |||\{\boldsymbol{u}, \hat{\boldsymbol{u}}\} - \{\boldsymbol{v}_h, \hat{\boldsymbol{v}}_h\}|||_h. \tag{105}$$

Form (105) and (100), we obtain (101). ■

We now consider the following problem with the homogeneous Neumann boundary condition on $\partial\Omega_N$: for every $\boldsymbol{f} \in L^2(\Omega)^2$, find $\boldsymbol{w} \in H_D^1(\Omega)^2$ such that

$$a(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{f})_\Omega \quad \forall \boldsymbol{v} \in H_D^1(\Omega)^2. \tag{106}$$

If the solution $\boldsymbol{w}$ of (106) belongs to $H^2(\Omega)^2$ for every $\boldsymbol{f} \in L^2(\Omega)^2$, then the closed graph theorem implies that there exists a positive constant C such that for every $\boldsymbol{f} \in L^2(\Omega)^2$,

$$\|\boldsymbol{w}\|_{2,\Omega} \leq C \|\boldsymbol{f}\|_\Omega. \tag{107}$$

The Aubin–Nitsche duality argument derives the following L^2-error estimate [5, 12, 13].

Theorem 3 *Let $\boldsymbol{u}$ and $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\}$ be the solutions of problems (4) and (16), respectively. Assume that $\boldsymbol{u} \in H^{3/2+\gamma}(\Omega)^2$ $(0 < \gamma \le 1/2)$ and that the solution of (106) belongs to $H^2(\Omega)^2$ for every $\boldsymbol{f} \in L^2(\Omega)^2$. Then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_\Omega \le C \max\{\eta^{-1},\, \eta^2\} h^{\frac{3}{2}+\gamma} \|\boldsymbol{u}\|_{\frac{3}{2}+\gamma,\Omega}, \tag{108}$$

where C is a positive constant independent of h, η, $\boldsymbol{u}$, and $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\}$.

Proof. Let $\boldsymbol{w} \in H^1_D(\Omega)$ be the solution of (106) where $\boldsymbol{f} := \boldsymbol{e}_h$. We have

$$\begin{aligned}
\|\boldsymbol{e}_h\|^2_\Omega &= a^\eta_h(\{\boldsymbol{e}_h,\, \hat{\boldsymbol{e}}_h\},\, \{\boldsymbol{w},\, \hat{\boldsymbol{w}}\}) \quad \text{(by the consistency (15))} \\
&= a^\eta_h(\{\boldsymbol{e}_h,\, \hat{\boldsymbol{e}}_h\},\, \{\boldsymbol{w},\, \hat{\boldsymbol{w}}\} - \{\boldsymbol{\chi}_h,\, \hat{\boldsymbol{\chi}}_h\}) \\
&\qquad \forall \{\boldsymbol{\chi}_h,\, \hat{\boldsymbol{\chi}}_h\} \in V^h_{k,D} \quad \text{(by the Galerkin orthogonality (102))} \\
&\le C \max\{1,\, \eta\} |||\, \{\boldsymbol{e}_h,\, \hat{\boldsymbol{e}}_h\}\, |||_h |||\, \{\boldsymbol{w},\, \hat{\boldsymbol{w}}\} - \{\boldsymbol{\chi}_h,\, \hat{\boldsymbol{\chi}}_h\}\, |||_h \\
&\qquad \text{(by the boundedness (87))} \\
&\le \max\{\eta^{-1},\, \eta^2\} h^{\frac{3}{2}+\gamma} \|\boldsymbol{u}\|_{\frac{3}{2}+\gamma,\Omega} \|\boldsymbol{w}\|_{2,\Omega} \quad \text{(by Theorem 2 and (100))} \\
&\le \max\{\eta^{-1},\, \eta^2\} h^{\frac{3}{2}+\gamma} \|\boldsymbol{u}\|_{\frac{3}{2}+\gamma,\Omega} \|\boldsymbol{e}_h\|_\Omega \quad \text{(by (107))}\,.
\end{aligned}$$

This implies that (108) holds. ■

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A. A formulation in Kikuchi–Ishii–Oikawa [10]

Subtracting the lifting term from a_h^η gives the following bilinear form:

$$b_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) := a_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) - d(R_h(\hat{\boldsymbol{u}} - \boldsymbol{u}),\, R_h(\hat{\boldsymbol{v}} - \boldsymbol{v})).$$

Kikuchi–Ishii–Oikawa [10] used the bilinear from b_h^η to formulate the following discrete problem: find $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\} \in V_{k,D}^h$ such that

$$b_h^\eta(\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\},\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) = F_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) \quad \forall\, \{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_{k,D}^h. \tag{109}$$

This discrete problem is also consistent, that is, the exact solution $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$ satisfies

$$b_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\, \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) = F_h(\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\})$$

for all $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L_D^2(\Gamma^h)^2$ $(\gamma > 0)$.

Boundedness holds for b_h^η as well.

Proposition 4 (Boundedness) *There exists a positive constant C such that for all $h \in (0,\, \bar{h}]$, for all $\eta > 0$, and for all $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$, $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\} \in H^{\frac{3}{2}+\gamma}(\mathcal{T}^h)^2 \times L^2(\Gamma^h)^2$ $(\gamma > 0)$,*

$$b_h^\eta(\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\},\; \{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}) \le C \max\{1,\, \eta\} |||\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}|||_h |||\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}|||_h,$$

where C is independent of h, η, $\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\}$, and $\{\boldsymbol{v},\, \hat{\boldsymbol{v}}\}$.

We can show the coreciveness for b_h^η.

Proposition 5 (Coreciveness) *There exists positive constants η_0 and C such that for all $h \in (0,\, \bar{h}]$, for all $\eta \ge \eta_0$, and for all $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h$,*

$$b_h^\eta(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}) \ge C |||\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}|||_h, \tag{110}$$

where η_0 and C are independent of h, η, $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}$.

Proof. We have, for all $\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\} \in V_k^h$,

$$\begin{aligned} & b_h^0(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2 + \eta I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2 \\ & = a_h^1(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2 + (\eta - 1) I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2 - d(R_h(\hat{\boldsymbol{v}}_h - \boldsymbol{v}_h))^2 \quad \text{(by (27))} \\ & \ge C |||\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\}|||_h^2 + (\eta - 1 - \overline{\alpha} C_r^2) I_h(\{\boldsymbol{v}_h,\, \hat{\boldsymbol{v}}_h\})^2. \end{aligned}$$

Thus if $\eta \ge 1 + \overline{\alpha} C_r^2 =: \eta_0$ then (110) holds. ■

Theorem 4 *Let $\boldsymbol{u}$ and $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\}$ be the solutions of problems (4) and (109), respectively. Assume that $\boldsymbol{u} \in H^{3/2+\gamma}(\Omega)^2$ $(0 < \gamma \le 1/2)$ and that the solution of (106) belongs to $H^2(\Omega)^2$ for every $\boldsymbol{f} \in L^2(\Omega)^2$. Then there exist positive constants η_0 and C such that for all $h \in (0,\, \bar{h}]$ and for all $\eta \ge \eta_0$,*

$$\begin{aligned} |||\{\boldsymbol{u},\, \hat{\boldsymbol{u}}\} - \{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\}|||_h &\le C \max\{1,\, \eta\} h^{\frac{1}{2}+\gamma} \|\boldsymbol{u}\|_{\frac{3}{2}+\gamma,\Omega}, \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_\Omega &\le C \max\{1,\, \eta^2\} h^{\frac{3}{2}+\gamma} \|\boldsymbol{u}\|_{\frac{3}{2}+\gamma,\Omega}, \end{aligned}$$

where η_0 and C are independent of h, η, $\boldsymbol{u}$, and $\{\boldsymbol{u}_h,\, \hat{\boldsymbol{u}}_h\}$.

Proof. Theorem 4 is proved in exactly the same way as in Theorems 2 and 3. ■