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Abstract

Factor inputs often generate joint products (by-product) that impair production. In some cases, called strong input-generated production externalities in this paper, these negative effects can be so strong that full use of factors becomes inefficient, and therefore factor use along the production possibility frontier (PPF) is endogenously determined. This paper examines monotonicity, continuity, convexity and other properties of the PPF in such situation. I show that the PPF is strictly decreasing and continuous, but may jump at the corner. The PPF is convex if the by-product generation function is quasi-concave. Moreover, the PPF is either entirely strictly convex or linear if the by-product generation function is linear.

Keywords: Production possibility frontier; joint product; input-generated; convexity

JEL classification: C02; D62; H41

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1 Introduction

Everyone has 24 hours a day. But it is unwise for anyone to work for 24 hours without rest to recover from fatigue. This inefficiency of fully using resources in production, hereafter called strong input-generated production externalities, may arise through two channels. First, there are intermediate processes, such as rest, that are essential to production and require factor inputs as well. Second, factor inputs generate joint products (by-product) that hamper production, such as congestion, resource depletion, and pollution. When the amount of factor is too large or these negative effects are strong enough, full use of factors becomes inefficient. Traffic jams is the example.

In the presence of strong input-generated production externalities, factor use along the production possibility frontier (PPF) is endogenously determined. The purpose of this study is to examine how such externalities affect the properties of the PPF, including monotonicity, continuity, convexity and others. I focus on the single-factor case to neutralize the effect of factor substitution that drives the PPF concave. I show that the PPF is strictly decreasing and continuous when all goods are produced, but may jump when a good stops to be produced. I also show that the PPF is (strictly) convex to the origin if the by-product generation function is (strictly) quasi-concave. I then analyze the model by assuming differentiability to obtain further insights. Among those, I derive the sufficient conditions for the set of factor use on the PPF to be convex-valued and, even stronger, single-valued. Moreover, I show that the PPF is either entirely strictly convex or linear given a linear by-production generation function.

In the literature, the analysis of the PPF under production externalities usually focuses on output-generated externalities (e.g., Herberg and Kemp, 1969; Herberg et al., 1982; Dalal, 2006). However, full employment holds on the PPF in the presence of output-generated production externalities, thus leaving no space for the focus of this study: how the trade-off between factor use and productivity, not rare phenomena in reality as the two examples above have suggested, affects the PPF. Another closely related literature is on public intermediate goods (e.g., Manning and McMillan, 1979; Tawada and Abe, 1984).[1]

[1]There are two types of public intermediate goods often referred to. The semi-public type, corresponding with Meade's (1952) "unpaid factors of production", enters the production function like a factor of

This study embraces the "constant returns to scale" case in Manning and McMillan (1979) as a special case, and extends their work by deriving similar results with less restrictive assumptions and by exploring new results about the PPF.[2] Our results also suggest that the PPF in their model is either entirely strictly convex or linear, thus not able to have mixed intervals as they implicitly suggest. The model in this study is static. But one can see it as the steady-state version of a dynamic model, so the results derived here remain valid for the steady-state PPF in corresponding dynamic models.

The rest of the paper is organized as follows. Section 2 presents the model and basic assumptions. Section 3 examines three basic properties of the PPF without using calculus. Section 4 characterizes calculus-based properties. Section 5 applies these results to several special cases. The last section concludes.

2 The Model

There is a single factor of production (v), two final goods (x and y), and a by-product (z). The technology satisfies

$$x = G^x(z)\, v_x, \tag{1a}$$

$$y = G^y(z)\, v_y, \tag{1b}$$

$$z = R(v_x, v_y), \tag{1c}$$

where $v_i \geq 0$ $(i = x, y)$ denotes the use of factor in good i, $R(v_x, v_y) \geq 0$ is the generation function of by-product, $G^i(z) \geq 0$ is the productivity function representing the product-specific relationship between the productivity and the amount of by-product.

The technology above formulates various scenarios in reality. For example, z can be regarded as the extraction of fishery resource, whose increase reduces fishery stock

production. The pure public type, corresponding with Meade's "creation of atmosphere", affects the total factor productivity.

[2]Tawada and Abe (1984) analyze a two-factor model of pure public intermediate goods and focus on the special case that industries have identical sensitivity to by-product. They find that the PPF is necessarily concave. Abe et al. (1986) obtain the same result while allowing non-separability of the production function and any number of factors. This study has very different focus from theirs and attempts to highlight the effect on the PPF of the difference in the sensitivity to by-product.

and thus makes fishing more difficult. Similarly, z can be the emission of pollution, which harms agriculture, tourism and many other industries sensitive to the environment. We can also interpret z as the use of factor in final production other than intermediate processes. So greater z implies less intermediates and consequently lower productivities. In this sense, Manning and McMillan's (1979) "constant returns to scale" (or pure public intermediate good) falls into a special case here.[3]

The PPF is defined by the maximum value function[4]

$$y = T(x) \equiv \max_{C(x)} G^y(z) v_y, \tag{2}$$

where $C(x)$ denotes the constraint set

$$C(x) \equiv \{(z, v_x, v_y); G^x(z) v_x \geq x, z = R(v_x, v_y), v_x + v_y \leq E\}. \tag{3}$$

The inequality $G^x(z) v_x \geq x$ means that free disposal is available, E is the factor endowment. Let $S(x)$ denote the solution set

$$S(x) = \arg\max_{C(x)} G^y(z) v_y. \tag{4}$$

To exclude trivial cases, assume that the feasible maximum outputs of x and y, denoted by $\bar{x}$ and $\bar{y}$, are positive. By the definition of $T(x)$, we have $T(0) = \bar{y}$ and $T(\bar{x}) = 0$.

The analysis proceeds by assuming that

(A1) $R(v_x, v_y)$ and $G^i(z)$ are continuous in all augments;

(A2) $R(v_x, v_y)$ is strictly increasing in all arguments;

(A3) $v_x + v_y \leq E$ is slack on the PPF.

[3]To see this, introduce a constant L, two variables L_r and r, and two functions $f_r(L_r)$ and $A^i(r)$. Let $L_r = L - z$, $r = f_r(L_r)$, $A^i(f_r(L - z)) = G^i(z)$, then $G^i(z) = A^i(f_r(L_r)) = A^i(r)$. Moreover, let $R(v_x, v_y) = v_x + v_y$, then $L_r = L - z = L - v_x - v_y$. So, (1) can be rewritten into $x = A^x(r) v_x$, $y = A^y(r) v_y$, $r = f_r(L_r)$ and $L = v_x + v_y + L_r$, which is exactly the "constant returns to scale" case in Manning and McMillan (1979).

[4]Defining the PPF as in (2) has a limitation. That is, if there is non-bijective mapping between x and y on the frontier, (2) describes only the upper locus of the PPF. But this limitation will not present a big problem here since Proposition 2 shows that (2) is strictly decreasing over its domain. This means that, at most, some vertical lines are degenerated to discontinuous jump points.

Assumption (A2) implies that given a level of z, there is a bijective mapping between v_x and v_y. Assumption (A1) and (A2) together imply

Lemma 1. *Given (A1) and (A2), the constraint* $G^x(z)\, v_x \geq x$ *binds on the PPF.*

Assumption (A3) is imposed so as to focus on strong input-generated production externalities. A simple example satisfying (A3) is that $x = (1 - v_x - v_y)\, v_x$, $y = (1 - v_x - v_y)\, v_y$, and the factor endowment $E = 1$. If (A3) fails to hold, the shape of the PPF depends on specific forms of $R(v_x, v_y)$ and $G^i(z)$.[5] Note that although (A3) excludes those by-products with only nonnegative effects ($dG^i(z)/dz \geq 0$ for all $z \geq 0$) such as knowledge spillover, it does not exclude positive externalities in certain ranges.

3 Monotonicity, Continuity and Convexity of the PPF

In this section, I establish monotonicity, continuity and convexity of the PPF. Define

$$\tilde{x} \equiv \inf \{x; T(x) = 0\}. \tag{5}$$

Thus, $\tilde{x} \in [0, \bar{x}]$. We can show that

Proposition 2 (Monotonicity)**.** *Given (A1) and (A2), the PPF,* $T(x)$*, is strictly decreasing over* $[0, \tilde{x}]$*, and satisfies* $T(x) = 0$ *over* $(\tilde{x}, \bar{x}]$*.*

Does $T(\tilde{x}) = 0$ hold? This depends on the continuity of $T(x)$ at $x = \tilde{x}$, which is established as follows. For convenience, let $Z(x)$ denote the set of by-product outputs at $(x, T(x))$ on the PPF.

Proposition 3 (Continuity)**.** *Given (A1) and (A2), the PPF,* $T(x)$*, is continuous over* $(0, \bar{x}]$*. Moreover,* $T(x)$ *is continuous at* $x = 0$ *if and only if*

$$\forall \sigma > 0, \exists z^0 \in Z(0) \text{ and } z' \in \left(z^0 - \sigma, z^0 + \sigma\right) \text{ so that } G^x(z') > 0. \tag{6}$$

[5] The following example clearly illustrates this point. Let $x = (3 - R(v_x, v_y))\, v_x$, $y = (3 - R(v_x, v_y))\, v_y$, and the factor endowment $E = 1$. It is easy to check that if $R(v_x, v_y) = v_x^2 + v_y^2$, then full employment holds on the PPF and the PPF is concave. If $R(v_x, v_y) = v_x^{1/2} + v_y^{1/2}$, full employment also holds on the PPF and the PPF is convex. If $R(v_x, v_y) = v_x^{1/2} + v_y^2$, full employment still holds on the PPF and now the PPF is concave when close to x axis and convex when close to y axis.

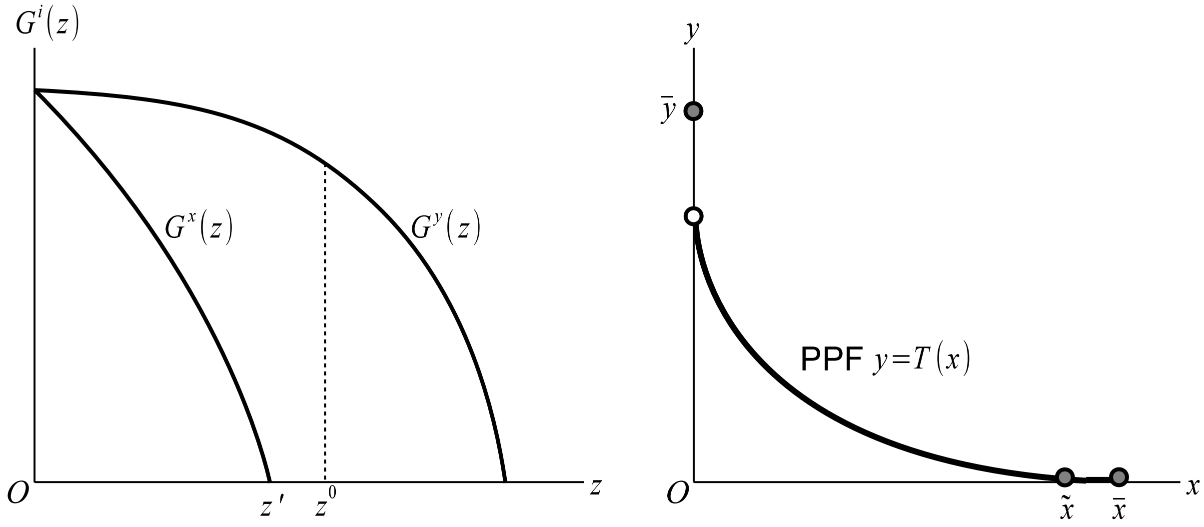

Figure 1: Jump discontinuity on the PPF

As well known, the PPF is discontinuous at the corner if there exists fixed cost. Proposition 3 indicates another channel through which the discontinuity may arise. As illustrated in the left diagram in Figure 1, y reaches the maximum $\bar{y}$ at $z = z^0$, but $z' < z^0$ and $G^x(z) = 0$ for $z \geq z'$. The PPF is therefore discontinuous at $x = 0$ according to Proposition 3, as shown in the right diagram. Note that $G^x(z^0) = 0$ for all $z^0 \in Z(0)$ does not necessarily mean that $T(x)$ is discontinuous at $x = 0$. As long as there exists $z^0 \in Z(0)$ so that $G^x(z) > 0$ in an arbitrarily small neighborhood of z^0, for example $G^x(z^0) = 0$ but $G^x(z) > 0$ for $z < z^0$, then $T(x)$ is continuous at $x = 0$.

The following proposition is about the convexity of the PPF. For detailed characterization of concavity and quasi-concavity, and pseudo-concavity that will arise later on, see, e.g., Diewert et al. (1981).

Proposition 4 (Convexity)**.** *Given (A1), (A2) and (A3), if* $R(v_x, v_y)$ *is quasi-concave, the PPF,* $T(x)$*, is convex over* $(0, \bar{x}]$*. If* $R(v_x, v_y)$ *is strictly quasi-concave,* $T(x)$ *is strictly convex over* $(0, \tilde{x})$*.*

Since quasi-concavity covers many functions used in economics such as the CES function, Proposition 4 suggests that the PPF tends to be convex in the presence of strong input-generated production externalities when there is only a single factor of production. The intuition of the proof is straightforward. If we fix z at certain level, feasible output bundles will lie on a locus convex to the origin due to the quasi-concavity of $R(v_x, v_y)$.

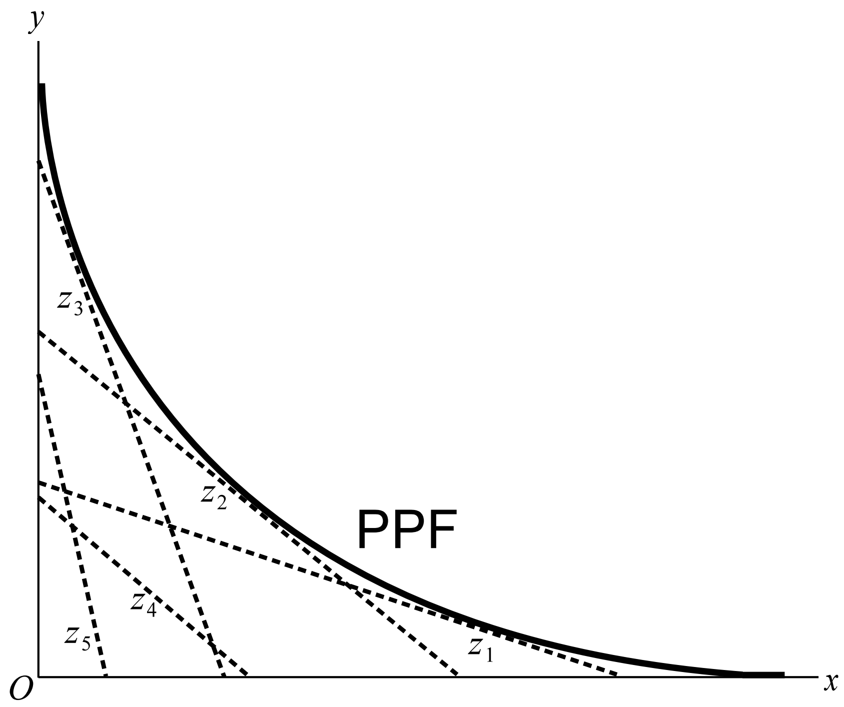

Figure 2: The PPF as the upper envelope

Fix z at another level, we can obtain another convex curve. Repeating this yields a family of loci convex to the origin. As shown in Figure 2, in which a linear $R(v_x, v_y)$ is assumed, five line segments are generated by changing z from z_1 to z_5. The PPF is the upper envelope of these convex loci and thus is also convex. Note that some values of z may generate loci not contributing to the PPF, such as z_4 and z_5 in the figure.

4 Calculus-based Properties

To derive richer results by exploiting calculus, replace assumption (A1) with

(A1′) $R(\cdot,\cdot)$ and $G^i(\cdot)$ are of class C^2.

It is convenient to define the sensitivity to by-product as the elasticity of productivity with respect to the level of by-product:

$$\varepsilon_i \equiv -\frac{d\ln G^i(z)}{d\ln z}, i = x, y.$$

To save notations, hereafter let G^i, $G^{i\prime}$ and $G^{i\prime\prime}$ denote respectively $G^i(z)$, $dG^i(z)/dz$ and $d^2G^i(z)/dz^2$ whenever no confusion arises. In what follows, we shall focus on the interval $(0, \tilde{x})$ for two reasons. First, $T(x) = 0$ over $[\tilde{x}, \bar{x}]$, which is of no special interest. Second, $x \in (0, \tilde{x})$ ensures positive outputs, which simplifies the first-order condition from the Kuhn–Tucker type to a system of equations.

Using Lemma 1 and (A3), rewrite the original problem (2) into

$$T(x) \equiv \max_{v_x, v_y} G^y (R(v_x, v_y)) v_y,$$

subject to $G^x (R(v_x, v_y)) v_x = x$. The Lagrangian can be written as

$$\mathcal{L} = G^y (R(v_x, v_y)) v_y + p (G^x (R(v_x, v_y)) v_x - x),$$

where p is the Lagrange multiplier representing the shadow price of x measured by y. For any $x \in (0, \tilde{x})$, the first-order condition yields

$$p = \frac{G^y R_x}{G^x R_y}, \tag{7a}$$

$$1 = \varepsilon_x \frac{R_x v_x}{z} + \varepsilon_y \frac{R_y v_y}{z}, \tag{7b}$$

where R_x and R_y denote respectively $\partial R(v_x, v_y) / \partial v_x$ and $\partial R(v_x, v_y) / \partial v_y$. We can also define

$$w \equiv p G^x / R_x, \tag{8}$$

which measures the marginal return associated with an increase in the by-product. To see this, consider a unit of increase in the by-product, which implies an increase in the use of factor in good x by $1/R_x$ units if factor use in good y is fixed. This yields a return of pG^x/R_x if the productivity remains unchanged. On the other hand, it follows from the definition of w and the first-order condition that

$$w = -\left(p G^{x\prime} v_x + G^{y\prime} v_y\right) = \frac{G^y}{R_y}. \tag{9}$$

Note that $(pG^{x\prime} v_x + G^{y\prime} v_y)$ measures the marginal loss due to declines in the productivity caused by an increase in the by-product. So the first equality in (9) means that the marginal return and loss must be equalized in optimum. Also note that G^y/R_y measures the marginal return associated with an increase in the by-product but through producing more good y, so the second equality in (9) indicates that the marginal return through producing more of either good must be equalized in optimum, too. Therefore, w is the

Pigouvian tax which can be imposed to producers to make a market-based economy operate on the PPF.[6]

Let $H \equiv \partial^2 \mathcal{L} / \partial (p, v_x, v_y)^2$ denote the Hessian matrix of $\mathcal{L}$, then the second-order necessary condition requires that $|H| \geq 0$. However, if $|H| = 0$, there is a kink on $T(x)$ and thus $T''(x)$ is not well-defined. To avoid such difficulty, in what follows we focus on the case of $|H| > 0$.

The following proposition provides a precise version of Proposition 4.

Proposition 5. *Given (A1′), (A2) and (A3), then*

$$T''(x) = \underbrace{\frac{wR_x}{(G^x)^2}}_{>0} Q + \underbrace{\frac{(wR_xR_y)^2}{z^2 |H|}}_{>0} D^2, x \in (0, \tilde{x}) \tag{10}$$

where

$$\begin{aligned} Q &\equiv 2\frac{R_{xy}}{R_xR_y} - \frac{R_{xx}}{R_x^2} - \frac{R_{yy}}{R_y^2} = q_1 + q_2, \\ D &\equiv \varepsilon_y (1 - R_y v_y q_2) - \varepsilon_x (1 - R_x v_x q_1), \\ q_1 &\equiv \frac{R_{xy}}{R_xR_y} - \frac{R_{xx}}{R_x^2}, q_2 \equiv \frac{R_{xy}}{R_xR_y} - \frac{R_{yy}}{R_y^2}. \end{aligned}$$

The implication of Proposition 5 comes by noting that Q has the same sign with the bordered Hessian matrix of $R(v_x, v_y)$:

$$Q = \frac{1}{(R_xR_y)^2} \begin{vmatrix} 0 & R_x & R_y \\ R_x & R_{xx} & R_{xy} \\ R_y & R_{xy} & R_{yy} \end{vmatrix}.$$

Hence, if $R(v_x, v_y)$ is quasi-concave, then $Q \geq 0$ and, according to (10), $T''(x) \geq 0$. So $T(x)$ is convex. If $R(v_x, v_y)$ is strictly quasi-concave, then $Q > 0$ and thus $T(x)$ is strictly convex. On the other hand, if $R(v_x, v_y)$ is quasi-convex, $Q \leq 0$ and the sign of $T''(x)$ becomes indeterminate. The curvature at each point on the PPF then depends on the relative magnitude of two terms in (10) valued at that point.

[6] To verify this, we can write the Lagrangian as $\mathcal{L} = G^y(z) v_y + p(G^x(z) v_x - x) - w(R(v_x, v_y) - z)$ and obtain (8) and (9) from the first-order condition. We do not write in this way only for simpler calculation.

Proposition 6. *Given assumption (A1′), (A2) and (A3), then along the PPF*

$$\frac{dz}{dx} = \underbrace{\frac{G^x G^y R_x R_y}{z\,|H|}}_{>0} D, x \in (0, \tilde{x}) \tag{11}$$

where D is defined as in Proposition 5.

The sign of dz/dx depends on that of D, and thus is ambiguous without further information on the specific forms of $R(v_x, v_y)$, $G^i(z)$, and the value of x.

In what follows, we show under what condition the solution set $S(x)$ defined in (4) is convex-valued. For this purpose, assume that

(A4) $G^i(z)$ $(i = x, y)$ is quasi-concave and $1/G^i(z)$ is convex.

Assumption (A4) is useful as shown subsequently. First, the quasi-concavity of $G^i(z)$ ensures the convexity of the domain of relevant functions. That is,

Lemma 7. *If* $G^i(z)$ $(i = x, y)$ *is quasi-concave, then* $\{z; G^i(z) > 0\}$ *and* $\{(z, v_i); G^i(z)\, v_i > 0\}$ *are open convex sets.*

On the other hand, the convexity of $1/G^i(z)$ implies that

Lemma 8. *Given (A1′), if* $1/G^i(z)$ *is (strictly) convex, then* $G^i(z)\, v_i$ *is (strictly) pseudo-concave with respect to* (z, v_i).

Note that the convexity of $1/G^i(z)$ is not as strong as it seems. For example, the (strict) concavity of $G^i(z)$ is sufficient for the (strict) convexity of $1/G^i(z)$. Finally, the following lemma is also useful.

Lemma 9. *If* $G^i(z)\, v_i$ $(i = x, y)$ *is (strictly) pseudo-concave with respect to* (z, v_i) *and if* $R(v_x, v_y)$ *is convex, then* $G^i(R(v_x, v_y))\, v_i$ *is (strictly) pseudo-concave with respect to* (v_x, v_y).

Using these lemmas, it can be shown that

Proposition 10. *Given (A1′), (A2), (A3) and (A4), if* $R(v_x, v_y)$ *is convex, then the solution set* $S(x)$ *is convex-valued for any* $x \in (0, \tilde{x})$*. Moreover, if either* $1/G^x(z)$ *or* $1/G^y(z)$ *is strictly convex, then* $S(x)$ *is single-valued and can be represented by a* C^1 *vector function.*

5 Applications

In this section, I apply the results above to several special cases of $R(v_x, v_y)$. First, consider a special form of $R(v_x, v_y)$ as follows.

$$R(v_x, v_y) = I(r_1 v_x + r_2 v_y), \tag{12}$$

where $r_1, r_2 > 0$ are constants and $I'(\cdot) > 0$ is a strictly increasing function. We can apply Proposition 5 and 6 to this special form of $R(v_x, v_y)$. It follows that $Q = q_1 = q_2 = 0$ and $D = \varepsilon_y - \varepsilon_x$. Substitute into (10) and obtain

$$T''(x) = \underbrace{\frac{(wr_1r_2)^2 I'^4}{z^2 |H|}}_{>0} (\varepsilon_y - \varepsilon_x)^2, \tag{13}$$

which implies directly the following corollary:

Corollary 11. *Given (A1′), (A2), (A3) and (12), the PPF, $T(x)$, is strictly convex for any $x \in (0, \tilde{x})$ if and only if $\varepsilon_x \neq \varepsilon_y$ there.*

The corollary highlights how the difference in the sensitivity between two goods renders the PPF convex. To see how the output of by-product changes along the PPF, substitute (12) into (11) and obtain

$$\frac{dz}{dx} = \underbrace{\frac{G^x G^y r_1 r_2 I'^2}{z |H|}}_{>0} (\varepsilon_y - \varepsilon_x), \tag{14}$$

which yields directly the following corollary:

Corollary 12. *Given (A1′), (A2), (A3) and (12), the sign of dz/dx on the PPF for any $x \in (0, \tilde{x})$ is determined by the sign of $(\varepsilon_y - \varepsilon_x)$ there.*

Corollary 11 and 12 are similar with Manning and McMillan's (1979) Proposition 5 and 6. In their model, the by-product generation function takes the form of $R(v_x, v_y) = v_x + v_y$, which is a special case of (12). In this sense, Corollary 11 and 12 are more general than their Proposition 5 and 6.

Second, consider another case of $R(v_x, v_y)$.

$$R(v_x, v_y) \text{ is linearly homogeneous and quasi-convex.} \tag{15}$$

Then we can obtain the following proposition by applying Proposition 10.

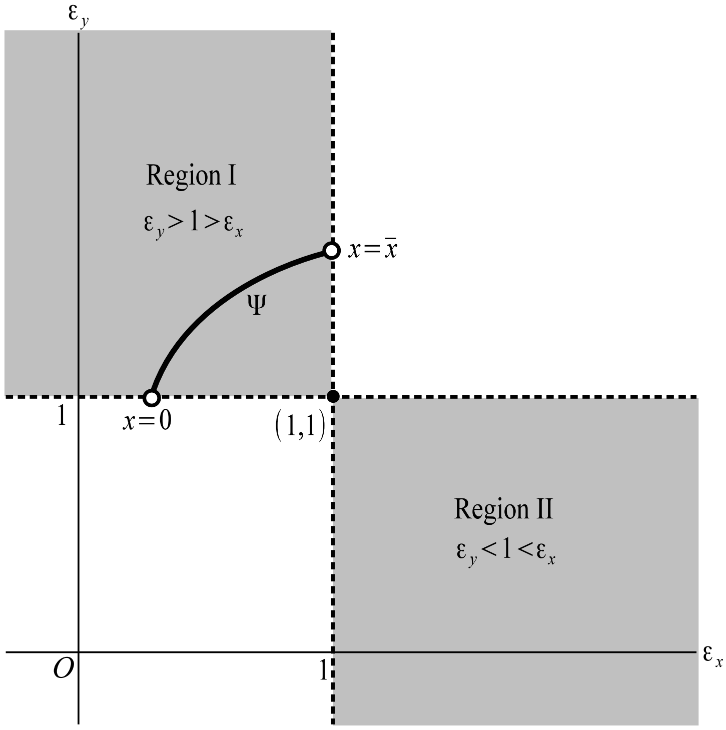

Figure 3: A possible path of $(\varepsilon_x, \varepsilon_y)$

Proposition 13. *Given (A1′), (A2), (A3), (A4) and (15), the sign of* $(\varepsilon_x - \varepsilon_y)$ *remains unchanged when moving along the PPF over* $(0, \tilde{x})$.

In Figure 3, the points in region I satisfy $\varepsilon_y > 1 > \varepsilon_x$; the points in region II satisfy $\varepsilon_y < 1 < \varepsilon_x$; point $(1, 1)$ corresponds with $\varepsilon_x = \varepsilon_y = 1$. The first-order condition together with the linearly homogeneity of $R(v_x, v_y)$ implies that, for any point on the PPF over $(0, \tilde{x})$, $(\varepsilon_x, \varepsilon_y)$ lies in region I, or region II, or at point $(1, 1)$. Proposition 13 moves one step forward by saying that, when moving along the PPF, $(\varepsilon_x, \varepsilon_y)$ must either remain in region I, or remain in region II, or stay at point $(1, 1)$, given certain condition. Figure 3 draws a possible path of $(\varepsilon_x, \varepsilon_y)$, which is labeled Ψ and located in region I.

Third, consider a specific form of $R(v_x, v_y)$ as follows.

$$R(v_x, v_y) = r_1 v_x + r_2 v_y. \tag{16}$$

Note that (16) satisfies both (12) and (15). It then follows directly from Proposition 13 that

Corollary 14. *Given (A1′), (A2), (A3), (A4) and (16), the PPF,* $T(x)$, *is either entirely strictly convex or entirely linear.*

Manning and McMillan's (1979) Proposition 6 implies that the strictly convex and linear intervals could coexist on the PPF in their model. Corollary 14 excludes this possibility.

Finally, consider a special case of (16) as follows.

$$R(v_x, v_y) = v_x + v_y. \tag{17}$$

So far, we do not consider how the total factor use $v \equiv v_x + v_y$ changes along the PPF since it depends on the specific forms of functions. But given (17), $z = v$ and it follows directly from (14) that

$$\frac{dv}{dx} = \frac{dz}{dx} = \underbrace{\frac{G^x G^y}{v|H|}}_{>0} (\varepsilon_y - \varepsilon_x).$$

According to Proposition 13, we have

Corollary 15. *Given (A1′), (A2), (A3), (A4) and (17), the total factor use v (also z) either increases uniformly, or decrease uniformly, or remains unchanged when moving along the PPF over $(0, \tilde{x})$.*

6 Conclusion

The properties of the PPF is an important issue in economic theory. For example, if the PPF of two ex-ante identical economies is convex, both economies can achieve higher efficiencies by specializing and trading with each other. This provides an explanation to the origin of comparative advantages. This study shows that, in the presence of strong input-generated production externalities, the PPF tends to be convex. Note that the assumption of a single factor of production is crucial. If there are more than one factors, the difference in factor intensities between goods works in driving the PPF concave, and the curvature at each point on the PPF depends on which force dominates there.[7] On the other hand, although the model has only two goods, Proposition 2, Proposition 3 and Proposition 4 remain valid even if there are more goods.

[7] Appendix A.12 gives a special case in which the factor intensity is identical among two goods. In such case, the PPF still tends to be convex.

A Appendix

A.1 Proof of Lemma 1

Let $(x', T(x'))$ denote a point on the PPF and $\left(z', v'_x, v'_y\right)$ denote the corresponding factor use and by-product output. Assume to the contrary that $G^x(z')\,v'_x > x'$. Then it is possible to find an input bundle $\left(v''_x, v''_y\right)$ satisfying $v''_x < v'_x$, $v''_y > v'_y$, $R\left(v''_x, v''_y\right) = z'$ and $G^x(z')\,v''_x \geq x'$. Hence (x', y'') is feasible, where $y'' = G^y(z')\,v''_y$. Note that $y'' > G^y(z')\,v'_y = T(x')$, which leads to a contradiction to the definition of $T(x')$.

A.2 Proof of Proposition 2

Note that the envelope theorem is not applicable because we do not assume the differentiability of $G^i(z)$ and $R(v_x, v_y)$. First, prove that $T(x)$ is strictly decreasing over $[0, \tilde{x}]$. There are two possible cases: $\tilde{x} = 0$ and $\tilde{x} > 0$. The case of $\tilde{x} = 0$ is trivial. Thus we can deal with only the case of $\tilde{x} > 0$. Assume to the contrary that there exist two values of $x \in [0, \tilde{x}]$, say x' and x'', so that $x' > x''$ and $T(x') \geq T(x'')$. Note that $x' > 0$ since $x' > x'' \geq 0$, and that $\tilde{x} > x''$ since $\tilde{x} \geq x' > x''$. Then we have $T(x'') > 0$ by the definition of $\tilde{x}$, and have $T(x') > 0$ since $T(x') \geq T(x'')$. Let $\left(v'_x, v'_y\right)$ denote the optimal factor input vector corresponding to x', then $x' = G^x(z')\,v'_x$ and $T(x') = G^y(z')\,v'_y$ where $z' = R\left(v'_x, v'_y\right)$. It follows $x' > 0$ and $T(x') > 0$ that $G^i(z') > 0$ $(i = x, y)$. Let $\left(v''_x, v''_y\right)$ denote the factor input vector so that $x'' = G^x(z')\,v''_x$ and $z' = R\left(v''_x, v''_y\right)$. Since $x' > x''$, we have $v''_x < v'_x$ and thus, by assumption (A2), $v''_y > v'_y$. This means $y'' \equiv G^y(z')\,v''_y > G^y(z')\,v'_y = T(x')$. Since (x'', y'') is a feasible production bundle, we have $T(x'') \geq y''$ by the definition of $T(x)$. This implies $T(x'') > T(x')$ and leads to a contradiction.

Second, prove that $T(x) = 0$ over $(\tilde{x}, \bar{x}]$. There are two cases: $\tilde{x} = \bar{x}$ and $\tilde{x} < \bar{x}$. The case of $\tilde{x} = \bar{x}$ is trivial since then $(\tilde{x}, \bar{x}] = \emptyset$. Thus focus only on the case of $\tilde{x} < \bar{x}$. Assume to the contrary there exists a value of x, say x', so that $x' \in (\tilde{x}, \bar{x}]$ and $T(x') > 0$. By similar procedures, we can show that $T(x) > T(x') > 0$ for any $x < x'$. This leads to a contradiction to the definition of $\tilde{x}$.

A.3 Proof of Proposition 3

The proof proceeds by first examining continuity over $(0, \bar{x}]$ and then moving on to the condition for continuity at $x = 0$.

Continuity over $(0, \bar{x}]$ According to Berge's theorem of the maximum, if the constraint set $C(x)$ defined by (3) is continuous over $(0, \bar{x}]$, then $T(x)$ is also continuous. To prove that $C(x)$ is continuous, we shall check both the upper semi-continuity and the lower semi-continuity of $C(x)$.

As for upper semi-continuity, take a sequence $\{x^n\} \to x' \in (0, \bar{x}]$ so that $x^n \in (0, \bar{x}]$, and take a sequence $\left\{\left(v_x^n, v_y^n\right)\right\} \to \left(v_x^s, v_y^s\right)$ so that $\left(z^n, v_x^n, v_y^n\right) \in C(x^n)$ for all n. By the sequential characterization, if $\left(z^s, v_x^s, v_y^s\right) \in C(x')$, then $C(x)$ is upper semi-continuous.

We first consider the case that $\left\{\left(z^n, v_x^n, v_y^n\right)\right\}$ or its subsequence (for simplicity, we use the same notation here) satisfies $G^x(z^n)\, v_x^n > x^n$, then it is obvious that

$$\exists N \text{ so that } \forall n > N, G^x\left(z^N\right) v_x^N > x',$$

which actually implies that $\left(z^s, v_x^s, v_y^s\right) \in C(x')$.

Second, we consider the case that $\left\{\left(z^n, v_x^n, v_y^n\right)\right\}$ or its subsequence satisfies $G^x(z^n)\, v_x^n = x^n$. Real analysis suggests that one of the following cases necessarily holds: (i) $\{x^n\}$ contains an increasing subsequence $\{x^{n_k}\}$; (ii) $\{x^n\}$ contains a decreasing subsequence $\{x^{n_k}\}$; (iii) $\{x^n\}$ contains both types of subsequences. In case (i), for any n_k and corresponding $\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right) \in C(x^{n_k})$, there are further two situations: $v_y^{n_k} > 0$ and $v_y^{n_k} = 0$. We first discuss the situation of $v_y^{n_k} > 0$. Since $\{x^{n_k}\} \to x'$ and $v_y^{n_k} > 0$, there exist a number N_1 and $\left(z^{n_k}, v_x^{N_1}, v_y^{N_1}\right) \in C(x')$ satisfying $G^x(z^{n_k})\, v_x^{N_1} = x'$, so that $\left\|\left(z^{n_k}, v_x^{N_1}, v_y^{N_1}\right), \left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right)\right\| \leq c_1(x' - x^{n_k})$ for all $n_k > N_1$. It is obvious that $v_x^{N_1} > v_x^{n_k}$ and $v_y^{N_1} < v_y^{n_k}$ for $x' > x^{n_k}$. Assumption (A1) ensures the existence of such constant $c_1 > 0$. The distance from $\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right)$ to $C(x')$ can be defined as follows.

$$d\left(\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right), C(x')\right) \equiv \inf_{\left(z', v_x', v_y'\right) \in C(X')} \left\|\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right), \left(z', v_x', v_y'\right)\right\|.$$

Then we have, for any $n_k > N_1$,

$$d\left(\left(z^{n_k} v_x^{n_k}, v_y^{n_k}\right), C(x')\right) \leq \left\|\left(z^{n_k}, v_x^{N_1}, v_y^{N_1}\right), \left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right)\right\| \leq c_1(x' - x^{n_k}). \tag{18}$$

Now we discuss the situation of $v_y^{n_k} = 0$ in case (i). In this situation, it is impossible to find an element in $C(x')$, as we did previously, by replacing $v_x^{n_k}$ with a larger value without changing z^{n_k}. However, there exists a number N_2 and $\left(z^{n_k}, v_x^{N_2}, 0\right) \in C(x')$ satisfying $x' = G^x(z^{n_k}) v_x^{N_2}$, so that $\left\|\left(z^{n_k}, v_x^{N_2}, 0\right), (z^{n_k}, v_x^{n_k}, 0)\right\| \leq c_1 (x' - x^{n_k})$ for all $n_k > N_2$. Again, (A1) ensures the existence of such constant $c_2 > 0$. The distance from $(z^{n_k}, v_x^{n_k}, 0)$ to $C(x')$ satisfies, for all $n_k > N_2$,

$$d\left((z^{n_k}, v_x^{n_k}, 0), C(x')\right) \leq \left\|\left(z^{n_k}, v_x^{N_2}, 0\right), (z^{n_k}, v_x^{n_k}, 0)\right\| \leq c_1 (x' - x^{n_k}). \tag{19}$$

Equations (18) and (19) together implies that $d\left(\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right), C(x')\right) \to 0$ when $x^{n_k} \to x'$. Hence, $\left\{\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right)\right\} \to \left(z^s, v_x^s, v_y^s\right) \in C(x')$.

The similar method can be applied to case (ii), which is rather simpler since now it is always possible, for sufficiently large n_k, to find an element in $C(x')$ by replacing $v_x^{n_k}$ by a smaller value without changing z^{n_k}. Because we have proved the upper semi-continuity in both case (i) and case (ii), case (iii) becomes trivial and requires no further discussion.

As for lower semi-continuity, take a sequence of $\{x^n\} \to x' \in (0, \bar{x}]$ and a point $\left(z', v_x', v_y'\right) \in C(x')$. By the sequential characterization, if there exists a sequence $\left\{\left(z^n, v_x^n, v_y^n\right)\right\}$ so that $\left(z^n, v_x^n, v_y^n\right) \in C(x^n)$ and $\left\{\left(z^n, v_x^n, v_y^n\right)\right\} \to \left(z', v_x', v_y'\right)$, then $C(x)$ is lower semi-continuous. To show this, we construct a sequence $\left\{\left(v_x^n, v_y^n\right)\right\}$ by letting $v_x^n = a(n) + v_x'$ and $v_y^n = b(n) + v_y'$. Then we choose a number N large enough, so that there exist $a(n)$ and $b(n)$ for all $n > N$ satisfying that $x^n = G^x(z') v_x^n$ and $R\left(v_x^n, v_y^n\right) = z' = R\left(v_x', v_y'\right)$. Hence $\left(z', v_x^n, v_y^n\right) \in C(x^n)$ for all $n > N$.

Note that $x^n = G^x(z') v_x^n = G^x(z')(a(n) + v_x') = x' + G^x(z') a(n)$ for all $n > N$. Thus we obtain $a(n) = (x^n - x') / G^x(z')$ since $G^x(z') > 0$ due to $x' > 0$. This implies that $a(n) \to 0$ when $x^n \to x'$. Furthermore, by assumption (A2) and $R\left(v_x^n, v_y^n\right) = z' = R\left(v_x', v_y'\right)$, $b(n) \to 0$ when $a(n) \to 0$. Therefore, $\left(z', v_x^n, v_y^n\right) \to \left(z', v_x', v_y'\right)$ when $x^n \to x'$.

Continuity at $x = 0$ First note that when $x = 0$, the optimal factor input in good x, $v_x^0 = 0$. Otherwise it is possible, according to (A2), to raise the output of y by reducing v_x and increasing v_y in such a way that z remains unchanged.

As for sufficiency of (6), it follows from (A1) that

$$\forall \varepsilon > 0 \text{ and } \forall z^0 \in Z(0), \exists \sigma > 0 \text{ so that } \forall z \in \left(z^0 - \sigma, z^0 + \sigma\right), \left|T(0) - G^y(z)\, v_y\right| < \frac{\varepsilon}{2}, \tag{20}$$

where v_y satisfies $z = R(0, v_y)$. The sufficient condition implies that there exists $z^{0\prime} \in Z(0)$ satisfying that

$$\forall \sigma > 0, \exists z' \in \left(z^{0\prime} - \sigma, z^{0\prime} + \sigma\right) \text{ so that } G^x(z') > 0. \tag{21}$$

Together with (20), we have

$$\forall \varepsilon > 0, \exists \sigma > 0 \text{ so that } \forall z' \in \left(z^{0\prime} - \sigma, z^{0\prime} + \sigma\right), \left|T(0) - G^y(z')\, v_y'\right| < \frac{\varepsilon}{2},$$

where v_y' satisfies $R\left(0, v_y'\right) = z'$. Provided a small enough value of x, say $x^1 > 0$, there exist $\left(v_x^1, v_y^1\right)$ so that $G^x(z')\, v_x^1 = x^1$ and $R\left(v_x^1, v_y^1\right) = z'$. It follows from the continuity of $R(\cdot, \cdot)$ that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ so that } \forall x' < \delta, \left|G^y(z')\, v_y' - G^y(z')\, v_y^1\right| < \frac{\varepsilon}{2}. \tag{22}$$

By the definition of $T(x)$, we have $T(x') \geq G^y(z')\, v_y^1$. On the other hand, by Proposition 2, we have $T(0) \geq T(x')$. Therefore,

$$|T(0) - T(x')| \leq \left|T(0) - G^y(z')\, v_y^1\right|.$$

By assumption (A2), $v_y^1 < v_y'$ since $v_x^1 > 0$, which means $G^y(z')\, v_y' > G^y(z')\, v_y^1$ and thus

$$\left|T(0) - G^y(z')\, v_y^1\right| = \left|T(0) - G^y(z')\, v_y'\right| + \left|G^y(z')\, v_y' - G^y(z')\, v_y^1\right|.$$

The two expressions imply

$$|T(0) - T(x')| \leq \left|T(0) - G^y(z')\, v_y'\right| + \left|G^y(z')\, v_y' - G^y(z')\, v_y^1\right|. \tag{23}$$

Together with (21) and (22) we obtain

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x < \delta,\ |T(0) - T(x)| < \varepsilon.$$

This establishes the continuity of $T(x)$ at $x = 0$.

As for necessity of (6), it is easier to prove the contrapositive: $T(x)$ is discontinuous at $x = 0$ if

$$\exists \sigma > 0 \text{ so that } \forall z^0 \in Z(0) \text{ and } \forall z \in \left(z^0 - \sigma, z^0 + \sigma\right), G^x(z) = 0.$$

For this purpose, define $m = \inf_z \left(T(0) - G^y(z)\, v_y^0\right)$ where z and v_y^0 satisfy that $z \notin (z^0 - \sigma, z^0 + \sigma)$ for any $z^0 \in Z(0)$ and $R\big(0, v_y^0\big) = z$. It is evident that $m > 0$, otherwise $z \in Z(0)$ and thus $z \in (z^0 - \sigma, z^0 + \sigma)$. For any $\delta > 0$, we can pick a number $x' \in (0, \delta)$. Let $z' \in Z(x')$, then $G^x(z') > 0$ for $x' > 0$, implying $z' \notin (z^0 - \sigma, z^0 + \sigma)$. Let v_y^1 satisfy that $R\left(0, v_y^1\right) = z'$, then

$$\left|T(0) - G^y(z')\, v_y^1\right| \geq m.$$

On the other hand, if we let $\left(v_x', v_y'\right)$ be the corresponding optimal factor inputs, then $G^y(z')\, v_y^1 > G^y\left(z^{x'}\right) v_y' = T(x')$ since $v_y' < v_y^1$ due to $v_x' > 0$. Therefore,

$$|T(0) - T(x')| > \left|T(0) - G^y(z')\, v_y^1\right| \geq m.$$

This implies that

$$\forall \delta > 0 \text{ and } \forall x \in (0, \delta), \exists m > 0 \text{ so that } |T(0) - T(x')| > m,$$

which establishes the discontinuity of $T(x)$ at $x = 0$.

A.4 Proof of Proposition 4

According to Lemma 1, the PPF can be equivalently defined by, instead of (2),

$$y = T(x) \equiv \max_z Y(z, x),$$

where $Y(z, x)$ is the output of good y given z and x, that is, $Y(z, x) = G^y(z)\, v_y$, $x = G^x(z)\, v_x$. There are two cases: $G^y(z) > 0$ and $G^y(z) = 0$. If $G^y(z) > 0$, we have $z = R\left(x/G^x(z), Y(z, x)/G^y(z)\right)$. Given any z, changing x in $\left(x/G^x(z), Y(z, x)/G^y(z)\right)$ gives a locus convex to the origin according to (A2) and the quasi-concavity of $R(v_x, v_y)$. This means that $Y(z, x)$ is a convex function of x. If $G^y(z) = 0$, $Y(z, x) = 0$. In both cases, $Y(z, x)$ is a convex function of x. Since $T(x)$ is the upper envelope of $Y(z, x)$ by changing z and since assumption (A3) ensures that all output bundles on this upper envelope are feasible, $T(x)$ is necessarily convex over $(0, \bar{x}]$.

If $R(v_x, v_y)$ is strictly quasi-concave, following similar arguments we show that $Y(z, x)$ is a strictly convex function of x for any z satisfying $G^y(z) > 0$. Let Ω denote the set of z's such that $Y(z, x)$ contributes to the upper envelope. According to Proposition 2,

$T(x) > 0$ for any $x \in (0, \tilde{x})$, which implies $G^y(z) > 0$ for any $z \in \Omega$. Since the upper envelope of $Y(z, x)$ by changing z within Ω gives $T(x)$, it is necessarily strictly convex over $(0, \tilde{x})$.

A.5 Proof of Proposition 5

By the envelope theorem, on the PPF

$$T'(x) = \frac{\partial \mathcal{L}}{\partial x} = -p = -\frac{G^y}{G^x}.$$

The local convexity of the PPF is then characterized by

$$T''(x) = -\frac{dp}{dx}.$$

Taking the total differentiation of first-order conditions and constraints yields

$$H\left(dp, dv_x, dv_y\right)' = (1, 0, 0)'\, dx. \tag{24}$$

$H \equiv \partial^2 \mathcal{L} / \partial \left(p, v_x, v_y\right)^2$ is the Hessian matrix of $\mathcal{L}$

$$H = \begin{bmatrix} 0 & G^{x\prime} R_x v_x + G^x & G^{x\prime} R_y v_x \\ G^{x\prime} R_x v_x + G^x & R_x^2 \left(B_1 + \frac{R_{xx}}{R_x^2} B_2 + B_3 + B_4\right) & R_x R_y \left(B_1 + \frac{R_{xy}}{R_x R_y} B_2 + B_3\right) \\ G^{x\prime} R_y v_x & R_x R_y \left(B_1 + \frac{R_{xy}}{R_x R_y} B_2 + B_3\right) & R_y^2 \left(B_1 + \frac{R_{yy}}{R_y^2} B_2 + B_3 - B_4\right) \end{bmatrix}$$

where

$$B_1 \equiv p G^{x\prime\prime} v_x + G^{y\prime\prime} v_y,$$
$$B_2 \equiv p G^{x\prime} v_x + G^{y\prime} v_y = -w,$$
$$B_3 \equiv \frac{p G^{x\prime}}{R_x} + \frac{G^{y\prime}}{R_y},$$
$$B_4 \equiv \frac{p G^{x\prime}}{R_x} - \frac{G^{y\prime}}{R_y}.$$

Since $|H| > 0$, by Cramer's rule,

$$\frac{dp}{dx} = \frac{|H_1|}{|H|},$$

where H_i is the matrix formed from H by replacing its i-th column by $(1,0,0)'$. It follows the definition of ε_i and the first-order condition that

$$G^{x\prime}R_x v_x + G^x = -\frac{G^{y\prime}R_y v_y}{G^y}G^x = \left(\varepsilon_y \frac{R_y v_y}{z}\right)G^x,$$
$$G^{x\prime}R_y v_x = \frac{G^{x\prime}R_x v_x}{G^x}\frac{G^x R_y}{R_x} = -\left(\varepsilon_x \frac{R_x v_x}{z}\right)\frac{G^x R_y}{R_x}.$$

Let $\lambda_x \equiv \varepsilon_x R_x v_x/z$, $\lambda_y \equiv \varepsilon_y R_y v_y/z$ and H_{ij} denote the entry in the i-th row and j-th column of H. Then

$$H = \begin{bmatrix} 0 & \lambda_y G^x & -\lambda_x \frac{G^x R_y}{R_x} \\ \lambda_y G^x & H_{22} & H_{23} \\ -\lambda_x \frac{G^x R_y}{R_x} & H_{23} & H_{33} \end{bmatrix}.$$

Do matrix transformation of H while keeping $|H|$ unchanged as follows.

$$|H| \overset{(\text{row } 1\times\frac{1}{G^x R_y},\ \text{col } 1\times\frac{1}{G^x R_y})}{=} (G^x R_y)^2 \begin{vmatrix} 0 & \frac{\lambda_y}{R_y} & -\frac{\lambda_x}{R_x} \\ \frac{\lambda_y}{R_y} & H_{22} & H_{23} \\ -\frac{\lambda_x}{R_x} & H_{23} & H_{33} \end{vmatrix}$$

$$\overset{(\text{row } 2\times R_y,\ \text{col } 2\times R_y)}{=} (G^x)^2 \begin{vmatrix} 0 & \lambda_y & -\frac{\lambda_x}{R_x} \\ \lambda_y & R_y^2 H_{22} & R_y H_{23} \\ -\frac{\lambda_x}{R_x} & R_y H_{23} & H_{33} \end{vmatrix}$$

$$\overset{(\text{row } 3\times R_x,\ \text{col } 3\times R_x)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & \lambda_y & -\lambda_x \\ \lambda_y & R_y^2 H_{22} & R_x R_y H_{23} \\ -\lambda_x & R_x R_y H_{23} & R_x^2 H_{33} \end{vmatrix}.$$

According to (7b), $\lambda_x + \lambda_y = 1$ on the PPF. Therefore,

$$|H| \overset{(\text{row } 2-\text{row } 3)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & \lambda_y & -\lambda_x \\ 1 & R_y^2 H_{22} - R_x R_y H_{23} & R_x R_y H_{23} - R_x^2 H_{33} \\ -\lambda_x & R_x R_y H_{23} & R_x^2 H_{33} \end{vmatrix}$$

$$\overset{(\text{col } 2-\text{col } 3)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & -\lambda_x \\ 1 & R_y^2 H_{22} + R_x^2 H_{33} - 2R_x R_y H_{23} & R_x R_y H_{23} - R_x^2 H_{33} \\ -\lambda_x & R_x R_y H_{23} - R_x^2 H_{33} & R_x^2 H_{33} \end{vmatrix}.$$

For the sake of notation, let $h_{22} \equiv R_y H_{22} + R_x H_{33} - 2R_x R_y H_{23}$, $h_{23} \equiv R_x R_y H_{23} - R_x H_{33}$ and $h_{33} \equiv R_x H_{33}$, then

$$|H| = \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & -\lambda_x \\ 1 & h_{22} & h_{23} \\ -\lambda_x & h_{23} & h_{33} \end{vmatrix} \overset{(\text{col } 3+\text{col } 2\times\lambda_x)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & h_{22} & h_{23}+\lambda_x h_{22} \\ -\lambda_x & h_{23} & h_{33}+\lambda_x h_{23} \end{vmatrix}$$

$$\overset{(\text{row } 3+\text{row } 2\times\lambda_x)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & h_{22} & h_{23}+\lambda_x h_{22} \\ 0 & h_{23}+\lambda_x h_{22} & h_{33}+2\lambda_x h_{23}+\lambda_x^2 h_{22} \end{vmatrix}$$

$$= -\left(\frac{G^x}{R_x}\right)^2 \left(h_{33}+2\lambda_x h_{23}+\lambda_x^2 h_{22}\right).$$

On the other hand, recalling the steps of matrix transformation, we have

$$|H_1| = \begin{vmatrix} H_{22} & H_{23} \\ H_{23} & H_{33} \end{vmatrix} = \frac{1}{(R_x R_y)^2} \begin{vmatrix} h_{22} & h_{23}+\lambda_x h_{22} \\ h_{23}+\lambda_x h_{22} & h_{33}+2\lambda_x h_{23}+\lambda_x^2 h_{22} \end{vmatrix}.$$

Therefore,

$$\frac{|H_1|}{|H|} = -\frac{h_{22}}{(G^x R_y)^2} - \frac{(h_{23}+\lambda_x h_{22})^2}{(R_x R_y)^2 |H|}.$$

Routine calculation gives that

$$\begin{aligned} h_{22} &= (R_x R_y)^2 wQ, \\ h_{23}+\lambda_x h_{22} &= \left[\lambda_x R_y^2 H_{22} - \lambda_y R_x^2 H_{33} + (\lambda_y - \lambda_x) R_x R_y H_{23}\right] \\ &= \frac{(R_x R_y)^2 w}{z} \left[\varepsilon_y (1 - R_y v_y q_2) - \varepsilon_x (1 - R_x v_x q_1)\right]. \end{aligned}$$

Substituting into the expression of $|H_1| / |H|$ yields (10).

A.6 Proof of Proposition 6

On the PPF, we have

$$\begin{aligned} \frac{dz}{dx} &= R_x \frac{dv_x}{dx} + R_y \frac{dv_y}{dx} \\ &= R_x \frac{|H_2|}{|H|} + R_y \frac{|H_3|}{|H|}. \end{aligned}$$

First calculate $|H_2|$,

$$|H_2| = \begin{vmatrix} 0 & 1 & -\lambda_x \frac{G^x R_y}{R_x} \\ \lambda_y G^x & 0 & H_{23} \\ -\lambda_x \frac{G^x R_y}{R_x} & 0 & H_{33} \end{vmatrix} \overset{(\text{col } 1 \times \frac{1}{G^x R_y})}{=} G^x R_y \begin{vmatrix} 0 & 1 & H_{13} \\ \frac{\lambda_y}{R_y} & 0 & H_{23} \\ -\frac{\lambda_x}{R_x} & 0 & H_{33} \end{vmatrix}$$

$$\overset{(\text{row } 2 \times R_y)}{=} G^x \begin{vmatrix} 0 & 1 & H_{13} \\ \lambda_y & 0 & R_y H_{23} \\ -\frac{\lambda_x}{R_x} & 0 & H_{33} \end{vmatrix} \overset{(\text{row } 3 \times R_x)}{=} \frac{G^x}{R_x} \begin{vmatrix} 0 & 1 & H_{13} \\ \lambda_y & 0 & R_y H_{23} \\ -\lambda_x & 0 & R_x H_{33} \end{vmatrix}.$$

Since $\lambda_x + \lambda_y = 1$,

$$|H_2| = \frac{G^x}{R_x^2} \begin{vmatrix} 0 & 1 & R_x H_{13} \\ \lambda_y & 0 & R_x R_y H_{23} \\ -\lambda_x & 0 & R_x^2 H_{33} \end{vmatrix} \overset{(\text{row } 2 - \text{row } 3)}{=} \frac{G^x}{R_x^2} \begin{vmatrix} 0 & 1 & R_x H_{13} \\ 1 & 0 & R_x R_y H_{23} - R_x^2 H_{33} \\ -\lambda_x & 0 & R_x^2 H_{33} \end{vmatrix}$$

$$\overset{(\text{row } 3 + \text{row } 2 \times \lambda_x)}{=} \frac{G^x}{R_x^2} \begin{vmatrix} 0 & 1 & R_x H_{13} \\ 1 & 0 & R_x R_y H_{23} + R_x^2 H_{33} \\ 0 & 0 & \lambda_x R_x R_y H_{23} + \lambda_y R_x^2 H_{33} \end{vmatrix} = -\frac{G^x}{R_x^2} \left(\lambda_x R_x R_y H_{23} + \lambda_y R_x^2 H_{33} \right).$$

$|H_3|$ can be calculated in a similar way,

$$|H_3| = \frac{G^x}{R_x} \begin{vmatrix} 0 & H_{12} & 1 \\ \lambda_y & R_y H_{22} & 0 \\ -\lambda_x & R_x H_{23} & 0 \end{vmatrix} \overset{(\text{col } 2 \times R_y)}{=} \frac{G^x}{R_x R_y} \begin{vmatrix} 0 & R_y H_{12} & 1 \\ \lambda_y & R_y^2 H_{22} & 0 \\ -\lambda_x & R_x R_y H_{23} & 0 \end{vmatrix}$$

$$\overset{(\text{row } 2 - \text{row } 3)}{=} \frac{G^x}{R_x R_y} \begin{vmatrix} 0 & R_y H_{12} & 1 \\ 1 & R_y^2 H_{22} - R_x R_y H_{23} & 0 \\ -\lambda_x & R_x R_y H_{23} & 0 \end{vmatrix}$$

$$\overset{(\text{row } 3 + \text{row } 2 \times \lambda_x)}{=} \frac{G^x}{R_x R_y} \begin{vmatrix} 0 & R_y H_{12} & 1 \\ 1 & R_y^2 H_{22} - R_x^2 H_{23} & 0 \\ 0 & \lambda_x R_y^2 H_{22} + \lambda_y R_x R_y H_{23} & 0 \end{vmatrix} = \frac{G^x}{R_x R_y} \left(\lambda_x R_y^2 H_{22} + \lambda_y R_x R_y H_{23} \right).$$

Therefore,

$$\begin{aligned} \frac{dz}{dx} &= \frac{G^x}{R_x |H|} \left[\lambda_x R_y^2 H_{22} - \lambda_y R_x^2 H_{33} + (\lambda_y - \lambda_x) R_x R_y H_{23} \right] \\ &= \frac{G^x R_x R_y^2 w}{z |H|} \left[\varepsilon_y (1 - R_y v_y q_2) - \varepsilon_x (1 - R_x v_x q_1) \right]. \end{aligned}$$

Substituting $w = G^y / R_y$ for w yields the result.

A.7 Proof of Lemma 7

The quasi-concavity $G^i(z)$ means that $\{z; G^i(z) > 0\}$ is an open connected interval, which is convex. As for $G^i(z)\, v_i$, note that $G^i(z)\, v_i > 0$ is equivalent to $G^i(z) > 0$ and $v_i > 0$ since $G^i(z) \geq 0$, i.e., $\{(z, v_i); G^i(z)\, v_i > 0\} = \{(z, v_i); G^i(z) > 0\} \cap \{(z, v_i); v_i > 0\}$, which is clearly an open convex set, too.

A.8 Proof of Lemma 8

The convexity of $1/G^i(z)$ is defined by, for any $z', z'' \in \{z; G^i(z) > 0\}$,

$$(z' - z'')\frac{d}{dz}\left(\frac{1}{G^i(z'')}\right) \leq \frac{1}{G^i(z')} - \frac{1}{G^i(z'')},$$

which can be rewritten into

$$(z' - z'')\, G^{i\prime}(z'') + G^i(z'')\left(\frac{G^i(z'')}{G^i(z')} - 1\right) \geq 0. \tag{25}$$

The pseudo-concavity of $G^i(z)\, v_i$ is defined by, for any $(z', v_i'), (z'', v_i'') \in \{(z, v_i); G^i(z)\, v_i > 0\}$,

$$G^i(z')\, v_i' > G^i(z'')\, v_i'' \Rightarrow [(z', v_i') - (z'', v_i'')]\, \nabla\left(G^i(z'')\, v_i''\right) > 0,$$

which can be simplified into

$$G^i(z')\, v_i' > G^i(z'')\, v_i'' \Rightarrow (z' - z'')\, G^{i\prime}(z'') + G^i(z'')\left(\frac{v_i'}{v_i''} - 1\right) > 0. \tag{26}$$

Note that

$$G^i(z')\, v_i' > G^i(z'')\, v_i'' \Leftrightarrow G^i(z'')\left(\frac{v_i'}{v_i''} - 1\right) > G^i(z'')\left(\frac{G^i(z'')}{G^i(z')} - 1\right). \tag{27}$$

Because (25) and (27) together imply (26), the convexity of $1/G^i(z)$ implies that $G^i(z)\, v_i$ is pseudo-concave.

Similarly, the strict convexity of $1/G^i(z)$ is defined by, provided $z' \neq z''$,

$$(z' - z'')\frac{d}{dz}\left(\frac{1}{G^i(z'')}\right) < \frac{1}{G^i(z')} - \frac{1}{G^i(z'')},$$

which can be rewritten into

$$(z' - z'')\, G^{i\prime}(z'') + G^i(z'')\left(\frac{G^i(z'')}{G^i(z')} - 1\right) > 0. \tag{28}$$

The strict pseudo-concavity of $G^i(z)\, v_i$ is defined by, provided $(z', v_i') \neq (z'', v_i'')$,

$$G^i(z')\, v_i' \geq G^i(z'')\, v_i'' \Rightarrow [(z', v_i') - (z'', v_i'')]\, \nabla \left(G^i(z'')\, v_i''\right) > 0,$$

which can be simplified into

$$G^i(z')\, v_i' \geq G^i(z'')\, v_i'' \Rightarrow (z' - z'')\, G^{i\prime}(z'') + G^i(z'') \left(\frac{v_i'}{v_i''} - 1\right) > 0. \tag{29}$$

There are two cases. If $z' \neq z''$, then (29) follows from (28). If $z' = z''$, it follows from $G^i(z')\, v_i' \geq G^i(z'')\, v_i''$ that $v_i' \geq v_i''$. Moreover, $z' = z''$ and $(z', v_i') \neq (z'', v_i'')$ together imply $v_i' \neq v_i''$. Hence we have $v_i' > v_i''$, which again means that (29) holds. This completes the proof.

A.9 Proof of Lemma 9

For concreteness, we prove the case of $i = x$. The case of $i = y$ can be proved similarly. Routine calculation gives that

$$\left[\left(v_x', v_y'\right) - \left(v_x'', v_y''\right)\right] \nabla \left(G^x\left(R\left(v_x'', v_y''\right)\right) v_x''\right) = A v_x'' + G^{x\prime}(z'')\, v_x'' B,$$

where

$$\begin{aligned}
&z' \equiv R\left(v_x', v_y'\right), z'' \equiv R\left(v_x'', v_y''\right), \\
&A \equiv (z' - z'')\, G^{x\prime}(z'') + G^x(z'') \left(\frac{v_x'}{v_x''} - 1\right), \\
&B \equiv (v_x' - v_x'')\, R_x\left(v_x'', v_y''\right) + \left(v_y' - v_y''\right) R_y\left(v_x'', v_y''\right) - (z' - z'').
\end{aligned}$$

Note that $B \leq 0$ since $R(v_x, v_y)$ is convex, while $G^{x\prime}(z'')$ could be either negative or non-negative. The pseudo-concavity of $G^x(z)\, v_x$ means that

$$G^x(z')\, v_x' > G^x(z'')\, v_x'' \Rightarrow A > 0.$$

If $G^{x\prime}(z'')$ is negative, then $G^{x\prime}(z'')\, v_x'' B \geq 0$. This simply implies that

$$G^x\left(R\left(v_x', v_y'\right)\right) v_x' > G^x\left(R\left(v_x'', v_y''\right)\right) v_x'' \Rightarrow A v_x'' + G^{x\prime}(z'')\, v_x'' B > 0,$$

which says that $G^x(R(v_x, v_y))\, v_x$ is pseudo-concave with respect to (v_x, v_y). If $G^{x\prime}(z'')$ is non-negative, then $G^{x\prime}(z'')\, v_x'' B \leq 0$. The alternative definition of pseudo-concavity requires

$$G^x(z')\, v_x' < G^x(z'')\, v_x'' \Rightarrow A < 0,$$

which implies

$$G^x\left(R\left(v_x',v_y'\right)\right)v_x' < G^x\left(R\left(v_x'',v_y''\right)\right)v_x'' \Rightarrow Av_x'' + G^{x\prime}\left(z''\right)v_x''B < 0.$$

This is equivalent to saying that $G^x\left(R\left(v_x,v_y\right)\right)v_x$ is pseudo-concave.

Now consider a strictly pseudo-concave $G^x\left(z\right)v_x$. The strict pseudo-concavity requires, provided that $\left(z',v_x'\right)\neq\left(z'',v_x''\right)$,

$$G^x\left(z'\right)v_x' \geq G^x\left(z''\right)v_x'' \Rightarrow A > 0.$$

The properties of $R\left(v_x,v_y\right)$, characterized by (A1) and (A2), ensure

$$\left(z',v_x'\right)\neq\left(z'',v_x''\right)\Leftrightarrow\left(v_x',v_y'\right)\neq\left(v_x'',v_y''\right).$$

Hence, provided that $\left(v_x',v_y'\right)\neq\left(v_x'',v_y''\right)$, if $G^{x\prime}\left(z''\right)$ is negative, then

$$G^x\left(R\left(v_x',v_y'\right)\right)v_x' \geq G^x\left(R\left(v_x'',v_y''\right)\right)v_x'' \Rightarrow Av_x'' + G^{x\prime}\left(z''\right)v_x''B > 0,$$

which says that $G^x\left(R\left(v_x,v_y\right)\right)v_x$ is strictly pseudo-concave with respect to $\left(v_x,v_y\right)$. If $G^{x\prime}\left(z''\right)$ is non-negative, then we can obtain the same conclusion by using the alternative definition of strict pseudo-concavity.

A.10 Proof of Proposition 10

Define

$$C'\left(x\right)\equiv\left\{\left(z,v_x,v_y\right);G^x\left(z\right)v_x-x\geq 0,z-R\left(v_x,v_y\right)\geq 0\right\}. \tag{30}$$

The constraint $C\left(x\right)$ of problem (2) can be replaced by $C'\left(x\right)$ without changing the solution set. This is because $z-R\left(v_x,v_y\right)\geq 0$ is binding in optimum and $v_x+v_y\leq E$ is not. Since $1/G^x\left(z\right)$ is convex, by Lemma 8, $G^x\left(z\right)v_x$ is pseudo-concave (thus quasi-concave) with respect to $\left(z,v_x\right)$, and thus with respect to $\left(z,v_x,v_y\right)$ as well. Since $R\left(v_x,v_y\right)$ is convex, $z-R\left(v_x,v_y\right)$ is concave. On the other hand, since $1/G^y\left(z\right)$ is convex, the objective function $G^y\left(z\right)v_y$ is pseudo-concave (thus quasi-concave) with respect to $\left(z,v_y\right)$, and thus with respect to $\left(z,v_x,v_y\right)$ as well. According to quasi-convex programming, the solution set $S\left(x\right)$ is convex.

On the other hand, the maximization problem (2) can be rewritten into

$$T\left(x\right)\equiv\max_{\left(v_x,v_y\right)\in C''\left(x\right)}G^y\left(R\left(v_x,v_y\right)\right)v_y,$$

where

$$C''(x) \equiv \{(v_x, v_y); G^x(R(v_x, v_y))v_x - x \geq 0\}.$$

Given that $1/G^i(z)$ $(i = x, y)$ and $R(v_x, v_y)$ are convex, and that either $1/G^x(z)$ or $1/G^y(z)$ is strictly convex, according to Lemma 8 and 9, $G^i(R(v_x, v_y))v_i$ $(i = x, y)$ is pseudo-concave and either one of them is strictly pseudo-concave. According to the results from quasi-convex programming, there is a unique optimal (v_x, v_y), denoted by (v_x^*, v_y^*), and the corresponding level of by-product is $z^* = R(v_x^*, v_y^*)$. Thus $S(x)=\{(z^*, v_x^*, v_y^*)\} = (z(x), v_x(x), v_y(x))$. It follows the implicit-function theorem that $S(x)$ is continuously differentiable.

A.11 Proof of Proposition 13

Define the set of good i's sensitivity at x on the PPF as

$$\varepsilon_i(x) \equiv \{\varepsilon_i; z \in Z(x)\},$$

and the set of the sensitivity bundles $(\varepsilon_x, \varepsilon_y)$ on the PPF over $(0, \tilde{x})$ as

$$\Psi \equiv \{(\varepsilon_x, \varepsilon_y); \varepsilon_x \in \varepsilon_x(x), \varepsilon_y \in \varepsilon_y(x), x \in (0, \tilde{x})\}.$$

Because $R(v_x, v_y)$ is linearly homogeneous and quasi-convex, it is also convex. According to Proposition 10, the solution set $S(x)$ is convex over $(0, \tilde{x})$. Thus $\varepsilon_i(x)$ is a connected set for any $x \in (0, \tilde{x})$. On the other hand, the constraint set $C(x)$ is continuous over $(0, \tilde{x})$ as shown in Appendix A.3. According to the theorem of the maximum, the solution set $S(x)$ is upper semi-continuous over $(0, \tilde{x})$, and so is $\varepsilon_i(x)$. Therefore, Ψ is also a connected set on the $(\varepsilon_x, \varepsilon_y)$ plane.

As given in (7b), $\varepsilon_x\theta_x + \varepsilon_y\theta_y = 1$, where $\theta_i \equiv R_i v_i/z$ satisfies $\theta_x + \theta_y = 1$ because of the linear homogeneity of $R(v_x, v_y)$. So at any point on the PPF either $\varepsilon_x > 1 > \varepsilon_y$, $\varepsilon_x = \varepsilon_y = 1$, or $\varepsilon_x < 1 < \varepsilon_y$ holds. Since Ψ is a connected set, if the sign of $(\varepsilon_x - \varepsilon_y)$ changes when moving along the PPF, then $\varepsilon_x > 1 > \varepsilon_y$, or $\varepsilon_x = \varepsilon_y = 1$, and $\varepsilon_x < 1 < \varepsilon_y$ must coexist on the PPF. This is impossible. To see this, let z' denote the value of z so that $\varepsilon_x = \varepsilon_y = 1$. Using this z', we can obtain a relationship between x and y as follows.

$$G^x(z')z' = R\left(x, \frac{G^x(z')}{G^y(z')}y\right), \tag{31}$$

where the linear homogeneity of $R(v_x, v_y)$ is used. Let $y = Y(x, z')$ denote the relationship (31). Clearly $(x, Y(x, z'))$ is feasible for any $x \in (0, \tilde{x})$. In the following, we shall show that $y = Y(x, z')$ is exactly the expression for the PPF, i.e. $T(x) = Y(x, z')$.

Assume to the contrary that there exists a feasible output bundle (x'', y'') lying outward to $y = Y(x, z')$. Let z'' denote the value of z corresponding with (x'', y''), then we have

$$G^x(z'')\, z'' = R\left(x'', \frac{G^x(z'')}{G^y(z'')} y''\right). \tag{32}$$

According to assumption (A2), $\partial Y(x, z')/\partial x < 0$. Hence, we find a point lying on $y = Y(x, z')$, say (x''', y'''), so that

$$x''' \left(\frac{G^x(z')}{G^y(z')} y'''\right)^{-1} = x'' \left(\frac{G^x(z'')}{G^y(z'')} y''\right)^{-1}. \tag{33}$$

Because (x'', y'') lies outward to $y = Y(x, z')$, we have $x''' < x''$. Since (x''', y''') lies on $y = Y(x, z')$,

$$G^x(z')\, z' = R\left(x''', \frac{G^x(z')}{G^y(z')} y'''\right).$$

Hence we have $G^x(z'')\, z'' > G^x(z')\, z'$ according to the linear homogeneity of $R(v_x, v_y)$.

On the other hand, according to Lemma 9, $G^i(z)\, v_i$ is pseudo-concave if $1/G^i(\cdot)$ is convex. This simply means, noting that $G^i(z)\, z$ can be obtained by letting $z = v_i$ in $G^i(z)\, v_i$, that $G^i(z)\, z$ is pseudo-concave as well. One of the properties of pseudo-concavity is that $G^i(z)\, z$ attains a global maximum when $d(G^i(z)\, z)/dz = 0$, i.e. $\varepsilon_i = 1$. Since $\varepsilon_x = \varepsilon_y = 1$ when $z = z'$, $G^i(z')\, z' \geq G^x(z'')\, z''$, this leads to a contradiction. Hence, there is no feasible output bundle lying outside of $y = Y(x, z')$, which means $T(x) = Y(x, z')$. Furthermore, along the PPF we have $z = z'$ and $\varepsilon_x = \varepsilon_y = 1$.

A.12 Two Factors: A Special Case

Here, instead of focusing on the general multi-factor case to derive the detailed condition for the PPF to be convex or concave, we examine a special two-factor model in which the factor intensity is identical between two goods. We begin by writing down the general two-good, two-factor model:

$$x = G^x(z) F^x(v_{1x}, v_{2x}), \tag{34a}$$

$$y = G^y(z) F^y(v_{1y}, v_{2y}), \tag{34b}$$

$$z = R(v_{1x}, v_{1y}, v_{2x}, v_{2y}), \tag{34c}$$

where v_{ji} $(j = 1, 2; i = x, y)$ is the use of factor j in good i. $R(v_{1x}, v_{1y}, v_{2x}, v_{2y})$ describes the relationship between the output of by-product and factor uses. $F^i(v_{1i}, v_{2i})$ has the standard properties of a Neoclassical production function that reflect the contribution of factors.

The PPF is defined by the following maximum value function

$$y = T(x) \equiv \max_{z, v_{1y}, v_{2y}} G^y(z) F^y(v_{1y}, v_{2y}), \tag{35}$$

subject to $G^x(z) F^x(v_{1x}, v_{2x}) = x$ and $z = R(v_{1x}, v_{1y}, v_{2x}, v_{2y})$. Again, assume that the factor constraint is slack on the PPF. It follows from the first-order conditions that

$$\frac{F_1^x}{F_2^x} = \frac{F_1^y}{F_2^y}.$$

Assume that two goods share the same factor intensity, i.e., F^x and F^y satisfy

$$\frac{F_1^x}{F_2^x} = \frac{F_1^y}{F_2^y} \text{ if } \frac{v_{1x}}{v_{2x}} = \frac{v_{1y}}{v_{2y}}. \tag{36}$$

Then, according to the first-order condition, $v_{1x}/v_{2x} = v_{1y}/v_{2y}$ on the PPF. Let c denote this ratio, then we have $v_{1x} = cv_{2x}$ and $v_{1y} = cv_{2y}$. The PPF can be expressed equivalently as

$$T(x) \equiv \max_c t(x, c),$$

where

$$t(x, c) \equiv \max_{z, v_{2y}} G^y(z) F^y(c, 1) v_{2y}, \tag{37}$$

subject to $G^x(z) F^x(c, 1) v_{2x} = x$ and $z = R(cv_{2x}, cv_{2y}, v_{2x}, v_{2y})$. For convenience, let $r(v_{2x}, v_{2y}) \equiv R(cv_{2x}, cv_{2y}, v_{2x}, v_{2y})$.

Clearly, the problem (37) is the single-factor case with a constant c. From Proposition 4, $t(x, c)$ is convex with respect to x, given that $r(v_{2x}, v_{2y})$ is quasi-concave with respect to (v_{2x}, v_{2y}). Since the upper envelope of $t(x, c)$ by changing c constructs $T(x)$, $T(x)$

is also convex. Therefore, in the two-factor case and given assumptions (A1′), (A2), (A3) and (A4), the PPF is convex if two goods share the identical factor intensity and $R(v_{1x}, v_{1y}, v_{2x}, v_{2y})$ is quasi-concave.

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